Forbidden zones for the expectation of a random variable. New version 1

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A forbidden zones theorem is deduced in the present article. Its consequences and applications are preliminary considered.

The following statement is proven: if some non-zero lower bound exists for the variance of a random variable, that takes on values in a finite interval, then non-zero bounds or forbidden zones exist for its expectation near the boundaries of the interval.

The article is motivated by the need of rigorous theoretical support for the practical analysis that has been performed for the influence of scattering and noise in the behavioral economics, decision sciences, utility and prospect theories.

If a noise can be one of possible causes of the above lower bound on the variance, then it can cause or broaden out such forbidden zones. So the theorem can provide new possibilities for mathematical description of the influence of such a noise.

The considered forbidden zones can evidently lead to some biases in measurements.

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Contents

1. Introduction

1.1. Moments, functions and bounds

The construction of bounds for functions of random variables is considered in a number of works. At that, information about the moments of random variables is widely used.

Bounds for the probabilities and expectations of convex functions of discrete random variables with finite support are studied in Prékopa (1990).

Inequalities for the expectations of functions are studied in Prékopa (1992). These inequalities are based on information of the moments of discrete random variables.

A class of lower bounds on the expectation of a convex function using the first two moments of the random variable with a bounded support is considered in Dokov and Morton (2005).

Bounds on the exponential moments of $\min(y, X)$ and $X I\{X < y\}$ using the first two moments of the random variable X are considered in Pinelis (2011).

1.2. Practical needs of consideration

There are a number of well-known basic problems concerned with the mathematical description of the behavior of a man. They are most important in behavioral economics in utility and prospect theories (and also in decision sciences, social sciences and psychology). Examples of the problems are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, the four-fold pattern paradox, etc.

The present article is motivated in large measure by the need of rigorous mathematical support for the already performed analysis of the influence of scattering and noisiness of data. The idea of the theorem considered here has explained, at least partially, the above problems (see, e.g., Harin 2012a, Harin 2012b, Harin 2015).

The above basic problems are pointed out, e.g., in Kahneman and Thaler (2006). In particular, see p. 222 paragraph 2:

> *"A long series of modern challenges to utility theory, starting with the paradoxes of Allais (1953) and Ellsberg (1961) and including framing effects, have demonstrated inconsistency in preferences"*

There are also general opinions in Kahneman and Thaler (2006): see p. 221, Abstract (the **boldface** is my own):

> *"Economics can be distinguished from other social sciences by the belief that most (all?) behavior can be explained by assuming that rational agents with stable, well-defined preferences interact in markets that (eventually) clear. An empirical result qualifies as an anomaly if* ***it is difficult to rationalize or if implausible assumptions are necessary to explain it*** *within the paradigm."*

and see ibid p. 221, Introduction, paragraph 1 (the **boldface** is my own):

> *"The assumption that utility is always maximized allows often surprising inferences about the nature of the desires that guide people's ever-rational choices. This methodology has had many uses and undeniably has charm for economists, but* ***it rests on the shaky foundation of an implausible and untested assumption****. In this column, we discuss a version of the utility maximization hypothesis that can be tested—and we find that* ***it is false****."*

Thaler declared similar opinions even more hard in Thaler (2016): see p. 1597, Conclusions, paragraph 1 (the **boldface** is my own):

> *"There is one central theme of this essay: it is time to fully embrace what I would call evidence-based economics. This should not be a hard sell. Economists use the most sophisticated statistical techniques of any social science, have access to increasingly large and rich datasets, and have embraced numerous new methods from experiments (both lab and field) to brain imaging to machine learning. Furthermore,* ***economics has become an increasingly empirical discipline****. Hamermesh (2013) finds that the percentage of* ***"theory"*** *papers in top economics journals has fallen from 50.7 percent in 1963 to 19.1 percent in 2011.* ***We are undeniably an empirical discipline****—so let's embrace that.*

So one can say that the need and aim of the present article are also, in a sense, to make economics finally be a little closer to the *"theory."*

The essence of some of the above problems consists in biases of preferences and decisions of a man in comparison with predictions of the probability theory. These biases are maximal near the boundaries of the probability scale, that is, at high and low probabilities.

For example, in Thaler (2016), p. 1582, paragraphs 1-4 (the **boldface** is my own):

> "*Problem 1.—Imagine that you face the following pair of concurrent decisions.*
>
> *First examine both decisions, and then indicate the options you prefer.*
>
> *Decision (i) Choose between:*
>
> *A. A sure gain of $240 [84%]*
>
> *B. 25% chance to gain $1,000 and 75% chance to gain or lose nothing [16%]*
>
> *Decision (ii) Choose between:*
>
> *C. A sure loss of $750 [13%]*
>
> *D. A 75% chance to lose $1,000 and a 25% chance to lose nothing [87%]*
>
> *The numbers in brackets indicate the percentage of subjects that chose that option.*
>
> *We observe a pattern that was frequently displayed: subjects were* ***risk averse*** *in the domain of gains* ***but risk seeking*** *in the domain of losses.*

These and similar examples will be simplified and considered below in the sections 3 and 4.

Note that subjects change their preferences from aversion to seeking and vice versa not only when the domain are changed from gains to losses but from high to low probabilities as well.

So the need of the present article is also to consider situations near the boundaries.

1.3. Two ways. Variance, expectation and forbidden zones

Many efforts were applied to solve the above basic problems of behavioral economics and other sciences.

