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Obregón, Carlos

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THE RECONSTRUCTION OF CAPITAL THEORY

The True Meaning of Capital in a Production Function

CARLOS OBREGON

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Abstract The purpose of the present article is to explore the possibilities of a reconstruction of a Theory of Capital capable of taking into account the Reswitching phenomenon. In Section 1 a new measure of capital-time, for neoaustrian processes of production, is introduced. The main outcome of the use of this proposed new measure of capital is this: it can be shown that, even when Reswitching occurs, there is still always an inverse relationship between the rate of interest or profit and the quantity of capital-time. In Section 2 the results of Section 1 are extended for the case of two good technologies examples. In Section 3 a surrogate production process is introduced. By developing this surrogate production process it can be shown that in general there is an inverse relationship between the interest rate and the quantity of surrogate capital per man, the surrogate capital/output ratio, and between the interest rate and the newly defined steady-state consumption per capita. Section 4 presents further comments on the results of the previous sections. Section 5 introduces numerical examples.

Keywords Reswitching · Capital · Capital theory · Theory of capital · Cambridge controversies · Summing Up · Capital Time · Neoclassical parables · Production Function

Preamble

Capital Theory is critical to economics because there are only two inputs of production, labor and capital. There have been several occasions in the history of economic thought in which controversies have risen as to: What capital really is? The most simply way to think about capital is to equate it to machines and all the other inputs of production excluding labor. As any good, these machines and other inputs of production have a set of prices inversely related to the quantities demanded of them. But there is one more price in the economy the interest

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CARLOS OBREGON E-mail: carlos.obregon4900@gmail.com

rate. And as any price it should in principle be inversely related to the quantity demanded of time. But is it? and What is time? If there are only two inputs of production at the aggregate level and time is not labor then it should be capital. This was long time ago Böhm-Bawerk proposal, for him capital is time. He showed his idea with a very simple wine production process involving only labor and time.

The last Capital Theory controversies happened in the 60s and early 70s of the last century. Hundreds of papers were published in the most prestigious journals. The controversy was critical for theoretical economic thinking. One of its contributions was to show that for the general case any other input of production excluding labor can be aggregated as time. If the quantity demanded of capital time behaves inversely to the interest rate, then upward and downward movements in the real interest rate must have an effect on productivity and economic growth. This first position was maintained by Samuelson, Solow and others in Cambridge USA and follows a long neoclassical tradition. It is behind, for example, the theory of economic growth of Solow for which he obtained the Nobel in Economics. However, if the quantity demanded of capital time does not behave inversely to the interest rate, then the interest rate is only a monetary phenomena, not linked to the real side of the economy. This second position was the proposal argued formally by Sraffa and others in Cambridge England. Sraffa had been a long term believer of this second position. He actually had convinced Keynes that the interest rate was only a monetary phenomena; as it is shown in the chapter in Keynes General Theory titled: Sundry Observations on the Nature of Capital.

Who was right? In a famous concluding article titled Summing Up, Samuelson concedes that Sraffa and others were right. He recognizes that the Reswitching phenomena is an economic reality and that it can not be shown that there is for the general case a well behaved demand for capital. Samuelson intellectual honesty closed down the controversy, but it did not convince most of the economic profession which continued teaching economics as if the controversy had never happened. The profession simply ignored Sraffa and others arguments as well as Samuelson recognition that they were right; and it has continue assuming - without theoretical support - that there is a well behaved demand for capital.

But theory is important and therefore we must understand theoretically: What is capital? Thus, we should reopen the controversy. We do that in this paper and we find a very surprising result. Sraffa and others were not right and Samuelson's acknowledge was not correct. Capital as time indeed have a well behaved demand function. However, capital as time can not be measured as proposed originally by Böhm-Bawerk.

For those not so familiar with the controversy we must recall that, using the trace of a matrix, Sraffa have proven for the general case that any input output matrix with n inputs an m outputs, where both n and m are very large numbers, can be transformed in a unique infinite mathematical series of dated labor containing only labor and time. Therefore, in this series there are indeed only two inputs of production labor and capital-time. It is not longer a simply wine production example. The series of dated labor represents a one to one unique transformation of a complex matrix of production of a real economy with many inputs and outputs. Moreover, he has shown that the Reswitching phenomena exists and that the economy as the interest rate goes down can change from a more capital intensity technique to a less capital intensity technique contradicting the possibility of a well behaved demand for capital. As we will see the Reswitching does exist, but

Sraffa is wrong in his measurement of capital as time. Once the right measure is introduced, the economy as the interest rate goes down always switches from a less to a more capital intensity technique of production. Thus, for the general case there is a well behaved demand for capital.

Introduction

In the traditional austrian model, labor was applied uniformly prior to the production of final output. In such a model it can be shown that lowering the interest or profit rate always leads to lenghthening of the arithmetic average time-period of production proposed by Böhm-Bawerk. Thus, as the interest rate declines, the competitive system switches to techniques or methods of production which are more and more roundabout and involve less and less total steady-state labor per time-period. The decline in the interest rate cheapens the utilization of capital or time and induces a substitution from labor-intensive to capital-intensive methods of production. As a consequence lowering the interest rate leads to increasing capital per man and to an increasing capital output ratio. Moreover, by using a more roundabout technique of production society enjoys a higher sustainable consumption per head. A key feature of such a simple austrian model is that as the interest rate declines the competitive system can never go back to methods of production which have been utilized previously.

By contrast Sraffa, Passinetti and others have built examples of production processes in which this key feature of the austrian model does not hold. In these examples Reswitching occurs: that is, the same technique of production is the most profitable of a number of techniques of production at more than one rate of interest of profit, even though other techniques are more profitable at values of the rate of interest in between.

The consequences of the Reswitching phenomenon for the Neoclassical Theory of Capital are most thoroughly summarized in Samuelson's "Summing Up". In this article Samuelson shows that the Reswitching phenomenon provides a dramatic instance where the well behaved neoclassical parables do not necessarily hold. Samuelson writes: "interest rates may bring lower steady-state consumption and lower capital output ratios, and the transition to such lower interest rates can involve denial of diminishing returns and entail reverse capital deepening in which current consumption is augmented rather that sacrificed" (1966, p. 582).

The economic intuition behind the possibility of Reswitching can be best understood by looking at the "Summing Up" example. In this example two techniques of production are introduced, and in one of them labor is not uniformly applied prior to the production of the final output. Champagne is the end product of both techniques A and B. In technique A seven units of labor make one unit of brandy which then ferments, by itself, into one unit of champagne in one more period. In technique B two units of labor make one grapejuice in one period which in one more period ripens, by itself, into one unit of wine. Then six units of labor shaking the wine produce one unit of champagne in one more period. The champagne produced in either technique is identical.

In summary; technique A uses seven units of labor that remain invested for two periods; technique B uses two units of labor that remain invested for three periods, along with six units of labor that remain invested for one period. The reader may wish to look at Table 5 in Section 5.1 in which techniques A and B are presented.

Here is the economic intuition behind the possibility of Reswitching. At zero or very low interest rates only labor and wage costs are relevant, so that technique B using eight units of labor is more expensive that A. But also at very high interest rates, above one hundred per cent per period, technique B is a more expensive method of production than technique A; the reason is that the compound interest applied to the two units of labor invested for three periods becomes very high. It can easily be shown that technique A becomes more expensive at rates of interest between fifty and one hundred per cent. Thus Reswitching does occur.

The discovery of Reswitching as a logical possibility and the analysis of its consequences for the behaviour of the neoclassical Capital Theory parables is a robust result of the Cambridge Controversies. The understanding of Reswitching is a fundamental development in the field of the Theory of Capital.

The Reswitching phenomenon presents new puzzles to theoretical research. Once it is shown that the neoclassical parables do not necessarily hold, there is no certainty as to what the impact of movements in the rate of interest or profit will be. However, within the perspective of economic theory it is important to understand what are the consequences of such movements. Thus, it would seem desirable to explore the possibilities of a reconstruction of a Theory of Capital throwing fresh light on the Reswitching phenomenon. This is the purpose of the present paper.

To start such a reconstruction effort, this paper begins by analyzing carefully Böhm-Bawerk's average time-period of production. This is an arithmetic weighted average of the time involved in the method of production; the weights are the units of labor used in the technique of production. For the discussed champagne example this measure will be: [7(2 periods)]/7 = a mean of 2 periods for technique A, and [2(3 periods) + 6(1 period)]/8 = 1.5 periods for technique B.

The consequences of Reswitching for the demand behaviour of Böhm-Bawerk's decision makers concerning capital-time can easily be appreciated. As the rate of interest goes down from more to less than 100% there is a switch from technique A to technique B. This switch does not follow the neoclassical intuition, since as the interest rate declines the competitive system moves from a more to a less roundabout technique of production; the relationship between the rate of interest and the quantity of capital as measured by Böhm-Bawerk's simple average period of investment –in some sense the "demand for capital"– is direct and not inverse as expected.

Böhm-Bawerk's primitive average time-period of production use indiscriminately as weights, units of labor that are invested in quite different time periods. As a consequence, the simple measure loses the effect of positive, compound interest rate on the value of such units of labor.

Given a positive interest rate, we should want to take its compound effects into consideration. Thus units of labor belonging to different time periods ought not to be used indiscriminately for numerical operations. It should be noted that society is not indifferent, as long as the rate of interest is positive, between investing equal amounts of labor into distinct periods.

The previous considerations clearly suggest that Böhm-Bawerk's arithmetic average time-period of production is not an adequate measure of the time involved in a method of production.

The purpose of Section 1 is to analyze the consequences of eliminating inconsistencies from the measure of the average time-period of production. In this section the units of labor used as weights in the calculation of a new average time-period of production are discounted at the relevant interest rate to bring them to a unique period. It is only after the labor units are reckoned at a common point in time that the new average period of production is estimated.

By using the measures described in the two previous paragraphs a new measure of capital-time is introduced in Section 1. This new measure of capital is a function of the interest rate because it takes into account the discounting effects of both the labor and the output units.

The main outcome of this proposed new measure of capital is: that even when Reswitching occurs, there is still always an inverse relationship between the rate of interest or profit and the quantity of capital-time.

In Section 2 the new measure of capital-time proposed in Section 1 is applied to the case of heterogeneous capital goods.

