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de la Fonteijne, Marcel R.

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Marcel R. de la Fonteijne

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Abstract

The book 'Capital in the Twenty-First Century' by the French economist Piketty about the inequality of income and wealth distribution is already quite a while in the spotlights. Jones in his paper 'Pareto and Piketty: The Macroeconomics of Top Income and Wealth Inequality' is describing the link between the empirical facts and macroeconomic theory. Jones derived a formula for the Pareto wealth coefficient where he focused on the influence of inheritance tax and the birth death process in a simple AK model with regard to Piketty's $r > g$, the birth rate n and the death rate d. We could not agree with him on his normalization process, although the Pareto coefficient stays the same. We show that the concept of normalized wealth, Jones is using, is wrong, because he is transferring the same concept to the driving power of wealth, which is not allowed. We conclude that due to the considered capital gain and inheritance process with an inheritance tax $0 < \tau_{inh} < 1$ there is an ongoing upward pressure toward maximum wealth inequality if there is no redistribution and an ongoing downward pressure towards no inequality if the redistribution is equal to the mean wealth.

JEL Classification E00 · E10 · E20 · E60 · H20· H30 · H60

Keywords: inequality, GDP growth, income, wealth, Pareto, Jones, Piketty

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The author is managing director at DLF Macro Economic Research, Delft, The Netherlands, marcel@delafonteijne.nl

1. Introduction

The book 'Capital in the Twenty-First Century' by the French economist Piketty (2013) about the inequality of income and wealth distribution is already quite a while in the spotlights. Let me start by saying that his work is impressive with regard to the years of hard work to collect and structure these historical data. One of the main points in his book is the influence of $r > g$ on inequality.

A general remark is needed when discussing Piketty's (2013) book. It often it not very clear what is exactly meant by the term capital, income, r, g and the capital to income ratio β. Does it include or exclude depreciation, is it per capita or normalized for population growth and technical growth. Sometimes Piketty is using r as capital return and subtract consumption to arrive at a net saving rate and compare it to g, but now probably with respect to a single consumer or subgroups of consumers. It is this un-clarity that make comparing Piketty, Jones and others a little bit tricky, but we will try to do so anyway.

Jones (2015) in his paper 'Pareto and Piketty: The Macroeconomics of Top Income and Wealth Inequality' is describing the link between the empirical facts and macroeconomic theory and I quote:

> 'I highlight some key empirical facts from this research and describe how they relate to macroeconomics and to economic theory more generally.
> One of the key links between data and theory is the Pareto distribution. The paper explains simple mechanisms that give rise to Pareto distributions for income and wealth and considers the economic forces that influence top inequality over time and across countries.'

In a simple example of the AK model with equal redistribution of a 100 percent inheritance tax Jones derives the Pareto wealth power coefficient

$$\eta_{Wealth} = \frac{n}{n+d} \tag{1}$$

using the concept of normalized individual wealth. This formula means that with small population growth n compared to death rate d this leads to very low inequality and with $n = 2d$ this results in $\eta_{Wealth} = .66$ and corresponding Gini coefficient of $Gini = .5$ which is independent of r and g. We will show that his formula is also valid in more general cases, but we had serious trouble with the normalization concept Jones is using. We will extend the validity of this formula to a wide class of models and production functions following an alternative method.

We will start with a description of the simple model and a step by step analysis of what are points of concern. We will show, just for clarity, the relation with a Pareto distribution and some practical formulas. Then we will discuss the normalization process Jones is using and go in detail through his example with several inheritance taxation options. We end with some conclusions.

2. Model

We start with a AK production function with only capital as factor of production and constant parameter A.

$$Y = AK \tag{2}$$

$$Y = wL + (r + \delta)K \tag{3}$$
$$Y = C + I + G \tag{4}$$
$$Y = C + S + T \tag{5}$$
$$G = T \tag{6}$$
$$C = \alpha K \tag{7}$$
$$T = \tau K \tag{8}$$
$$\dot{L} = nL \tag{9}$$
$$\dot{K} = I - \delta K \tag{10}$$

Where
Y is income with growth g_Y and productivity $y = \frac{Y}{L}$

K is capital with growth g_K and capital deepening $k = \frac{K}{L}$

L is population with growth n
$C = \alpha K$ is consumption with rate α expressed in K
G is governmental expenditure
I is investment
S is savings
$T = \tau K$ is governmental revenue with wealth tax rate τ expressed in K
w=0
δ rate of capital depreciation
A is gross return on capital
r is net return on capital
and **.** is denoting the time derivative.

The investments can be calculated from

$$I = Y - C - T = AK - \alpha K - \tau K = (r - \delta - \alpha - \tau)K \tag{11}$$

And the net investment is

$$\dot{K} = I - \delta K = (r - \alpha - \tau)K \tag{12}$$

The growth of capital K is

$$g_K = \frac{\dot{K}}{K} = r - \alpha - \tau \tag{13}$$

Because A is a constant $g_Y = g_K$ and population growth is n, the mean per capita growth is

$$g_y = g_Y - n = g_K - n = r - \alpha - \tau - n \tag{14}$$

also known as the productivity growth.

