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10 January 2018

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January 10, 2018

Abstract

Using a notion of convexity in discrete convex analysis, we introduce a discrete analogue of the Kuhn-Tucker theorem. We apply it to an auction model and show that existing iterative auctions can be viewed as the process of finding a saddle point of the Lagrange function.

JEL classification: C78, D44

1 Introduction

Economists often encounter a maximization problem under constraints. To solve this problem, the Kuhn-Tucker theorem (henceforth KT theorem) is a fundamental mathematical tool. This theorem is applicable to functions with continuous variables, but recent economic problems often deal with discrete variables. Examples include iterative auctions (see Cramton et al. (2006) for a survey) and matching problems (see Roth and Sotomayor (1990) and Kojima (2015) for surveys). The purpose of the present paper is to introduce a discrete analogue of the KT theorem.

The key idea of the KT theorem is to translate a solution to the maximization problem under constraints into a saddle point of the Lagrange function. This translation is possible if both the objective and the constraint functions satisfy the convexity assumption. To describe a convexity assumption in discrete settings, we utilize the notion of $M^\natural$-concavity in discrete convex analysis (Murota 2003). We first consider a maximization problem under constraints with discrete variables where the objective and constraint functions are $\mathrm{M}^\natural$-concave. It

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turns out that a discrete version of the KT theorem does not hold for this problem. To overcome this difficulty, we make additional assumptions on constraint functions. Our main result shows that the discrete KT theorem holds under the following two conditions: (i) the objective function is $\mathrm{M}^\natural$-concave; and (ii) the set of the constraint functions is a *hierarchical set of affine functions*. More specifically, the constraint functions are of the form $x \mapsto a - b \cdot x$, with a being a constant and b being a characteristic vector, and the characteristic vectors form a discrete structure called *hierarchy*. The notion of a hierarchy was previously employed by Budish et al. (2013), who considered the problme of randomly allocating indivisible items under constraints. They prove that if the constraints have a hierarchical structure, any random allocation is implementable. They also provide real-world examples in which the constraints have a hierarchical structure. We reinforce the advantage of a hierarchy by showing that it is sufficient to recover a discrete KT theorem.

We apply the discrete KT theorem to an auction model of Gul and Stacchetti (1999).[1] Consider the problem of maximizing the sum of utilities of agents under the constraint that each item has a single unit. We show that the Lagrange function corresponding to this problem coincides with the Lyapunov function proposed by Ausubel (2006). In particular, the competitive price vectors appear as a solution to minimizing the Lagrange function, i.e., a Lagrangian multiplier. Our result provides a mathematical foundation to Ausubel's (2006) auction, which proceeds by minimizing the Lyapunov function. Moreover, as many existing iterative auctions can be embedded into Ausubel's (2006) auction (see Murota et al. (2016)), our result provides a unified approach to existing auctions.

The rest of the paper is organized as follows. Section 2 presents preliminaries. Section 3 presents the KT theorem for continuous and discrete cases. Section 4 presents an application of the discrete KT theorem to auctions. Section 5 concludes. All proofs are provided in Section 6.

2 Preliminaries

Let K be a finite set. Let $\mathbb{R}^K$ denote the real vector space indexed by the elements in K. Let $\mathbb{Z}^K \subseteq \mathbb{R}^K$ be the set of vectors with integer coordinates. For a function $f : \mathbb{R}^K \to \mathbb{R} \cup \{-\infty\}$, we define the **effective domain of f** by

$$\mathrm{dom} f = \{x \in \mathbb{R}^K : f(x) > -\infty\}.$$

[1]This model is a special case of the Kelso and Crawford (1982) model of job matching.

For each $A \subseteq K$, let $\chi_A \in \{0,1\}^K$ denote the **characteristic vector of $\boldsymbol{A}$**, i.e.,

$$(\chi_A)_k = \begin{cases} 1 & \text{if } k \in A, \\ 0 & \text{otherwise.} \end{cases}$$

For a singleton set $\{k\} \subseteq K$, we write χ_k for $\chi_{\{k\}}$.

For $x \in \mathbb{Z}^K$, we define

$$\mathrm{supp}^+ x = \{k \in K : x_k > 0\}, \ \mathrm{supp}^- x = \{k \in K : x_k < 0\}.$$

A function $f : \mathbb{R}^K \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is **concave** if, for any $x, y \in \mathbb{R}^K$ and $\lambda \in (0,1)$, we have

$$\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x + (1-\lambda) y).$$

A function $f : \mathbb{Z}^K \to \mathbb{Z} \cup \{-\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is **M$^\natural$-concave** (Murota 2003) if, for any $x, y \in \mathbb{Z}^K$ and $k \in \mathrm{supp}^+(x-y)$, we have

(i) $f(x) + f(y) \leq f(x - \chi_k) + f(y + \chi_k)$, or

(ii) there exists $\ell \in \mathrm{supp}^-(x-y)$ such that $f(x) + f(y) \leq f(x - \chi_k + \chi_\ell) + f(y + \chi_k - \chi_\ell)$.