In the case of two good technologies examples, time at first sight appears not to be relevant. The examples presented are such that the process of production lasts only one period from the point in time in which labor and other capital inputs enter into the process to the point in time in which output is obtained. See Section 5 for examples of Bruno-Burmeister, Garegnani and others. In such examples the time structure of the two alternative techniques of production appears to be identical. However, there is an implicit distinct series of dated labor which can be obtained from the price solution corresponding to each technique.

Total labor in the infinite series of dated labor is equal to the discounted value of the direct labor plus the direct labor of the inputs directly used in the first round of production plus the discounted value of direct labor of the second round of factors needed to produce the first round factors, and so on. The series do converge under quite unrestricted assumptions.

The rounds of which we speak do not take place in calendar time, however they can be interpreted as showing-going backward in time how much production must be started many periods back to meet the consumption targets of the last period.

Thus the infinite series of dated labor shows the initial conditions of production required for the competitive system to be able to produce the capital goods that it requires to be able to maintain the targeted levels of production and consumption along the steady state.

The transformation of a process of production of heterogeneous capital goods into an infinite series of dated labor allows us to determine the indirect labor or labor embodied in the capital goods used as input along the steady state.

In two good technologies processes of production the time structure of total labor, including direct and indirect labor is different for the two alternative techniques of production. When an extension of the measure of capital obtained in Section 1 is applied for the series of dated labor of each technique, it can be shown that it maintains an inverse relationship with respect to interest rate movements.

In Section 3, using the newly proposed measure of capital developed in Sections 1 and 2, a surrogate production process is obtained. In this surrogate production process, it is shown that in general, even when Reswitching occurs, as the economy moves from one technique to another, there is an inverse relationship between the interest rate and the quantity of capital per man, the capital/output ratio, and between the interest rate and the newly defined steady-state consumption per capita. As a consequence of the conclusions of Section 3, it seems to be possible to specify certain one directional effects in the economy of movements in the rate of interest or profit.

In Section 4 we present further comments in the results of Sections 1 to 3.

In Section 5 we have calculated numerical examples for the set of relationships introduced in Sections 1 to 4. Samuelson's Summing Up example is calculated, also the following two good technologies examples are estimated: Bruno-Burmeister (1966), Garegnani (1966), Morishima (1966). and Pasinetti's (1966).

1 A New Measure of Capital

In this section the units of labor used as weights in the calculation of a new average time-period of production, are discounted at the relevant interest rate to bring them to a unique period. It is only after the labor units are at a common point in time that the new average period of production is estimated. This is the methodology used for $K_L(r)$ in equation (2); $K_L(r)$ represents the average time that a unit of labor remains invested in the technique of production.

In addition to $K_L(r)$, time enters in the method of production through the dates in which each unit of output is obtained; the average time that a unit of output is held is estimated in equation (3), and it is obtained with a similar methodology as $K_L(r)$ in (2).

Because of the previous comments capital-time intensity, $K(r)$, is measured in equation (1) in terms of $K_L(r)$ and in terms of $K_Q(r)$.

1.1 A Measure of the Capital Intensity of a Technique

Let,

$$K(r) = K_L(r) - K_Q(r) \tag{1}$$

$K_L(r)$ is defined as:

$$K_L(r) = \frac{\sum (1+r)^{-t}(N-t)L_t}{\sum (1+r)^{-t}L_t} = \frac{\sum (1+r)^{N-t}(N-t)L_t}{\sum (1+r)^{N-t}L_t}$$

$$= \frac{(1+r)f'(r;L)}{f(r;L)} \tag{2}$$

We observe that $K_L(r)$ is well defined as $f(r;L)$ is strictly positive and that $K_L(r)$ is continuous and has derivatives of all orders. We further note that $K_L(r)$ is invariant in relation to the point in time at which it is valued. Finally, $K_L(0)$ represents Böhm-Bawerk's measure. [For the definition of the function f(r:.) see appendix one].

$K_Q(r)$ measured as:

$$K_Q(r) = \frac{\sum (1+r)^{-t}(N-t)(\frac{P}{w}^{*} q_t)}{\sum (1+r)^{-t}(\frac{P}{w^*} q_t)} = \frac{(1+r)f'(r;\frac{P}{w^*}q)}{f(r;L)} \tag{3}$$

Where the output is measured in units of labor in relation to the frontier wage, and the last inequality follows from the fact that $f(r; a^*) = 0$. [For the definition of a^* the reader should see Section 1.2.] $K_Q(r)$ has similar properties to those of $K_L(r)$ with respect to it being well defined, continuous and invariant.

Using equations (1) to (3), we obtain:

$$K(r) = \frac{(1+r)f'(r; a^*)}{f(r; L)} \tag{4}$$

$K(r)$ is also well defined, continuous, has derivatives of all orders and is invariant. Moreover, it can be prove that $K_L \geq K_Q$ thus $K(r) \geq 0$. Further, as $0 \leq K_L(r) \leq N-1$ and $0 \leq K_Q(r) \leq N-1$, we have that the measure of capital is bounded by $0 \leq K(r) \leq (N-1)$.

1.2 Characterization of a Production Technique

The characterization of the technique is given by $T[N, L, q]$ where N is the number of periods during which the process lasts, $L = (L_1, L_2, ..., L_N)$ is the vector of labor units used per period and $q = (q_1, q_2, ..., q_N)$ is the vector of output units produced per period.

Given a price p of the product and a wage w, and making the assumption that all the payments are made at the end of each period, the present value of the output is:

$$Q(r) = p \sum (1+r)^{-t} q_t = (1+r)^{-N} f(r; pq) \tag{5}$$

(See the appendix for the definition and properties of the function $f(r : .)$). [1]

In the other hand, the present value of the labor input is

$$D(r, w) = w \sum (1+r)^{-t} L_t = (1+r)^{-N} f(r; wL). \tag{6}$$

Hence the net present value (of the output) of the techniques is

$$\begin{aligned} T(r, w) &= Q(r) - D(r, w) \\ &= (1+r)^{-N} f(r; pq - wL) \\ &= -w(1+r)^{-N} f(r; a), \end{aligned} \tag{7}$$

where $a = L - \frac{p}{w} q$, is the vector of net units of labor used, for $-\frac{p}{w} q$ is the equivalent in unit of labor of the output vector.

Given the price p of the output the factor price frontier is defined in terms of the maximum wage w^* which the technique can pay whilst maintaining the present value non-negative; that is, w^* is such that $T(r, w^*) = 0$. This occurs if an only if $f(r; pq - w^* L) = 0$. From this we have

$$w^* = w^*(r) = \frac{f(r; pq)}{f(r; L)} > 0, \tag{8}$$

[1] Note that all the summations along this work range from 1 to N.

because $f(r;pq)$ and $f(r;L)$ are strictly positive. We now define $a^* = L - \frac{p}{w^*}q$, the vector of net units of labor in the frontier and note that the condition $f(r;a^*) = 0$, holds as $T(r,w^*) = 0$.

We shall say that a technique is preferable to another at a given rate of interest if its frontier wage is greater.

1.3 Reswitching: The Possibility of Switching Points

We shall consider two production techniques, $T_A[N, L_A, q_A]$ and $T_B[N, L_B, q_B]$. We observe that choosing the same number of periods, N, for both techniques, offers no loss of generality for if they were different we could always take the value of the largest and extend the vectors L and q of the other technique by inserting zeros.

T_A will be preferred in those regions of interest rate where $w_A^*(r) > w_B^*(r)$, and analogously, T_B will be preferred for those values for which $w_A^*(r) < w_B^*(r)$. We shall denote this preference by $T_A > T_B$ and $T_A < T_B$, respectively.

In those points for which $w_A^*(r) = w_B^*(r)$ we are indifferent to either technique and we will write $T_A = T_B$. Then, if we define the function

$$w(r) = w_A^*(r) - w_B^*(r) = \frac{f(r;pq_A)}{f(r;L_A)} - \frac{f(r;pq_B)}{f(r;L_B)}, \tag{9}$$

we will have that $T_A \gtreqless T_B \Longleftrightarrow w(r) \gtreqless 0$.

$w(r)$ is well defined, continuous and has derivatives of all orders. The continuity of $w(r)$ means that for a change of sign to occur it must pass through zero. Hence the points in which a re-switching takes place must necessarily be points of indifference between T_A and T_B. Therefore, if r_0 is a switching point, we will have that $w(r_0) = 0$, which provides us with a necessary condition. Inserting this into equation (9) gives rise to the following for a switching point r_0,

$$\frac{f(r_0;pq_A)}{f(r_0;L_A)} = \frac{f(r_0;pq_B)}{f(r_0;L_B)} \Longleftrightarrow \frac{f(r_0;q_A)}{f(r_0;L_A)} = \frac{f(r_0;q_B)}{f(r_0;L_B)}. \tag{10}$$

We observe that

$$w(r) = 0 \Longleftrightarrow g(r) = f(r;q_A)f(r;L_B) - f(r;q_B)f(r;L_A) = 0.$$

However, $g(r)$ is a polynomial in r of degree at most $2(N-1)$ and so has a maximum number of $2(N-1)$ possible changes between T_A and T_B. This means that re-switching is a logical possibility. Amongst the points for which $w(r) = 0$, are also included inflexion points. These points are distinguished by the following condition: r_0 is an inflexion point for $w(r)$ if

$$w^j(r_0) = 0, \;\; j = 1, ..., 2n \;\; and \;\; w^{2n+1}(r_0) = 0$$

We shall exclude these points from our analysis and make some further remarks about their behaviour at the end of 1.4. Let us now consider the derivative of w, that is:

$$w'(r) = {w_A^*}'(r) - {w_B^*}'(r).$$

In general

$$\begin{aligned} w^{*\prime}(r) &= \frac{f(r;L)f'(r;pq) - f(r;pq)f'(r;L)}{f(r;L)^2} \\ &= \frac{f'(r;pq) - f'(r;w^*L)}{f(r;L)} = \frac{-w^* f'(r;a^*)}{f(r;L)}. \end{aligned} \tag{11}$$

On making use of the relation $f(r;a^*) = 0$. It follows from this that

$$w'(r) = \frac{w_B^*(r)f'(r;a_B^*)}{f(r;L_B)} - \frac{w_A^*(r)f'(r;a_A^*)}{f(r;L_A)}.$$

In a switching point r_0 we have that $w_A^*(r_0) = w_B^*(r_0) = w^*(r_0)$, so we arrive to the following:

$$w'(r) = w^*(r_0)\frac{f'(r_0;a_B^*)}{f(r_0;L_B)}\frac{f'(r_0;a_A^*)}{f(r_0;L_A)}. \tag{12}$$

We can suppose without any loss of generality that T_A is in use and that the interest rate move through a switching point r_0. $w(r)$ must decrease (passing from positive to negative) when passing through r_0. Hence, if r_0 is increasing we must have $w'(r_0) < 0$. Analogously, if r_0 is decreasing we must have $w'(r_0) > 0$.

