Indeterminacy in a Matching Model of Money with Productive Government Expenditure

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Abstract

This study explores the effects of inflation on economic growth in a two-sector monetary search-and-matching model with productive government expenditure. We find that when labor intensity of production in the centralized market is below a threshold, the economy features a unique balanced growth path along which inflation reduces growth so long as capital intensity of production in the decentralized market is positive. When labor intensity in the centralized market is above the threshold however, the economy may feature multiple balanced growth paths. Multiple equilibria arise when the matching probability in the decentralized market is above a threshold. In this case, the high-growth equilibrium features a negative effect of inflation on economic growth whereas the low-growth equilibrium may feature a negative, a positive or a non-monotonic effect of inflation on growth. When the matching probability is above the threshold but not too high, the low-growth equilibrium is locally determinate whereas the high-growth equilibrium is locally indeterminate and subject to sunspot-driven business cycles around it. Finally, when the matching probability is sufficiently high, both equilibria are locally determinate, and hence, either equilibrium may emerge in the economy.

Keywords: Economic growth; inflation; money; random matching; indeterminacy

JEL Classification: E30, E40, O42

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1 Introduction

This study explores the effects of inflation on economic growth in a monetary search-and-matching model with equilibrium indeterminacy in the form of sunspot-driven business cycles. We consider a two-sector search-and-matching model from Lagos and Wright (2005) and follow Aruoba *et al.* (2011) and Waller (2011) to incorporate endogenous capital accumulation into the model. The novelty of our study is that we allow for productive government spending as in the seminal study by Barro (1990) in order to generate endogenous growth. The resulting monetary search-and-matching model with productive government spending features equilibrium indeterminacy and sunspot-driven business cycles that are absent in the Barro model and the Lagos-Wright model.

Our results can be summarized as follows. When labor intensity of production in the centralized market is below a threshold value, the economy features a unique and determinate balanced growth equilibrium in which an increase in the money growth rate leads to a lower growth rate of output. Given that the threshold value on labor intensity of production in the model is 0.5, empirical values of labor income share, which approximates labor intensity, are often above this threshold. When labor intensity in the centralized market is above the threshold, the economy either features multiple balanced growth equilibria or exhibits no equilibrium. Multiple equilibria arise when the matching probability in the decentralized market is above a threshold value. When the matching probability is above this threshold but not too high, the low-growth equilibrium is locally determinate whereas the high-growth equilibrium is locally indeterminate and subject to sunspot-driven business cycles around it. When the matching probability in the decentralized market is sufficiently high, both equilibria are locally determinate. In this case, either equilibrium could emerge in the economy. When multiple equilibria are present, the high-growth equilibrium always features a negative effect of inflation on economic growth whereas the low-growth equilibrium may feature a negative effect, a positive effect or a non-monotonic effect of inflation on growth.

The intuition behind the different effects of inflation on growth can be explained as follows. A higher inflation rate increases the cost of consumption in the decentralized market where consumption requires the use of money as a medium of exchange. Here we interpret the decentralized market as an informal market, where transactions rely on fiat money. Due to the negative effect of inflation on the demand for consumption goods in the decentralized market, individuals have less incentives to accumulate physical capital, which is a factor input for the production of consumption goods given a positive capital intensity in the decentralized market. As a result, higher inflation reduces capital accumulation and causes a negative effect on economic growth. This negative *capital-accumulation* effect of inflation is also present in previous studies, such as Aruoba *et al.* (2011), Waller (2011) and Chu *et al.* (2014). However, with the presence of productive government spending, inflation has an additional positive *labor-market* effect on growth. When inflation reduces the demand for consumption in the decentralized market, it also shifts the demand for consumption to the centralized market, where money is not needed for transaction purposes. This increase in consumption causes the individuals to also want to consume more leisure and reduce their supply of labor in the centralized market. Given that the labor demand curve may become upward sloping in the presence of productive government spending, the shift in labor supply in this case leads to a surprising increase in equilibrium labor input, which in turn increases the levels of output and capital investment in the centralized market. In the low-growth equilibrium, both this positive labor-market effect and the negative capital-accumulation effect are present to generate ambiguous effects of inflation on economic growth.

