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Less is More: Capital Theory and Almost Irregular-Uncontrollable Actual Economies*

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Capital theory and the associated with it price effects consequent upon changes in the distributive variables hold centre stage when it comes to the internal consistency of both classical and neoclassical theories of value. This paper briefly reviews the literature and then focuses on the detected skew eigenvalue distribution of the vertically integrated technical coefficients matrices of actual economies. The findings prompt the use of the Schur triangularization theorem for the construction even of a single industry from the input-output structure of the entire economy. Such a hyper-basic industry, in combination with hyper-non-basic industries, embodies properties that may capture the behaviour of the entire economic system. Thus, we can derive some meaningful results consistent with the available empirical evidence, which finally suggest that actual economies tend to respond as 'irregular-uncontrollable' systems.

Key words: Almost irregular-uncontrollable economy, Capital theory, Eigenvalue distribution, Hyper-basic industry, Effective rank

JEL classifications: B21, B51, C67, D46, D57

1. Introduction

One of the enduring, puzzling and still not from the fully resolved issues in economic theory is the effects of changes in income distribution on commodity prices. We know that Ricardo ([1821] 1951, pp. 30-43) was from the first to formulate the question and to argue that a definitive answer can be only obtained with the possession of an "invariable measure of value". That is, a commodity whose value would, under all technological and distributional circumstances, remain the same and using this as the *numéraire* commodity, we could identify the source of changes in the prices of all other commodities. Ricardo devoted in vain his entire intellectual life to defining either

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analytically or practically such a standard of value, which would remain invariant to both changes in income distribution and production conditions. Marx ([1894] 1959, Chap. 11) also faced a similar problem and proposed a solution on the basis of the difference of an industry's capital-intensity from the economy-wide average capital-intensity.

The advent of neoclassical theory at the end of the nineteenth century defined relative prices as indexes of relative scarcity. As a consequence, the prices of factors of production were theorized to move monotonically in the upward or downward direction with changes in income distribution. However, the determination of the price of a unit of capital in a way which would be consistent with the premises of the neoclassical theory was very hard to pinpoint. Robinson (1953) inspired by Piero Sraffa's teaching and writings exposed the inconsistencies in the neoclassical theorization of capital as a 'factor of production'. Subsequently, Sraffa (1960) changed fundamentally the established ideas on the relations between commodity prices and income distribution.

The underlying idea in Sraffa's (1960) analysis is that an industry's capital-intensity depends on changes in income distribution which may initiate complex movements in relative prices that may even alternate the characterization of an industry from capital to labour intensive and *vice versa*. Consequently, the old classical rule according to which the change in relative prices is strictly related to the capital-intensity of the industry relative to others or some kind of invariable average does not in general hold. Furthermore, Sraffa showed that it is possible that a capital-intensive technique may be chosen for both low and high rates of profit, a result that runs contrary to the neoclassical theory of scarcity prices. Under these circumstances, the determination of a well-behaved demand for capital schedule is in question, and if such a core schedule is questioned, then the presence of interdependency rules out the possibility of confidently determining the remaining important demand and supply schedules. The consequences for neoclassical analysis are thus quite upsetting (Tsoulfidis, 2010, p. 207).

Thus, the famous Cambridge capital controversies of the 1960s and 1970s have shown that long-period prices do not necessarily display monotonic paths with respect to changes in income distribution; as a consequence, the profit rate could not be taken as a consistent index of the relative scarcity of capital. These theoretical findings, however, were not corroborated by analogous empirical evidence, either because such research was extremely difficult to pursue at that time, or for the reason that, if a theory is found logically inconsistent, then there is no any pressing reason to test it empirically.

In each case, those controversies were conducted on purely theoretical grounds without necessarily having any contact with actual economic data, as this can be judged by the numerical examples utilized on both sides of the debate. For instance, on the US side Cambridge, there is Samuelson's (1962) parable of a one-commodity world, the associated with it strictly linear wage-profit rate (WPR) curves and the well-behaved supply of capital schedules. And this was sharply contrasted to the multi-commodity-world of the UK side Cambridge Sraffian economists, whose numerical examples and WPR curves are characterized by 'any' number of curvatures and possible shapes of supply of capital schedules. These 'exotic' shapes of the WPR curves indicate that the price-profit rate (PPR) curves can display extremes and inflection points rendering untenable the neoclassical theorization of prices as scarcity indexes and proving that the capital-intensity could not be defined in any uncontroversial way.

Summing-up, the theoretical findings were more in favour of the Sraffa-inspired critique as Samuelson (1966), the leading figure from the neoclassical camp, admitted. The same is true with Robert M. Solow and Charles E. Ferguson, while the list could be extended to include Lucas (1988), who opined that the debate was won from the Cambridge UK side and so did Mas Colell (1989), by noting that the relationship between capital-intensity and profit rate could take 'any' possible shape. Although major neoclassical economists have admitted the weakness of their theory to come to terms with the results of the capital theory controversies, they, however, characterized them 'paradoxical', in the sense that these results contradict with the widely accepted principles of neoclassical theory which are put on par with 'common sense'. This characterization might be justifiable, at least partly, for phenomena that may be exempted from the 'law of consumer's demand', such as, for instance, the well-known 'Giffen paradox'. But, it is hard to accept it for those that arise in international trade and came to be known as 'Leontief paradox' (see, e.g. Metcalfe and Steedman, 1979; Paraskevopoulou et al., 2016) and much more difficult to accept it for those findings of the capital theory that undermine the core propositions of the neoclassical theory. It seems, however, that, despite these serious unsolved complications, the neoclassical economists gradually lost interest in the capital theory debates, while the newer generations of neoclassical economists rarely refer to these issues and continue using various forms of 'production functions' as if there was no problem with the theory they are based on.

In the meantime, the production price-wage-profit rate system of actual economies (but, *ex hypothesis*, linear, closed and single-product) has been examined in a relatively large number of studies. From Sekerka et al. (1970), Krelle (1977) and Shaikh (1984, 1998) onwards, the key stylized findings in these empirical studies are that:

(i). The vectors of vertically integrated labour coefficients, or labour values, and 'actual production prices' are close to each other, as judged by alternative measures of deviation.[1] The estimated deviations are not too sensitive to the type of measure used for their evaluation (Mariolis and Tsoulfidis, 2010, 2014a; Mariolis, 2011; Mariolis and Soklis, 2011).

(ii). The 'actual profit rate' is usually no greater than 50% of its maximum feasible value and, most of the time, is in the range of 30% to 40%. Therefore, the polynomial approximation (Steedman, 1999) of the actual production prices, expressed in terms of Sraffa's (1960, Chaps. 4-5) Standard commodity (SSC), through 'dated quantities of embodied labour' requires the inclusion of just a few terms (Tsoulfidis and Mariolis, 2007).

(iii). Non-monotonic PPR curves, expressed in terms of SSC, are not only relatively rare (i.e. not significantly more than 20% of the tested cases) but also have no more than one extreme point. Cases of reversal in the direction of deviation between production prices and labour values ('price-labour value reversals') are rarer. In fact, the price-movement is, more often than not, governed by the 'capital-intensity effect', i.e. by the difference between the industry's vertically integrated capital-intensity and the capital-intensity of the Sraffian Standard system (SSS), where the latter equals the reciprocal of the maximum feasible value of the profit rate. However, this 'traditional flavour' condition *can* be modified by the 'price effect', i.e. the revaluation of the industry's vertically integrated capital, which depends on the entire economic system and, therefore, is not predictable at the level of any single industry (Sraffa 1960, pp. 14-15; Pasinetti 1977, pp. 82-84; Mariolis et al. 2015). Empirical evidence associated with quite diverse economies, and spanning different time periods, showed that the capital-intensity effect overshadows the price effect, although there are cases where the latter effect is strong enough that it can supersede the former giving rise to extrema and 'price-labour value reversals' (also see Tsoulfidis and Mariolis, 2007, Tsoulfidis, 2008, Mariolis and

[1] The terms 'actual production prices' and 'actual profit rate' are used to signify production prices and profit rate that correspond to the 'actual' real wage rate. The latter is estimated on the basis of the available input-output data.

Tsoulfidis, 2009). It then follows that the idea of representing the PPR curves through linear or, *a fortiori*, quadratic approximations is absolutely justifiable and empirically powerful (Bienenfeld, 1988; Shaikh, 2012; Iliadi et al., 2014).