1.4 Capital Intensity in Switching Points

We clearly have that

$$K_A(r) \gtreqless K_B(r) \Longleftrightarrow K_A(r) - K_B(r) \gtreqless 0. \tag{13}$$

However,

$$K_A(r) - K_B(r) = (1+r)\left(\frac{f'(r;a_A^*)}{f(r;L_A)} - \frac{f'(r;a_B^*)}{f(r;L_B)}\right),$$

and at a switching point we conclude that

$$K_A(r_0) - K_B(r_0) = \frac{-(1+r_0)w'(r_0)}{w^*(r_0)} \gtreqless 0 \tag{14}$$

$$K_A(r_0) \gtreqless K_B(r_0) \Longleftrightarrow -w'(r_0) \gtreqless 0. \tag{15}$$

In view of the remarks made in the previous section, we have that for r increasing (i.e. $w'(r_0) < 0$) the change is to a technique which is less intensive in capital (i.e. $K_A(r_0) > K_B(r_0)$). In a similar way, for r decreasing (i.e. $w'(r_0) > 0$) the change is to a technique which is more intensive in capital (i.e. $K_A(r_0) < K_B(r_0)$).

Hence, there is a well-behaved (inverse) relation between the interest rate and the quantity of capital demanded.

Finally, in the particular case in which the switching point is an inflexion point we have that $w'(r_0) = 0$ and $K_A(r_0) = K_B(r_0)$. in this way, even in these points, the re-switching goes to a technique equally intensive in capital

1.5 Conclusion of Section 1

We have proven that for the general case $K(r)$ has an inverse relationship with the rate of interest or profit; there is a well-behaved demand for capital.

2 Heterogeneous Capital Goods

As it has been shown by other authors[2], a process of production containing heterogeneous capital goods can be transformed into a convergent series of dated labor.

In this section we will show that the measure of capital proposed in the previous section, when applied to a convergent series of dated labor, also converges. Thus, in fact it is possible to find a unique finite value for the average period of time which a unit of indirect an direct labor remains invested in the production process. Moreover, it will be shown that the measure of capital-time of an infinite series of dated labor has the same properties than the one of the austrian model introduced in the previous section, that is, it behaves inversely with respect to movements in the level of the rate of interest, even when Reswitching occurs.

2.1 Heterogeneous Capital Goods and Capital-Time

In general a process of production containing heterogeneous capital goods can be expressed by

$$P = (1+u)a_0[I-(d+r)a]^{-1}W \tag{16}$$

where P is the vector of prices, W is the nominal wage, a_0 is the labor vector, a is the matrix of fixed proportions technology, d is equal to the depreciation rate of capital and its value is between zero and one, and u is equal to r if wages are paid at the beginning of the period and equal to zero if they are paid at the end of the period.

Note that in what follows we will assume for the sake of simplicity of exposition that $u = 0$ and $d = 1$. The proof can be easily extended for other values of u and d.

It should be observed that P can be expressed in dated quantities of labor, that is:

$$P = a_0\left[I + (1+r)a + (1+r)^2a^2 + \ldots\right]W. \tag{17}$$

Now let z be a point in the infinite past, then from (17) and using (2) we obtain (18), (19) and (20),

$$\sum(1+r)^{-t}L_{t_i} = \left[a_0(1+r)^{-z-1} + a_0a(1+r)^{-z-1+1} + a_0a^2(1+r)^{-z-1+2} + \ldots\right]e_i \tag{18}$$

$$\sum(1+r)^{-t}(n-t)L_{t_i} = \left[a_0a(1+r)^{-z-1+1} + 2a_0a^2(1+r)^{-z-1+2} + 3a_0a^3(1+r)^{-z-1+3} + \ldots\right]e_i \tag{19}$$

$$K_{La_i} = \frac{\left[a_0a(1+r) + 2a_0a^2(1+r)^2 + 3a_0a^3(1+r)^3 + \ldots\right]e_i}{\left[a_0 + a_0a(1+r) + a_0a^2(1+r)^2 + a_0a^3(1+r)^3 + \ldots\right]e_i} \tag{20}$$

[2]See Sraffa (1962) and Pasinetti (1977).

where $i = 0, 1, 2, \ldots$, and $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $e_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Expression (20) gives the value of capital for the economic system as a whole when $i = 0$, and gives the value of capital for each sector as i takes the values corresponding to each sector, 1 and 2.

It is convenient to recall that the value of K_{La_i} converges. In fact, the convergence of the denominator is a necessary condition for the existence of a viable economic solution of the price system. It can be shown that the denominator converges whenever the spectral radius (the maximum characteristic root) of the matrix $a(1+r)$ is less than one[3]. With the denominator converging, the convergence of K_{La_i} will be guaranteed if the numerator converges.

The numerator in (20) converges because the scalar series $1 + q + 2q^2 + \ldots$ converges whenever $|q| < 1$[4]. Thus, the numerator converges whenever the maximum characteristic root of the matrix $a(1+r)$ is less than one. Again, this condition must be satisfied for the price system to have a viable economic solution.

By a well-known theorem, if $f(q) = c_m q^m$ converges for $|q| < 1$, then if $A \in C_{n \times n}$ is such that $p(a) < 1$, the matrix series, $f(A) = \sum_{m=0}^{\infty} C_m A^m$, converges[5]. Where $C_{n \times n}$ denotes the set of the complex numbers and $p(A)$ denotes the spectral radius of A or $p(A) = \max \{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_t|\}$, where $|\lambda_i|$ denotes the distinct characteristic roots of A. Moreover, by another theorem of matrix calculus if $\sum_{n=0}^{\infty} A_n = S$, then $\sum_{n=0}^{\infty} P A_n Q = PSQ$, where P and Q are vectors.

Thus, the condition for the convergence of the numerator of K_{La_i} is that the maximum characteristic root of the matrix $a(1+r)$ be less than one[3]. This condition can also be stated as $(1+r) < \frac{1}{\lambda_m}$, where λ_m is the maximum characteristic root of the matrix a. Notice that this last condition is also the condition for the convergence of the denominator. So we can conclude that whenever the denominator, that is the series of dated labor, converges the measure of capital-time proposed in this article also converges.

2.2 Two Good Technologies

Whenever we compare two economic systems adjacent to each other in the technological frontier, the two alternative systems at a given rate of profit can differ in only one equation because they must necessarily yield the same solution for all prices and there is in general only one degree of freedom in the solution. Thus the corresponding matrices will differ in general in only one method of production[6]. Thus, to show theoretically how the newly proposed measure of capital behaves it is only necessary to refer to the case of two good technologies.

The factor price frontier of each technique is defined by

$$1 = x_0 \left[I - (1+r)x\right]^{-1} e_1 W \tag{21}$$

where $x = a, b$. e_1 is the unit vector corresponding to the commodity used as numeraire. W is expressed in term of this numeraire.

[3] See Pasinetti (1977, p.264.).

[4] See Courant (1963, p.380).

[5] See Cullen, Charles (1972, p.257).

[6] See Pasinetti (1977, p.163).

Let us now define

$$W(r) = W_a^*(r) - W_b^*(r) \tag{22}$$

and

$$G_i(r;a) = a_0\left[I - (1+r)a\right]^{-1} e_i;\; G_i(r;b) = b_0\left[I - (1+r)b\right]^{-1} e_i;\; i = 1,2 \tag{23}$$

Since W must be equal across sectors it holds that

$$\begin{aligned} W_a^*(r) &= \frac{1}{G_1(r;a)} = \frac{P_2}{G_2(r;a)}; \\ W_b^*(r) &= \frac{1}{G_1(r;b)} = \frac{P_2}{G_2(r;b)} \end{aligned} \tag{24}$$

Thus, from (22) and (24) $W^*(r)$ denoted in (25) by $W(r)$, is equal to

$$W(r) = \frac{1}{G_1(r;a)} - \frac{1}{G_1(r;b)} = \frac{P_2}{G_2(r;a)} - \frac{P_2}{G_2(r;b)}. \tag{25}$$

Now we observe that

$$W_a^{*\prime}(r) = \frac{G_1{}'(r;a)}{G_1(r;a)^2} = -\frac{P_2 G_2{}'(r;a)}{G_2(r;a)^2} = -\frac{W_a^*(r) G_i{}'(r;a)}{G_i(r;a)}; \qquad i = 1,2. \tag{26}$$

Obtaining $W_b^{*\prime}$ from an expression similar to (26) we get

$$W'(r) = \frac{W_b^*(r) G_i{}'(r;b)}{G_i(r;b)} - \frac{W_a^*(r) G_i{}'(r;a)}{G_i(r;a)}; \qquad i = 1,2. \tag{27}$$

And since at a switching point (r_0) $W_b^*(r) = W_a^*(r)$, it holds that

$$W'(r_0) = W^{*\prime}(r_0)\left[\frac{G_i{}'(r;b)}{G_i(r;b)} - \frac{G_i{}'(r;a)}{G_i(r;a)}\right]; \qquad i = 1,2. \tag{28}$$

Again we can suppose without loss of generality, that T_a is in use and that the interest rate moves through a switching point r_0. $W(r)$ must decrease (passing from positive to negative) when passing through r_0. Hence, if r_0 is increasing we must have $W'(r_0) < 0$. Analogously, if r is decreasing, we have $W'(r_0) > 0$.

Now at a switching point we know that

$$a_0\left[I - (1+r)a\right]^{-1} e_i = b_0\left[I - (1+r)b\right]^{-1} e_i. \tag{29}$$

This implies that the denominator of K_{La_i} is equal to the denominator K_{Lb_i}. Thus,

$$K_{La_i} \lesseqgtr K_{Lb_i} \Longleftrightarrow \text{Numerator of } K_{La_i} \lesseqgtr \text{Numerator of } K_{Lb_i}$$

Moreover, we can observe that

$$\begin{aligned} \text{Numerator of } K_{L_{x_i}} &= x_0\left[(1+r)x + 2(1+r)^2x^2 + 3(1+r)^3x^3 \ldots\right] e_i \\ &= (1+r) G_i{}'(r;x), \end{aligned} \tag{30}$$

with $x = a, b$.