For an interpretation of M$^\natural$-concavity, see Section 3 of Kojima et al. (2017).

3 Kuhn-Tucker theorem

For $X \subseteq \mathbb{R}^K$, let $ri(X)$ denote the relative interior of X. We begin with the KT theorem with continuous variables.

Theorem (Kuhn-Tucker theorem). *Let $f : \mathbb{R}^K \to \mathbb{R} \cup \{-\infty\}$ be a concave function and $g_1, \ldots, g_q : \mathbb{R}^K \to \mathbb{R}$ be concave functions. Suppose there exists $x \in \mathbb{R}^K$ such that*

$$x \in ri(dom f) \text{ and } g_j(x) > 0 \text{ for all } j = 1, \ldots, q.$$

Then, for $x^* \in \mathbb{R}^K$, the following are equivalent:

1. *x^* is a solution to $\max f(x)$ subject to $g_j(x) \geq 0$ for all $j = 1, \ldots, q$.*

2. *There exists $(\lambda_1^*, \ldots, \lambda_q^*) \in \mathbb{R}_+^q$ such that*

$$L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda) \text{ for all } x \in \mathbb{R}^K, \lambda \in \mathbb{R}_+^q,$$

where $L(\cdot,\cdot) : \mathbb{R}^K \times \mathbb{R}^q_+ \to \mathbb{R} \cup \{-\infty\}$ *is given by*

$$L(x,\lambda) = f(x) + \sum_{j=1}^{q} \lambda_j g_j(x) \textit{ for all } x \in \mathbb{R}^K, \lambda \in \mathbb{R}^q_+.$$

Keeping this theorem in mind, a discrete KT theorem is a statement that replaces all of the "$\mathbb{R}$" with "$\mathbb{Z}$", and "concave" with some notion of discrete concavity. One may consider assuming that f and $g_1, \ldots, g_q$ are M$^\natural$-concave, but this does not work, as shown in the next counterexample.

Example 1. Let $K = \{1,2,3\}$ and $f(x) = x_1 + x_2 + x_3$ for all $x \in \mathbb{Z}^K$. Consider three constraint functions given by

$$g_1(x) = 2 - x_1 - x_2,\ g_2(x) = 2 - x_1 - x_3,\ g_3(x) = 2 - x_2 - x_3 \text{ for all } x \in \mathbb{Z}^K.$$

We remark that f and g_1, g_2, g_3 are M$^\natural$-concave.

One easily verifies that $x^* = (1,1,1)$ is a solution to $\max f(x)$ subject to $g_j(x) \geq 0$ for $j = 1,2,3$. Consider the Lagrange function: for all $x \in \mathbb{Z}^K_+$ and $\lambda \in \mathbb{Z}^3_+$,

$$L(x,\lambda) = x_1 + x_2 + x_3 + \lambda_1(2 - x_1 - x_2) + \lambda_2(2 - x_1 - x_3) + \lambda_3(2 - x_2 - x_3). \tag{1}$$

Suppose there exists $\lambda^* \in \mathbb{Z}^3_+$ such that $L(x,\lambda^*) \leq L(x^*,\lambda^*)$ for all $x \in \mathbb{Z}^K_+$. This is true only if the coefficients for x_1, x_2, x_3 in (1) are 0, i.e.,

$$1 - \lambda^*_1 - \lambda^*_2 = 0,\ 1 - \lambda^*_1 - \lambda^*_3 = 0,\ 1 - \lambda^*_2 - \lambda^*_3 = 0.$$

This is true only if $\lambda^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, a contradiction to $\lambda^* \in \mathbb{Z}^3_+$. □

This example illustrates that a discrete KT theorem does not hold, even if each constraint function $g_j(\cdot)$ is given by an affine function $g_j(x) = a - x \cdot \chi_A$ for some $a \in \mathbb{Z}$ and $A \subseteq K$. We further restrict the class of constraint functions by using a concept in the combinatorial optimization literature.

We say that $\mathcal{A} \subseteq 2^K$ is a **hierarchy** if, for every pair of elements A and A' in $\mathcal{A}$, we have $A \subseteq A'$ or $A' \subseteq A$ or $A \cap A' = \emptyset$.[2] We say that a set of functions $g_1, \ldots, g_q : \mathbb{Z}^K \to \mathbb{Z}$ is a **hierarchical set of affine functions** if

1. For each $j = 1, \ldots, q$, there exist $a_j \in \mathbb{Z}$ and $A_j \subseteq K$ with $A_j \neq \emptyset$ such that $g_j(x) = a_j - x \cdot \chi_{A_j}$ for all $x \in \mathbb{Z}^K$; and

2. The set $\{A_j : j = 1, \ldots, q\}$ is a hierarchy.

[2]Hierarchies are also called laminar families in the literature.