(iv). Although the actual economies deviate considerably from the Ricardo-Marx-Dmitriev-Samuelson 'equal value compositions of capital' case, the WPR curves are near-linear, i.e. the correlation coefficients between the distributive variables tend to be above 99%, and their second derivatives change sign no more than once or, very rarely, twice, irrespective of the *numéraire* chosen (Leontief, 1985; Ochoa, 1989; Petrović, 1991; Han and Schefold, 2006).

All these findings imply that, although the actual economies *cannot* be analyzed on the basis of 'neoclassical parables', the role of price-feedback effects is actually of limited quantitative significance.

In the late-2000s the relevant research took a new direction on the basis of the modern classical theory of value corollaries and the spectral representation (or, in more general terms, the 'modern state variable representation') of linear systems.[2] It has been particularly pointed out that the functional expressions of the price-wage-profit rate relationships admit lower and upper norm bounds, while their monotonicity could be connected to the characteristic value distribution of the matrix of vertically integrated technical coefficients and, therefore, to the 'effective rank (or dimensions)' of this matrix. Since nothing can be said a priori about this crucial factor in real-world economies, the examination of actual input-output data became absolutely necessary.

Thus, it has been well-ascertained that, across countries and over time, the moduli of the eigenvalues as well as the singular values of actual economies follow exponentially decaying trends. Moreover, when the capital stock matrices are taken into account, they are characterized by a nearly 'L-shaped' pattern. Namely, in the latter, more realistic case, the decay of the characteristic values is remarkably faster (see Mariolis and Tsoulfidis, 2016b). This new stylized fact implies that only a few eigenvalues really matter for the observed shapes of the P-WPR curves, which is another way to say that these curves tend to be similar to those of low-dimensional systems. In effect, it seems that matrix similarity transformations of the price system that result in only a few industries extract the essential features contained in the original-actual

[2] See Schefold (2008, 2013a), Mariolis and Tsoulfidis (2009, 2011, 2014b). For further analytical investigations, see Schefold (2013b, 2016), Mariolis (2015a), Mariolis and Tsoulfidis (2016a), Shaikh (2016, Chap. 9).

system and provide the basis for constructing reliable approximations of the observed relationships.

The objective of this paper is to provide a unified treatment of both the theoretical and empirical fundamentals of this recently developed research line that suggests – not the irrelevance of Sraffian analysis but – a *new logic* approach for (i) revealing the essential properties of the static and dynamic behaviour of a linear, closed and single-product system as a whole; (ii) determining the extent to which these properties deviate from those predicted by the traditional theories of value; and (iii) deriving meaningful theoretical results consistent with the available empirical evidence. Supported also by new empirical evidence, the present paper shows that the effective rank of actual economies is rather low and, therefore, their price characteristic features tend to be similar to those of "uncontrollable" (Kalman, 1961) and "irregular" (Schefold, 1971) systems. Hence, actual economies may be described by just a few, or even a single, 'hyper-basic' industries without significant loss of information.

The remainder of the paper is structured as follows. Section 2 treats the theoretical and empirical fundamentals of the new research line and, thus, points out the uncontrollable-irregular features of actual economies. Section 3 provides new evidence on the spectral properties of actual input-output structures using data from the US and other major economies. Finally, Section 4 concludes the paper.

2. Spectral Decomposition of the Price System and Actual Economies

2.1. Preliminary relations

Let us suppose a linear circulating capital model of production described by the irreducible $n \times n$ matrix of direct technical coefficients, **A**, whose Perron-Frobenius eigenvalue is less than one, and the surplus produced is distributed between profits and wages. Let $\mathbf{l}$ be the $1 \times n$ vector of direct labour coefficients, w the uniform money wage paid *ex post*, and r the economy-wide profit rate.[3] On the basis of these assumptions we can write the vector of production prices, $\mathbf{p}$, as follows

[3] The transpose of a $1 \times n$ vector $\mathbf{y} \equiv [y_j]$ is denoted by $\mathbf{y}^{\mathrm{T}}$. Furthermore, $\lambda_{\mathbf{A}1}$ denotes the Perron-Frobenius eigenvalue of a semi-positive $n \times n$ matrix $\mathbf{A} \equiv [a_{ij}]$, and $(\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}}, \mathbf{y}_{\mathbf{A}1})$ the corresponding eigenvectors, while $\lambda_{\mathbf{A}k}$, $k = 2,...,n$ and $|\lambda_{\mathbf{A}2}| \geq |\lambda_{\mathbf{A}3}| \geq ... \geq |\lambda_{\mathbf{A}n}|$, denote the non-dominant

$$\mathbf{p} = w\mathbf{l} + (1+r)\mathbf{pA} \tag{1}$$

After rearrangement, equation (1) becomes

$$\mathbf{p} = w\mathbf{v} + r\mathbf{pH}$$

or

$$\mathbf{p} = w\mathbf{v} + \rho\mathbf{pJ} \tag{2}$$

where $\mathbf{v} \equiv \mathbf{l}[\mathbf{I}-\mathbf{A}]^{-1}$ denotes the vector of vertically integrated labour coefficients, or labour values, and $\mathbf{H} \equiv \mathbf{A}[\mathbf{I}-\mathbf{A}]^{-1}$ the vertically integrated technical coefficients matrix. Moreover, $\rho \equiv rR^{-1}$, $0 \le \rho \le 1$, denotes the relative profit rate, which equals the share of profits in the SSS, and $R \equiv \lambda_{\mathbf{A}1}^{-1} - 1 = \lambda_{\mathbf{H}1}^{-1}$ the maximum possible profit rate (i.e. the profit rate corresponding to $w = 0$ and $\mathbf{p} > \mathbf{0}$), which equals the ratio of the net product to the means of production in the SSS (see Sraffa, 1960, pp. 21-23). Finally, $\mathbf{J} \equiv R\mathbf{H}$ denotes the normalized vertically integrated technical coefficients matrix, $\lambda_{\mathbf{J}1} = R\lambda_{\mathbf{H}1} = 1$, and the moduli of the normalized eigenvalues of system (2) are less than those of system (1), i.e. $|\lambda_{\mathbf{J}k}| < |\lambda_{\mathbf{A}k}|\lambda_{\mathbf{A}1}^{-1}$ holds for all k (see, e.g. Mariolis and Tsoulfidis, 2014b, pp. 213-214).

If SSC is chosen as the *numéraire*, i.e. $\mathbf{pz}^{\mathrm{T}} = 1$, where $\mathbf{z}^{\mathrm{T}} \equiv [\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}}$ and $\mathbf{lx}_{\mathbf{A}1}^{\mathrm{T}} = 1$, then the WPR curve is the following linear relation

$$w = 1 - \rho \tag{3}$$

and, if $\rho < 1$,

$$\mathbf{p} = (1-\rho)\mathbf{v}[\mathbf{I}-\rho\mathbf{J}]^{-1} = (1-\rho)\mathbf{v}[\mathbf{I} + \rho\mathbf{J} + (\rho\mathbf{J})^2 + (\rho\mathbf{J})^3 + ...] \tag{4}$$

which gives the production prices, expressed in terms of SSC, as polynomial functions of ρ.[4] From equations (2), (3) and (4) it follows that:

(i) $w(0) = 1$;

(ii) $\mathbf{p}(0) = \mathbf{v}$;

(iii) $w(1) = 0$;

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eigenvalues, and $(\mathbf{x}_{\mathbf{A}k}^{\mathrm{T}}, \mathbf{y}_{\mathbf{A}k})$ the corresponding eigenvectors. Finally, $\mathbf{I}$ denotes the $n \times n$ identity matrix, and $\mathbf{e}_j$ the jth unit vector.

[4] If wages are paid *ex ante*, then the WPR curve is non-linear, i.e. $w = (1+R\rho)^{-1}(1-\rho)$, and ρ is no greater than the share of profits in the SSS; however, equation (4) holds true. In the case of fixed capital *à la* Leontief (1953)-Bródy (1970), $\mathbf{H}$ should be replaced by $\mathbf{K}[\mathbf{I}-\mathbf{A}]^{-1}$, where $\mathbf{K}$ denotes the matrix of capital stock coefficients.