Thus,

$$K_{La_i} \lesseqgtr K_{Lb_i} \Longleftrightarrow G_i{}'(r;a) \lesseqgtr G_i{}'(r;b). \tag{31}$$

Using (28), (29) and (31) we obtain

$$K_{La_i}(r_0) \lesseqgtr K_{Lb_i}(r_0) \Longleftrightarrow -W'(r_0) \lesseqgtr 0. \tag{32}$$

It must be noted that in the case of two good technologies $K_Q(r) = 0$, thus the previous expression implies:

$$K_{a_i} \lesseqgtr K_{b_i}(r_0) - W'(r_0) \lesseqgtr 0; \qquad i = 1, 2. \tag{33}$$

Using the definition of (20) for $i = 0$, and (21), $K_{L_{x_0}}$, the average period of investment of the total labor used in both sectors can be defined as:

$$K_{L_{x_0}} = K_{L_{x_1}} \frac{G_1(r;x)}{G_0(r;x)} + K_{L_{x_2}} \frac{G_2(r;x)}{G_0(r;x)}; \qquad x = a, b \tag{34}$$

Moreover, we know that at a switching point $G_1(r;a) = G_2(r;b)$ and since $G_0(r;x) = G_1(r;x) + G_2(r;x)$, then at a switching point $G_0(r;a) = G_0(r;b)$.

Thus,

$$K_{L_{a_i}} \lesseqgtr K_{L_{b_i}} \Longleftrightarrow K_{L_{a_0}} \lesseqgtr K_{L_{b_0}}; \qquad i = 1, 2 \tag{35}$$

In view of the remarks made in the previous section, we have that for r increasing (i.e. $W'(r_0) < 0$) the change is to a technique which is less intensive in capital (i.e. $K_{a_i}(r_0) > K_{b_i}(r_0)$, with $i = 1, 2$). In a similar manner, for r decreasing (i.e. $W'(r_0) > 0$) the change is to a technique which is more intensive in capital (i.e. $K_{a_i}(r_0) < K_{b_i}(r_0)$, with $i = 1, 2$).

Hence, there is a well-behaved (inverse) relation between the interest rate and the quantity of capital demanded.

Finally, in the particular case in which the switching point is an inflexion point we have $W'(r_0) = 0$, and so $K_{a_i}(r_0) = K_{b_i}(r_0)$, with $i = 1, 2$. In this way, even in these points the Reswitching takes place to a technique equally intensive in capital.

2.3 Conclusion of Section 2

It has been shown that in general in processes of production of two good technologies, as the economy switches from one technique to the other, $K_i(r)$ for $i = 0, 1, 2, \ldots$, has an inverse relationship with the rate of interest or profit.

3 A Surrogate Production Process

The surrogate production process (SPP) conveys the same present discounted value of total output, uses the same present discounted value of labor input, and has the same average period of production than the original technique of production. Moreover, the SPP's of two alternative techniques of production preserve the switching point condition that at these points both techniques allow for the payment of the same wage rate.

The SPP is purely a conceptual device. It does not exist in a physical sense, and it is not an alternative physical process of production. The SPP is an accounting procedure, which at a given rate of interest, mimics in value terms and in labor units the original process of production according to the specified transformation.

Section 3.1 develops the surrogate transformation of both the neoaustrian and the two good technologies process of production. Section 3.2 shows that the neoclassical Capital Theory parables do behave according to traditional intuition in the SPP. In section 3.3 we explore the meaning of the previously proposed definition of surrogate capital. Finally sections 3.4 and 3.5 develop a steady-state surrogate production process.

3.1 The Transformation to a Surrogate Production Process

3.1.1 The Neoaustrian Case

Given the value of $K(r)$ obtained in Section 1 there is a unique value of n which satisfies the condition that labor is applied uniformly in the process of production and which also satisfies the condition that the average period of investment of one unit of labor is identical to the original $K(r)$.

It must be observed that if $l(r)$ labor enters uniformly into the process of production, then equation (2) becomes: [7]

$$K_1(r) = n - \frac{\sum t(1+r)^{-t}}{\sum (1+r)^{-t}}. \tag{36}$$

Now, if we substitute $K(r)$, the value of capital-time in the original neoaustrian technique, for $K_i(r)$ in the previous expression, we find

$$K(r) = n - \frac{\sum t(1+r)^{-t}}{\sum (1+r)^{-t}}. \tag{37}$$

The expression in the right hand-side of this expression is a monotonically increasing function in n, thus as n increases the whole expression increases. Since the value of $K(r)$ is given (from section 1), there must be a unique value of n at which (37) holds. Thus it shows that once we know $K(r)$, the average period of investment per unit of labor, we can always find the length n of a SPP in which labor is applied uniformly all through the n periods.

The second condition that the SPP must fulfill is that the discounted value of total labor must be equal to that in the original process of production. Taking as a reference point the end period in which output is actually obtained, the total discounted value of total labor in the original technique of production is given by:

$$L_d = \frac{1}{W} D(r; L)(1+r)^N = \sum (1+r)^{-t} L_t (1+r)^N \tag{38}$$

At a given r, we know the value of L_d in the original technique of production, and using L_d and n we can find $l(r)$, the value of labor that is invested each period in the SPP.

$$l(r) = \frac{L_d}{\sum_{t=1}^{n} (1+r)^{-t} (1+r)^n} \tag{39}$$

[7] n and l will be used for the SPP, while N and L will be preserved to denote the original process of production.

Table 1 SPP.

PERIOD	LABOR	OUTPUT
1	l(r)	
2	l(r)	
.		
.		
n	l(r)	P(r)Q

It must be observed that $\sum_{t=1}^{n}(1+r)^{-t}(1+r)^{n}$ is again a monotonic increasing function in n. Thus given d and r, the higher is n, the lower is $l(r)$.

The SPP is depicted in Table 1.

The SPP satisfies a third condition, which is that the discounted value of the output produced is equal to the one of the original technique of production.

Since both the original technique and the SPP have the same discounted value of labor input and produce the same discounted value of output, it follows that they both generate the same wage rate.

3.1.2 Two Good Technologies

In the case of two good technologies, the final output of the SPP can be thought of as a composite commodity which contains one unit of each final good produced by each sector in the original technique of production.

Assuming without loss of generality, that $P_2 = 1$ is the price of the consumption good, we have:

$$P_1 + 1 = P \tag{40}$$

Where P is the price of the composite commodity produced in the SPP. The discounted value of total labor input in the SPP will be equal to the discounted sum of the labor input in the convergent series of dated labor corresponding to each sector. And the average period of investment of one unit of labor will be given by $K_{La0}(r)$ in equation (20).

Given $K_{la0}(r)$, it is possible to find n from a similar equation to equation (37). Notice that the value n is always finite, thus the convergent infinite series of dated labor can always be transformed into a SPP of n finite periods.

In the case of a convergent series of dated labor, the discounted value of total labor, L_d taking as a reference the point at which output is obtained is equal to:

$$L_d = x_0[I + (1+r)x + (1+r)^2x^2 + ...]q \tag{41}$$

Where the notation comes from section 2, and q is a column vector of the quantities produced. At a given r, we know the value of L_d and introducing L_d and n in (39) we can obtain the value of labor $l(r)$, which is invested each period in the SPP.

3.2 Capital Theory Parables

3.2.1 Output Per Man in the SPP

The value of output in the SPP,must be equal to the wages paid to the labor invested in the period in which the output is obtained plus the discounted value of the wages paid to labor invested in previous periods:

$$PQ = Wl(r) + Wl(r)\sum_{t=1}^{n-1}(1+r)^{n-t} \tag{42}$$

Dividing (42) by $l(r)$, we can obtain:

$$\frac{PQ}{l(r)} = W[1 + \sum_{t=1}^{n-1}(1+r)^{n-t}] \tag{43}$$

From (37) we know that the higher is $K(r)$ the higher is n. Moreover,in (43) $\sum_{t=1}^{n-1}(1+r)^{n-t}$ is a monotonic increasing function in n. Thus, the higher is $K(r)$ or $K_{LaO}(r)$, for a given W, the higher the value of output per man in (43) will be. Moreover at a switch point the wage rate and the price of output must be equal for both techniques of production, hence:

$$K_a(r) \lesseqgtr K_b(r) \Longleftrightarrow \frac{P(r)Q_a}{l_a(r)} \lesseqgtr \frac{P(r)Q_b}{l_b(r)} \tag{44}$$

[in (44) and in what follows $K_i(r)$ refers also to $K_{Li0}(r)$]. (44) can be conceptualized as expressing that if the competitive system were to consume the total discounted output produced at the end of the production process, the more time intensive technique of production will also allow for a higher level of consumption per capita.

3.2.2 Capital Per Man in the SPP

Using the Von Neumann method the "goods in process" in the SPP could be treated as different surrogate commodities. Thus the labor invested in each period could be thought of as producing diverse types of surrogate machines. Using the Von Neumann method, we can define n activities in the SPP such that:

Activity 1:	l(r) workers produce one machine of type 1.
Activity 2:	l(r) workers using a machine of type 1 produce a machine of type 2.
.	
.	
Activity n:	l(r) workers using a machine of type n-1 produce the final output of the SPP.

Each activity could be represented by a column vector of an input matrix, where the rows, from top to bottom, correspond to inputs of labor machines of type 1, machines of type 2,..., machines of type n, and the final commodity

$$\left[\frac{A_0}{A}\right] = \begin{bmatrix} l(r) & l(r) & . & . & . & l(r) \\ 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 1 \\ 0 & & 0 & 0 & . & 0 \end{bmatrix}$$

The corresponding output matrix is the $n \times n$ unit diagonal matrix; and let the row vector P designate the prices of machines of type 1 to $n-1$ and of the final commodity. Then we may proceed to calculate the equilibrium price vector as:

$$P = WA_0[Q_m - (1+r)A]^{-1}$$

From which we can obtain the following set of equations:

$$\begin{aligned} P_{m1} &= Wl(r) \\ P_{m2} &= Wl(r)(1+r) + Wl(r) \\ &. \\ &. \\ P_{m(n-1)} &= Wl(r)(1+r)^{n-2} + Wl(r)(1+r)^{n-3} + ... + Wl(r) \\ P &= Wl(r)(1+r)^{n-1} Wl(r)(1+r)^{n-2} + Wl(r)(1+r)^{n-3} + ... + Wl(r) \end{aligned} \tag{45}$$

From (45) P is equal to:

$$P = Wl(r) + Wl(r)\sum_{t=1}^{n-1}(1+r)^{n-t} = Wl(r) + (1+r)P_{m(n-1)} \tag{46}$$

Moreover since in the SPP we are producing one unit of each type of machine and one unit of the final (composite) commodity, (47) follows from (46):

$$PQ = Wl(r) + Wl(r)\sum_{t=1}^{n-1}(1+r)^{n-t} = Wl(r) + (1+r)P_m K_m \tag{47}$$

Where K_m denotes the machine produced in period $n-1$, and P_m denotes, from now on, its price.