One of possible ways to solve them is widely discussed, e.g., in Schoemaker and Hershey (1992), Hey and Orme (1994), Chay et al (2005), Butler and Loomes (2007). The essence of this way consists in a proper attention to uncertainty, imprecision, noise, incompleteness and other reasons that might cause dispersion, scattering and spread of data.

Another possible way to solve these problems is to consider the vicinities of the borders of the probability scale, e.g. at $p \sim 1$. Steingrimsson and Luce (2007) and Aczél and Luce (2007) emphasized a fundamental question: whether Prelec's weighting function $W(p)$ (see Prelec, 1998) is equal to 1 at $p=1$.

In any case, one may suppose that a synthesis of the above two ways can be of interest. This idea of the synthesis turned out to be useful indeed. It has been successful to explain, at least partially, the underweighting of high and the overweighting of low probabilities, risk aversion, and some other problems (see, e.g., Harin 2012a, Harin 2012b and Harin 2015). There exist also a more general approach (see, e.g., Harin 2007 and Harin 2014b) and works about experimental support of the obtained results (see, e.g., Harin 2014a and Harin 2016).

In the present article some information about the variance of a random variable that takes on values in a finite closed interval is used to estimate bounds on its expectation. It is proven that if there is a non-zero lower bound on the variance of the variable, then non-zero bounds or forbidden zones for its expectation exist near the boundaries of the interval.

The role of a noise, as a possible cause of these forbidden zones and their possible influence on results of measurements near the boundaries of intervals are preliminary considered as well.

Keeping in mind the above bounds on functions of random variables Prékopa (1990), Prékopa (1992), Dokov and Morton (2005) and Pinelis (2011), functions of the expectation of a random variable can be further investigated.

Due to the convenience of abbreviations and consonant with the usage in previous works, here the terms "bound" and "forbidden zones" will sometimes be referred to with the term "restriction," especially in mathematical expressions, using its first letter "r" or "R," for example "r_{Expect}" or "r_μ" or "R."

2. Theorem

2.1. Preliminaries

The practical need of the article is a discrete random variable taking the finite number of values. This corresponds to usual finite numbers of measurements in the behavioral economics. A general case will be considered here nevertheless.

Let us consider a probability space *(Ω, Æ, P)* and a random variable *X*, such that *Ω → R*. Suppose that the support of X is an interval $[a,b]: 0 < (b-a) < \infty$. Suppose that X can have a continuous part and a discrete part and at least one of these parts is not identically equal to zero.

Let us denote the possible discrete values of X as $\{x_k\}$, $k = 1,2,...,K$, where $K \geq 1$, and $a \leq x_k \leq b$, and the possible continuous values of X as $x \in [a,b]$.

Under the condition

$$\sum_{k=1}^{K} f_X(x_k) + \int_{-\infty}^{+\infty} f(x)dx = \sum_{k=1}^{K} f_X(x_k) + \int_{a}^{b} f(x)dx = 1,$$

let us consider the expectation of X

$$E[X] \equiv \sum_{k=1}^{K} x_k f_X(x_k) + \int_{a}^{b} xf(x)dx \equiv \mu,$$

its variance

$$E[X-\mu]^2 = \sum_{k=1}^{K} (x_k - \mu)^2 f_X(x_k) + \int_{a}^{b} x^2 f(x)dx \equiv \sigma^2$$

and possible interrelationships between them.

2.2. Conditions of the variance maximality

The maximal value of the variance of a random variable of any type is intuitively equal to the variance of the discrete random variable whose probability mass function has only two non-zero values located at the boundaries of the interval. This statement is nevertheless proven in lemmas in the Appendix.

Such a probability mass function can be represented by the two values $f_X(a) = (b-\mu)/(b-a)$ and $f_X(b) = (\mu-a)/(b-a)$. The following inequality is consequently true for the variance of the considered random variable X

$$E[X-\mu]^2 \leq (\mu-a)^2 \frac{b-\mu}{b-a} + (b-\mu)^2 \frac{\mu-a}{b-a} = (\mu-a)(b-\mu). \qquad (1)$$

2.3. Existence theorem

Theorem. Suppose a random variable X takes on values in an interval $[a, b]$, $0 < (b-a) < \infty$. If there is some minimal non-zero variance $\sigma^2_{Min} > 0 : E[X-\mu]2 \geq \sigma^2_{Min}$, then some non-zero bounds (restrictions) $r_\mu \equiv r_{Expect} \equiv r_{Restrict.Expect} > 0$ exist on its expectation $\mu \equiv E[X]$ near the boundaries of the interval $[a, b]$, that is

$$a < (a + r_\mu) \leq \mu \leq (b - r_\mu) < b. \qquad (2).$$

Proof. It follows from (1) and the hypotheses of the theorem that

$$0 < \sigma^2_{Min} \leq E[X-\mu]^2 \leq (\mu - a)(b - \mu).$$

For the boundary a this leads to the inequalities $\sigma^2_{Min} \leq (\mu - a)(b - a)$ and

$$\mu \geq a + \frac{\sigma^2_{Min}}{b-a}. \qquad (3).$$

For the boundary b the consideration is similar and gives the inequality

$$\mu \leq b - \frac{\sigma^2_{Min}}{b-a}. \qquad (4).$$

So, if we consider the image I_μ of the values of the expectation μ, then we see that this image coincides with the interval $[a, b]$ if the minimal variance σ^2_{Min} is equal to zero in the above inequalities (3) and (4). If the minimal variance σ^2_{Min} is more than zero, then I_μ is divided into the three zones.