(iv) $\mathbf{p}(1)$ is the left Perron-Frobenius eigenvector of $\mathbf{J}$, expressed in terms of SSC, i.e.

$$\mathbf{p}(1) = (\mathbf{y}_{\mathbf{J}1}\mathbf{z}^{\mathrm{T}})^{-1}\mathbf{y}_{\mathbf{J}1} = (\mathbf{y}_{\mathbf{J}1}[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}})^{-1}\mathbf{y}_{\mathbf{J}1}$$

or, since $[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}} = (1-\lambda_{\mathbf{A}1})\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}}$ and matrices $\mathbf{A}$ and $\mathbf{J}$ have the same eigenvectors,

$$\mathbf{p}(1) = [(1-\lambda_{\mathbf{A}1})\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}]^{-1}\mathbf{y}_{\mathbf{J}1} \tag{5}$$

(v) excluding the trivial and unrealistic case of equal value compositions of capital, where $\mathbf{p}(0) = \mathbf{p}(1)$ and, therefore, prices are constant and equal to the labour values, as well as the case of two-industry systems, where the PPR curves are necessarily monotonic, changes in income distribution may activate complex capital revaluation effects, which imply that the direction of relative price-movements *cannot* be known a priori.

All traditional statements with respect to the exact price movements cannot, in general, be extended beyond a world where (i) there are no produced means of production; or (ii) there are produced means of production, while the profit rate on the value of those means of production is zero; or, finally, (iii) that profit rate is positive, while the economy produces one and only one, single or composite, commodity (see Samuelson 1953-1954, pp. 17-19; Sraffa, 1960, Chap. 6; Salvadori and Steedman, 1985). Consequently, the conceptual and analytical difficulties of the traditional theories of value and distribution arise from the existence of complex interindustry linkages in the realistic case of production of commodities *and* positive profits by means of commodities.

2.2. Turning to the outside world

It should, however, be taken into account that the *empirical* results usually give quasi-linear price movements in terms of SSC. These finding could be explained by the shape of the eigenvalue distribution: the eigenvalues of actual matrices $\mathbf{J}$ follow a rectangular hyperbola-like distribution in the case of circulating capital, and a nearly L-shaped form in the – more realistic – case of the presence of fixed capital stocks. In other words, the stylized facts show that (i) the non-dominant eigenvalues of $\mathbf{J}$ are, as a statistical mean, by far lower than 1; and (ii) the large gap between the second and dominant eigenvalues of $\mathbf{J}$ allow pretty accurate approximations of the PRP trajectories through low order spectral approximations (Mariolis and Tsoulfidis, 2016a, 2016b).

Thus, although the actual matrices $\mathbf{J}$ appear to have full rank, the particular distribution of their eigenvalues gives, however, rise to an effective rank much lower than the actual rank. It then follows that even an effective rank (or dimensionality) of 1 is sufficient for a satisfactory approximation to the PPR trajectories.[5]

2.3. Finding the 'Archimedean point'

In order to zero in on this fundamental point, which is supported by the available empirical evidence, we decompose matrix $\mathbf{J}$ to its 'spectral representation' (see, e.g. Meyer 2001, 517-518)

$$\mathbf{J} = (\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1} + \sum_{k=2}^{n} \lambda_{\mathbf{J}k}(\mathbf{y}_{\mathbf{J}k}\mathbf{x}_{\mathbf{J}k}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}k}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}k} \tag{6}$$

If there are strong quasi-linear dependencies amongst the technical conditions of production in all the vertically integrated industries, then $rank[\mathbf{J}] \approx 1$, or $|\lambda_{\mathbf{J}k}| \approx 0$ for all k, and, therefore, equation (6) implies that $\mathbf{J} \approx \mathbf{J}^{\mathrm{A}} \equiv (\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1}$. Hence, from equation (4) it follows that

$$\mathbf{p} \approx \mathbf{p}^{\mathrm{A}} \equiv (1-\rho)\mathbf{p}(0)[\mathbf{I} - \rho\mathbf{J}^{\mathrm{A}}]^{-1}$$

or, by applying the Sherman-Morrison formula,[6]

$$\mathbf{p} \approx \mathbf{p}^{\mathrm{A}} \equiv (1-\rho)\mathbf{p}(0)[\mathbf{I} + (1-\rho)^{-1}\rho(\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1}]$$

or, invoking equations (5) and $\mathbf{p}(0)\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}} = (1-\lambda_{\mathbf{A}1})^{-1}$,

$$\mathbf{p} \approx \mathbf{p}^{\mathrm{A}} = (1-\rho)\mathbf{p}(0) + \rho\mathbf{p}(1) \tag{7}$$

namely, $\mathbf{p}^{\mathrm{A}}$ is a linear ('convex') combination of the extreme, economically significant, values of the price vector, $\mathbf{p}(0)$ and $\mathbf{p}(1)$.

This eigenvalue decomposition *rank-one* approximation for the price vector has the following properties:

(i). It is linear and exact at the extreme values of ρ.

(ii). Its accuracy is directly related to the magnitudes of $|\lambda_{\mathbf{J}k}|^{-1}$.

(iii). When $rank[\mathbf{J}] = 1$, it becomes exact for all ρ.

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[5] For the corresponding treatment of the WRP curves; see Mariolis (2015a, 2015b) and Mariolis and Tsoulfidis (2016a, Chap. 5).

[6] Let $\boldsymbol{\chi}$, $\boldsymbol{\psi}$ be arbitrary $n-$vectors. Then $\det[\mathbf{I} - \boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\psi}] = 1 - \boldsymbol{\psi}\boldsymbol{\chi}^{\mathrm{T}}$ and, iff $\boldsymbol{\psi}\boldsymbol{\chi}^{\mathrm{T}} \neq 1$, $[\mathbf{I} - \boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\psi}]^{-1} = \mathbf{I} + (1 - \boldsymbol{\psi}\boldsymbol{\chi}^{\mathrm{T}})^{-1}\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\psi}$ (see, e.g. Meyer 2001, p. 124).

In that latter, ideal-type (in the Weberian sense) case, i.e. $\mathbf{J}=\mathbf{J}^{\mathrm{A}}$, the economy exhibits the following two essential characteristics:

(i). Irrespective of the direction of the labour value vector $\mathbf{p}(0)=\mathbf{v}$, it holds that

$$\mathbf{p}(0)\mathbf{J}^{h}=\mathbf{p}(0)(\mathbf{J}^{\mathrm{A}})^{h}=[(1-\lambda_{\mathbf{A}1})\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}]^{-1}\mathbf{y}_{\mathbf{J}1}=\mathbf{p}(1),\ h=1,2,...$$

since

$$(\mathbf{J}^{\mathrm{A}})^{h}=(\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-h}(\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{h-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1}=\mathbf{J}^{\mathrm{A}}$$

Hence, the nxn 'Krylov matrix'

$$[\mathbf{p}^{\mathrm{T}}(0),\mathbf{J}^{\mathrm{T}}\mathbf{p}^{\mathrm{T}}(0),...,[\mathbf{J}^{\mathrm{T}}]^{n-1}\mathbf{p}^{\mathrm{T}}(0)]^{\mathrm{T}}$$

has rank equal to 2 and, therefore, the economy is said to be 'irregular' or, more specifically, 'regular of rank 2'.[7] This means that the price vectors relative to any 3 distinct values of the profit rate ($0\leq\rho<1$) are linearly dependent (see Bidard and Salvadori, 1995).

By contrast, an $n-$economy is said to be 'regular of rank n' or 'completely regular' iff the aforementioned Krylov matrix has rank equal to n or, equivalently, iff no right eigenvector of $\mathbf{J}$ is orthogonal to $\mathbf{v}$. In that case, the price vectors relative to any n distinct values of the profit rate are linearly independent. The concepts of 'regularity/irregularity' have been introduced by Schefold (1971), who argued that irregular systems are not generic:

> [T]he price vector of a [completely] regular Sraffa system is not only not constant, but its variations in function of the rate of profit result in a complicated twisted curve such that the n price vectors belonging to n different levels of the rate of profit […] span a $(n-1)-$dimensional hyperplane which never contains the origin […]. [T]he [completely] regular systems are the rule from a mathematical point of view […] the set of irregular Sraffa systems with $(n,n)-$input-output-matrices is of measure zero in the set of all Sraffa systems with the same number of commodities and industries. But this observation taken by itself does not mean much. The set of all semi-positive decomposable $(n,n)-$matrices is also of measure zero in the set of all semi-positive $(n,n)-$matrices, and yet it is quite clear that the analysis of the "exceptional" decomposable matrices is of greatest economic interest, although they are more difficult to handle than indecomposable matrices. There is an excellent economic

[7] Obviously, in the theoretical case of equal value compositions of capital, the economy is regular of rank 1, irrespective of the rank of $\mathbf{J}$.

reason why decomposable systems are important: pure consumption goods and other non-basics exist; therefore decomposable systems exist. I should like to argue that matters are quite different with irregular systems. I believe that there is no economic reason why real systems should not be [completely] regular or why irregular systems should exist in reality; irregularity is only a fluke, or, at best, an approximation. (Schefold, 1976, p. 27)

In order to complete the picture, it is necessary to explicate that these concepts are algebraically equivalent to those of 'controllability/uncontrollability' that have been introduced by Kalman (1961) and apply to the following dynamic version of the price system:

$$\mathbf{p}_{t+1} = w_t\mathbf{v} + \bar{\rho}\mathbf{p}_t\mathbf{J}\,,\; t = 0,1,...$$

where $\bar{\rho}$ denotes the exogenously given nominal relative profit rate, and $\mathbf{p}_0 = \mathbf{0}$ (Mariolis, 2003). Iff the aforementioned Krylov matrix has rank equal to n, then this dynamic price system is said to be completely controllable, which means that the initial state $\mathbf{p}_0$ can be transferred, by application of w_t, to any state, in some finite time.