This study relates to the literature on inflation and economic growth; see for example, Wang and Yip (1992), Gomme (1993), Dotsey and Ireland (1996), Ho *et al.* (2007), Chang *et al.* (2007), Chen *et al.* (2008) and Chu and Cozzi (2014). Some studies, such as Farmer (1997), Itaya and Mino (2003) and Lai and Chin (2010), also explore the effects of inflation on equilibrium indeterminacy.[1] Studies in this literature model money demand using the classical approaches, such as cash-in-advance constraints, money in utility and transaction costs, without considering search and matching. This study provides a novel attempt to relate this literature to the literature on search-theoretic models of fiat money in order to highlight the implications of random matching on the effects of inflation on economic growth and sunspot-driven business cycles.[2] Our analysis contributes to this direction by showing how the emergence of equilibrium indeterminacy in the presence of endogenous growth driven by productive government spending can be shown in a standard search-and-matching model and how it is affected by parameters that determine matching frictions. Specifically, we find that the degree of labor intensity of production in the centralized market and the matching probability in the decentralized market are the key determinants of the dynamic properties of the equilibria, whereas the degree of capital intensity of production in the decentralized market determines whether inflation affects economic growth.

This study also relates to the literature on matching models of money and capital; see for example, Shi (1999), Menner (2006), Williamson and Wright (2010), Aruoba *et al.* (2011), Bencivenga and Camera (2011) and Waller (2011). Our study differs from these studies by allowing for *endogenous* economic growth in the long run. Chu *et al.* (2014) also consider the effects of inflation on endogenous economic growth in a matching model of money and capital. Their model generates endogenous growth via capital externality and does not exhibit equilibrium indeterminacy due to the absence of productive government expenditure. Our model generates endogenous growth via productive government expenditure and features a unique equilibrium with the same comparative static effects of inflation as in Chu *et al.* (2014) under one parameter space but also multiple equilibria with different comparative static effects of inflation under another parameter space that is empirically more relevant. In other words, the analysis in this study nests the analysis in Chu *et al.* (2014) as a special case. Furthermore, we generalize the model to the case of asymmetric degrees of capital intensity in the two markets and find that they have different implications on equilibrium dynamics and the effects of inflation.

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[1] See Benhabib and Farmer (1994, 1996), Farmer and Guo (1994) and Schmitt-Grohe and Uribe (1997) for seminal studies on equilibrium indeterminacy. Benhabib and Farmer (1999) and Mino (2017) provide a survey of this literature. For studies that explore various mechanisms for the emergence of indeterminacy, see for example Wen (1998, 2001) and Goenka and Poulsen (2005). For studies that explore fiscal policy and indeterminacy, see for example Goenka (1994) and Park and Philippopoulos (2004).

[2] Previous studies, such as Giammarioli (2003), Hashimzade and Ortigueira (2005), Krause and Lubik (2011) and Dong *et al.* (2016), also explore equilibrium indeterminacy in matching models of the labor market and the credit market, but not matching models of fiat money. Furthermore, our analysis of this Benhabib-Farmer-Guo type of indeterminacy (i.e., sunspot-driven business cycles) is different from the multiplicity of monetary equilibria (i.e., multiple equilibria on the price level and nominal/real money holdings) in matching models discussed in Jean *et al.* (2010).

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 studies dynamics. Section 4 examines the effects of inflation. Section 5 explores the relationship between taxation and growth. The final section concludes.

2 The model

We consider an economy that consists of a unit continuum of identical and infinitely-lived individuals in discrete time. In each period, there are economic activities in two markets: individuals first enter a decentralized market (hereafter DM) and then a centralized market (hereafter CM). We interpret the DM as an informal market, in which transactions rely on fiat money and it is also easy for vendors to evade taxes, so that the government can only levy taxes on wage and capital income in the CM, where transactions rely on credit. Following the literature, we assume that there is no discounting within each period, while the discount factor is $\beta \in (0,1)$ between any two consecutive periods.

2.1 Individuals' optimization in the CM

In the CM, individuals consume a general good or invest it to accumulate physical capital in order to maximize their lifetime discounted utility. Their instantaneous utility function is represented by[3]

$$u_t = \theta \ln x_t - \gamma h_t,$$

where x_t is the consumption of the general good, h_t is the supply of labor, and the parameters $\gamma > 0$ and $\theta > 0$ determine respectively the disutility of labor supply and the importance of consumption in the CM. Let's denote $W(m_t, k_t)$ and $V(m_t, k_t)$ as the period-t value functions for individuals in the CM and the DM, respectively. For the maximization problem of individuals in the CM, we have