These zones are the two forbidden zones R_μ or simply R and the residual obtainable or open zone O_μ (or simply O). The forbidden zones are located near the boundaries of the interval $[a, b]$, they can be denoted as R_a and R_b. They are restricted for the values of the expectation μ. The residual obtainable zone O is obtainable, open for the values of the expectation μ.

Denoting the bounds (restrictions r_μ) on the expectation μ as

$$r_\mu \equiv \frac{\sigma^2_{Min}}{b-a},$$

and using (3) and (4), we obtain the generalized inequalities

$$a + r_\mu \leq \mu \leq b - r_\mu .$$

Therefore, if the inequalities $0 < (b-a) < \infty$ and $\sigma^2_{Min} > 0$ are true, then the non-zero bounds (restrictions) $r_\mu > 0$ exist, such that the inequalities (2)

$$a < (a + r_\mu) \leq \mu \leq (b - r_\mu) < b$$

are satisfied, which proves the theorem.

3. Consequences of the theorem. Examples

3.1. Practical need and general implication

The initial reason of the above theorem was to describe and explain the practical experiments in behavioral economics.

Due to the need of financial incentives for subjects of the experiments and to the finiteness of financial possibilities of experimenter's teams, the numbers of experimental results are necessarily finite.

The theorem meets this practical need. In addition to its practical value, the theorem proves that this result is true for any random variable.

3.2. Minimal variance. Data scattering. Noise

The theorem states that the factor, which leads to the forbidden zones and determines their widths, is the non-zero minimal variance. It is exactly the minimal variance, not the variance itself.

There can be a wealth of causes of this non-zero minimal variance. It can be caused evidently by any non-zero scattering and spread of data. The list of such causes is rather wide. It includes a noise, imprecision, errors, incompleteness, various types of uncertainty, etc. Such causes are considered in a lot of works, e.g., Schoemaker and Hershey (1992), Hey and Orme (1994), Chay et al (2005), Butler and Loomes (2007).

A noise can be one of usual sources of the non-zero minimal variance.

There are many types and subtypes of noise. A hypothetic task of determining of an exact relationship between a level of noise and a non-zero minimal variance of random variables can be a rather complicated one.

If, nevertheless, a noise leads to some non-zero minimal variance of the considered random variable, then such a noise leads evidently to the above non-zero forbidden zones. If a noise leads to some increasing of the value of this minimal variance then the value of these zones increase as well.

So the theorem can provide a new mathematical tool for description of the influence of at least some types of a noise near the boundaries of intervals.

3.3. Practical example of existence. Ships and waves

Suppose the calm or mirror-like sea. Suppose a small rigid boat or any other small rigid floating body that is at rest in the mirror-like sea. Suppose that this boat or the body rests in the mirror-like sea right against (or be constantly touching) the moorage wall (that is also rigid).

As long as the sea is calm, the expectation of its side can touch the wall.

Suppose the heavy sea. Suppose a small rigid boat or any other small rigid floating body that oscillates on waves in the heavy sea. Suppose that this boat or the body oscillates on waves near the rigid moorage wall.

When the boat is oscillated by sea waves, then its side oscillates also (both up-down and left-right) and it can touch the wall only in the nearest extremity of the oscillations. Therefore, the expectation of the side cannot touch the wall (if the oscillations are non-zero). Therefore, the expectation of the side is biased from the wall.

So, one can say that, in the presence of the waves, a forbidden zone exists between the expectation of the side and the wall.

This forbidden zone biases and separates the expectation from the wall. The width of the forbidden zone is roughly about a half of the amplitude of the oscillations.

3.4. Practical examples of existence. Washing machine, drill, …

Suppose a washing machine that can vibrate when pressing bed linen. Suppose the washing machine near a rigid wall. Suppose an edgeless side of a drill or any other rigid body that can vibrate is located near a rigid surface or wall.

If the washing machine or the drill is at rest, then the expectation of its edgeless side can be located right against (be constantly touching) the wall.

If the washing machine or the drill vibrates, then the expectation of its edgeless side is biased and kept away from the wall due to its vibrations.

3.5. General example. Rigidness. Pressing. Sure outcomes

The same is true for any other rigid body near any rigid surface or wall.

If the body is at rest, then the expectation of its side can be located right against the wall (be constantly touching the wall). If the body vibrates, then the expectation of its side is biased and kept away from the wall by the vibrations of the body.

In other words, a forbidden zone arises between the rigid wall (surface) and the expectation of the side of the rigid body, when the body vibrates. The width of the forbidden zone is roughly about a half of the amplitude of the vibrations.

The above rigid boat near rigid moorage wall, rigid washing machine near rigid wall and rigid drill near rigid surface were the examples of a rigid body that can vibrate or oscillate near a rigid boundary (a rigid surface).

What do the conditions of "rigid" body and "rigid" boundary mean?

If either the body or the boundary or the both are not rigid, then the vibrations and oscillations can be suppressed partially or even totally. Hence the forbidden zone can be suppressed also.

If a vibrating rigid body is pressed by some pressing force or pressure plate to a rigid surface, then the forbidden zone can be suppressed either partially or even totally, depending on the parameters of the pressure.

This suppression corresponds to the case of sure outcomes in the behavioral economics.