The constraint functions in Example 1 violate this condition because $\{\{1,2\},\{1,3\},\{2,3\}\}$ is not a hierarchy. We are in a position to state our main result.

Theorem 1 (Discrete Kuhn-Tucker theorem)**.** *Let $f : \mathbb{Z}^K \to \mathbb{Z} \cup \{-\infty\}$ be an $M^\natural$-concave function and $g_1, \ldots, g_q : \mathbb{Z}^K \to \mathbb{Z}$ be a hierarchical set of affine functions. Then, for $x^* \in \mathbb{Z}^K_+$, the following are equivalent:*

1. *x^* is a solution to $\max f(x)$ subject to $g_j(x) \geq 0$ for all $j = 1, \ldots, q$.*
2. *There exists $(\lambda_1^*, \ldots, \lambda_q^*) \in \mathbb{Z}^q_+$ such that*

$$L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda) \text{ for all } x \in \mathbb{Z}^K, \lambda \in \mathbb{Z}^q_+, \tag{2}$$

where $L(\cdot,\cdot) : \mathbb{Z}^K \times \mathbb{Z}^q_+ \to \mathbb{Z} \cup \{-\infty\}$ is given by

$$L(x, \lambda) = f(x) + \sum_{j=1}^{q} \lambda_j g_j(x) \text{ for all } x \in \mathbb{Z}^K, \lambda \in \mathbb{Z}^q_+. \tag{3}$$

The proof is provided in Section 6.

4 Application to an auction model

We apply the discrete KT theorem to an auction model. Let K be a set of items and N be a set of agents. Each agent $i \in N$ has a valuation function $v_i : \{0,1\}^K \to \mathbb{Z}$; we identify a subset of items $A \subseteq K$ with a characteristic vector $\chi_A \in \{0,1\}^K$.

For each $i \in N$, we define the **demand correspondence** $D_i : \mathbb{R}^K_+ \to \{0,1\}^K$ by

$$D_i(p) = \left\{x \in \{0,1\}^K : v_i(x) - p \cdot x \geq v_i(y) - p \cdot y \text{ for all } y \in \{0,1\}^K\right\} \text{ for all } p \in \mathbb{R}^K_+.$$

We say that v_i is **monotonic** if for any $A, A' \subseteq K$ with $A \subseteq A'$, we have $v(\chi_A) \leq v(\chi_{A'})$. We say that v_i satisfies the **gross substitutes condition** (Kelso and Crawford 1982) if for any $p, q \in \mathbb{R}^K_+$ with $p \leq q$ and $x \in D_i(p)$, there exists $y \in D_i(q)$ such that $x_k \leq y_k$ if $p_k = q_k$.

Lemma 1 (Fujishige and Yang 2003)**.** *Suppose v is monotonic. Then v satisfies the gross substitutes condition if and only if v is $M^\natural$-concave.*

For each $i \in N$, we define the **indirect utility function** $V_i : \mathbb{R}^K_+ \to \mathbb{Z}$ by

$$V_i(p) = \max_{x \in \{0,1\}^K} \left(v_i(x) - p \cdot x\right) \text{ for all } p \in \mathbb{R}^K_+$$

We define the **Lyapunov function** (Ausubel 2006) $L : \mathbb{Z}^K_+ \to \mathbb{Z}$ by

$$L(p) = \sum_{i \in N} V_i(p) + p \cdot \chi_K \text{ for all } p \in \mathbb{Z}^K_+.$$

We consider the set of $|N \times K|$-dimensional 0-1 vectors $\{0,1\}^{N\times K}$. For $x \in \{0,1\}^{N\times K}$ and $(i,k) \in N \times K$, $x_{(i,k)} = 1$ is intended to mean that agent i consumes one unit of item k. For $x \in \{0,1\}^{N\times K}$ and $i \in N$, let x_i denote the projection of x on $\{0,1\}^{\{i\}\times K}$.

We define $f : \mathbb{Z}^K \to \mathbb{Z}$ by

$$f(x_1, \dots, x_n) = \begin{cases} \sum_{i \in N} v_i(x_i) & \text{if } (x_1, \dots, x_n) \in \{0,1\}^{N\times K}, \\ -\infty & \text{otherwise.} \end{cases}$$

As recognized in the literature, if v_i is M$^\natural$-concave for all $i \in N$, $f(\cdot)$ is also M$^\natural$-concave.[3]