In our present case, i.e. $\mathbf{J} = \mathbf{J}^{\mathrm{A}}$, $\mathbf{p}_{t+1}$ is a linear ('conical') combination of $\mathbf{v}$ and $\mathbf{y}_{\mathbf{J}1}$, irrespective of the input sequence, w_t, i.e.

$$\mathbf{p}_{t+1} = w_t\mathbf{v} + (w_0\bar{\rho}^t + w_1\bar{\rho}^{t-1} + ... + w_{t-1}\bar{\rho})b\mathbf{y}_{\mathbf{J}1}$$

where $b \equiv (\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}(\mathbf{v}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})$ (compare with equation (7)). This uncontrollable system (or, more generally, the low-rank controllable systems) seems to have some correspondence with the 'autopoietic systems' of living and social systems theory (see, e.g. Dekkers, Chap. 7).

(ii). The Schur triangularization theorem (see, e.g. Meyer, 2001, pp. 508-509) implies that $\mathbf{J}^{\mathrm{A}}$ can be transformed, via a semi-positive similarity matrix $\mathbf{T}$, into

$$\tilde{\mathbf{J}}^{\mathrm{A}} \equiv \mathbf{T}^{-1}\mathbf{J}^{\mathrm{A}}\mathbf{T} = \begin{bmatrix} 1 & [\tilde{\mathbf{J}}_{12}^{\mathrm{A}}]_{1\times(n-1)} \\ \mathbf{0}_{(n-1)\times 1} & \mathbf{0}_{(n-1)\times(n-1)} \end{bmatrix} \tag{8}$$

where the first column of $\mathbf{T}$ is $\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}$, the remaining columns are arbitrary, and the vector $\tilde{\mathbf{J}}_{12}^{\mathrm{A}}$ is necessarily positive (Mariolis, 2013). If, for instance,

$$\mathbf{T} = [\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}, \mathbf{e}_2^{\mathrm{T}}, ..., \mathbf{e}_n^{\mathrm{T}}] \tag{8a}$$

then

$$\tilde{\mathbf{J}}_{12}^{\mathrm{A}} = (\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}[y_{2\mathbf{J}1}, y_{3\mathbf{J}1}, ..., y_{n\mathbf{J}1}] \tag{8b}$$

The similarity matrix, $\mathbf{T}$, defines a new coordinate system in which the original system matrix, $\mathbf{J} = \mathbf{J}^{\mathrm{A}}$, is represented by a semi-positive triangular matrix, $\tilde{\mathbf{J}}^{\mathrm{A}}$, which has the eigenvalues along its main diagonal. Thus, the original price system (2) is decomposed as follows:

$$\mathbf{p} = w\mathbf{v} + \rho\mathbf{p}(\mathbf{T}\tilde{\mathbf{J}}^{\mathrm{A}}\mathbf{T}^{-1})$$

or, post-multiplying by $\mathbf{T}^{-1}$,

$$\boldsymbol{\pi} = w\boldsymbol{\omega} + \rho\boldsymbol{\pi}\tilde{\mathbf{J}}^{\mathrm{A}} \tag{9}$$

where $\boldsymbol{\pi} \equiv \mathbf{pT}$, $\boldsymbol{\omega} \equiv \mathbf{vT}$ denote the transformed vectors of price and labour values, respectively, $\pi_1 = \mathbf{p}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}$ and $\omega_1 = \mathbf{v}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}$. The first equation in the transformed price system (9) corresponds to an industry producing a composite pure capital good, which is no more than the SSS, whereas the remaining equations correspond to non-uniquely determined industries producing pure consumption goods. It then follows that, even when the matrix $\mathbf{J}$ is *indecomposable*, the original system is equivalent to an economically significant and generalized (1 by $n-1$) Marx-Fel'dman-Mahalanobis (or, in more traditional terms, 'corn-tractor') system. Hence, the transformed industry producing the pure capital good can be characterized as 'hyper-basic'.

2.4. Matching the pieces

In the ideal-type case $rank[\mathbf{J}] = 1$ the economy is regular-controllable of rank 2 and economically equivalent to a decomposable system with one basic commodity and $n-1$ non-self-reproducing non-basics. Thus, on the one hand, the price side of the economy is 'a little more' complex than that of a pure labour theory of value economy and, at the same time, much simpler than that of a completely regular-controllable economy. In fact, its price side corresponds to that of the traditional neoclassical theory of value. On the other hand, the economy can be fully described by a triangular matrix with only n positive technical coefficients and, therefore, its production structure is 'a little' more complex than that of 'Austrian'-type economies, where the technical coefficients matrix is, by assumption, *strictly* triangular (see, e.g. Burmeister, 1974).

When $rank[\mathbf{J}] \geq 2$, the spectral representation of the system matrix continues to be a powerful tool for constructing higher-rank approximations of the PPR curves that may involve more than one hyper-basic industry and, thus, various ideal-types for analyzing the actual system.[8]

Finally, when, as is the case with actual economies, $rank[\mathbf{J}] = n$ *but* a particular eigenvalue distribution gives rise to an effective rank much lower than the actual rank, the economy tends to respond as an irregular-uncontrollable system. Consequently, the real, after-Sraffa paradox, in the sense of knowledge vacuum, is not the 'paradoxes in capital theory' but the very fact that, in correspondence to the rectangular hyperbola-like distribution of their eigenvalues, actual economies constitute 'almost irregular-uncontrollable' systems.

3. Empirical Evidence

Starting off with the distribution of eigenvalues for eight major economies, we provide new empirical evidence that supports the previous analysis. We restrict ourselves to a single year (2011) provided that the detected configuration of eigenvalues is pretty much the same for all the actual economies that have been tested so far (Mariolis and Tsoulfidis, 2016a, Chaps. 5-6, 2016b). We use data from the World Input-Output Database (http://www.wiod.org), where the number of industries is not different across economies and the data are compiled with the same methods and they are expressed in dollars thereby facilitating inter-country comparisons (also see Timmer et al., 2015).

Table 1 reports the moduli of the eigenvalues of $\mathbf{J}$ (sorted in descending order) and three metrics of distribution of moduli of the non-dominant eigenvalues, namely, (i) the arithmetic mean, *AM*, that assigns equal weight to all moduli; (ii) the geometric mean, *GM*, that assigns more weight to lower moduli, and, therefore, is more appropriate in detecting the central tendency of an exponential set of numbers; and (iii) the so-called spectral flatness, *SF*, defined as the ratio of the geometric mean to the arithmetic mean, and shows how spiky or flat is the distribution under consideration.

[8] Spectral higher-rank approximations can also be derived from the singular values of $\mathbf{J}$, i.e. from the square roots of the eigenvalues of the symmetric matrix $\mathbf{J}\mathbf{J}^{\mathrm{T}}$. For the relationships between these spectral approximations and Bienenfeld's (1988) and Steedman's (1999) polynomial approximations, see Mariolis and Tsoulfidis (2016a, Chap. 5).