The term "sure" presumes here that some additional efforts are applied to guarantee this sure outcome in comparison with the probable ones. This leads to some qualitative difference between these probable and sure outcomes. This can lead to some quantitative difference between the widths of the forbidden zones for the expectations of data in these probable and sure outcomes.

Due to the additional guaranteeing efforts, the width of the forbidden zones for the expectations of data in the sure outcomes can be less than the width for the probable outcomes. The width for the sure outcomes can even be equal to zero, what means that the cause of the forbidden zones is too weak to overcome the additional guaranteeing efforts.

4. Applications of the theorem. Newness

4.1. Practical applications in behavioral economics and decision sciences

The idea of the considered forbidden zones was applied, e.g., in Harin (2012b). This work was devoted to the well-known problems of behavioral economics, decision sciences, utility and prospect theories. Such problems were pointed out, e.g., in Kahneman and Thaler (2006).

In Harin (2012b), some examples of typical paradoxes were studied. The studied and similar paradoxes may concern problems such as the underweighting of high and the overweighting of low probabilities, risk aversion, etc.

The dispersion and noisiness of the initial data can lead to the forbidden zones for the expectations of these data. This should be taken into account when dealing with these kinds of problems. The described above forbidden zones explained, at least partially, the analyzed examples of paradoxes.

4.2. Practical numerical examples of applications. Gains

The above example of Thaler (2016) can be simplified similar to Harin (2012b).

Imagine that you face the following pair of concurrent decisions.

Choose between:

A) A sure gain of \$99.

B) 99% chance to gain \$100 and 1% chance to gain or lose nothing.

Ideal case

In the ideal case without the forbidden zones, the probable gain has the probability *99%* and the expected values for the probable and sure outcomes are

$$\$99 \times 100\% = \$99 ,$$

$$\$100 \times 99\% = \$99 .$$

Here, the expected values are exactly equal to each other

$$\$99 = \$99 .$$

Forbidden zones

Let us consider the case of the forbidden zones.

Suppose that the width of the forbidden zones for the expectations of data in the probable outcome is equal to, say, *$2*.

Let us consider the case when the width of the forbidden zones for the expectations of data in the sure outcome can be less than the width for the probable outcome and is equal to, say, *$1*. We have

$$\$99 \times 100\% - \$1 = \$99 - \$1 = \$98,$$

$$\$100 \times 99\% - \$2 = \$99 - \$2 = \$97.$$

Here, the both expected values are biased, but the sure expected value is biased less than the probable one and we have

$$\$98 > \$97.$$

Here, the sure gain is (due to the difference and obvious preference between the expected values) more preferable than the probable one.

Let us consider the case when the width of the forbidden zones for the expectations of data in the sure outcome is equal to zero. We have

$$\$99 \times 100\% = \$99,$$

$$\$100 \times 99\% - \$2 = \$99 - \$2 = \$97.$$

Here, the probable expected value is biased but the sure expected value is not

$$\$99 > \$97.$$

Here, the sure gain is all the more preferable than the probable one.

So, the forbidden zones and their natural difference for probable and sure outcomes can predict the experimental fact that the subjects are risk averse in the domain of gains. They explain, at least qualitatively or partially, the analyzed example of Thaler (2016) and many other similar results.

The theorem provides the mathematical support for the solution in the domain of gains.

4.3. Practical numerical examples of applications. Losses

The case of gains has been explained many times in a lot of ways. The uniform explanation for both gains and losses, without additional suppositions, such as Kahneman and Tversky (1979), has not been recognized by the present article's author nevertheless.

Let us consider the case of losses.

Imagine that you face the following pair of concurrent decisions. Choose between:

A) A sure loss of $99.

B) 99% chance to loss $100 and 1% chance to gain or lose nothing.

Ideal case

In the ideal case without the forbidden zones, the probable loss has the probability *99%* and the expected values for the probable and sure outcomes are

$$-\$99 \times 100\% = -\$99,$$

$$-\$100 \times 99\% = -\$99.$$

Here, the expected values are exactly equal to each other

$$-\$99 = -\$99.$$

Forbidden zones

Let us consider the case of the forbidden zones.

Suppose that the width of the forbidden zones for the expectations of data in the probable outcome is equal to, say, *$2*.

Let us consider the case when the width of the forbidden zones for the expectations of data in the sure outcome can be less than that for the probable outcome and is equal to, say, *$1*.

Note that the forbidden zone biases the expectation from the boundary of the interval to its middle. The width of the forbidden zone is subtracted from the absolute value of the gain/loss therefore. That is the width of the forbidden zone is subtracted from the value of the gain and added to the value of the loss.

We have

$$-\$99 \times 100\% + \$1 = -\$99 + \$1 = -\$98\,,$$

$$-\$100 \times 99\% + \$2 = -\$99 + \$2 = -\$97\,.$$

Here, the both expected values are biased, the sure expected value is biased less than the probable one as in the case of gains, but here the bias increases the advantage (preferability) of the outcome

$$-\$98 < -\$97$$

and the probable loss is (due to the difference and obvious preference between the expected values) more preferable than the sure one.

Let us consider the case when the width of the forbidden zones for the expectations of data in the sure outcome is equal to zero. We have

$$-\$99 \times 100\% = -\$99\,,$$

$$-\$100 \times 99\% + \$2 = -\$99 + \$2 = -\$97\,.$$

Here, the probable expected value is biased but the sure expected value is not and we have

$$-\$99 < -\$97\,.$$

Here, the probable loss is all the more preferable than the sure one.