$$W(m_t, k_t) = \max_{x_t, h_t, m_{t+1}, k_{t+1}} \{\theta \ln x_t - \gamma h_t + \beta V(m_{t+1}, k_{t+1})\}, \tag{1}$$

subject to a sequence of budget constraints given by

$$k_{t+1} + \frac{m_{t+1}}{p_t} = (1-\tau_{h,t}) w_t h_t + (1-\tau_{k,t}) r_t k_t + (1-\delta) k_t - x_t + T_t + \frac{m_t}{p_t}, \tag{2}$$

where p_t is the price of general good x_t, w_t is the real wage rate, r_t is the real rental price of capital, $\tau_{h,t}, \tau_{k,t} \in (0,1)$ denote the tax rates of labor income and capital income, respectively, k_t denotes the capital stock owned by an individual, and m_t is the nominal money balance in period t. The parameter $\delta \in (0,1)$ is the depreciation rate of capital. T_t denotes a real lump-sum transfer from the government.

[3]Due to separable utility in x_t and h_t, we must consider log utility in order to be consistent with the balanced growth path along which x_t grows at a constant rate and h_t remains stationary.

If we use the budget constraint to substitute h_t into equation (1), then standard dynamic optimization leads to the following first-order conditions:

$$\frac{\theta}{x_t} = \frac{\gamma}{(1 - \tau_{h,t})\, w_t}, \tag{3}$$

$$\frac{\theta}{x_t} = \beta V_k (m_{t+1}, k_{t+1}), \tag{4}$$

$$\frac{\theta}{p_t x_t} = \beta V_m (m_{t+1}, k_{t+1}). \tag{5}$$

Equation (3) represents a horizontal labor supply curve. Furthermore, equations (3) to (5) imply that all individuals enter the DM in the next period with the same holdings of capital and money because x_t is the same across individuals, due to their quasi-linear preference, as shown in (3). Finally, the envelope conditions are given by

$$W_k (m_t, k_t) = \frac{\theta [1 - \delta + (1 - \tau_{k,t})\, r_t]}{x_t}, \tag{6}$$

$$W_m (m_t, k_t) = \frac{\theta}{p_t x_t}. \tag{7}$$

2.2 Individuals' optimization in the DM

In the DM, firms do not operate, and a special good is produced and traded privately among individuals. We denote $\sigma \in (0, 0.5)$ as the probability of an agent becoming a buyer. Similarly, with probability σ an agent becomes a seller, and with probability $1 - 2\sigma$ he is a nontrader. Following Lagos and Wright (2005), one buyer meets one seller randomly and anonymously with a matching technology and buyers pay money in trade. Given this matching setup, the value of entering the DM is given by

$$V (m_t, k_t) = \sigma V^b (m_t, k_t) + \sigma V^s (m_t, k_t) + (1 - 2\sigma)\, W (m_t, k_t), \tag{8}$$

where $V^b (m_t, k_t)$ and $V^s (m_t, k_t)$ are the values of being a buyer and a seller, respectively.

To analyze $V^b(.)$ and $V^s(.)$, we consider the following functional forms for the buyers' preference and the sellers' production technology. In the DM, each buyer's utility $\ln q_t^b$ is increasing and concave in the consumption of the special good. Each seller produces special good q_t^s by combining her capital k_t and effort e_t subject to the following Cobb-Douglas production function:

$$q_t^s = F(k_t, Z_t e_t) = A k_t^{\eta} (Z_t e_t)^{1-\eta}, \tag{9}$$

where $A > 0$ is a Hicks-neutral productivity parameter. The parameter $\eta \in (0, 1)$ determines capital intensity η and labor intensity $1 - \eta$ of production in the DM whereas Z_t is the level of labor productivity. As in the seminal study by Barro (1990), labor productivity is determined by productive government expenditure; i.e., we assume that $Z_t = G_t$.[4] Rewriting equation (9), we can express the utility cost of production in terms of effort as

$$e\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right) = A^{-1/(1-\eta)} \left(\frac{q_t^s}{G_t}\right)^{1/(1-\eta)} \left(\frac{k_t}{G_t}\right)^{-\eta/(1-\eta)}. \tag{10}$$

Buyers purchase special good q_t^b by spending money d_t^b, whereas sellers earn money d_t^s by producing special good q_t^s. Given these terms of trade, the values of being a buyer and a seller are respectively

$$V^b(m_t, k_t) = \ln q_t^b + W(m_t - d_t^b, k_t), \tag{11}$$

$$V^s(m_t, k_t) = -e\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right) + W(m_t + d_t^s, k_t). \tag{12}$$