3. Some special values for the consumption rate α when using the AK model

If we assume that disinvestment in not an option then we require investment I greater than zero.
If we choose or log optimize $\alpha = A - \tau$ then $I = 0$ which will lead to

$$\dot{K} = I - \delta K = -\delta K \tag{15}$$

with solution

$$K = K_0 e^{-\delta t} \tag{16}$$

and with growth rate $g_Y =$ -δ the economy will vanish.

The maximum value for α is

$$\alpha_{max} = A - \tau \tag{17}$$

For $I = \delta K$ the economy is not growing $K = K_0$ and $Y = AK_0$. The corresponding value for α is

$$\alpha_{K=K_0} = A - \tau - \delta \tag{18}$$

and per capita income is reducing with the growth rate of the population.

$$y = y_0 e^{-nt} \tag{19}$$

For $I = \delta K + nK$ the economy is growing and per capita capital and per capita income is remaining the same.

$$y = Ak = Ak_0 \tag{20}$$

$$Y = Y_0 e^{nt} = AK_0 e^{nt} \tag{21}$$

The corresponding value for α is

$$\alpha_{k=k_0} = A - \tau - \delta - n \tag{22}$$

Choosing or log optimizing $\alpha < \alpha_{k=k_0}$ will result in a growing per capita income.

Taking in account the thought of Phelps (1961) with care for future generations $\alpha_{k=k_0}$should be considered as the maximum value.

Although the AK model sometimes can serve as a theoretical example, we must admit that we are not an advocate in general using this kind of robotized 'Matrix Reloaded' stand-alone capital generating economy, probably only mastered by a computer program, where one programming mistake in e.g. time preference can be fatal.

4. Relation with Pareto distribution

So far these are macroeconomics measures in the aggregate and it is in fact impossible to come to a measure of inequality without going into details of individual behavior or at least the knowledge over distributions on such behavior. Jones is using the fact that if a quantity Y is growing exponential in relation to another quantity X and X is exponential distributed in time then Y is Pareto distributed. In formula form:

If the age distribution is

$$\Pr\,[\text{Age} > x] = e^{-dx} \tag{23}$$

where d denotes the death rate and income rises exponential with age

$$y = e^{\mu x} \tag{24}$$

then Y is Pareto distributed

$$\Pr\,[\text{Income} > y] = \left(\frac{y}{y_{min}}\right)^{-\alpha_{par}} \tag{25}$$

and the Pareto exponent

$$\eta_{par} = \frac{1}{\alpha_{par}} = \frac{\mu}{d} \tag{26}$$

Important is that you need y_{min} direct or indirect to know the distribution.

The complementary distribution is

$$F(y) = 1 - \left(\frac{y}{y_{min}}\right)^{-\alpha_{par}} \tag{27}$$

With probability density function

$$f(y) = \frac{\alpha_{par}}{y_{min}} \left(\frac{y}{y_{min}}\right)^{-\alpha_{par}} \quad y_{min} < y < \infty \tag{28}$$

The mean value of y is

$$y_{mean} = E(y) = \int_{y_{min}}^{\infty} y f(y) dy = \int_{y_{min}}^{\infty} \alpha_{par} \left(\frac{y}{y_{min}}\right)^{-\alpha_{par}+1} dy = \frac{\alpha_{par}}{\alpha_{par}-1} y_{min} \tag{29}$$

And expressing y_{min} in y_{mean} and α_{par}

$$y_{min} = \frac{\alpha_{par}-1}{\alpha_{par}} y_{mean} \tag{30}$$

or alternatively expressing α_{par} in y_{min} and y_{mean}

$$\alpha_{par} = \frac{1}{1-\frac{y_{min}}{y_{mean}}} \tag{31}$$

or

$$\eta_{par} = 1 - \frac{y_{min}}{y_{mean}} \tag{32}$$

Notice that α_{par} has to be greater than $\alpha_{par} > 1$ to ensure the existence of a mean value in the distribution and thus the validity of the formulas, in which case the Pareto coefficient can be calculated only knowing the ratio of the minimum value of income y_{min} and the mean value y_{mean}.

The same formulas can also be used when using wealth instead of income. Though the difference is in the fact that income ends after being death and in case of wealth we have to come up with a destination for the accumulated stock of wealth. Another important thing to realize is that the value of y_{min} has to grow at the same pace as y_{mean} or at least the ratio has to convert in the long run in order to keep a stationary Pareto distribution in equilibrium.

In general, to generate inequality we need a driving force and a mechanism.

In general g_y is the per capita growth in the economy which is equal to the capital growth g_k per capita on the long term.. To come to the distribution of income or wealth we assume the individual growth rate is equal to the mean per capita growth rate g_y and g_k. Be aware that this assumed knowledge is very specific.