The theorem provides the mathematical support for the solution in the domain of losses.

4.4. Practical application. Newness

So, the theorem provides the mathematical support for the solution in the domains of both gains and losses.

Due to, e.g., Harin (2012b), the forbidden zones and their natural difference for probable and sure outcomes can predict the experimental fact that the subjects are risk seeking in the domain of gains but risk seeking in the domain of losses. They explain, at least qualitatively or partially, the analyzed examples of Thaler (2016) and many other similar results.

The important feature is that, due to, e.g., Harin (2012b), the described forbidden zones can solve the problems and explain experimental results not only in the domains of the gains and losses. The important feature is also that these solution and explanation are uniform in all the domains and need not additional suppositions. Hence the forbidden zones and their natural difference for probable and sure outcomes can qualitatively or, at least, partially predict the experimental facts and solve the problems in various domains.

The mathematical description of the above forbidden zones has been done in recent years. Unfortunately, these zones were not described in mathematics before.

The analysis of the literature, comments of comments of journals' editors and reviewers on similar articles and on the previous versions of the present article and more than 10-years experience of the editorship in NEP reports on utility and prospect theories (see Harin 2005-2017) allow to suppose the following.

The mathematical support for the above solution, that is presented by the theorem and its consequences, is a new one.

Why did not this evident and widespread phenomenon be mathematically described before? The long absence of such a description can be explained by the evidence of its practical applications. That is these forbidden zones can be easily estimated as approximately a half of the amplitude of the oscillations and there is no need in more detailed analysis and calculation. The phenomena that are similar to the forbidden zones between ships boards and moorage wall, washing machines and walls, etc. are evident and are usually taken into account in the cases of their evident and essential influence.

The problems and paradoxes of the behavioral economics, utility and prospect theories are, probably, the first field where such phenomena are hidden by other details of experiments and hence are non-evident.

4.5. Possible applications. Noise. Biases of measurements' data

4.5.1. Noise

Let us preliminary consider possible applications of the theorem to a noise.

If a noise leads to some non-zero minimal variance of the considered random variable, then this non-zero minimal variance and, consequently, this noise leads to the above non-zero forbidden zones for the expectation of this variable. If a noise leads to some increasing of the value of this minimal variance then the width of these forbidden zones increases also.

The presented theorem allows to make a step to a possible new mathematical tool for description of the possible influence of noise near the boundaries of finite intervals. In particular, if a noise leads to a non-zero minimal variance $\sigma^2_{Min} : \sigma^2 > \sigma^2_{Min} > 0$ of a random variable, then the theorem predicts the forbidden zones having the width r_{Noise} that is not less than

$$r_{Noise} \geq \frac{\sigma^2_{Min}}{b-a}.$$

So, the presented theorem is the first preliminary step to a general mathematical description of the possible influence of noise near the boundaries of finite intervals.

4.5.2. Biases of measurements' data

Let us preliminary consider possible applications of the theorem to possible biases of measurements' data.

The considered forbidden zones can evidently lead to some biases in measurements. We can preliminary consider this a bit closer. Suppose some measurements are preformed on a finite interval and its result is the expectation of measurements' data. Suppose some forbidden zones arise near the boundaries of the interval due to the minimal variance of the data.

The expectations of the data of the measurements cannot be indeed located inside the forbidden zones. They cannot be located closer to the boundaries of the interval than the width of the forbidden zone.

The above forbidden zones can cause biases for the expectations of the data of measurements. The biases are directed from the boundaries to the middle of the interval. The biases have the opposite signs near the opposite boundaries of the interval. The absolute values of the biases decrease from the boundaries to the middle of the interval.

When the minimal variance of the data is equal to zero, then the expectations of the data of measurements can touch the boundaries of the interval. When the above forbidden zones are not taken into the consideration then the estimated results are also located closer to the boundaries than the real case.

In particular, if the minimal variance of the data σ^2_{Min} is non-zero, that is if $\sigma^2 > \sigma^2_{Min} > 0$, then the theorem predicts that near the boundaries of intervals, the absolute value Δ_{Bias} of the biases is not less than

$$|\Delta_{Bias}| \geq \frac{\sigma^2_{Min}}{b-a}.$$

So, the presented theorem, its consequences and applications can be considered as the first preliminary step to a general mathematical description of the biases of measurements' data near the boundaries of finite intervals.

5. Conclusions

The article can be concluded by the following three statements:

1) Problems. There are well-known problems of behavioral economics (see, e.g., Hey and Orme 1994, Kahneman and Thaler 2006, Thaler 2016). Typical problems consist in comparison of sure and probable outcomes. They are the most pronounced near the boundaries of intervals.

For example, Thaler in Thaler (2016) noted (the **boldface** is my own):

"*We observe a pattern that was frequently displayed: subjects were **risk averse** in the domain of gains **but risk seeking** in the domain of losses.*"

The examples of Thaler (2016) can be simplified similar to Harin (2012b):

Domain of gains. Choose between: A) A sure gain of \$99. B) 99% chance to gain \$100 and 1% chance to gain or lose nothing. The expectations are

$$\$99 \times 100\% = \$99 = \$99 = \$100 \times 99\% ,$$

Domain of losses. Choose between: A) A sure loss of \$99. B) 99% chance to loss \$100 and 1% chance to gain or lose nothing.

$$-\$99 \times 100\% = -\$99 = -\$99 = -\$100 \times 99\% ,$$

In the both cases the expected values are exactly equal to each other. Nevertheless the preferences of the subjects are essentially biased in the opposite directions for gains and losses. This is the well-known and fundamental paradox of the behavioral economics.