From (47) we can obtain the productive power of K_m in terms of labor units, which we shall denote by K_m^*:

$$K_m^* = \frac{P_m K_m}{W} = l(r)\sum_{t=1}^{n-1}(1+r)^{n-t-1} \tag{48}$$

Thus PQ can also be expressed by (49),

$$PQ = Wl(r) + (1+r)WK_m^* \tag{49}$$

Dividing (47) and (59) by $l(r)$ we obtain (50) and (51):

$$\frac{PQ}{W} = W + (1+r)[\frac{P_k K_m}{l(r)}] \tag{50}$$

$$\frac{PQ}{W} = W[1 + (1+r)[\frac{K_m^*}{l(r)}] \tag{51}$$

From (44) we know that, at a switch point, the technique with a higher capital-time will also have a higher output per man; and since at a switch W and r are the same for both techniques, (50) and (51) imply that the technique with higher capital-time will also have a higher capital per man ratio. Thus, given (50) and (51), (52) holds

$$\begin{aligned} K_a(r) \lesseqgtr K_b(r) &\iff \frac{P_{ma}K_{ma}}{l_a(r)} \lesseqgtr \frac{P_{mb}K_{mb}}{l_b(r)} \\ &\iff \frac{K_{ma}^*}{l_a(r)} \lesseqgtr \frac{K_{mb}^*}{l_b(r)} \end{aligned} \tag{52}$$

3.2.3 Capital/Output Ratio in the SPP

The capital/output ratio can be obtained by dividing (47) by $P_m K_m$ and obtaining the inverse of this quotient:

$$\frac{P_m K_m}{PQ} = \frac{\frac{P_m K_m}{l(r)}}{W + (1+r)\frac{P_m K_m}{l(r)}} \tag{53}$$

(53) shows that at a switch point the technique with a higher capital per man in value terms will also have a higher capital/output ratio in value terms. The capital/output ratio can also be expressed in labor units, using (47) we can obtain:

$$\frac{K_m^*}{PQ/W} = \frac{K_m^*/l(r)}{1 + (1+r)[K_m^*/l(r)]} \tag{54}$$

From (53) and (54) and using (52), it follows that:

$$\begin{aligned} K_a(r) \lesseqgtr K_b(r) &\iff \frac{P_{ma}K_{ma}}{PQ} \lesseqgtr \frac{P_{mb}K_{mb}}{PQ} \\ &\iff \frac{K_{ma}^*}{PQ/W} \lesseqgtr \frac{K_{mb}^*}{PQ/W} \end{aligned} \tag{55}$$

3.3 The Meaning of Surrogate Capital

K_m is an accounting measure of the degree of indirectedness of a given technique of production. K_m does not exist in a physical sense and does not have a market of its own.

K_m measures the degree of indirectedness associated with a surrogate process of production in which in each period the labor invested is equal to the actual discounted average invested in the original process of production. In this sense K_m is a measure of indirectedness associated to the original process of production. K_m is defined using $l(r)$, which in turn is defined based upon the actual technological information contained in the original process of production.

3.4 Two Good Technologies Along the Steady State

If the competitive system is conceived as operating along the steady state, we must take into account the amount of direct labor needed in each period along the steady state. Also it is necessary to take as a datum, from the original blue print technology, the amount of each good that can be used each period for consumption purposes. To further explore the meaning of conceiving the competitive system as being along the steady state, let us recall that the sum of total, direct and indirect labor is equal to:

$$x_0 q + x_0[(1+r)x + (1+r)^2 x^2 + ...]q \tag{56}$$

Moreover, from (17), the following equations hold:

$$P(r)q = W(r)x_0 q + P(r)(1+r)xq = W(r)x_0 q + W(r)[(1+r)x + (1+r)^2 x^2 + ...]q \tag{57}$$

$$P(r)C = W(r)x_0 q + P(r)rxq \tag{58}$$

$$P(r)q = P(r)C + P(r)xq \tag{59}$$

From (56) and (57) we can obtain:

$$W(r) = \frac{P(r)q}{x_0 q + x_0[(1+r)x + (1+r)^2 x^2 + ...]q} \tag{60}$$

(60) shows that $W(r)$ indicates the level of consumption that can be obtained by one unit of total discounted labor if total output was to be consumed in one period.

However conceiving the competitive system as being in steady state implies that total output is not consumed in one period but instead part of the output is used as input of production. The steady state discounted consumption of one unit of total labor is given by:

$$\frac{P(r)C[\frac{1}{1-\frac{1}{1+r}}]}{x_0 q[\frac{1}{1-\frac{1}{1+r}}] + x_0[(1+r)x + (1+r)^2 x^2 + ...]q} \tag{61}$$

Table 2 Steady state SPP

PERIOD	LABOR	OUTPUT
1	$l_s(r)$	
2	$l_s(r)$	
.	.	
n	$l_s(r)$	P(r)C
n+1	$l_s(r)$	P(r)C
n+2	$l_s(r)$	P(r)C
.	.	.
.	.	.

which using (57) and (58) and multiplying numerator and denominator by $r/1+r$, can be shown to be equal to:

$$\frac{W(r)x_0q + P(r)rxq}{x_0q + P(r)rxq/W(r)} = W(r) \tag{62}$$

(62) shows that $W(r)$ also indicates the level of discounted consumption that can be obtained by one unit of total discounted labor when the competitive system is operating in the steady state. Thus, as the competitive system optimizes along the factor price frontier by choosing the technique that yields a higher wage, it is in fact optimizing the discounted consumption per unit of total discounted labor.

By analyzing carefully (60) and (61), the reader will appreciate that up to now the development of the SPP has been based upon (60). Our purpose in what follows is to develop a SPP based upon (61); that is, a SPP that will mimic in labor units and in value terms the operation of the original blue print technology along the steady state.

(61) in opposition to (60) specifies the technological requirements for production to take place all along the steady state in the original blue print technology. (61) defines the amount of direct labor required along the steady state. Also (61) defines the level of consumption of each good that can be obtained in each period, maintaining the level of re-investement required in each period according to the technological specifications of the original blue print technology.

Using (61) instead of (60) implies some modifications in the concepts and measures introduced up to now. To obtain the average labor invested per period we have to take into account the direct labor invested each period along the steady state. Thus we have to develop a steady state SPP. This steady state SPP will be conceived as it is show in Table 2.

In Table 2, $l_s(r)$ takes into consideration the direct labor invested in the original technology along the steady state. Thus $l_s(r)$ in the steady state SPP is the actual discounted average labor per period invested in the original steady state technology. PC is the steady state consumption allowed in both the original technique and the steady state SPP.

n can be obtained from equation (63), which performs the role previously undertaken by (37):

$$K(r) = \frac{\sum_{t=1}^{n-1}(n-t)(1+r)^{-t}}{\sum_{t=1}^{n-1}(1+r)^{-t} + \frac{1}{1-\frac{1}{1+r}}} \tag{63}$$

Again the expression in the right side of (63) is an increasing monotonic function in n; thus knowing K(r) we can obtain n; and knowing n, $l_s(r)$ can be obtained from (64):

$$l_s(r) = \frac{x_0 q[\frac{1}{1+\frac{1}{1+r}}] + x_0[(1+r)x + (1+r)^2 x^2 + ...]q}{\frac{1}{1+\frac{1}{1+r}} + \sum_{t=1}^{n-1}(1+r)^{n-t}} \tag{64}$$

3.4.1 Consumption Per Capita in the Steady-State SPP

Using (64) and (61) it can be shown that the consumption per unit of average discounted labor per period is given by:

$$\frac{P(r)C}{l_s(r)} = W(r) + \frac{W(r)}{1+r}\sum_{t=1}^{n-1}(1+r)^{n-t} \tag{65}$$

Where recalling that $\sum_{t=1}^{n-1}(1+r)^{n-t}$ is a monotonic increasing function in n. We can obtain:

$$K_{sa}(r) \lesseqgtr K_{sb}(r) \Longleftrightarrow \frac{P(r)C_a}{l_{sa}(r)} \lesseqgtr \frac{P(r)C_b}{l_{sb}(r)} \tag{66}$$

And dividing by (65) by W we can obtain (67):

$$K_{sa}(r) \lesseqgtr K_{sb}(r) \Longleftrightarrow \frac{C*_a}{l_{sa}(r)} \lesseqgtr \frac{C*_b}{l_{sb}(r)} \tag{67}$$

(66) states that there is an inverse relationship between the rate of interest and the value of consumption per capita along the steady state. (67) states the same relationship in labor units.

3.4.2 Capital Per Man and Capital/Output Ratio in the Steady-State SPP

Surrogate capital can be used as a measure of the indirectedness connected with a given steady state technique of production. Steady state surrogate capital will be denoted by K_{ms}.

From (61) and (64) we can obtain:

$$P(r)C[\frac{1}{1-\frac{1}{1+r}}] = W(r)l_s(r)[\frac{1}{1-\frac{1}{1+r}}] + W(r)l_s(r)\sum_{t=1}^{n-1}(1+r)^{n-t} \tag{68}$$

From which we can derive:

$$P(r)C = W(r)l_s(r) + W(r)l_s(r)[\frac{r}{1+r}]\sum_{t=1}^{n-1}(1+r)^{n-t} \tag{69}$$

We could apply the Von Neuman method to the initial steady state SPP periods (before the first unit of output is produced). Thus, with a similar methodology to the one used previously to obtain (45), we could obtain:

$$P_{ms(n-1)}K_{ms} = W(r)l_s(r)\sum_{t=1}^{n-1}(1+r)^{n-t-1} \tag{70}$$

Substituting (70) in (69) we get:

$$P(r)C = W(r)l_s(r) + rP_{ms}K_{ms} \tag{71}$$

recalling that $P(r)C$ is equal to net output, Q_n

$$Q_n = W(r)l_s(r) + rP_{ms}K_{ms} \tag{72}$$

Dividing by W, Q_n can also be expressed in labor units as:

$$Q_n^* = l_s(r) + rK_{ms}^* \tag{73}$$

Thus (74) will hold as the competitive system switches from one technique to the other:

$$Q_n^* = l_s(r) + r\,K_{ms}^* \tag{74}$$

Now, dividing (72) and (73) by $l_s(r)$ we can obtain (75) and (76):

$$\frac{K_{ms}}{l_s(r)} = \frac{[P(r)C/l_s(r)] - W(r)}{r} \tag{75}$$

$$\frac{K_{ms}^*}{l_s(r)} = \frac{[P(r)C^*/l_s(r)] - 1}{r} \tag{76}$$

And using (66) and (67) we obtain:

$$\begin{aligned} K_{sa}(r) \lesseqgtr K_{sb}(r) &\iff \frac{K_{msa}}{l_{sa}(r)} \lesseqgtr \frac{K_{msb}}{l_{sb}(r)} \\ &\iff \frac{K_{msa}^*}{l_{sa}(r)} \lesseqgtr \frac{K_{msb}^*}{l_{sb}(r)} \end{aligned} \tag{77}$$

(77) shows that surrogate capital per average labor invested per period has an inverse relationship to movements in the rate of interest both in value terms and in labor units.