It seems reasonable to assume that no capital will be destroyed when people die and we further assume that this capital is equally redistributed to the newborns, i.e. they inherit the fraction $\frac{d}{n+d}$ of the mean capital per capita and thus

$$k_{min} = \frac{d}{n+d} k_{mean} \tag{33}$$

Together with equation (32) but now for wealth gives us the Pareto coefficient

$$\eta_{par} = 1 - \frac{k_{min}}{k_{mean}} = 1 - \frac{d}{n+d} = \frac{n}{n+d} \tag{34}$$

So far the only assumptions are the same growth rate for each individual g_y and g_k, a constant ratio $\frac{k_{min}}{k_{mean}}$ and equal redistribution of inheritances, which leads to the conclusion that it holds for every economy and every production function as long as these assumptions are true, and this is the only solution. Remember that k_{mean} is growing with g_y and so is k_{min}.

Clearly in our model the assumption of each individual growth rate to be g_k is the same as saying that the saving rate *s* expressed in income, with capital to income ratio β, is the same for each individual. This has the advantage that we can leave out the restrictions for individual income, taxation, etc.

Where Jones is claiming that in this case the result for inequality is tax invariant, we like to emphasize that in his example taxation is also done by a 100 % inheritance tax and a 100% equal redistribution.

We can generalize the conclusion:
If each individual savings rate in an economy is *s* then with an equal redistribution of all the inheritances the wealth distribution will be Pareto with coefficient

$$\eta_{par} = \frac{n}{n+d} \qquad \text{for } n, n+d \geq 0 \tag{35}$$

However, values for $0 > n \geq -d$ resulting in severe inequality coefficients with $\eta_{par} > 1$ and non-existing distributions are ruled out, because $n < 0$ results in extinction of the population on the long run.

And, that the individual savings rate would be of major importance that's what you thought already.

Let's try another approach and follow Jones.

5. The normalization of the individual wealth accumulation

We will now turn to the normalization of individual wealth accumulation and we quote Jones (2014):

> In the case of wealth inequality, this exponential growth is fundamentally tied to the interest rate, *r*: in a standard asset accumulation equation, the return on wealth is a key determinant of the growth rate of an individual's wealth. On the other hand, this growth in an individual's wealth occurs against a backdrop of economic growth in the overall economy. To obtain a variable that will exhibit a stationary distribution, one must normalize an individual's wealth level by average wealth per person or income per person in the economy. If average wealth grows at rate g — which in standard models will equal the growth rate of income per person and capital per person—the normalized wealth of an individual then grows at rate $r - g$. This logic underlies the key $r - g$ term for wealth inequality that makes a frequent appearance in Piketty's book. Of course, *r* and *g* are potentially endogenous variables in general equilibrium so—as we will see—one must be careful in thinking about how they might vary independently.

We have several points where we see things different, except from the fact that *r* and *g* are endogenous linked.
First, in our model but also in general, for *r* we like to use the term net return rate (without depreciation) on capital used to generate GDP, which includes firm capital and (a part of) governmental capital. This net return rate has a very loose relation with interest rates with low risk and not a fundamental tied one.
Second, net macroeconomic return on investment *r* is not a key determinant of the growth rate of an individual's wealth, but economic growth is, as well as household investment in private property. Especially here we have the impression that *r* is used as an individual return on investment.
Third, this growth in an individual's wealth does not occur against a backdrop of economic growth in the overall economy, at least not necessarily.
Forth, to obtain a variable that will exhibit a stationary distribution, one must normalize an individual's wealth level by average wealth per person or income per person in the economy. We do agree on that. Jones and Kim (2014) are using the concept of income normalization to normalize individual income with respect to average individual growth *g*. Nothing wrong with that and you can also apply the same procedure to wealth. However, we like to point out that scaling in this normalization process is done in wealth in order to remove the mean growth of an individual wealth distribution, but it is not allowed in general to subtract mean per capita macroeconomic growth of wealth from the wealth growth (not per capita) to arrive at the net driving force for inequality that will lead to inequality in combination with a birth death process. In the model without normalizing the parameter for individual exponential growth is

$$g_y = g_k = g_Y - n = r - \alpha - \tau - n \qquad (36)$$

in combination with a birth death process with parameter $(n + d)$ this will lead to a Pareto distribution with parameter

$$\eta_{par} = \frac{g_Y - n}{n+d} = \frac{r - \alpha - \tau - n}{n+d} \tag{37}$$

if e.g. k_{min} would have been a constant value and the rest of the inheritance would have been destroyed, but that is obviously for good reason not the case.

Jones is following a harder to grasp way by normalization of the individual growth rate, interesting to see that he start with the growth in the economy in total and subtract mean per capita growth. We motivate this as follows. The total driving power of growth in the economy is g_Y and if we subtract the individual mean growth g_y then all what remains is the net driving power for inequality, which is of course always equal to n, which is true if the inheritance tax is 100 % and is equally distributed.

$$g_{net} = g_Y - g_y = n \tag{38}$$

And again resulting in

$$\eta_{par} = \frac{n}{n+d} \tag{39}$$

From a normalizing point of view not very logical, because then I would suggest to start with the individual driving power g_y of equation (36) and subtract again g_y resulting in $\eta_{par} = 0$, which would be wrong.