2) Solution of the problems. There is a solution of these problems (see, e.g., Harin 2012a, Harin 2012b, Harin 2015). It consists in the idea of forbidden zones near the boundaries of finite intervals. These forbidden zones allow to solve the above problems.

3) Mathematical support for the solution. The presented theorem, its consequence and applications provide the mathematical support for the solution.

The presented theorem proves that, for a finite interval $[a, b]$ under the condition of existence of the non-zero minimal variance $\sigma^2_{Min} : \sigma^2 > \sigma^2_{Min} > 0$, the expectation μ of measurements' data is bounded as

$$a + \frac{\sigma^2_{Min}}{b-a} \le \mu \le b - \frac{\sigma^2_{Min}}{b-a} .$$

In other words, the theorem proves the existence of the forbidden zones.

The forbidden zone biases the expectation from the boundary of the interval to its middle. The width of the forbidden zone is subtracted from the absolute value of the gain/loss therefore. That means that the width of the forbidden zone is subtracted from the value of the gain and added to the value of the loss.

Suppose that the width of the forbidden zones is equal to *$2* for the probable outcomes and, for the simplicity, zero for the sure outcomes.

In the case of gains we have

$$\$99 \times 100\% = \$99,$$

$$\$100 \times 99\% - \$2 = \$99 - \$2 = \$97.$$

Here, the probable expected value is biased but the sure expected value is not

$$\$99 > \$97.$$

The sure gain is more preferable than the probable one, as is supported by a wealth of experiments.

In the case of losses we have

$$-\$99 \times 100\% = -\$99,$$

$$-\$100 \times 99\% + \$2 = -\$99 + \$2 = -\$97.$$

Here, the probable expected value is biased but the sure expected value is not and we have

$$-\$99 < -\$97.$$

The probable gain is more preferable than the sure one, as is supported by a wealth of experiments.

So the theorem, its consequence and applications provide the uniform mathematical support for the solution in more than one domain. This is, at least, a rare if not unique result for the considered problems.

Main particular contribution. The main particular contribution of the present article is this mathematical support for the solution of the above problems of behavioral economics.

Possible additional general contributions. Two more possible additional general contributions can be preliminary mentioned:

4) Possible general addition. Noise. In addition, the presented theorem is the first preliminary step to a general mathematical description of the possible influence of noise near the boundaries of finite intervals. In particular, if a noise leads to a non-zero minimal variance $\sigma^2_{Min} : \sigma^2 > \sigma^2_{Min} > 0$ of a random variable, then the theorem predicts the forbidden zones having the width r_{Noise} that is not less than

$$r_{Noise} \geq \frac{\sigma^2_{Min}}{b-a}.$$

5) Possible general addition. Biases of the data of measurements. In addition, the presented theorem is the first preliminary step to a general mathematical description of the biases of measurements' data near the boundaries of finite intervals. In particular, if the minimal variance of the data σ^2_{Min} is non-zero, that is if $\sigma^2 > \sigma^2_{Min} > 0$, then the theorem predicts the biases of measurements' data. The biases have the opposite signs near the opposite boundaries, are maximal near the boundaries and tend to zero in the middles of the intervals. Near the boundaries of intervals, the absolute value Δ_{Bias} of the biases is not less than

$$|\Delta_{Bias}| \geq \frac{\sigma^2_{Min}}{b-a}.$$

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Appendix. Lemmas of variance maximality conditions

Preliminaries

The initial particular need is the mathematical support for the solution (see, e.g., Harin 2012a, Harin 2012b and Harin 2015) of the problems of behavioral economics. These problems take place for the discrete finite random variables. Nevertheless let us give the support for the general case.

In the general case, we have (see subsection 2.1)

$$E[X-\mu]^2 = \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k) + \int_a^b (x-\mu)^2 f(x)dx .$$

under the condition that either the probability mass function or probability density function or alternatively both of them are not identically equal to zero, or

$$\sum_{k=1}^{K} p(x_k) + \int_a^b f(x)dx = 1 .$$

Pairs of values that mean value coincides with the expectation of the random variable were used, e.g., in Harin (2013). More arbitrary choice of pairs of values was used in Harin (2017). Here every discrete and infinitesimal value will be divided into the pair of values in the following manner:

Let us divide every value $p(x_k)$ into the two values located at a and b

$$p(x_k)\frac{b-x_k}{b-a} \quad \text{and} \quad p(x_k)\frac{x_k-a}{b-a} .$$

The total value of these two parts is evidently equal to $p(x_k)$. The center of gravity of these two parts is evidently equal to x_k.

Let us divide every value of $f(x)$ into the two values located at a and b

$$f(x)\frac{b-x}{b-a} \quad \text{and} \quad f(x)\frac{x-a}{b-a} .$$

The total value of these two parts is evidently equal to $f(x)$. The center of gravity of these two parts is evidently equal to x.

Let us prove that the variances of the divided parts are not less than those of the initial parts.