Table 1. Distribution of the moduli of eigenvalues; Australia, Brazil, P.R. China, France, Germany, India, Japan, and USA, year 2011[9]

Eigenvalues Ranking	AUS	BRZ	CHN	FRC	GER	IND	JPN	USA
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.359	0.379	0.409	0.437	0.526	0.408	0.472	0.488
3	0.289	0.359	0.316	0.333	0.399	0.408	0.472	0.488
4	0.240	0.311	0.291	0.292	0.399	0.264	0.422	0.429
5	0.212	0.254	0.247	0.257	0.348	0.245	0.406	0.293
6	0.184	0.254	0.221	0.224	0.260	0.232	0.272	0.293
7	0.184	0.230	0.220	0.224	0.218	0.232	0.251	0.235
8	0.154	0.230	0.220	0.221	0.218	0.167	0.230	0.207
9	0.136	0.216	0.208	0.221	0.209	0.122	0.214	0.207
10	0.136	0.216	0.194	0.193	0.209	0.122	0.155	0.147
11	0.115	0.174	0.099	0.184	0.180	0.122	0.155	0.117
12	0.110	0.162	0.084	0.157	0.180	0.122	0.121	0.106
13	0.104	0.103	0.084	0.152	0.171	0.096	0.121	0.105
14	0.104	0.103	0.067	0.152	0.171	0.096	0.076	0.105
15	0.088	0.097	0.058	0.122	0.150	0.085	0.063	0.085
16	0.088	0.050	0.039	0.122	0.124	0.063	0.063	0.085
17	0.067	0.050	0.039	0.109	0.112	0.054	0.049	0.077
18	0.067	0.024	0.035	0.082	0.112	0.054	0.037	0.077
19	0.058	0.018	0.035	0.075	0.093	0.028	0.037	0.051
20	0.045	0.016	0.025	0.068	0.093	0.026	0.036	0.051
21	0.043	0.016	0.019	0.068	0.091	0.026	0.036	0.049
22	0.043	0.014	0.014	0.056	0.069	0.014	0.027	0.049
23	0.039	0.014	0.014	0.042	0.042	0.014	0.017	0.049
24	0.039	0.003	0.012	0.035	0.034	0.014	0.017	0.032
25	0.022	0.003	0.012	0.032	0.034	0.009	0.016	0.032
26	0.020	0.000	0.010	0.024	0.031	0.009	0.016	0.024
27	0.013	0.000	0.010	0.024	0.031	0.005	0.012	0.024
28	0.013	0.000	0.005	0.019	0.030	0.001	0.012	0.023
29	0.011	0.000	0.005	0.008	0.030	0.001	0.012	0.017
30	0.005	0.000	0.004	0.008	0.025	0.001	0.009	0.008
31	0.002	0.000	0.004	0.008	0.015	0.000	0.008	0.005
32	0.002	0.000	0.003	0.002	0.007	0.000	0.006	0.003
33	0.000	0.000	0.000	0.002	0.005	0.000	0.006	0.001
34	0.000	0.000	---	0.000	0.005	0.000	0.000	0.001
AM	0.117	0.126	0.121	0.146	0.165	0.119	0.143	0.146
GM	0.024	0.001	0.022	0.060	0.085	0.012	0.048	0.058
SF	0.201	0.006	0.179	0.415	0.517	0.099	0.339	0.397

[9] China's industry 19 (i.e. "Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel" contains no data; thus, the number of eigenvalues for this economy is 33. It is also noted that the last eigenvalues for India and Brazil are almost indistinguishable from zero, and this results in smaller geometric means and spectral flatness.

From these findings and the analytical numerical results it follows that:

(i). The moduli of the first non-dominant eigenvalues fall markedly, whereas the rest constellate in much lower values forming a 'long tail', and the empirical evidence so far suggests that they would not play any significant role in the observed shapes of P-WPR curves of the economies. After experimentation with various possible functional forms, we found that a single exponential functional form fits all the moduli data pretty well, as this can be judged by the high R-square–and the fact that all the estimated coefficients are statistically significant, with zero probability values. This form is displayed in Figure 1 and given by[10]

$$y = \alpha_0 + \alpha_1 \exp(x^{-0.2}), \; \alpha_0 < 0 \text{ and } \alpha_1 > 0$$

where y stands for the moduli of the eigenvalues (displayed on the vertical axis for each of our eight countries in Figure 1), whereas x stands for the respective ranking of the moduli of eigenvalues (displayed on the horizontal axis), and α_0, α_1 are parameters to be estimated. Finally, the regression equations are displayed inside the graphs for each of the countries under examination. This functional form is surprisingly similar to that associated with the findings of previous studies for a number of diverse economies and years quite distant from each other (see Mariolis and Tsoulfidis, 2016a, Chaps. 5-6, 2016b).

(ii). The complex (as well as the negative) eigenvalues tend to appear in the lower ranks, i.e. their modulus is relatively small. However, even in the cases that they appear in the higher ranks, i.e. second or third rank, the real part has been found to be much larger than the imaginary part, which is equivalent to saying that the imaginary part may even be ignored. Moreover, in the fewer cases that the imaginary part of an eigenvalue exceeds the real one, not only their ratio is relatively small but also the modulus of the eigenvalue can be considered as a negligible quantity. Finally, by inspecting all of our eigenvalues, we observe that, in *general*, the imaginary part gets progressively smaller. Consequently, the already detected distributions of the moduli can be viewed as fair representation of the distributions of the eigenvalues, and the complex eigenvalues play no perceptible role in the question at hand (however, they may be crucial in other topics; see, e.g. Rodousakis, 2012, 2016).

[10] In fact, we tried an optimization procedure to find the best possible form, and from the many possibilities, we opted for a simple but, at the same time, general enough to fit the moduli of the eigenvalues of all economies under consideration.

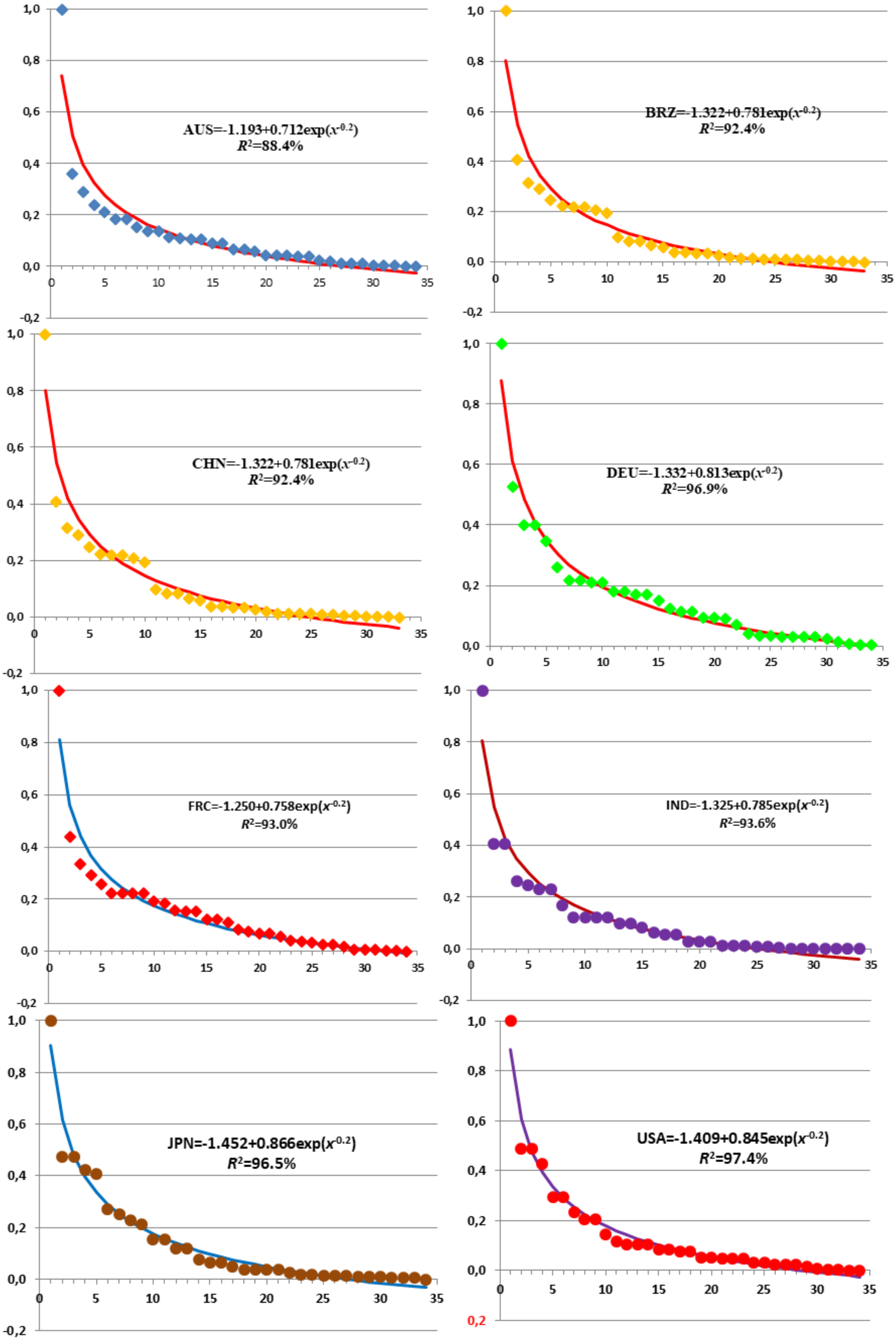


Figure 1. Exponential fit of the distribution of the moduli of the eigenvalues; Australia, Brazil, P.R. China, France, Germany, India, Japan, and USA, year 2011