The concept of normalization the distribution of wealth cannot be replaced by the normalization of the individual growth that is driving inequality by simply subtracting the mean growth level. Normalization is not the same as the process of inheritance taxation. It is in fact a growth deflated wealth per individual. It is not negative growth. You only can represent it as equivalent to negative growth. In this case you can consider n as a net equivalent growth value driving inequality.

Next we will derive formulas in case of 100 % inheritance tax of which fraction γ is equally redistributed.

6. Inequality with 100 % inheritance tax of which fraction γ is equally redistributed

We will now turn to the situation that the inheritance tax is 100% of the inheritance, of which fraction γ is equally redistributed to the newborns and we will examine what it means for inequality in the considered model. Suppose $f(k)$ is describing the equilibrium density function of individual wealth k with mean wealth k_{mean}.
If k_{min} is the minimum value in the distribution then

$$k_{min} \le \gamma \frac{d}{n+d} k_{mean} \tag{40}$$

and

$$\gamma \frac{d}{n+d} k_{mean} \le k_{mean} \tag{41}$$

After enough periods there comes a time *t* that all people with wealth lower than $\gamma\frac{d}{n+d}k_{mean}$ have died and because under this limit there has been no replacement $f\left(k \leq \gamma\frac{d}{n+d}k_{mean}\right) = 0$, the minimum value for k is

$$k_{min} = \gamma\frac{d}{n+d}k_{mean} \tag{42}$$

So the minimum value is equal to the redistributed wealth to newborns and in that case we already know the exact form of the distribution, i.e. Pareto. We use equation (30) to derive the relation between k_{min}, k_{mean} and α to be

$$k_{min} = \frac{\alpha_{par}-1}{\alpha_{par}}k_{mean} \tag{43}$$

And substituting eq. (42) into eq. (43) this results in

$$\gamma\frac{d}{n+d} = \frac{\alpha_{par}-1}{\alpha_{par}} \tag{44}$$

or

$$\alpha_{par} = \frac{1}{1-\gamma\frac{d}{n+d}} \tag{45}$$

And the Pareto coefficient

$$\eta_{par} = 1-\gamma\frac{d}{n+d} \tag{46}$$

To calculate the Gini coefficient we use

$$Gini = \frac{1}{2\alpha_{par}-1} \tag{47}$$

In case $\gamma\frac{d}{n+d} = 0$ then $\alpha_{par} = 1$ and $\eta = 1$ and there is maximum inequality.
In case $\gamma\frac{d}{n+d} = .5$ then $\alpha_{par} = 2$ and $\eta = .5$ with Gini=.33 and there is inequality.
In case $\gamma\frac{d}{n+d} = 1$ then $\alpha_{par} = \infty$ and $\eta = 0$ and there is no inequality.

Important to notice, however, we did not have put to work the part of the inheritance tax not redistributed i.e. $(1-\gamma)$. But what can you do with this part. Destroying capital does not seems a smart decision, although possible. Converting capital to consumption is probably not possible if we assume that capital consists of capital goods and if it were possible it would probably violate the log optimized consumption rate *c*. We end up with the only smart way to use this capital and to redistribute this part as well, meaning that $\gamma = 1$. In the considered case it holds for all kind of production functions and for all kinds of model and that the outcome for inequality is independent of *r* and *g* All what matter is the ratio $\frac{k_{min}}{k_{mean}}$. It is not even restricted to depreciable firm capital.

Also worthwhile noticing is the fact that for inequality it is irrelevant what you do with the destroyed $(1-\gamma)$ fraction as long as it does not change the ratio $\frac{k_{min}}{k_{mean}}$. That leads us to the introduction of an inheritance tax rate τ_{inh}.

7. Inequality with inheritance tax rate τ_{inh} and equal distribution

The next logical step is to introduce an inheritance tax rate τ_{inh}. The remaining fraction $1-\tau_{inh}$ will keep on growing. Wealth accumulation can now take place over several generations, i.e. a longer period of time. Just for the sake of the experiment part γ of the inheritance tax will be redistributed equally and the rest will be destroyed. The fraction Newborns inherit

$$k_{inh} = \gamma\tau_{inh}\frac{d}{n+d}k_{mean} + (1-\tau_{inh})\,\frac{d}{n+d}k_{actual} \tag{48}$$

and the procedure is neutral for the government. In equilibrium we have

$$k_{min} \leq \gamma\tau_{inh}\frac{d}{n+d}k_{mean} + (1-\tau_{inh})\frac{d}{n+d}k_{actual} \tag{49}$$

The value of k_{actual} is not fixed, but varying and the simple relationship does not exist any longer. For $\tau_{inh} = 1$ we have the Pareto distribution again.
For $\tau_{inh} = 0$ the initial inequality will not change. Due to the fact that there is no defined family relationship and no structured birth death process this is a bit forced example.
For $0 < \tau_{inh} < 1$ the distribution is somewhere between log-normal and Pareto.