A1. Lemma 1

Lemma 1. Discrete part lemma. If the support of a random variable X, is an interval $[a,b]: 0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2=\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx\equiv\sigma^2,$$

where p is the probability mass function of X, $a\le x_k\le b$, $k=1,2,...,K$, where $K\ge 1$ and $\mu\equiv E[X]$ and

$$\sum_{k=1}^{K}p(x_k)\ge 0,$$

then the inequality

$$\begin{aligned}&\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)\ge\\&\ge\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)\end{aligned}. \tag{16}$$

is true.

Proof. Let us find the difference between the transformed

$$\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)$$

and initial

$$\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)$$

expressions for the variance.

Let us consider separately the cases of $x_k\ge\mu$ and $x_k\le\mu$.

Case of $x_k \ge \mu$,

If $x_k \ge \mu$, then the expression in the square brackets can be simplified

$$\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]\ge$$

$$\ge\left[(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]=$$

$$=(b-\mu)^2\left[\frac{x_k-a}{b-a}-\left(\frac{x_k-\mu}{b-\mu}\right)^2\right]$$

Due to $x_k \le b$ and

$$0\le\frac{x_k-\mu}{b-\mu}\le 1,$$

it holds true that

$$\left(\frac{x_k-\mu}{b-\mu}\right)^2\le\frac{x_k-\mu}{b-\mu}$$

and

$$\frac{x_k-a}{b-a}-\left(\frac{x_k-\mu}{b-\mu}\right)^2\ge\frac{x_k-a}{b-a}-\frac{x_k-\mu}{b-\mu}$$

and then

$$\frac{x_k-a}{b-a}-\frac{x_k-\mu}{b-\mu}\equiv\frac{(x_k-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)}-\frac{x_k-\mu}{b-\mu}.$$

Due to

$$0\le\frac{x_k-a}{b-a}\le 1 \quad \text{and} \quad \mu-a\ge 0,$$

the inequality

$$\frac{(x_k-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)}\ge\frac{x_k-\mu}{b-\mu}$$

is true and

$$(b-\mu)^2\left[\frac{x_k-a}{b-a}-\left(\frac{x_k-\mu}{b-\mu}\right)^2\right]\ge 0.$$

So in the case of $x_k \ge \mu$ the difference between the transformed and initial expressions for the variance is non-negative.

Case of $x_k \le \mu$,

If $x_k \le \mu$, then

$$\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]=$$

$$=\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(\mu-x_k)^2\right]\ge$$

$$\ge\left[(\mu-a)^2\frac{b-x_k}{b-a}-(\mu-x_k)^2\right]=$$

$$=(\mu-a)^2\left[\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\right]$$

Due to

$$0\le\frac{\mu-x_k}{\mu-a}\le 1,$$

we have

$$\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\ge\frac{b-x_k}{b-a}-\frac{\mu-x_k}{\mu-a}.$$

Then

$$\frac{b-x_k}{b-a}-\frac{\mu-x_k}{\mu-a}\equiv\frac{(b-\mu)+(\mu-x_k)}{(b-\mu)+(\mu-a)}-\frac{\mu-x_k}{\mu-a}.$$

Due to

$$0\le\frac{\mu-x_k}{\mu-a}\le 1 \quad \text{and} \quad b-\mu\ge 0$$

we have

$$\frac{(b-\mu)+(\mu-x_k)}{(b-\mu)+(\mu-a)}\ge\frac{\mu-x_k}{\mu-a}$$

and

$$(\mu-a)^2\left[\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\right]\ge 0.$$

So in the case of $x_k \le \mu$ the difference between the transformed and initial expressions for the variance is non-negative.

So the difference

$$(a-\mu)^2 p(x_k)\frac{b-x_k}{b-a}+(b-\mu)^2 p(x_k)\frac{x_k-a}{b-a}-(x_k-\mu)^2 p(x_k)=$$
$$=p(x_k)\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right].$$

is non-negative.

Let us calculate the difference between the transformed and initial expressions of the discrete part of the variance

$$E_{Discr.Transformed}[X-\mu]^2-E_{Discr.Initial}[X-\mu]^2=$$
$$=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)-\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)=.$$
$$=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]p(x_k)$$

Every member of a sum is non-negative, as in the above expression. Hence the total sum is non-negative as well.

So for the discrete case the variance is maximal when the probability mass function is concentrated at the boundaries of the interval.

$$E_{Discr.Transformed}[X-\mu]^2-E_{Discr.Initial}[X-\mu]^2=$$
$$=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)-\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)=.$$
$$=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]p(x_k)$$

If every member of a sum is non-negative, as in the above expression, then the total sum is non-negative as well.

Also the expression

$$(a-\mu)^2(b-x_k)+(b-\mu)^2(x_k-a)\,.$$

can be identically rewritten to

$$[(x_k-a)+(\mu-x_k)]^2(b-x_k)+$$
$$+[(b-x_k)+(x_k-\mu)]^2(x_k-a)=$$

$$=[(x_k-a)^2+2(x_k-a)(\mu-x_k)+(\mu-x_k)^2](b-x_k)+$$

$$+[(b-x_k)^2+2(b-x_k)(x_k-\mu)+(x_k-\mu)^2](x_k-a)$$

and

$$[(x_k-a)^2+2(x_k-a)(\mu-x_k)+(\mu-x_k)^2](b-x_k)+$$
$$+[(b-x_k)^2+2(b-x_k)(x_k-\mu)+(x_k-\mu)^2](x_k-a)=$$

$$=(x_k-\mu)^2(b-a)+$$

$$+(x_k-a)^2(b-x_k)+(b-x_k)^2(x_k-a)+$$

$$+2(x_k-a)(b-x_k)[(\mu-x_k)+(x_k-\mu)]$$

This can be transformed to

$$(x_k-\mu)^2+$$
$$(a-\mu)^2(b-x_k)+(b-\mu)^2(x_k-a)$$

that is analogous to the theorem of Huygens-Steiner (The general possibility of application of the Huygens-Steiner theorem was helpfully pointed out by one of the anonymous referees when the preceding version of the present article was refereed)

A2. Lemma 2

Let the probability density function is not identically equal to zero.