Finally since $P(r)C = Q_n$, the capital/output ratio will be equal to $P_{ms}K_{ms}/P(r)C$. Manipulating in (72) we obtain:

$$\frac{P_{ms}K_{ms}}{Q_n} = \frac{P_{ms}K_{ms}}{P(r)C} = \frac{[P(r)/l_s(r)] - W(r)}{\frac{P(r)C}{l_s(r)}} \tag{78}$$

And from (78), it is easy to see that (79) holds:

Table 3 A steady state neoaustrian technique.

PERIOD	1	2	3	4	5	6	7	8
LABOR FIRST REP.	L	L		L				
OUTPUT FIRST REP.					Q			
LABOR SECOND REP.		L	L		L			
OUTPUT SECOND REP.						Q		
LABOR THIRD REP.			L	L		L		
OUTPUT THIRD REP.							Q	
LABOR FOURTH REP.				L	L		L	
OUTPUT FOURTH REP.								Q

$$K_{sa}(r) \lesseqgtr K_{sb}(r) \iff \frac{P_{msa}K_{msa}}{Q_n} \lesseqgtr \frac{P_{msb}K_{msb}}{Q_n} \iff \frac{K^*_{msa}}{Q^*_{na}} \lesseqgtr \frac{K^*_{msb}}{Q^*_{nb}} \tag{79}$$

(79) shows that the surrogate capital/output ratio has an inverse relationship to movements in the rate of interest, both in value terms and in labor units.

3.5 The Neoaustrian Economy Along the Steady State

The steady state neoaustrian technique of production will present some initial periods in which labor enters the process of production and output is not yet being obtained, and an infinite number of periods in which each period labor enters as input and output is obtained. In Table 3 the initial periods will be 1 to 4. Along the steady state $K(r)$ will still be equal to equation (2). This is so because the steady state consists in simply repetitions of the original technique; thus the length of time that each unit of labor remains invested does not change and the average period of investment is the same. See example in Section 5.1. To obtain a steady state SPP we must take into account the labor invested each period along the steady state in the original neoaustrian technology. Because of this reason n should be derived from (67). $l_s(r)$ can be obtained from (80):

$$l_s(r) = \frac{L_{id} + L_{s0}[\frac{1}{1-\frac{1}{1+r}}]}{\sum_{t=1}^{n-1}(1+r)^{-t} + \frac{1}{1-\frac{1}{1+r}}} \tag{80}$$

Where L_{id} is the value of the labor units invested in the initial periods in which output is not produced, discounted to the period in which the first unit of output is produced (period 5 in Table 3). And L_{s0} is the amount of labor invested each period along the steady state.

3.5.1 Consumption Per Capita

PQ is equal to

$$
\begin{aligned}
P(r)Q &= \frac{W(r)[L_{id} + L_{s0}(\frac{1}{1-\frac{1}{1+r}})]}{\frac{1}{1-\frac{1}{1+r}}} \\
&= \frac{W(r)l_s(r)[\sum_{t=1}^{n-1}(1+r)^{n-t} + \frac{1}{1-\frac{1}{1+r}}]}{\frac{1}{1-\frac{1}{1+r}}}
\end{aligned} \tag{81}
$$

Recalling that in the neoaustrian case total output is consumed, consumption per unit of labor in the steady state is given by:

$$
\frac{PQ}{l_s(r)} = \frac{W(r)[\sum_{t=1}^{n-1}(1+r)^{n-t} + \frac{1+r}{r}]}{\frac{1+r}{r}} \tag{82}
$$

Since $\sum_{t=1}^{n-1}(1+r)^{n-t}$ is a monotonic increasing function in n, (83) holds:

$$
\begin{aligned}
K_a(r) \lesseqgtr K_b(r) &\Longleftrightarrow \frac{P(r)Q_a}{l_{sa}(r)} \lesseqgtr \frac{P(r)Q_b}{l_{sb}(r)} \\
&\Longleftrightarrow \frac{Q_a^*}{l_{sa}(r)} \lesseqgtr \frac{Q_b^*}{l_{sb}(r)}
\end{aligned} \tag{83}
$$

3.5.2 Capital Per Man and Capital/Output Ratio

From (81) we get:

$$
P(r)Q = W(r)l_s(r) + \frac{r}{1+r}W(r)l_s(r)\sum_{t=1}^{n-1}(1+r)^{n-t} \tag{84}
$$

As the reader will appreciate, (84) has similarities with (69). Using (46) and considering the surrogate capital K_{ms} at the unit level:

$$
Q_n = P(r)Q = W(r)l_s(r) + rP_{ms}K_{ms} \tag{85}
$$

By manipulating (85), as we did previously with (72), it is easy to show that (77) also holds in neoaustrian processes of production. Finally (85) can also be used to see that (79) also holds in the neoaustrian cases.

3.6 Conclusion of Section 3

Traditionally the average labor invested per period has been assumed to be adequately measured by the amount of direct labor that enters each period in the process of production along the steady state. However conceiving the competitive systems as being in the steady state does not eliminate the initial conditions which define the time structure of the alternative techniques of production. In neoaustrian examples along the steady state, labor is producing for distinct periods in time. See our reconstruction of the time structure in Samuelson's steady state example in section 5.1.

In the examples of two good technologies provided by Garegnani, Bruno-Burmeister, Morishima and others, there is no longer an explicit initial time structure, the only information provided is the labor and capital goods requirements along the steady state. And traditionally the labor requirement along the steady state has been used as a measure of the amount of labor invested per period.

However, as we have shown, in two good technologies there is an implicit time structure, information of which can be obtained from the formal conditions required to solve for the set of prices. For the competitive systems to be able to have the capital inputs required to maintain production along the steady state, it is necessary to produce them first. These initial requirements have to be taken into account in the final measure of the average labor per period. One way to do so is the SPP in which, at a given rate of interest, the average labor invested per period includes not only the direct labor along the steady state bu also the direct labor required in previous rounds of productions of the capital goods needed as inputs along the steady state.

The SPP provides an alternative measure of the average labor invested per period that does include the initial labor required in the production of the initial capital goods.

SPP also provides a measure of surrogate capital. Surrogate capita is an accounting measure of the degree of indirectness of a given technique of production, which is closely related to the average discounted labor per period. Surrogate capital measures the degree of indirectness in the SPP, but since the labor invested in each period in SPP is equal to the actual average discounted labor of the original technique, surrogate capita is in fact an associated measure of the degree of indirectness of the original process of production.

4 Final Comments

Comparing the choice of techniques at $r = 0$ and at $r > 0$, provides further insight into the newly defined measure of capital and labor.

The main indicator of the optimality of a given production technique is the wage rate. Thus, of a given pair of production techniques, the one that allows for the payment of a higher wage will be chosen. The wage rate, however, indicates distinct economic relationships at $r = 0$ than at $r > 0$.

From (17) and making $r = 0$, we get

$$P = Wx_0(I - x)^{-1}; \qquad x = a, b. \tag{86}$$

which dual is given by

$$q = (I - x)^{-1}C. \tag{87}$$

It can be easily shown that W is equal to

$$W = \frac{PC}{x_0 q} - \frac{Pxq}{x_0 q} - \frac{Pq}{x_0 q + x_0(x + x^2 + \ldots)q}. \tag{88}$$

Now, if we make $r > 0$, instead of (88) we will obtain (60) and (61). By comparing (88) with (60) and (61) we can construct Table 4, shown below.

Table 4 shows that at $r > 0$, the competitive system is optimizing a discounted measure of consumption per capita. Also it shows that at $r > 0$, the wage rate

Table 4 W is an indicator of the choice of techniques

THE ECONOMY IS:	r=0, W IS EQUAL TO:	r>0, W IS EQUAL TO:
CONSUMING TOTAL OUTPUT IN THE FINAL PERIOD	THE LEVEL OF CONSUMPTION OF ONE UNIT OF TOTAL (DIRECT AND INDIRECT) LABOR	THE LEVEL OF CONSUMPTION OF ONE UNIT OF TOTAL DISCOUNTED (DIRECT AND INDIRECT) LABOR
OPERATING ALONG THE STEADY STATE	THE LEVEL OF CONSUMPTION OF ONE UNIT OF DIRECT LABOR	THE LEVEL OF DISCOUNTED CONSUMPTION OF ONE UNIT OF TOTAL DISCOUNTED (DIRECT AND INDIRECT) LABOR

is related to the consumption per unit of total labor; it does not indicate, as at $r = 0$, the leve of consumption of one unit of direct labor.

The newly defined measure of average labor per period is a geometric average of total, direct and indirect, labor invested in the production process. And the newly proposed measure of capita time is the average period of investment of one unit of total (direct and indirect) labor. Thus, our newly defined measures of capital and labor are closely linked to the competitive system optimizing rule of choosing the technique of production, which yields the higher wage rate.

In what follows we will explore the distinctions between the choice of techniques when $r = 0$ and $r > 0$.