To see the effect of the inheritance tax τ_{inh}, redistribution part γ and the birth rate we did a simulation. In fig. 1 you see the probability density function (pdf) of wealth with $\tau_{inh} = .2$ and $\gamma = 0$

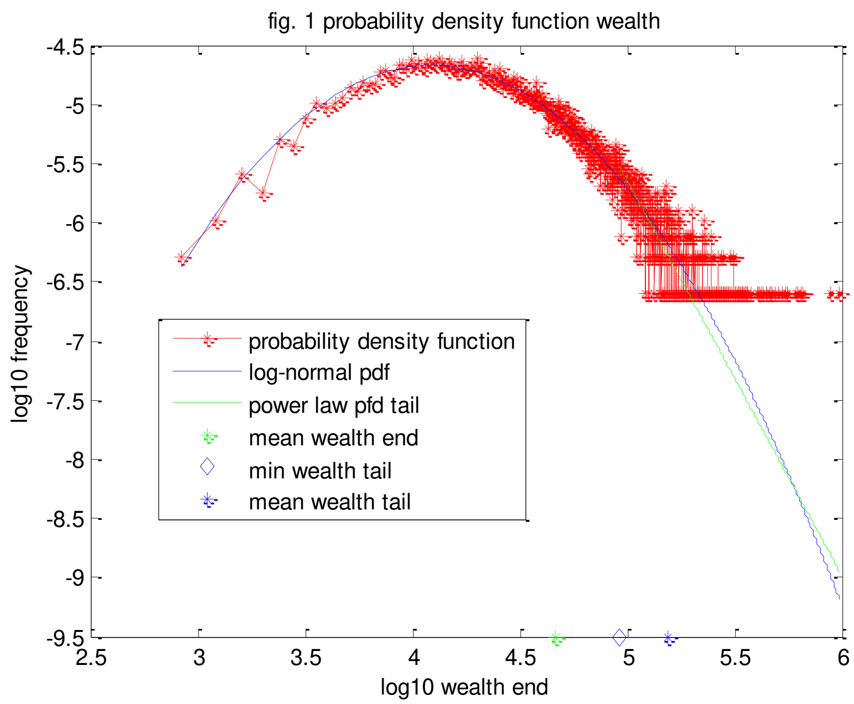

Fig. 1 Simulated pdf of wealth after 250 years, $\tau_{inh} = .2$, $g_Y = .02$, $d = .015$, $\gamma = 0$ and initial wealth distribution as shown by the Lorenz line in fig. 3 (black).

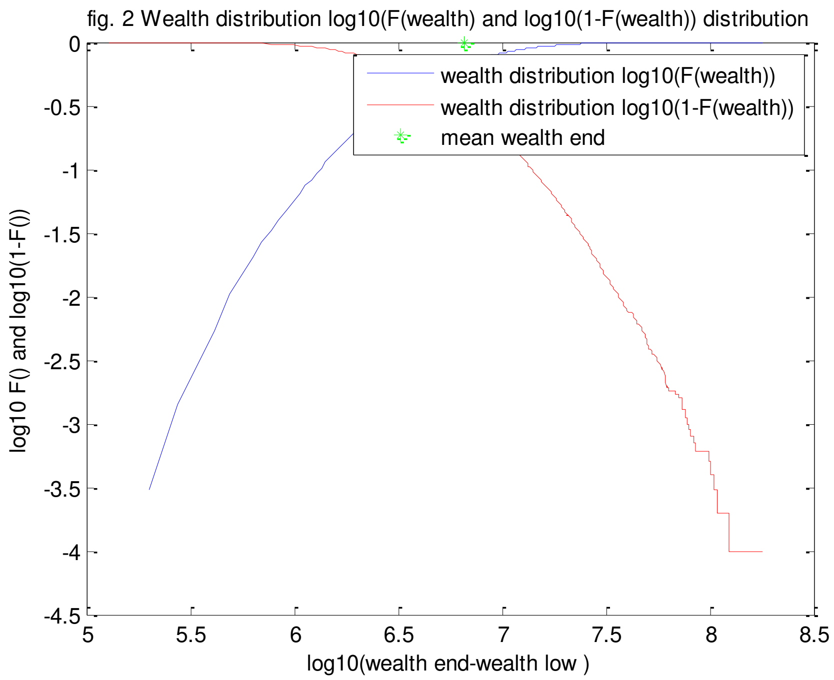

Fig. 2 Wealth distribution, same parameters as in fig. 1

From this wealth distribution in fig. 2 we can estimate the power α_{par} power law distribution for the high end wealth values (e.g. , which result in a bad approximation because for low values of τ_{inh} the distribution is more log-normal, for high values of τ_{inh} the approximation is quite good.