We consider the problem of maximizing $f(\cdot)$ (i.e., the sum of valuations) under the constraint that, for each item k, the total amount consumed over agent is at most 1. To describe this constraint, consider $g_k : \{0,1\}^{N\times K} \to \mathbb{Z}$ defined by

$$g_k(x) = 1 - x \cdot \chi_{N \times \{k\}} \text{ for all } x \in \{0,1\}^{N\times K},$$

where $\chi_{N\times\{k\}} \in \{0,1\}^{N\times K}$ is the characteristic vector defined by

$$(\chi_{N\times\{k\}})_{(i,k')} = \begin{cases} 1 & \text{if } k' = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x^* \in \{0,1\}^{N\times K}$ be a solution to the maximization problem. As the set $\{\chi_{N\times\{k\}} : k \in K\}$ is a hierarchy, we can apply the discrete KT theorem. There exists $p^* \in \mathbb{Z}^K_+$ such that (x^*, p^*)

[3]This preservation of M$^\natural$-concavity was previously discussed by Fujishige and Tamura (2007) (see equation (15)).

is a saddle point of the Lagrange function,[4] i.e.,

$$
\begin{aligned}
L(x^*, p^*) &= \min_{p \in \mathbb{Z}^K} \max_{x \in \{0,1\}^{N \times K}} L(x, p) \\
&= \min_{p \in \mathbb{Z}^K} \max_{x \in \{0,1\}^{N \times K}} \Big\{ \sum_{i \in N} v_i(x_i) + \sum_{k \in K} p_k g_k(x) \Big\} \\
&= \min_{p \in \mathbb{Z}^K} \max_{x \in \{0,1\}^{N \times K}} \Big\{ \sum_{i \in N} v_i(x_i) + \sum_{k \in K} p_k - \sum_{k \in K} p_k \sum_{i \in N} (x_i)_k \Big\} \\
&= \min_{p \in \mathbb{Z}^K} \max_{x \in \{0,1\}^{N \times K}} \Big\{ \sum_{i \in N} v_i(x_i) + \sum_{k \in K} p_k - \sum_{i \in N} p \cdot x_i \Big\} \\
&= \min_{p \in \mathbb{Z}^K} \max_{x \in \{0,1\}^{N \times K}} \Big\{ \sum_{i \in N} \big(v_i(x_i) - p \cdot x_i \big) + \sum_{k \in K} p_k \Big\} \\
&= \min_{p \in \mathbb{Z}^K} \Big\{ V_i(p) + \sum_{k \in K} p_k \Big\} \\
&= \min_{p \in \mathbb{Z}^K} L(p).
\end{aligned}
$$

This means that finding a saddle point of the Lagrange function corresponds to Ausubel's (2006) auction, the process of finding a minimizer of the Lyapunov function. Note that the above argument implies that p^* is a competitive price vector if and only if p^* minimizes $L(\cdot)$, which was previously proved by Ausubel (2006).

5 Conclusion

A maximization problem under constraints requires us to consider many functions (objective and constraint functions) at the same time, which makes the problem complicated. The usefulness of the KT theorem is to simplify the problem by aggregating relevant information into one function (Lagrange function). Note that Ausubel's (2006) Lyapunov function also plays a similar role in the context of auctions. The original problem of finding a competitive price vector concerns agents' valuations and the capacity of items, which are all aggregated into the minimization of the Lyapunof function. The contrast between the two provides some intuition for why the auction algorithm is connected to the KT theorem.

We can expand the Lagrange function approach to the Kelso and Crawford (1982) model of job matching and provide new insight into the competitive price vectors. We will discuss this issue in an updated version of this paper.

[4] We remark that any saddle point (x^*, p^*) satisfies $L(x^*, p^*) = \max_x \min_p L(x, p) = \min_p \max_x L(x, p)$.

6 Proof of Theorem 1

6.1 Preliminaries

We enumerate the definitions, theorems and claims used in the proof.

Definition 1. For $x \in \mathbb{Z}^K$ and $A \subseteq K$, we define $x(A) = \sum_{k \in A} x_k$.

Definition 2. We say that $X \subseteq \mathbb{Z}^K$ with $X \neq \emptyset$ is an **M$^\natural$-convex set** (Murota 2003) if, for any $x, y \in X$ and $k \in \mathrm{supp}^+(x-y)$, we have

(i) $x - \chi_k \in X$, $y + \chi_k \in X$, or

(ii) there exists $\ell \in \mathrm{supp}^-(x-y)$ such that $x - \chi_k + \chi_\ell \in X$, $y + \chi_k - \chi_\ell \in X$.