By focusing on the US economy, the general picture remains the same for the much larger in dimensions input-output tables, which have been published by the Bureau of Economic Analysis (http://www.bea.gov), for the benchmark years 1997 ($n=488$), 2002 ($n=426$) and 2007 ($n=389$). Figure 2 displays the location of the eigenvalues in the complex plane for the year 2007, while Figure 3 displays the exponential fit of the distribution of the moduli of the eigenvalues (the axes are as in Figure 1).[11] A visual inspection of eigenvalues displayed in Figure 2 makes it abundantly clear that the majority of the non-dominant eigenvalues are crowded at very low values and bounded in a relatively small region of the unit circle.

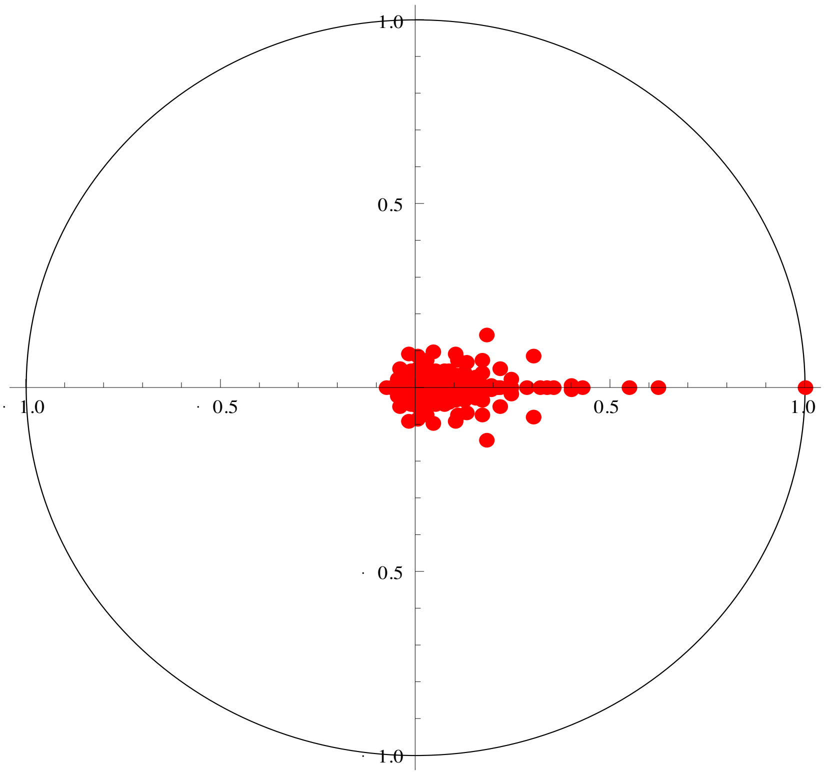


Figure 2. The location of the eigenvalues in the complex plane; USA, year 2007, $n=389$

[11] These results are rather similar to those for the years 1997 and 2002, that is $\alpha_0=-0.754$, $\alpha_1=0.641$, $\alpha_2=-0.3$, $R^2=98.7\%$ (year 1997) and $\alpha_0=-0.800$, $\alpha_1=0.678$, $\alpha_2=-0.3$, $R^2=99.1\%$ (year 2002); see Mariolis and Tsoulfidis (2014b, pp. 216 and 218).

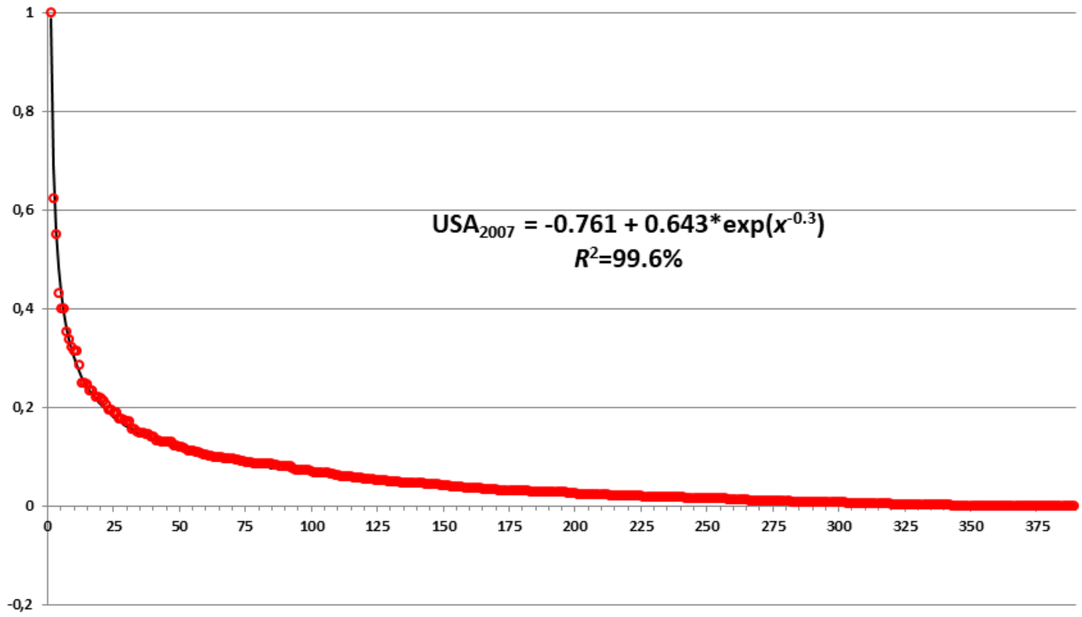


Figure 3. Exponential fit of the moduli of the eigenvalues; USA, year 2007, $n = 389$

Although the larger dimensions input-output tables provide the basis for a more detailed analysis of the P-WPR curves, the hitherto empirical evidence suggests, however, that they do not lead to essentially different conclusions regarding the features of the eigenvalue distributions and, therefore, the shape of those curves.[12] Thus, our next focus is on the data of the last available 15 x 15 input-output table of the US economy for the year 2014 (https://www.bea.gov/industry/io_annual.htm; provided that for the other years the results are not very different).[13] Figure 4 displays the exponential fit of the distribution of the moduli of the eigenvalues (the axes are as in Figure1), and Figure 5 displays the 15 trajectories of the production price-labour value ratios, expressed in terms of SSC and measured on the vertical axis, as functions of the relative profit rate, $0 \le \rho \le 1$, which is measured on the horizontal axis (see equations (4) and (5)).

[12] See Mariolis and Tsoulfidis (2011, pp. 105-109, 2014b, pp. 215-219); Gurgul and Wójtowicz, (2015); Pires and Shaikh (2015) and Shaikh (2016, Chap. 9).
[13] For the nomenclature of the industries, see Table A.1 in the Appendix.