In fig. 3 you can see the similarity between the Lorenz line of the simulation (red line) and a theoretical log-normal (blue line) distribution. Also shown is the power distribution calculated as a best estimate from the tail in the wealth distribution for values $k > 2k_{mean}$ as an example.

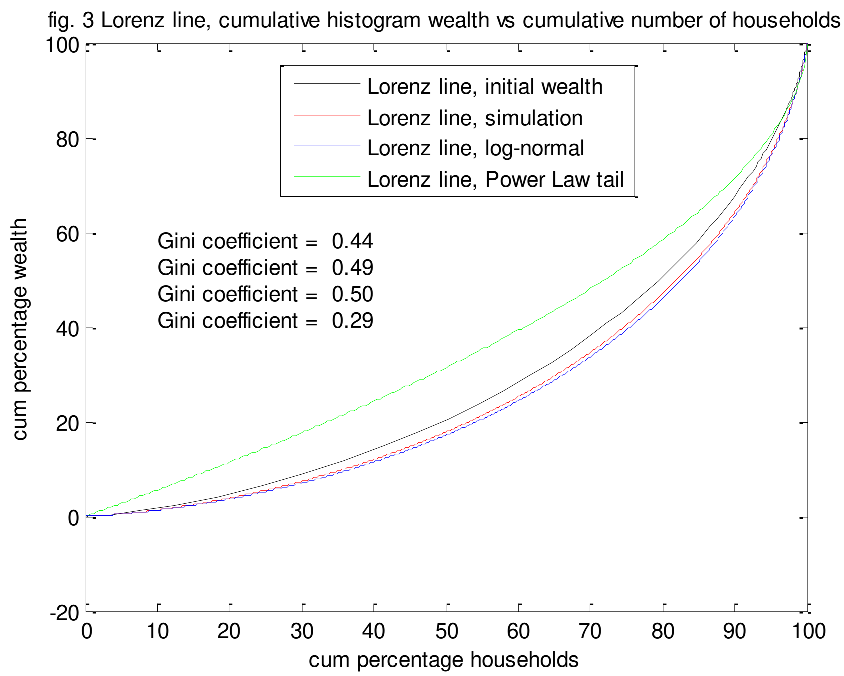

Fig. 3 Lorenz line for simulation: red line, same parameter as in fig. 1, the estimated power law for wealth values greater than $2k_{mean}$.

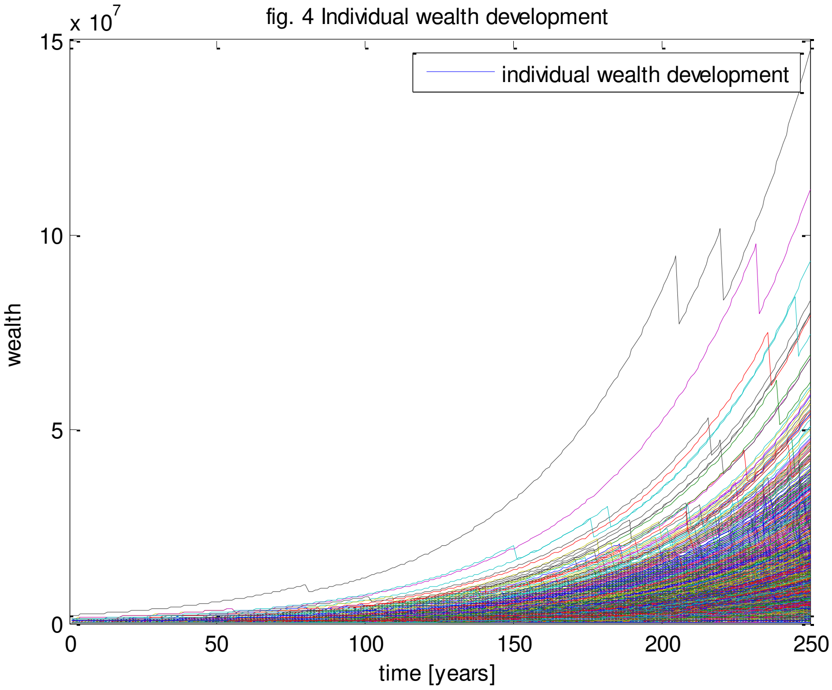

Fig. 4 Individual wealth development as a function of time, same parameters as in fig. 1

We calculate the Gini coefficient for several inheritance tax rates, starting with a wealth distribution with Gini coefficient $Gini = .42$, the Lorenz curve of this distribution is shown in black in fig. 3.