Definition 3. We say that $X \subseteq \mathbb{Z}^K$ with $X \neq \emptyset$ is an **L$^\natural$-convex set** (Murota 2003) if, for any $x, y \in X$ with $\mathrm{supp}^+(x-y) \neq \emptyset$, we have

$$x - \chi_A \in X \text{ and } y + \chi_A \in X \text{ for } A = \arg\max_{k \in A}\{x_k - y_k\}.$$

Definition 4. A set function $\rho : 2^K \to \mathbb{R} \cup \{+\infty\}$ is **submodular** if

$$\rho(A) + \rho(A') \geq \rho(A \cup A') + \rho(A \cap A') \text{ for all } A, A' \in 2^K.$$

Definition 5. For $X \subseteq \mathbb{R}^K$, we define the **indicator function of $\boldsymbol{X}$**, $\delta_X : \mathbb{R}^K \to \mathbb{Z} \cup \{-\infty\}$, by

$$\delta_X(x) = \begin{cases} 0 & \text{if } x \in X, \\ -\infty & \text{otherwise.} \end{cases}$$

Definition 6. For $f : \mathbb{Z}^K \to \mathbb{Z} \cup \{-\infty\}$ and $x^* \in \mathrm{dom} f$, we define the **supergradient of $\boldsymbol{f}$ at $\boldsymbol{x^*}$** (in $\mathbb{Z}$) by

$$\partial_{\mathbb{Z}} f(x^*) = \{\hat{x} \in \mathbb{Z}^K : f(x^*) + \hat{x} \cdot (x - x^*) \geq f(x) \text{ for all } x \in \mathbb{Z}^K\}.$$

For $f : \mathbb{R}^K \to \mathbb{R} \cup \{-\infty\}$ and $x^* \in \mathrm{dom} f$, we define the **supergradient of $\boldsymbol{f}$ at $\boldsymbol{x^*}$** (in $\mathbb{R}$) by

$$\partial_{\mathbb{R}} f(x^*) = \{\hat{x} \in \mathbb{R}^K : f(x^*) + \hat{x} \cdot (x - x^*) \geq f(x) \text{ for all } x \in \mathbb{R}^K\}.$$

Definition 7. For $X, Y \subseteq \mathbb{R}^K$, we define

$$-X = \{x \in \mathbb{R}^K : -x \in X\},\ X - Y = \{x - y \in \mathbb{R}^K : x \in X, y \in Y\}.$$

Definition 8. For $X \subseteq \mathbb{Z}^K$, let $\overline{X} \subseteq \mathbb{R}^K$ denote the **convex hull of** $\boldsymbol{X}$.

Definition 9. Let $\mathcal{K} \subseteq 2^K$. We define the **cone generated by** $\boldsymbol{(\chi_A)_{A\in\mathcal{K}}}$ (in $\mathbb{Z}$) by

$$cone_{\mathbb{Z}}(\mathcal{K}) = \left\{x \in \mathbb{Z}^K : x = \sum_{A\in\mathcal{K}} \lambda_A \cdot \chi_A \text{ for some } (\lambda_A)_{A\in\mathcal{K}} \in \mathbb{Z}_+^{\mathcal{K}}\right\}.$$

We define the **cone generated by** $\boldsymbol{(\chi_A)_{A\in\mathcal{K}}}$ (in $\mathbb{R}$) by

$$cone_{\mathbb{R}}(\mathcal{K}) = \left\{x \in \mathbb{R}^K : x = \sum_{A\in\mathcal{K}} \lambda_A \cdot \chi_A \text{ for some } (\lambda_A)_{A\in\mathcal{K}} \in \mathbb{R}_+^{\mathcal{K}}\right\}.$$

Definition 10. For $X \subseteq \mathbb{Z}^K$, we define the **polar of** $\boldsymbol{X}$ by

$$X^\circ = \{y \in \mathbb{Z}^K : y \cdot x \leq 0 \text{ for all } x \in X\}.$$

Theorem 2 (Murota 2003, p.117)**.** Let $X \subseteq \mathbb{Z}^K$ be such that

$$X = \{x \in \mathbb{Z}^K : x(A) \leq \rho(A) \text{ for all } A \subseteq K\}$$

for some submodular set function ρ with $\rho(\emptyset) = 0$. Then, X is an M$^\natural$-convex set.

Theorem 3 (Murota 2003, Theorem 8.17 (M$^\natural$-concave intersection theorem))**.** For M$^\natural$-concave functions $f_1, f_2 : \mathbb{Z}^K \to \mathbb{Z} \cup \{-\infty\}$ and a point $x^* \in \mathrm{dom} f_1 \cap \mathrm{dom} f_2$, we have

$$f_1(x^*) + f_2(x^*) \leq f_1(x) + f_2(x) \text{ for all } x \in \mathbb{Z}^K$$

if and only if there exists $\hat{x} \in \mathbb{Z}^K$ such that

$$\hat{x} \in \partial_{\mathbb{Z}} f_1(x^*) \text{ and } -\hat{x} \in \partial_{\mathbb{Z}} f_2(x^*).$$

Theorem 4 (Murota 2003, (5.8) (Convexity in intersection for L$^\natural$-convex sets))**.** For L$^\natural$-convex sets $X_1, X_2 \subseteq \mathbb{Z}^K$, we have

$$\overline{X_1} \cap \overline{X_2} \neq \emptyset \Longrightarrow X_1 \cap X_2 \neq \emptyset.$$

Claim 1. Let $\mathcal{K} \subseteq 2^K$ be a hierarchy and $(a_A)_{A\in\mathcal{K}} \in \mathbb{Z}^{\mathcal{K}}$. Then, the set

$$X = \{x \in \mathbb{Z}^K : x(A) \leq a_A \text{ for all } A \in \mathcal{K}\}$$

is an M$^\natural$-convex set.