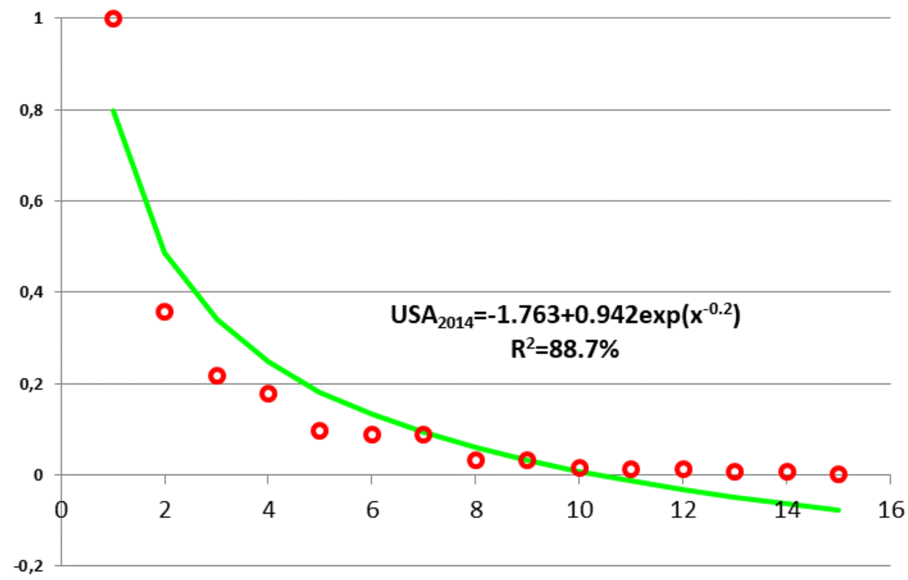


Figure 4. Exponential fit of the moduli of the eigenvalues; USA, year 2014, $n = 15$

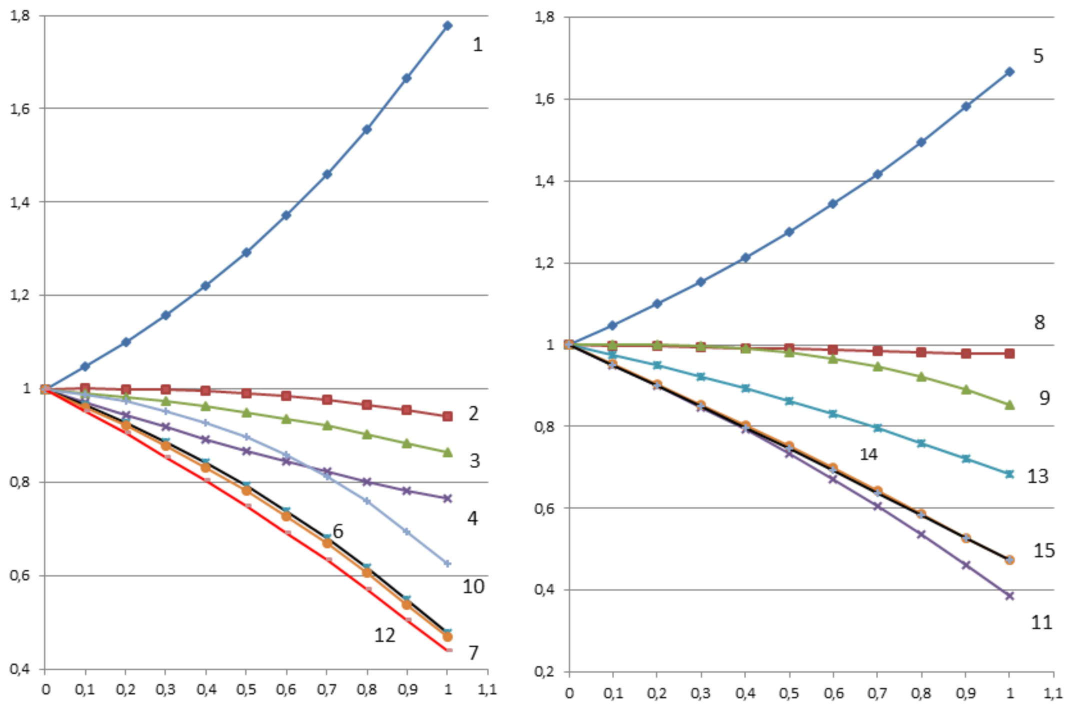
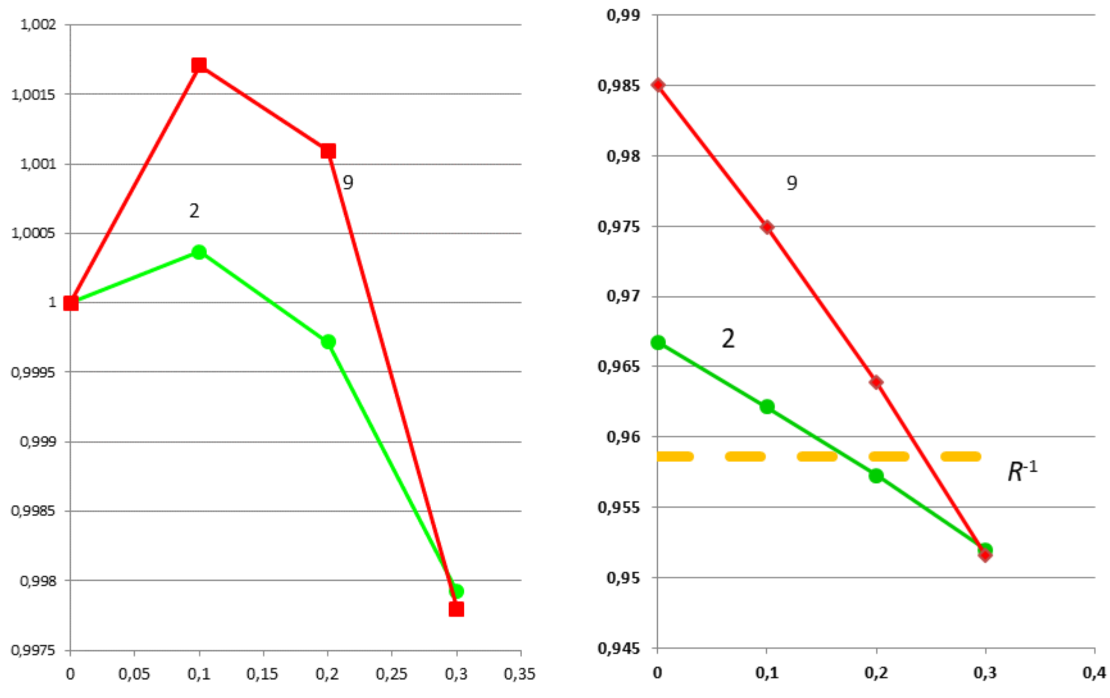


Figure 5. Production price-labour value ratios as functions of the relative profit rate; USA, year 2014, $n = 15$

From these results, which are in close accord with those of all past studies on this topic, it follows that:

(i). At first sight, the trajectories of prices appear that pretty much move monotonically (Figure 5). Thus, the capital-intensity effects overshadow the price effects, and this finding seems to match the underlying eigenvalue distribution (Figure 4).

(ii). However, a more detailed examination reveals that in two out of fifteen industries (or 13.3% of the cases under consideration) there is price-labour value reversals, that is, the production price-labour value curves associated with the vertically integrated industries 2 and 9 display maxima and then cross the line of equality between production prices and labour values at a positive value (and less than unity) of the relative profit rate. As the left-hand side panel of Figure 6 shows (where we restricted the relative profit rate up to 0.30 for reasons of visual clarity), the line of price-value equality is crossed at a relative profit rate in the range of 0.15 to 0.20 (industry 2) or 0.20 to 0.25 (industry 9). This phenomenon is reflected in the right-hand side panel of Figure 6, which depicts the relevant capital-intensities (measured on the vertical axis) and indicates that, as the relative profit rate increases, these two industries are *transformed* to labour intensive relative to the SSS.[14]

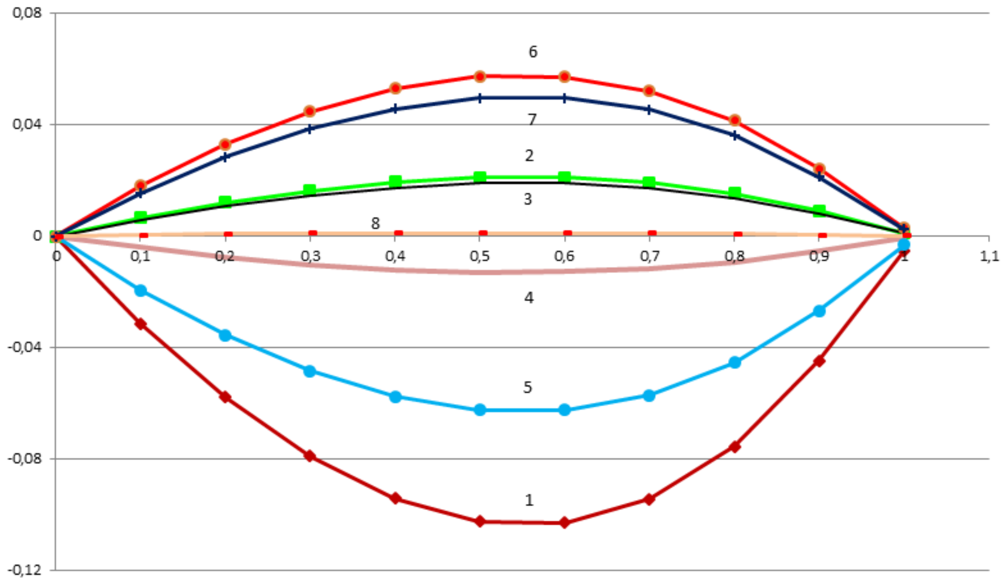


(a) Production price-labour value ratios (b) Capital-intensities

Figure 6. (a) Production price-labour value ratios; and (b) capital-intensities of vertically integrated industries 2 and 9, as functions of the relative profit rate; USA, year 2014, 15 industries

[14] The capital-intensity of the vertically integrated industry producing commodity j is estimated by $\mathbf{pHe}_j^{\mathrm{T}} v_j^{-1}$, while the capital-intensity of the SSS equals the reciprocal of the Standard ratio, $R^{-1} = \lambda_{\mathbf{H}1}$, which in our case is approximately equal to 0.959.