Differentiating (11) and (12) and substituting them into (8), we can obtain the following envelope condition for m_t:

$$\begin{aligned} V_m(m_t, k_t) \quad &= \quad (1-2\sigma)W_m(m_t, k_t) + \sigma\left[\frac{1}{q_t^b}\frac{\partial q_t^b}{\partial m_t} + W_m(m_t - d_t^b, k_t)\left(1 - \frac{\partial d_t^b}{\partial m_t}\right)\right] \\ &+\sigma\left[-e_1\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right)\frac{1}{G_t}\frac{\partial q_t^s}{\partial m_t} + W_m(m_t + d_t^s, k_t)\left(1 + \frac{\partial d_t^s}{\partial m_t}\right)\right], \end{aligned} \tag{13}$$

where $W_m(m_t, k_t) = W_m(m_t - d_t^b, k_t) = W_m(m_t + d_t^s, k_t) = \theta/(p_t x_t)$ from (7). Similarly, we can obtain the following envelope condition for k_t:

$$\begin{aligned} V_k(m_t, k_t) \quad &= \quad (1-2\sigma)W_k(m_t, k_t) + \sigma\left[\frac{1}{q_t^b}\frac{\partial q_t^b}{\partial k_t} - W_m(m_t - d_t^b, k_t)\frac{\partial d_t^b}{\partial k_t} + W_k(m_t - d_t^b, k_t)\right] \\ &+\sigma\left[-e_1\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right)\frac{1}{G_t}\frac{\partial q_t^s}{\partial k_t} - e_2\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right)\frac{1}{G_t} + W_m(m_t + d_t^s, k_t)\frac{\partial d_t^s}{\partial k_t} + W_k(m_t + d_t^s, k_t)\right], \end{aligned} \tag{14}$$

where $W_k(m_t, k_t) = W_k(m_t - d_t^b, k_t) = W_k(m_t + d_t^s, k_t) = \theta\left[(1-\tau_{k,t}) r_t + (1-\delta)\right]/x_t$ from (6).

To solve the marginal value of holding money (13) and capital (14), we consider a competitive equilibrium with price taking as in Aruoba *et al.* (2011) and Waller (2011).[5] Under price taking, once buyers and sellers are matched, they both act as price takers. Given the price $\widetilde{p}_t$ of the special good, buyers choose q_t^b to maximize

$$V^b(m_t, k_t) = \max_{q_t^b}[\ln q_t^b + W(m_t - \widetilde{p}_t q_t^b, k_t)] \tag{15}$$

subject to the budget constraint

$$d_t^b = \widetilde{p}_t q_t^b \le m_t. \tag{16}$$

[4]It is useful to note that Barro (1990) considers inelastic labor supply whereas we consider elastic labor supply, which interacts with productive government spending to generate indeterminacy. The constant returns to scale with respect to k_t and G_t is necessary to generate endogenous long-run growth but not for equilibrium indeterminacy.

[5]We cannot consider bargaining in this model because the bargaining condition is incompatible with endogenous growth; see Appendix A in Chu *et al.* (2014) for a detailed discussion.

It can be shown that in the DM, buyers spend all their money, so that the money constraint implies that

$$q_t^b = m_t/\widetilde{p}_t. \tag{17}$$

As for sellers' maximization problem in the DM, it is given by

$$V^s(m_t, k_t) = \max_{q_t^s} \left[-e\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right) + W(m_t + \widetilde{p}_t q_t^s, k_t) \right]. \tag{18}$$

Sellers' optimal supplies of the special good can be obtained from the following condition:

$$e_1\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right)\frac{1}{G_t} = \widetilde{p}_t W_m(m_t + \widetilde{p}_t q_t^s, k_t) \Leftrightarrow \frac{1}{1-\eta} e\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right) = \theta \frac{\widetilde{p}_t q_t^s}{p_t x_t}, \tag{19}$$

where e_1 denotes the derivative of $e(.)$ with respect to its first argument. The second equality of (19) makes use of (7) and (10).