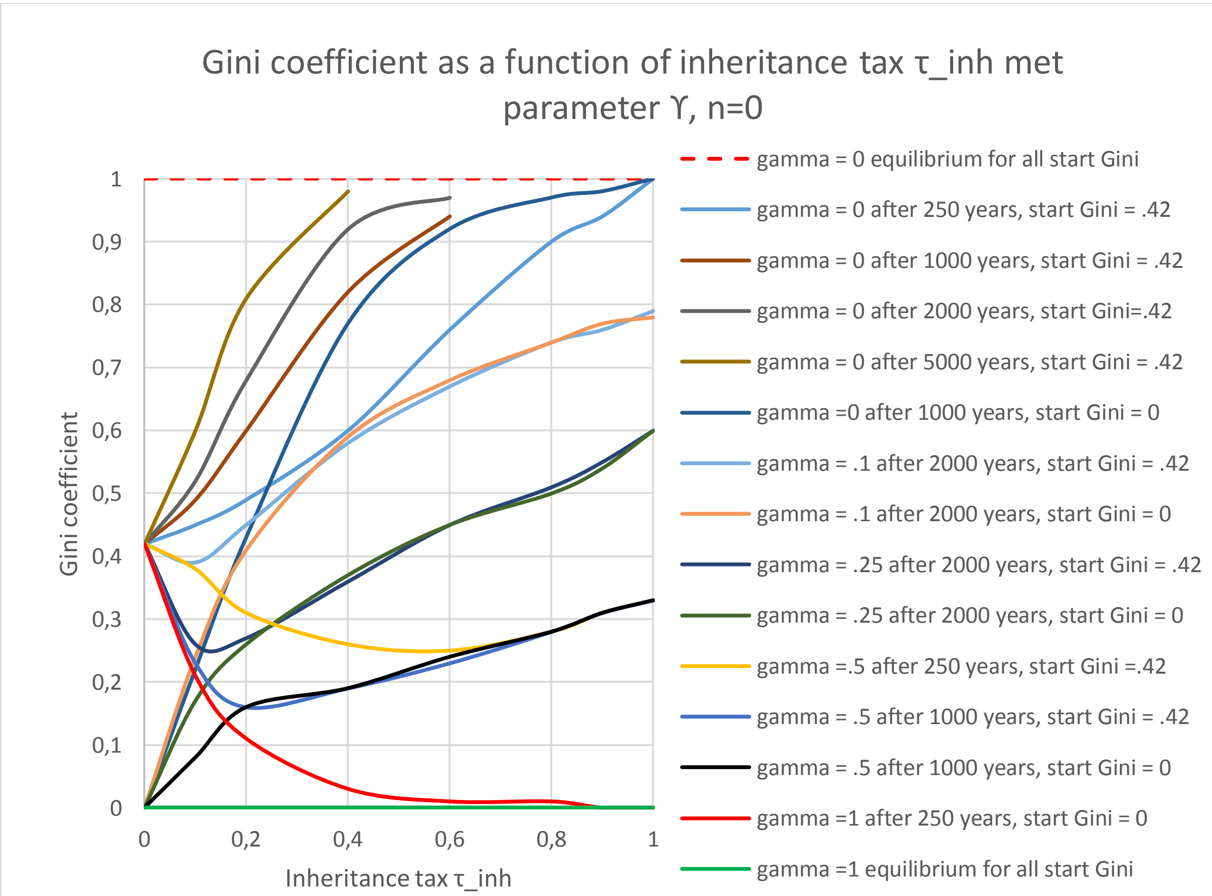

Fig. 5 Gini coefficient as function of the inheritance tax τ_{inh} and parameter γ for n=0

In case all the tax is redistributed $\gamma = 1$ and $n = 0$ the result is shown in fig. 5 by the curve in green, indicating that there is no inequality. To show how slow convergence take place we also show the Gini coefficient after 250 years in red. The red dashed line represents $\gamma = 0$ for which $Gini = 1$ in equilibrium.

The formula $\eta_{Wealth} = \frac{n}{n+d}$ derived for the AK model and used by Jones for inequality is a nice mathematical result under the condition that

- each individual has the same exponential capital growth
- a birth death process exists as described
- there is a 100 % inheritance tax that is equally redistributed

We showed that the formula is valid for all economy's under the mentioned conditions and restated like this the result is trivial, though important to realize. These conditions introduce severe limitations if you consider practical situations. To my knowledge inheritance taxes on average are closer to zero than to 100 % and typically 0 to 20 % and redistribution is not done directly and the tax is in general used for general governmental purposes like infra-structure and education.

We can conclude that due to the considered capital gain and inheritance process with inheritance tax $0 < \tau_{inh} < 1$ there is an ongoing upward pressure toward maximum wealth inequality if there is no redistribution and an ongoing downward pressure towards no inequality if the redistribution is equal to the mean wealth, which is the case for $n = 0$ and/or $\gamma = 1$. Only if $n \neq 0$ and $\gamma \neq 1$ then the Gini coefficient will stabilize between 0 and 1. See e.g. the dark green and dark blue line in fig. 5, for $\tau_{inh} = 1, n = 0$ and $\gamma = .25$ the Gini coefficient is $Gini = .6$.

What is not surprisingly, but interesting to notice, is the slow convergence from one state to another equilibrium, especially when τ_{inh} is low.