At $r = 0$ we have

$$W_a \gtrless W_b \Longleftrightarrow \frac{P_a C_a}{a_0 q} \gtrless \frac{P_b C_b}{b_0 q} \tag{89}$$

But since $W_a \gtrless W_b \Longleftrightarrow P_a \gtrless P_b$ we can conclude from (89) that the level of consumption per unit of direct labor is higher in the technique with a higher W, at any set of prices P such that $P_a = P_b$. Now recall that when $r > 0$, at a switch point prices are equal for both techniques; then (90) holds:

$$W_a \gtrless W_b \Longleftrightarrow \frac{P(r)C_a}{a_0 q} \gtrless \frac{P(r)C_b}{b_0 q} \tag{90}$$

(Notice that prices related to $r > 0$ are denoted as functions of r to distinguish them form prices related to $r = 0$). Moreover, since $P(r)C_x/x_0q = w(r) + P(r)rxq/x_0q$, and since at a switch point $W(r)$ is equal for both techniques, (91) also holds:

$$W_a \gtrless W_b \Longleftrightarrow \frac{P(r)aq}{a_0 q} \gtrless \frac{P(r)bq}{b_0 q} \Longleftrightarrow \frac{rP(r)aq}{a_0 q} \gtrless \frac{rP(r)bq}{b_0 q} \tag{91}$$

Now assume, without loss generality, that $W_a > W_b$. Also assume that the rate of interest is going down and that the competitive system is switching from technique a to b. Because of (91) the switching, in this case, will be towards a technique of production which is less capital intensive. Moreover, because of (90) the switching will be towards a technique of production which generates less consumption per unit of direct labor. (91) also shows that technique b also has a lower level of consumption out of profits per unit of direct labor.

Thus as the competitive system switches from technique a to b here is, along the steady state, a real sacrifice of non-discounted consumption per unit of direct labor. Moreover from Table 4 and our initial assumption ($W_a > W_b$), it follows that if consumption were to take place in the final period, there would be a sacrifice of consumption per non-discounted unit of labor.

However, if there is an economic opportunity cost of receiving the output produced later rather than sooner, the sacrifice of non-discounted consumption could in principle be compensated in discounted terms by receiving the output sooner.

Thus as the competitive system optimizes along the factor price frontier, it is in fact imposing an opportunity cost upon the passage of time, the rate of interest. At $r > 0$, selecting the technique of production that yields the higher wage, $W(r)$, implies optimizing the discounted value of consumption per unit of discounted total (direct and indirect) labor. Thus the switch from technique a to b is explained by the fact that technique b yields higher consumption per unit of total labor in discounted terms.

With this background it is much easier to understand why it is possible to reconstruct the traditional Capital Theory parables in Section 3. The measure of average labor per period proposed in Section 3 is an average of total discounted (direct and indirect) labor, and so it is related to the choice of the optimal wage $W(r)$, to the choice of technique. Recall capital-time is also measured in relationship to total discounted labor.

In economic terms there is a very important difference between optimizing consumption per unit of direct labor, as with $r = 0$, or optimizing discounted consumption per unit of discounted total labor, as with $r > 0$. Discounting implies that the competitive system is imposing an opportunity cost upon the passage of time.

Whenever a positive interest rate is used in the solution for prices of the system of production, the implicit time structure of the indirect labor embodied in the capital goods becomes relevant and it is necessary to take it into account in the measurement of the average labor invested per period, as well as in the measure of the average period of investment of one unit of labor.

If we use the rate of interest -as the opportunity cost- to perform our discounted calculations of capital-time and of the average total labor in the SPP, we are able to reconstruct the traditional parables.

However, if we do not take into account the distinct implicit time structure of the alternative techniques of production, and we define capital and labor in the traditional fashion of the Cambridge Capital Theory Controversies, we will find that all along diverse levels of r, one of the two techniques gives higher consumption per capita and has higher capital per man; in fact this is the same technique that yields a higher wage rate at $r = 0$.

Thus, the clue of the matter is whether or not one should use the positive interest rate to perform discounting operations in the measurement of labor and capital-time and the answer will depend upon the extent to which the positive interest rate reflects or not a real opportunity cost.

If the rate of interest truly reflects an economic opportunity cost, this means that society is indifferent between c consumption today or $c(1 + r)$ tomorrow. If this is the case, the relevant measures of capital and labor must involve discounting operations.

It must be pointed out that in real economies the rate of interest might be above or below the real opportunity cost of the society. If it is above it implies that long term projects are unnecessarily expensive. If the rate of interest is too

Table 5 Alternative Production Techniques

	TECHNIQUE A		TECHNIQUE B	
PERIOD	UNITS OF LABOR REQUIRED	OUTPUT BOTTLES OF CHAMPAGNE OR EQUIVALENT IN DOLLARS	UNITS OF LABOR REQUIRED	OUTPUT BOTTLES OR CHAMPAGNE OR EQUIVALENT IN DOLLARS
1			2	
2	7			
3			6	
4		1		1

Table 6 Transition from A to B technique, and back again.

LABOR TIME	1	2	3	4	5	6	7	...	20	21	22	23	24	25	...
STAGE 3	0	0	(14)	14	14	14	14	...	(14)	0	0	0	0	0	...
STAGE 2	56	56	(42)	(42)	0	0	0	...	0	(14)	(14)	56	56	56	...
STAGE 1	0	0	0	0	(42)	42	42	...	42	42	(42)	0	0	0	...
FINAL OUTPUT	.	.	8	8	6	(6+7)	7	...	7	7	7	(2+7)	(2)	8	...

high, it is precluding the attainment of higher levels of consumption per capita that could be obtained by selecting the techniques according to the real economic opportunity cost. On the other side a too low interest rate implies the usage of time intensive projects which are not the more productive ones in discounted terms of the given higher economic opportunity cost that society really has.

The discussion of how to determine the true economic opportunity cost in a given economy lies outside of the scope of this article, but we should mention that we do not think that in real economies time has a price because of the prevalence of production processes in which time is directly productive as in the champagne example, but rather we think that it has a price because by receiving the output sooner rather than later society enjoys more economic opportunities.

These economic opportunities are related not only to the possibility of consuming sooner but also to other possibilities of producing sooner and more that could be related to technological development and other economic dimensions that are not captured in the two good technologies steady state comparison.

Finally, since in a given real economy the rate of interest or profit might be higher or lower than the relevant economic opportunity cost that society faces, it seems clear that one cannot explain distributional conflicts in purely productive terms. However, the point remains that whatever the level of the rate of interest or profit is, it has important consequences in the production side of the economy.

5 Examples

5.1 Samuelson's Summing Up Example

The example is presented in Table 5 above.

By setting $w_A = w_B$ and solving for r, it can be shown that Reswitching occurs at $r = 1$ and $r = 5$. Samuelson introduced a table (reproduced here above as Table 6) in order to analyze the transition from technique A to B and back again.

Using this table Samuelson points out that as the interest rate goes down, at 100% the competitive system changes from technique A to B. And he points

Table 7 Production Techniques and Transition Techniques.

TIME	UNITS OF LABOR REQUIRED	OUTPUT	UNITS OF LABOR REQUIRED	OUTPUT	UNITS OF LABOR REQUIRED	OUTPUT	UNITS OF LABOR REQUIRED	OUTPUT
1							14	
2			14+42		14		14	
3	56		42				14+42	
4			42	6	42			2+7
5		8		6+7		7		2

Table 8 Reestimated Results

CONCEPT	W=.06349 r=.5			W=.03571 r=1		
	T E C H N I Q U E			T E C H N I Q U E		
	B	BA	A	A	AB	B
$K(r)$	1.86	1.9	2	2	2.04	2.14
n	3.87	3.91	4.04	3.77	3.8	3.9
$l(r)$	16.72	31.59	13.37	18.34	55.49	14.36
$P(r)Q/l(r)$	0.42	0.49	0.6	0.44	0.45	0.49
$Q^*/l(r)$	6.59	7.72	9.42	12.21	12.61	13.65
$P_mK_m/l(r)$	0.24	0.28	0.36	0.2	0.21	0.23
$K_m^*/l(r)$	3.73	4.49	5.62	5.61	5.81	6.33
$P_mK_m/P(r)Q$	0.556	0.58	0.6	0.46	0.4604	0.463
K_m^*/Q^*	0.556	0.58	0.6	0.46	0.4604	0.463

out that in the transition period, form time 3 through 6, the system generates $8 + 8 + 6 + 13 = 35$ units of champagne output, which is definitely greater than $32 = 8 + 8 + 8 + 8$. Thus, as the rate of interest goes down society moves from a technique which produces more output per man, 8/56, to one which produces less, 7/56, but not only that-in the transition society enjoys a surplus of current consumption. As the interest rate declines it involves a disaccumulation of capital (rather than accumulation), and in the transition period there is a surplus (rather than a sacrifice) of current consumption.

The first step to reestimate Samuelson's results is to obtain the new measure of capital $K(r)$ for techniques A and B, and for the transition periods between A and B and between B and A. In Table 6 we have identified by parenthesis the quantities of labor that are relevant for the analysis of the transition production process between both techniques. With this information and our previous knowledge of techniques A and B it is possible to construct Table 7 above.

In Table 7 the reader will observe that the labor units which produce the output of the transition periods belong to distinct periods. They did not enter into the production process at the same point in time in which the output is produced. Table 8 above presents $K(r)$ for the two techniques A and B and for the two transition periods.

As the reader can see, the transition from technique A to B at $r = 1$ behaves adequately. The period of transition between the techniques A and B implies the necessity of increasing the waiting time which finally results in the fact that the technique B has the larger waiting time. There is a clear inverse relationship with respect to the interest rate. This same inverse relation is maintained in the transition form technique B to technique A at $r = 0.5$.

As we can observe, when the system changes from the technique A to B as the interest rate goes below one, transition from the time 3 to 6 implies the necessity

Table 9 Techniques A and B in the Steady State

TECHNIQUE A						TECHNIQUE B						
LABOR TIME	1	2	3	4	...	LABOR TIME	1	2	3	4	5	...
STAGE 2	56	56	56	56	...	STAGE 1			42	42	42	...
						STAGE 3	14	14	14	14	14	...
FINAL OUTPUT			8	8	...	FINAL OUTPUT				7	7	...