Proof. We define $\rho : 2^K \to \mathbb{Z} \cup \{+\infty\}$ by

$$\rho(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ a_j & \text{if } A = A_j, \\ a_j + a_{j'} & \text{if } A = A_j \cup A_{j'} \text{ for some } j, j' \in \{1, \ldots, q\} \text{ with } j \neq j', \\ +\infty & \text{otherwise.} \end{cases}$$

Then, ρ is submodular and

$$X = \{x \in \mathbb{Z}^K : x(A) \leq \rho(A) \text{ for all } A \subseteq K\}.$$

By Theorem 2, X is an M$^\natural$-convex set. □

Claim 2. Let $\mathcal{K} \in 2^K$ be a hierarchy. Then, the set

$$cone_{\mathbb{R}}(\mathcal{K}) \cap \mathbb{Z}^K$$

is an L$^\natural$-convex set.

Proof. This theorem follows from Claim 1 and the discrete conjugacy theorem (see Murota (2003), Theorem 8.12). □

Claim 3. Let $\mathcal{K} \in 2^K$. Then,

$$\overline{cone_{\mathbb{Z}}(\mathcal{K})} = cone_{\mathbb{R}}(\mathcal{K}).$$

Proof. We define the $|K|$-dimensional unit simplex by

$$\Delta = \Big\{\mu \in \mathbb{R}^{|K|+1} : \mu_j \geq 0 \text{ for all } j = 1, \ldots, |K|+1, \sum_{j=1}^{|K|+1} \mu_j = 1\Big\}.$$

Proof of $\subseteq$: Let $x \in \overline{cone_{\mathbb{Z}}(\mathcal{K})}$. By Carathéodory's theorem, there exist $x_1, \ldots, x_{|K|+1} \in cone_{\mathbb{Z}}(\mathcal{K})$ and $\mu \in \Delta$ such that $x = \sum_j \mu_j x_j$. For each $x_j \in cone_{\mathbb{Z}}(\mathcal{K})$ with $j = 1, \ldots, |K|+1$, there exists $(\lambda_{A,j})_{A\in\mathcal{A}} \in \mathbb{Z}_+^{\mathcal{K}}$ such that $x_j = \sum_{A\in\mathcal{K}} \lambda_{A,j}\chi_A$. Then,

$$x = \sum_{j=1}^{|K|+1} \mu_j \sum_{A\in\mathcal{K}} \lambda_{A,j}\chi_A = \sum_{A\in\mathcal{K}} \sum_{j=1}^{|K|+1} (\mu_j \cdot \lambda_{A,j})\chi_A.$$

Hence, $x \in cone_{\mathbb{R}}(\mathcal{K})$.

Proof of $\supseteq$: Let $x \in cone_{\mathbb{R}}(\mathcal{K})$. Then, there exists $(\lambda_A)_{A\in\mathcal{K}} \in \mathbb{R}_+^{\mathcal{K}}$ such that $x = \sum_{A\in\mathcal{K}} \lambda_A\chi_A$. For each $A \in \mathcal{K}$, by $\mathbb{R}_+^{\mathcal{K}} = \overline{\mathbb{Z}_+^{\mathcal{K}}}$ and Carathéodory's theorem, there exist $z_{A,1}, \ldots, z_{A,|K|+1} \in \mathbb{Z}_+$ and $\mu \in \Delta$ such that $\lambda_A = \sum_j \mu_j z_{A,j}$. Then,

$$x = \sum_{A\in\mathcal{K}} \Big(\sum_{j=1}^{|K|+1} \mu_j z_{A,j} \Big) \chi_A = \sum_{j=1}^{|K|+1} \mu_j \sum_{A\in\mathcal{K}} z_{A,j}\chi_A.$$

Hence, $x \in \overline{cone_{\mathbb{Z}}(\mathcal{K})}$. □

Claim 4. For any hierarchy $\mathcal{K} \subseteq 2^K$,

$$cone_{\mathbb{R}}(\mathcal{K}) \cap \mathbb{Z}^K = cone_{\mathbb{Z}}(\mathcal{K}).$$

Proof. One easily verifies that $\supseteq$ holds. We prove $\subseteq$ by induction on $|\mathcal{K}|$.

induction base: Suppose $|\mathcal{K}| = 1$. Let $x \in cone_{\mathbb{R}}(\mathcal{K}) \cap \mathbb{Z}^K$. Assuming $\mathcal{K} = \{A\}$, $x = \lambda_A \cdot \chi_A$ for some $\lambda_A \in \mathbb{R}_+$. Since $x \in \mathbb{Z}^K$, $\lambda_A \in \mathbb{Z}_+$. Hence, $x \in cone_{\mathbb{Z}}(\mathcal{K})$.