Using (17) and (19), we can obtain $\partial q_t^b/\partial m_t = 1/\widetilde{p}_t$, $\partial d_t^b/\partial m_t = 1$, and $\partial d_t^s/\partial k_t = \widetilde{p}_t\left(\partial q_t^s/\partial k_t\right)$, whereas the other partial derivatives, $\partial q_t^b/\partial k_t$, $\partial d_t^b/\partial k_t$, $\partial q_t^s/\partial m_t$ and $\partial d_t^s/\partial m_t$, in (13) and (14) are zero. Substituting these conditions, $q_t^b = q_t^s = q_t$ and (19) into (13) and (14), we can derive the following conditions:

$$V_m(m_t, k_t) = \frac{(1-\sigma)\,\theta}{p_t x_t} + \frac{\sigma}{\widetilde{p}_t q_t}, \tag{20}$$

$$V_k(m_t, k_t) = \frac{\theta\left[(1-\tau_{k,t})\, r_t + (1-\delta)\right]}{x_t} - \frac{\sigma}{G_t} e_2\left(\frac{q_t}{G_t}, \frac{k_t}{G_t}\right), \tag{21}$$

where e_2 denotes the derivative of $e(.)$ with respect to its second argument. The intuition behind these two conditions can be explained as follows. The marginal value of money holding is the expected gain in utility by either consuming more special good q_t in the DM with probability σ or consuming more general good x_t in the CM with probability $1-\sigma$. The marginal value of capital holding is the gain in utility by consuming more general good x_t in the CM with the after-tax net capital income $(1-\tau_{k,t})\, r_t + 1 - \delta$ plus the expected gain in utility by incurring less production effort as a seller in the DM with probability σ.[6]

2.3 Firms' optimization in the CM

In the CM, there is a large number of identical firms. In each period, each firm produces the general good using capital K_t and labor H_t. The production function is given by

$$Y_{x,t} = AK_t^{\alpha}\left(Z_t H_t\right)^{1-\alpha}, \tag{22}$$

where the parameter $\alpha \in (0,1)$ determines labor intensity $1-\alpha$ of production in the CM. Labor productivity is determined by productive government spending as before; i.e., $Z_t = G_t$. Taking factor prices and the government's expenditure as given, the representative firm

[6] Recall that $e_2(q_t/G_t, k_t/G_t) < 0$; see equation (10).

chooses H_t and K_t to maximize its profits. Interior solutions of the firm's problem are characterized by the first-order conditions as follows:

$$r_t = \alpha A K_t^{\alpha-1} (G_t H_t)^{1-\alpha}, \tag{23}$$

$$w_t = (1-\alpha) A K_t^{\alpha} H_t^{-\alpha} G_t^{1-\alpha}. \tag{24}$$

In equilibrium, $K_t = k_t$ and $H_t = h_t$.

2.4 Government

In this economy, the government plays the following two roles: it implements fiscal and monetary policies. In each period, the government's public expenditure is financed by imposing a tax on individuals' wage and capital income in the CM. Therefore, the government's budget constraint can be expressed as[7]

$$G_t = \tau_{h,t} w_t h_t + \tau_{k,t} r_t k_t = \hat{\tau}_t Y_{x,t}, \tag{25}$$

where we denote $\hat{\tau}_t \equiv (1-\alpha)\tau_{h,t} + \alpha \tau_{k,t}$. The government also issues money at an exogenously given rate at $\mu_t = (m_{t+1} - m_t)/m_t$ to finance a lump-sum transfer that has a real value of $T_t = (m_{t+1} - m_t)/p_t = \mu_t m_t / p_t$. We separate the fiscal and monetary components of the government in order to allow for monetary policy independence. In other words, we do not consider the case in which the government can use the central bank to finance its fiscal spending.[8]

2.5 Equilibrium

The equilibrium is defined as a sequence of allocations $\{G_t, x_t, h_t, Y_{x,t}, q_t, d_t, m_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, a sequence of prices $\{r_t, w_t, p_t, \tilde{p}_t\}_{t=0}^{\infty}$ and a sequence of policies $\{\mu_t, \tau_{h,t}, \tau_{k,t}, T_t\}_{t=0}^{\infty}$, with the following conditions satisfied in each period.