Table 10 Reestimated Steady State Results

	W=.06349 r=.5		W=.03571 r=1	
CONCEPT	TECHNIQUE		TECHNIQUE	
	B	A	A	B
$K(r)$	1.86	2	2	2.14
n_s	4.43	4.6	4.02	4.24
$l_s(r)$	27.95	29.75	26.1	20.39
$P(r)Q/l_s(r)$	0.25	0.27	0.31	0.343
$Q^*/l_s(r)$	3.94	4.24	8.58	9.61
$P_{ms}K_{ms}/l_s(r)$	0.25	0.274	0.135	0.154
$K^*_{ms}/l_s(r)$	3.93	4.31	3.79	4.31
$P_{ms}K_{ms}/P(r)Q$	0.995	1.02	0.442	0.448
K^*_{ms}/Q^*	0.995	1.02	0.442	0.448

of increasing the waiting time and as a consequence there is a greater consumption per capita. Table 8 shows that there is an inverse relationship between the interest rate and: capital-time, surrogate capital/output ratio, surrogate capital per man and per capita consumption. We could also reestimate Samuelson's Summing up results by comparing the alternative techniques, A and B, along the steady state. The techniques A and B along the steady state are presented in Table 9 above.

Consider briefly the steady states in Table 9. If we calculate the PDV of the steady state consumption stream and we divide it by the PDV labor stream we obtain the following results. If we include the initial points (1 and 2 for A, and 1, 2 and 3 for B) in which no output is produced we obtain, according to (82), that discounted consumption per capita is equal to $W(r)$.

But if we do not include the initial points we will obtain that discounted consumption per capita is equal to 1/7 for A and 1/8 for B. Notice that 1/7 and 1/8 are the same results that one obtains by comparing the two alternative technique of production at $r = 0$.

Using Table 9 we can estimate the steady state SPP. Table 10 above presents the results.

Comparing the results in Table 10 with the ones in Table 8 will be useful to further appreciate the differences between the SPP and the steady state SPP. The reader will observe that consumption per capita is, in a given technique, higher in Table 8 than in Table 10. The reason can easily be understood by looking at Table 11 below.

In Table 11 the SPPs of technique A corresponding to Table 8 and Table 10 are presented. The differences are as follows. In Table 8, we are measuring consumption per unit of surrogate labor invested in the last period; output is obtained only after 4.04 periods. In Table 10 we are measuring consumption per unit of labor invested in each period; and output along the steady state is obtained also in each period.

Table 11 SPPs and Original Processes of Production for Technique A.

S P P					
PERIOD	1	1.04	2.04	3.04	4.04
LABOR	.04(13.37)	13.37	13.37	13.37	13.37
OUTPUT					8

ORIGINAL TECHNIQUE			
PERIOD	1	2	3
LABOR	56		
OUTPUT			8

S P P							
PERIOD	1	1.6	2.6	3.6	4.6	5.6	...
LABOR	.6(29.8)	29.8	29.8	29.8	29.8	29.8	
OUTPUT					8	8	

ORIGINAL TECHNIQUE					
PERIOD	1	2	3	4	...
LABOR	56	56	56	56	...
OUTPUT			8	8	

5.2 Bruno-Burmeister's Example

Bruno-Burmeister (1966) introduced the following example:

$$\begin{array}{cc} Technique\, a: & Technique\, b: \\ a_0 = \begin{bmatrix} .66 & 1 \end{bmatrix} \quad a = \begin{bmatrix} .3 & .1 \\ .02 & 0 \end{bmatrix} & b_0 = \begin{bmatrix} .01 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 & .1 \\ .71 & 0 \end{bmatrix} \end{array}$$

Table 12 presents the reestimated results for the the steady state SPP:

Table 12 Reestimated Steady State Results for Bruno-Burmeister's Example

	W=.5772 r=.466		W=.1847 r=1.669	
CONCEPT	T E C H N I Q U E		T E C H N I Q U E	
	A	B	B	A
$K(r)$	1.57	1.87	3.67	4.97
n_s	4.03	4.49	5.44	6.67
$l_s(r)$	0.89	0.58	0.15	0.05
$P(r)C/l_s(r)$	1.81	2.13	14.17	47.87
$C^*/l_s(r)$	3.13	3.7	76.73	259.2
$P_{ms}K_{ms}/l_s(r)$	1.8	2.28	3.14	10.7
$K^*_{ms}/l_s(r)$	3.12	3.95	17	57.97
$P_{ms}K_{ms}/Q_n$	0.997	1.07	0.222	0.224
K^*_{ms}/Q^*_n	0.997	1.07	0.222	0.224

Table 13 Reestimated Steady State Results for Garegnani's Example

	W=.039 r=.1		W=.024 r=.2	
CONCEPT	TECHNIQUE		TECHNIQUE	
	B	A	A	B
$K(r)$	4.68	4.74	7.33	7.44
n_s	10	10.1	12.25	12.37
$l_s(r)$	8.7	7.79	2.84	2.96
$P(r)C/l_s(r)$	0.064	0.065	0.177	0.18
$C^*/l_s(r)$	1.63	2.65	7.25	7.42
$P_{ms}K_{ms}/l_s(r)$	0.224	0.365	0.636	0.652
$K^*_{ms}/l_s(r)$	5.73	14.96	26.06	26.8
$P_{ms}K_{ms}/Q_n$	3.516	5.66	3.59	3.61
K^*_{ms}/Q^*_n	3.516	5.66	3.59	3.61

5.3 Garegnani's Example

Garegnani (1966) introduces the following example:

$$\begin{array}{cc} Technique\, a: & Technique\, b: \\ a_0 = \begin{bmatrix} 8.9 & \frac{9}{50} \end{bmatrix} \quad a = \begin{bmatrix} 0 & .5 \\ \frac{379}{423} & \frac{1}{10} \end{bmatrix} & b_0 = \begin{bmatrix} 8.9 & \frac{3}{2} \end{bmatrix} \quad b = \begin{bmatrix} 0 & .25 \\ \frac{379}{423} & \frac{5}{12} \end{bmatrix} \end{array}$$

The reestimated results are presented in Table 13 above.

5.4 Morishima's Example

Morishima (1966) introduces the following example:

$$\begin{array}{cc} Technique\, a: & Technique\, b: \\ a_0 = \begin{bmatrix} .5 & 1 \end{bmatrix} \quad a = \begin{bmatrix} .5 & .2 \\ .1 & 0 \end{bmatrix} & b_0 = \begin{bmatrix} .2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 & .2 \\ 1 & 0 \end{bmatrix} \end{array}$$

Morishima's example requires some manipulations before we can transform the techniques of production into dated labor. In Morishima (1966) the depreciation rate of the capital input is equal to zero, thus the price equation will be: $P(r) = x0[I + rx + r^2x^2 + ...)q$, which letting $rx = (1 + r)y$ can be transformed into: $P(r) = x_0[I + (1 + r)y + (1 + r)y)^2 + ...]q$, which is already an equation expressed in dated labor. In Morishima's example it can be shown that $P(r)C = W(r)x_0q + \frac{r^2}{1+r}P(r)xq$.

The reestimated results are presented in Table 14 below.

Finally as a further reference for the interested reader, Table 15 below presents the measure of capital-time per sector for the three previous examples.

5.5 Pasinetti's Example

Pasinetti introduced the following numerical example:

Technique A: $20w_A(1 + r)^8 + .8P(1 + r) = YP$

Technique B: $w_B(1 + r)^{25} + .8P(1 + r) = 24w_B = YP$

Table 14 Reestimated Steady State Results for Morishima's Example

CONCEPT	W=.9308 r=.5025		W=.5319 r=1.478	
	TECHNIQUE		TECHNIQUE	
	A	B	B	A
$K(r)$	0.24	0.41	2.11	3.38
n_s	1.84	2.24	3.9	5.18
$l_s(r)$	1.15	0.86	0.24	0.08
$P(r)C/l_s(r)$	1.3	1.53	7.65	23
$C^*/l_s(r)$	1.4	1.64	14.38	43.24
$P_{ms}K_{ms}/l_s(r)$	0.495	0.79	1.94	6.14
$K^*_{ms}/l_s(r)$	0.532	0.85	3.66	11.54
$P_{ms}K_{ms}/Q_n$	0.38	0.52	0.254	0.267
K^*_{ms}/Q^*_n	0.38	0.52	0.254	0.267

Table 15 capital-time Per Sector

	Bruno-Burmeister			
	Technique A		Technique B	
	.466	1.67	.466	1.67
SECTOR I	1.85	5.56	2.35	4.04
SECTOR II	1.28	3.82	1.36	3.05

	Morishima			
	Technique A		Technique B	
	.5	1.48	.5	1.48
SECTOR I	1.44	4.03	1.82	2.41
SECTOR II	.1	2.35	.126	1.61

	Garegnani			
	Technique A		Technique B	
	.1	.2	.1	.2
SECTOR I	4.33	6.89	4.28	6.99
SECTOR II	5.4	7.97	5.32	8.1

Pasinetti shows that there is Reswitching at $r = .036$ and $r = .162$.

Reducing the example to dated labor and using (17) to (20) independently for each technique Table 16 below can be estimated. As the reader can appreciate paradoxical behavior does not arise.

A P P E N D I X

Definition and Properties of the Function f(r;)

Let $q = (q_1, ..., q_N)$. For $r > -1$ we define the function $f(r; q)$ by:

$$f(r; q) = \sum (1 + r)^{N-t} q_t$$

Properties:

(i) $f(r; q)$ is a polynomial of degree at most $N - 1$ in r.
Hence it has derivatives of all orders and at most $N - 1$ real roots.

Table 16 Reestimated Results for Pasinetti's Example

CONCEPT	PQ=.1719 W=.006516 r=.035184 T E C H N I Q U E B	 A	PQ=.0707 W=.001067 r=.161574 T E C H N I Q U E A	 B
$K(r)$	7.07	12.82	21.14	29.08
n	14.02	23.51	27.9	36.13
$l(r)$	1.49	0.74	0.167	0.048
$P(r)Q/l(r)$	0.116	0.233	0.424	1.47
$Q^*/l(r)$	17.73	35.65	397.62	1377.39
$P_m K_m/l(r)$	0.105	0.218	0.364	1.26
$K_m^*/l(r)$	16.16	33.48	341.45	1184.94
$P_m K_m/P(r)Q$	0.912	0.939	0.859	0.8602
K_m^*/Q^*	0.912	0.939	0.859	0.8602

(ii) $f'(r;q) = (1+r)^{-1}[Nf(r;q) - \sum t(1+r)^{N-t}q_t]$
Hence,

$$\sum t(1+r)^{N-t}q_t = Nf(r;q) - (1+r)f'(r;q)$$

(iii) $f(r;q)$ is a linear operator in q i.e.
$cf(r;q) + df(r;q') = f(r;cq+dq')$

(iv) If $q \geq 0$ i.e. $q_i \geq 0$ $i = 1,..,N$, then
$f(r;q) \geq 0$ and $f'(r;q) \geq 0$

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