Induction step: Suppose the result holds for $|\mathcal{K}| = t$, and we prove the result for $|\mathcal{K}| = t+1$, where $t \geq 1$.

Let $x \in cone_{\mathbb{R}}(\mathcal{K}) \cap \mathbb{Z}^K$ and $A' \in \mathcal{K}$. Then, there exists $(\lambda_A)_{A\in\mathcal{K}} \in \mathbb{R}_+^{\mathcal{K}}$ such that

$$x = \sum_{A\in\mathcal{K}\setminus\{A'\}} \lambda_A \cdot \chi_A + \lambda_{A'} \cdot \chi_{A'}.$$

This implies that

$$cone_{\mathbb{R}}(\mathcal{K}\setminus\{A'\}) \cap \big(\{x\} - cone_{\mathbb{R}}(\{A'\})\big) \neq \emptyset. \tag{4}$$

By Claim 3,

$$cone_{\mathbb{R}}(\mathcal{K}\setminus\{A'\}) = \overline{cone_{\mathbb{Z}}(\mathcal{K}\setminus\{A'\})}. \tag{5}$$

We also have

$$\begin{aligned}\{x\} - cone_{\mathbb{R}}(\{A'\}) &= \{x\} - \overline{cone_{\mathbb{Z}}(\{A'\})} \\ &= \overline{\{x\} - cone_{\mathbb{Z}}(\{A'\})}, \end{aligned} \tag{6}$$

where the first inequality follows from Claim 3 and the second inequality follows from Proposition 3.17(4) of Murota (2003).

By (4)-(6),

$$\overline{cone_{\mathbb{Z}}(\mathcal{K}\backslash\{A'\})} \cap \overline{\{x\} - cone_{\mathbb{Z}}(\{A'\})} \neq \emptyset.$$

By the induction hypothesis and Claim 2, $cone_{\mathbb{Z}}(\mathcal{K}\backslash\{A'\})$ is L$^\natural$-convex. One easily verifies that $\{x\} - cone_{\mathbb{Z}}(\{A'\})$ is also L$^\natural$-convex. By Theorem 4,

$$cone_{\mathbb{Z}}(\mathcal{K}\backslash\{A'\}) \cap \Big(\{x\} - cone_{\mathbb{Z}}(\{A'\})\Big) \neq \emptyset.$$

This implies that $x \in cone_{\mathbb{Z}}(\mathcal{K})$. □

6.2 Proof of 2 ⇒ 1:

We mimic the proof of the KT theorem (for continuous settings) by Tiel (1984, p.103).

By the latter inequality in (2), $L(x^*, \lambda^*) \leq L(x^*, \lambda)$ for all $\lambda \in \mathbb{Z}^q_+$. Together with (3),

$$\begin{aligned}\sum_{j=1}^{q} \lambda_j^* g_j(x^*) &\leq \sum_{j=1}^{q} \lambda_j g_j(x^*) \text{ for all } \lambda \in \mathbb{Z}^q_+, \\ 0 &\leq \sum_{j=1}^{q} (\lambda_j - \lambda_j^*) g_j(x^*) \text{ for all } \lambda \in \mathbb{Z}^q_+. \end{aligned} \tag{7}$$

Since (7) holds for all $\lambda \in \mathbb{Z}^q_+$,

$$g_j(x^*) \geq 0 \text{ for all } j = 1, \ldots, q. \tag{8}$$

Letting $\lambda = \mathbf{0}$ in (7),

$$\sum_{j=1}^{q} \lambda_j^* g_j(x^*) \leq 0.$$

Combining this inequality with (8) yields

$$\sum_{j=1}^{q} \lambda_j^* g_j(x^*) = 0.$$

This equation and the former inequality in (2), $L(x, \lambda^*) \leq L(x^*, \lambda^*)$ for all $x \in \mathbb{Z}^K$, imply

$$f(x) + \sum_{j=1}^{q} \lambda_j^* g_j(x) \leq f(x^*) \text{ for all } x \in \mathbb{Z}^K.$$

This means that $f(x) \leq f(x^*)$ whenever $g_j(x) \geq 0$ for all $j = 1, \ldots, q$. Together with (8), we obtain the desired condition.