(iii). There are good grounds to test the empirical validity of the rank-1 approximation for the production price vector, which is based on matrix $\mathbf{J}^{\mathrm{A}} \equiv (\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1}$ and defined by equation (7). Since this approximation is linear and exact at the extreme, economically significant, values of the relative profit rate, its absolute errors, $p_j^{\mathrm{A}} - p_j$, are maximized at intermediate values of this distributive variable. Indeed, as Figure 7 shows, in the negative direction, the maximal error is almost 0.103 at $\rho \cong 0.60$ (and associated with industry 1) and, in the positive direction, the maximal error is 0.089 at $\rho \cong 0.60$ (and associated with industry 10). Moreover, the 'average absolute deviation', without the extreme values of ρ, is 2.67%. Thus, it can be concluded that, although higher-rank approximations would lead to more accurate price curves, the linear approximation works well.



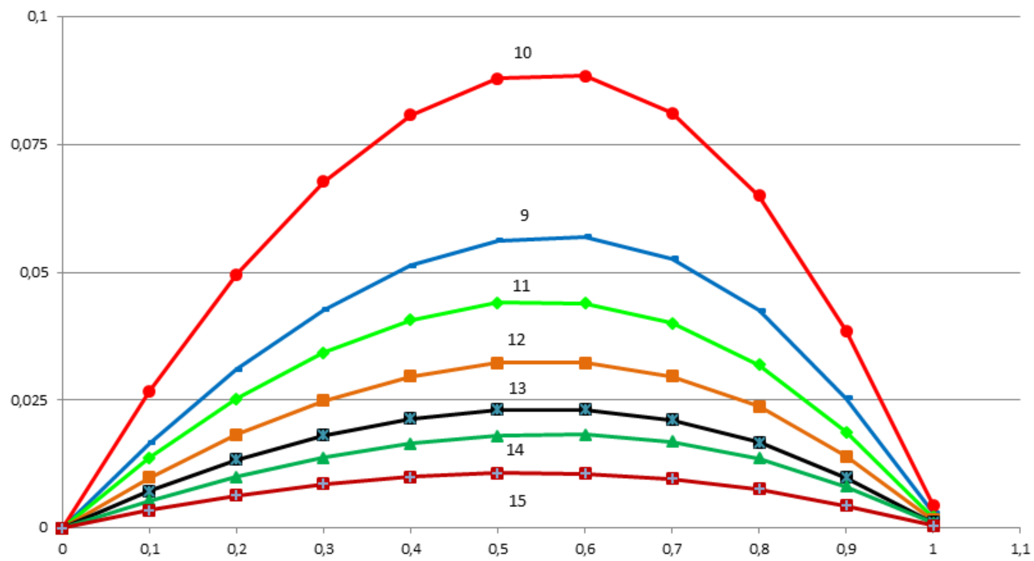


Figure 7. The absolute errors in the linear approximation of production prices as functions of the relative profit rate

Having established that $\mathbf{J}^{\mathrm{A}}$ is a good *first* approximation of $\mathbf{J}$, in the sense that both matrices give rise to price trajectories close to each other, we apply the similarity transformation, defined by equations (8) and (8a, b), to these matrices. It then follows that:

(i). The first row of the triangular, semi-positive and rank-1, matrix

$$\tilde{\mathbf{J}}^{\mathrm{A}} \equiv \mathbf{T}^{-1}\mathbf{J}^{\mathrm{A}}\mathbf{T} = \mathbf{T}^{-1}[\mathbf{J} - \mathbf{E}]\mathbf{T}$$

where $\mathbf{E} \equiv \sum_{k=2}^{n} \lambda_{\mathbf{J}k} (\mathbf{y}_{\mathbf{J}k}\mathbf{x}_{\mathbf{J}k}^{\mathrm{T}})^{-1} \mathbf{x}_{\mathbf{J}k}^{\mathrm{T}} \mathbf{y}_{\mathbf{J}k}$ denotes the error matrix (see equation (6)), is:

1	0.139	0.136	0.230	0.385
0.114	0.123	0.266	0.182	0.097
0.127	0.151	0.191	0.147	0.158

(ii). The first row of the non-triangular, non-semi-positive and rank-15, matrix $\tilde{\mathbf{J}} \equiv \mathbf{T}^{-1}\mathbf{J}\mathbf{T}$ is:

1	0.124	0.121	0.204	0.343
0.101	0.110	0.237	0.162	0.086
0.113	0.134	0.170	0.131	0.140

(iii). The ‘mean absolute percentage deviation’ between these two rows is 11.5%, whereas the ‘normalized d – distance’ (Steedman and Tomkins, 1998; Mariolis and Soklis, 2010, p. 94) is 2.3%.

Since, on the one hand, $\mathbf{J}^{\mathrm{A}}$ is a good approximation of $\mathbf{J}$ and, on the other hand, the first row of $\tilde{\mathbf{J}}^{\mathrm{A}}$ is a good approximation of the first row of $\tilde{\mathbf{J}}$, it follows that the essential PRP-information embedded in the original-actual economy is captured by the first row of $\tilde{\mathbf{J}}$ and extracted in the first row of $\tilde{\mathbf{J}}^{\mathrm{A}}$. Hence, the economy defined by the rank-1 matrix $\tilde{\mathbf{J}}^{\mathrm{A}}$ and producing a pure capital good and $n-1$ pure consumption goods, can be considered as a fairly representative simulacrum of the actual economy.

It goes without saying that, due to the nearly L-shaped form of the eigenvalue distribution, such a representation would be, in general, much more powerful in the presence of fixed capital stocks.

4. Concluding Remarks

The input-output data of actual single-product economies suggested that the majority of the non-dominant eigenvalues of the normalized vertically integrated technical coefficients matrices are crowded at very low values and bounded in a relatively small region of the unit circle. This stylized fact implies that the economically relevant movements of prices in function of the profit rate do not follow erratic patterns and, therefore, their approximation through low-order formulae (ranging from linear to quadratic) give accurate results, which can be improved only marginally by the inclusion of higher order terms. Consequently, actual economies constitute almost irregular-uncontrollable systems, which can be adequately described by a single or just a few hyper-basic industries.

The theoretical and empirical findings indicate that many of issues still lie hidden underneath the surface of capital theory debates and may be discovered through a combination of proper economic theory and use of data derived from the structure of actual economies. In such a direction, it appears that, although a lot is lost by one-commodity world postulations, embedded, explicitly or otherwise, in the traditional theories of value, there is room for using models with at most three basic commodities and many non-self-reproducing non-basics as surrogates for actual single-product systems. Given that little is gained by considering higher dimensions, it can be boldly stated that: less is more.

Future research work should (i) seek for possible relationships between measures of almost irregularity and monotonicity of price curves; (ii) delve into the proximate determinants of the irregular-uncontrollable aspects of actual economies; and (iii) incorporate the cases of many primary inputs (such as non-competitive imports) and joint-products.

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Appendix

The 15 industries input-output structure of the USA is reported in Table A.1. For the estimation procedures we refer to Mariolis and Tsoulfidis (2016a, Chap. 3) and the literature therein.

Table A.1. Nomenclature of 15 Industries, USA economy

1	Agriculture, forestry, fishing, and hunting
2	Mining
3	Utilities
4	Construction
5	Manufacturing
6	Wholesale trade
7	Retail trade
8	Transportation and warehousing
9	Information
10	Finance, insurance, real estate, rental, and leasing
11	Professional and business services
12	Educational services, health care, and social assistance
13	Arts, entertainment, recreation, accommodation, and food services
14	Other services, except government
15	Government