- In the CM, individuals choose $\{x_t, h_t, m_{t+1}, k_{t+1}\}$ to maximize (1) subject to (2), taking $\{r_t, w_t, p_t\}$ and $\{\mu_t, \tau_{h,t}, \tau_{k,t}, T_t\}$ as given;
- In the DM, buyers and sellers choose $\{q_t, d_t\}$ to maximize (11) and (12) respectively, taking $\{\tilde{p}_t\}$ as given;
- Firms in the CM produce $\{Y_{x,t}\}$ competitively to maximize profit taking $\{r_t, w_t\}$ and $\{G_t\}$ as given;

[7] Our results are also robust to the case of government spending being financed by a lump-sum tax $G = T$, so long as we assume that government spending G is proportional to output (i.e., $G = \kappa Y_x$) in order to ensure balanced growth.

[8] In the case of seigniorage, higher inflation would increase tax revenue for productive government spending, and hence, it would have an additional positive effect on economic growth.

- The real aggregate consumption includes consumption in CM and DM such that

$$c_t = \left(p_t x_t + \sigma \tilde{p}_t q_t\right)/p_t;$$

- The real aggregate output includes output in CM and DM such that

$$Y_t = \left(p_t Y_{x,t} + \sigma \tilde{p}_t q_t\right)/p_t;$$

- The capital stock accumulates through investment from the general good such that

$$k_{t+1} = Y_{x,t} - x_t - G_t + (1-\delta)\, k_t;$$

- The government balances its budget in every period such that $G_t = \hat{\tau}_t Y_{x,t}$ and $T_t = \mu_t m_t / p_t$.

- All markets clear in every period.

3 Equilibrium indeterminacy

In the rest of the paper, we assume stationary monetary and tax policies, i.e., $\mu_t = \mu$, $\tau_{h,t} = \tau_h$, and $\tau_{k,t} = \tau_k$, which implies $\hat{\tau}_t = \hat{\tau} \equiv (1-\alpha)\,\tau_h + \alpha\tau_k$. The stationary money growth rate has a lower bound, i.e., $\mu \geq \beta - 1$.[9] The dynamical system can be derived as follows. First, we define two transformed variables $\Phi_t \equiv m_t / \left(p_t x_t\right)$ and $\Omega_t \equiv x_t / k_t$. Φ_t represents the ratio of real money balance to consumption in the CM, whereas Ω_t represents the consumption-capital ratio in CM. Note that Φ_t and Ω_t are both *jump* variables and they are stationary on a balanced growth path. From equations (5) and (20), we obtain an autonomous dynamical system for Φ_t, which is given by the following difference equation:

$$\Phi_{t+1} = \frac{1+\mu}{\beta(1-\sigma)}\Phi_t - \frac{\sigma}{\theta(1-\sigma)} \equiv f(\Phi_t). \tag{26}$$

Figure 1 shows that the money-consumption ratio Φ_t jumps immediately to a unique and saddle-point stable steady-state equilibrium Φ.

[9] It can be shown that this lower bound is equivalent to a zero lower bound on the nominal interest rate.

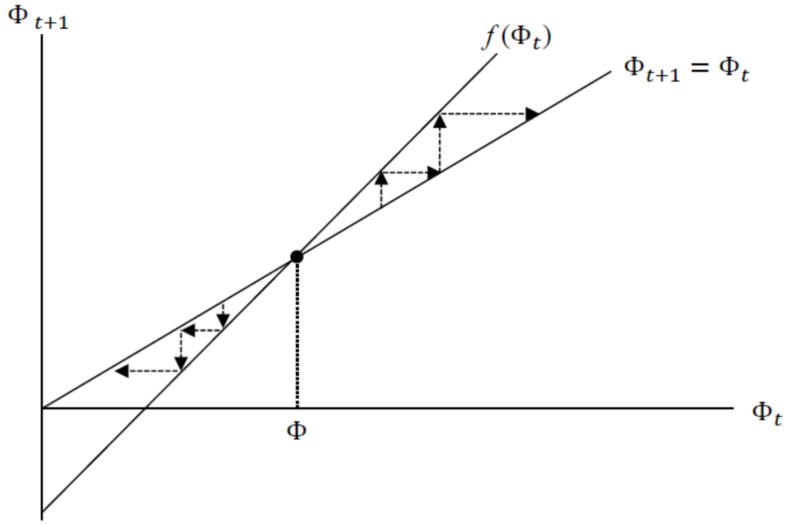

Figure 1: Phase diagram of Φ_t

Manipulating equations (22) and (25) yields $G_t = (\hat{\tau} A)^{1/\alpha} k_t h_t^{(1-\alpha)/\alpha}$, which is increasing in labor h_t. We then use this condition to rearrange (23) and (24) as

$$r_t = \alpha A^{1/\alpha} (\hat{\tau} h_t)^{(1-\alpha)/\alpha}, \tag{23a}$$

$$w_t = (1-\alpha) \hat{\tau}^{(1-\alpha)/\alpha} A^{1/\alpha} k_t h_t^{(1-2\alpha)/\alpha}. \tag{24a}$$