Lemma 2. Continuous part lemma. If the support of a random variable X, is an interval $[a,b]$: $0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2=\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx \equiv \sigma^2,$$

where f is the probability density function of X and $\mu \equiv E[X]$ and

$$\int_a^b f(x)dx \geq 0,$$

then the inequality

$$\int_a^b \left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx \geq \int_a^b (x-\mu)^2 f(x)dx. \qquad (18)$$

is true.

Proof. Let us find the difference between the transformed

$$\int_a^b \left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx$$

and initial

$$\int_a^b (x-\mu)^2 f(x)dx$$

expressions for the variance.

Let us consider separately the cases of $x \geq \mu$ and $x \leq \mu$.

Case of $x \geq \mu$

If $x_k \geq \mu$, then the difference can be simplified as

$$\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}-(x-\mu)^2\right]\geq$$

$$\geq\left[(b-\mu)^2\frac{x-a}{b-a}-(x-\mu)^2\right]= \qquad .$$

$$=(b-\mu)^2\left[\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\right]$$

Due to $x \leq b$ and

$$0\leq\frac{x-\mu}{b-\mu}\leq 1,$$

it holds true that

$$\left(\frac{x-\mu}{b-\mu}\right)^2\leq\frac{x-\mu}{b-\mu}$$

and

$$\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\geq\frac{x-a}{b-a}-\frac{x-\mu}{b-\mu}$$

and then

$$\frac{x-a}{b-a}-\frac{x-\mu}{b-\mu}\equiv\frac{(x-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)}-\frac{x-\mu}{b-\mu}.$$

Due to

$$0\leq\frac{x-a}{b-a}\leq 1 \quad \text{and} \quad \mu-a\geq 0,$$

we have

$$\frac{(x-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)}\geq\frac{x-\mu}{b-\mu}.$$

and

$$(b-\mu)^2\left[\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\right]\geq 0.$$

Case of $x \le \mu$,

If $x \le \mu$, then the difference can be simplified as

$$\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}-(\mu-x)^2\right]\ge$$

$$\ge\left[(\mu-a)^2\frac{b-x}{b-a}-(\mu-x)^2\right]=$$

$$=(\mu-a)^2\left[\frac{x-a}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\right]$$

.

Due to

$$0\le\frac{\mu-x}{\mu-a}\le 1,$$

we have

$$\frac{b-x}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\ge\frac{b-x}{b-a}-\frac{\mu-x}{\mu-a}.$$

Then

$$\frac{b-x}{b-a}-\frac{\mu-x}{\mu-a}\equiv\frac{(b-\mu)+(\mu-x)}{(b-\mu)+(\mu-a)}-\frac{\mu-x}{\mu-a}.$$

Due to

$$0\le\frac{\mu-x}{\mu-a}\le 1 \quad \text{and} \quad b-\mu\ge 0$$

we have

$$\frac{(b-\mu)+(\mu-x)}{(b-\mu)+(\mu-a)}\ge\frac{\mu-x}{\mu-a}$$

and

$$(\mu-a)^2\left[\frac{b-x}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\right]\ge 0.$$

Maximality

Let us calculate the difference between the transformed and initial expressions of the continuous part of the variance

$$E_{Contin.Transformed}[X-\mu]^2 - E_{Contin.Initial}[X-\mu]^2 =$$
$$= \int_a^b \left[(a-\mu)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} \right] f(x)dx - \int_a^b (x-\mu)^2 f(x)dx = .$$
$$= \int_a^b \left[(a-\mu)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} - (x-\mu)^2 \right] f(x)dx$$

If the integrand of an integral is non-negative for every point in the scope of the limits of integration as in the above expression, then the complete integral is non-negative as well. The difference is therefore non-negative.

So for the continuous case the variance is maximal when the probability density function is concentrated at the boundaries of the interval.

A3. Lemma 3

Let the probability mass function is not identically equal to zero.

Lemma 3. General mixed case lemma. If the support of a random variable X, is an interval $[a,b]: 0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2=\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx \equiv \sigma^2,$$

where p is the probability mass function of X, $a \le x_k \le b$, $k=1,2,...,K$, where $K \ge 1$ and f is the probability density function of X and $\mu \equiv E[X]$ and

$$\sum_{k=1}^{K} p(x_k)+\int_a^b f(x)dx=1,$$

then the inequality

$$\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)+$$
$$+\int_a^b\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx \ge,$$
$$\ge \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx$$

is true.

Proof. The general mixed case is compiled from the discrete and continuous parts under the condition that at least one of them is not identically equal to zero. The conclusions concerned to these parts are true for their sum as well.

So in any case both for the probability mass function and/or probability density function and/or their mixed case, the variance is maximal when the probability mass function and/or probability density function are concentrated at the boundaries of the interval in the form of the probability mass function that has only the two values located at the boundaries of the interval.