8. Optimization of consumption using time preference

In case we have still a degree of freedom left we have the possibility to optimize consumption.
We define the utility function as

$$u(c) = \frac{c^{\theta-1}-1}{\theta-1} \tag{50}$$

where $c = \frac{C}{L}$ is consumption per capita.
We maximize with time preference ρ the discounted sum of consumption

$$\max \int_0^\infty e^{-\rho t}\, u(c)dt \tag{51}$$

In our example if you also take into account the death rate d and with $\theta = 0$ this will lead to consumption rate (in parts of *K*)

$$\alpha = n + d \tag{52}$$

as mentioned by Jones. We only have to take in consideration the limiting value for α as calculated earlier.

The same procedure applied to a Cobb-Douglas production function will lead to

$$r - g = \rho \tag{53}$$

and capital to income ratio of firms

$$\beta_F = \frac{\alpha_{CD}}{g+\rho+\delta} \tag{54}$$

where α_{CD} is the power coefficient of capital in the Cobb-Douglas production function and g is economic growth.
Once again this gives us no clue that $r - g$ could be responsible for any kind of inequality from a fundamental point of view.
The corresponding consumption rate c_ρ expressed in Y results in

$$c_\rho = 1 - \frac{g+\delta}{g+\rho+\delta}\alpha_{CD} \tag{55}$$

$$Y = c_\rho C \tag{56}$$

If we assume the consumers time preference ρ constant then the net return rate on capital depends on g

$$r = \rho + g \tag{57}$$

We feel that also firms have a say in this decision. Take e.g. as an alternative r is constant then we can calculate the capital to income ratio β_F and c_ρ

$$\beta_F = \frac{\alpha_{CD}}{r+\delta} \tag{58}$$

The capital to income ratio is constant too and the consumption rate is

$$c_\rho = 1 - (g + \delta)\beta_F \tag{59}$$

It is exactly this way of thinking we used to arrive at the formulas presented in de la Fonteijne (2014-1), where we showed an alternative way by choosing $c_\rho(g = 0)$ to determine β_F and thus indirect r and ρ. It is clear that by doing so we divert from Piketty's approach.

We believe that it is more appropriate to hold the net return rate on capital constant at a desired value then using consumer time preference, but a more thorough examination is needed. In general we do not feel very comfortable with the concept of maximization of discounted consumer value, despite 'The Golden Rule' or 'The modified Golden rule'.

9. Statistical fluctuation due to birth death

Due to statistical fluctuations of the birth death process there will remain a low level of inequality and wealth K deflated for exponential growth will show a random walk movement as shown in fig. 6. Also inaccuracies in fig. 5 are caused by statistical fluctuations, due to a limited amount of samples.

Fig. 6 Random walk of mean wealth deflated for growth, same parameters as in fig. 1.

10. Conclusion

The formula $\eta_{Wealth} = \frac{n}{n+d}$ derived for the AK model and used by Jones for inequality is a nice mathematical result under the condition that

- each individual has the same exponential capital growth
- a birth death process exists as described
- there is a 100 % inheritance tax that is equally redistributed

We showed that the formula is valid for all economy's under the mentioned conditions and restated like this the result is trivial, though important to realize. These conditions introduce severe limitations if you consider practical situations. To my knowledge inheritance taxes are closer to zero than to 100 % and typically 0 to 20 % and redistribution is not done directly and the tax is in general used for general governmental purposes like infra-structure and education.

Although there is a close relation between r and g, the difference $r - g$ has not necessarily a meaning with respect to inequality and this is in accordance with De la Fonteijne (2014-1) where we analyzed and disagreed with the conclusion of Piketty on this subject.

We like to point out that the abstract concept of normalization with respect to the driving force, Jones is presenting, has nothing to do with growth itself and is not allowed. The subtraction of a growth component represents at best the 100 % inheritance tax which is equally redistributed to newborns. Of course, there is nothing wrong with normalizing an individual wealth distribution.

We even can conclude that if the savings rate is equal for all individuals and with a certain inheritance tax rate greater than zero, which is equally redistributed in total, all existing wealth inequality will vanish on the long run, except from the induced inequality caused by a death birth process following the formula of Jones. This conclusion is independent of the economy model you are using and holds for any type of production function you prefer.

The AK model is too simple to understand the different causes of wealth inequality, nor does it give insight whether or not inequality does harm the economy in total. It is better to discriminate between the different kinds of capital and their return on investment. Especially if you want to differentiate between wealth saved by capital returns or by income savings you needs to consider labor as well as capital.

The use of the variable *r* as net or gross return on capital is not always clear and misleading, which is also the case in the work of Piketty.

As a final point, focusing on the presented Pareto distributions generating mechanism is a nice concept from a mathematical point of view, but probably not from a practical point of view, in which case you can divert from exponential age distributions, equal individual driving force rate, etc., even if you restrict Pareto to the top income or top wealth distribution.

11. Acknowledgement

This paper is part of a study to reduce unemployment in a sustainable way whilst keeping governmental debt within sustainable limits and improve prosperity. In our opinion this is one of the most important things to achieve in society from a macro economic and from a participation society point of view. We are convinced we can provide a feasible solution to this problem as part of a sustainable society.

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