6.3 Proof of 1 $\Rightarrow$ 2:

By assumption, for all $j = 1, \ldots, q$, there exist $a_j \in \mathbb{Z}$ and $A_j \subseteq K$ with $A_j \neq \emptyset$ such that

$$g_j(x) = a_j - x(A_j) \text{ for all } x \in \mathbb{Z}^K.$$

Our purpose is to find $\lambda^* \in \mathbb{Z}_+^q$ that satisfies the statement in 2. Suppose $A_j = A_{j'}$ for some $j, j' \in \{1, \ldots, q\}$ with $a_j \leq a_{j'}$. Then, $g_{j'}(\cdot)$ is a redundant constraint function. In the proof below, we can ignore such j' by letting $\lambda_{j'}^* = 0$. Hence, w.l.o.g., we assume

$$A_j \neq A_{j'} \text{ for all } j, j' \in \{1, \ldots, q\}.$$

Set $C = \{x \in \mathbb{Z}^K : g_j(x) \geq 0 \text{ for all } j = 1, \ldots, q\}$. By Claim 1, $\delta_C(\cdot)$ is an M$^\natural$-concave function. Since x^* is a solution to the maximization problem under constraints,

$$f(x^*) + \delta_C(x^*) \geq f(x) + \delta_C(x) \text{ for all } x \in \mathbb{Z}^K.$$

By Theorem 3, there exists $\hat{x} \in \mathbb{Z}^K$ such that

$$\hat{x} \in \partial_{\mathbb{Z}} f(x^*), \tag{9}$$

$$-\hat{x} \in \partial_{\mathbb{Z}} \delta_C(x^*). \tag{10}$$

By (10) and the definition of a supergradient, $-\hat{x} \in \partial_{\mathbb{R}} \delta_{\overline{C}}(x^*)$. As recognized in the literature on convex analysis (see, for example, Rockafellar (1970), Section 23), $-\partial_{\mathbb{R}} \delta_{\overline{C}}(x^*)$ is the normal cone to $\bar{C}$ at x^*. i.e.,

$$-\partial_{\mathbb{R}}\delta_{\overline{C}}(x^*) = \{y \in \mathbb{Z}^K : y \cdot (x - x^*) \leq 0 \text{ for all } x \in \overline{C}\}.$$

By Proposition 5.2.4 of Hiriart-Urruty and Lemaréchal (2001), the normal cone is the polar of the tangent cone. This fact and (10) imply $\hat{x} \in cone_{\mathbb{R}}(\mathcal{K})$, where

$$\mathcal{K} = \big\{A \subseteq K : A = A_j \text{ for some } j \in \{1, \dots, q\} \text{ with } g_j(x^*) = 0\big\}.$$

Since $\mathcal{K}$ is hierarchy, by Claim 4, $\hat{x} \in cone_{\mathbb{Z}}(\mathcal{K})$. Hence, there exists $(\lambda^*_A)_{A \in \mathcal{K}} \in \mathbb{Z}^{\mathcal{K}}_+$ such that

$$\hat{x} = \sum_{A \in \mathcal{K}} \lambda^*_A \chi_A.$$

For each $j \in \{1, \dots, q\}$ with $g_j(x^*) = 0$, set $\lambda^*_j = \lambda^*_{A_j}$ for $A_j \in \mathcal{K}$. For each $j \in \{1, \dots, q\}$ with $g_j(x^*) > 0$, set $\lambda^*_j = 0$.

We prove that (x^*, λ^*) is a saddle point of $L(\cdot, \cdot)$ defined by (3). We first fix λ^* and regard $L(\cdot, \lambda^*)$ as a function on $\mathbb{R}^K$. Then,

$$\begin{aligned}
\partial_{\mathbb{Z}} L(x^*, \lambda^*) &= \partial_{\mathbb{Z}}\Big(f(x^*) + \sum_{j=1}^{q} \lambda^*_j g_j(x^*)\Big) \\
&= \partial_{\mathbb{Z}} f(x^*) - \sum_{j=1}^{q} \lambda_j \chi_{A_j} \\
&\ni \hat{x} - \hat{x} \\
&= 0,
\end{aligned}$$

where the second equality, the decomposition of the supergradient, follows from the fact that $g_j(\cdot)$, $j = 1, \dots, q$, are affine functions. This means that $L(\cdot, \lambda^*)$ is maximized at x^*.

Next fix x^* and regard $L(x^*, \cdot)$ as a function on $\mathbb{Z}^q_+$. As x^* satisfies the constraints, $g_j(x^*) \geq 0$ for all $j = 1, \dots, q$. Hence, for any $\lambda \in \mathbb{Z}^q_+$,

$$\sum_{j=1}^{q} \lambda_j g_j(x^*) \geq 0. \tag{11}$$

Moreover, by the construction of λ^*,

$$\sum_{j=1}^{q} \lambda^*_j g_j(x^*) = 0. \tag{12}$$

We conclude

$$L(x^*, \lambda) = f(x^*) + \sum_{j=1}^{q} \lambda_j g_j(x^*) \geq f(x^*) + \sum_{j=1}^{q} \lambda_j^* g_j(x^*) = L(x^*, \lambda^*) \text{ for all } \lambda \in \mathbb{Z}_+^q,$$

where the inequality follows from (11) and (12). □

Acknowledgement

This work was supported by JSPS Grant-in-Aid for Research Activity Start-up Grant Number 17H07179.

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