It is useful to note that (24a) represents the labor demand curve in the CM, which is upward sloping if and only if $\alpha < 1/2$ (i.e., labor intensity $1-\alpha > 1/2$ in the CM). Combining labor demand in (24a) and labor supply in (3), we derive that the following equilibrium relationship between labor h_t and the consumption-capital ratio Ω_t:

$$h_t = \left[\frac{\theta}{\gamma} (1-\tau_h) (1-\alpha) \hat{\tau}^{(1-\alpha)/\alpha} A^{1/\alpha}\right]^{\alpha/(2\alpha-1)} \Omega_t^{\alpha/(1-2\alpha)}, \tag{27}$$

which shows a positive relationship between labor h_t and the consumption-capital ratio Ω_t if and only if $\alpha < 1/2$ (i.e., labor intensity $1-\alpha > 1/2$ in the CM). This positive relationship captures the case in which a decrease in labor supply (i.e., an *upward* shift in the horizontal labor supply curve) leads to an increase in equilibrium labor input due to an upward-sloping labor demand curve in the CM.

Combining equations (4), (10), (19), (21), (23a) and (27), we obtain the dynamical equation of consumption in the CM:

$$\frac{x_{t+1}}{x_t} = \beta \left[1 - \delta + \alpha (1-\tau_k) D \Omega_{t+1}^{\epsilon} + \eta \sigma \Phi_{t+1} \Omega_{t+1}\right], \tag{28}$$

where we define two composite parameters $\{D, \epsilon\}$ as follows.

$$D \equiv \hat{\tau}^{(1-\alpha)/\alpha} A^{1/\alpha} \left[\frac{\theta}{\gamma} (1-\tau_h) (1-\alpha) \hat{\tau}^{(1-\alpha)/\alpha} A^{1/\alpha}\right]^{(1-\alpha)/(2\alpha-1)} > 0,$$

and $\epsilon \equiv (1-\alpha)/(1-2\alpha)$. For convenience, we plot the value of ϵ against α in Figure 2.

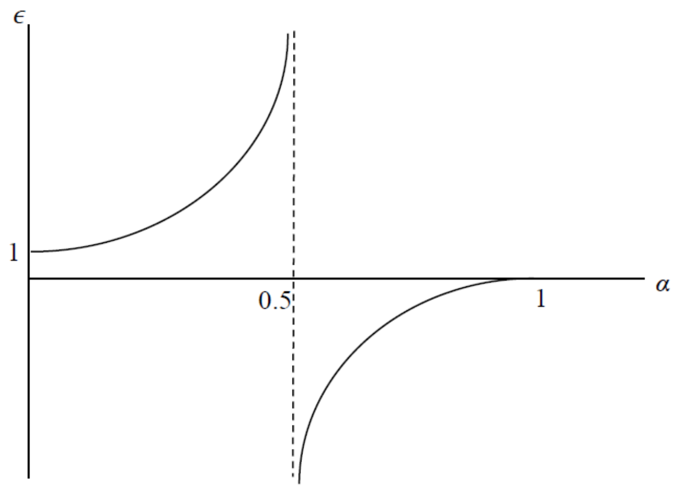

Figure 2: Relationship between α and ϵ

The resource constraint implies the following dynamics of the capital stock k_t:

$$\frac{k_{t+1}}{k_t} = (1-\hat{\tau})\, D\Omega_t^{\epsilon} - \Omega_t + 1 - \delta, \tag{29}$$

where we have used (22), (27) and $G_t = (\hat{\tau}A)^{1/\alpha} k_t h_t^{(1-\alpha)/\alpha}$. Combining equations (28) and (29), we derive the dynamics of $\Omega_t \equiv x_t/k_t$ as follows.

$$\frac{\Omega_{t+1}}{\Omega_t} = \frac{\beta\left[1-\delta+\alpha\,(1-\tau_k)\, D\Omega_{t+1}^{\epsilon}+\eta\sigma\Phi_{t+1}\Omega_{t+1}\right]}{(1-\hat{\tau})\, D\Omega_t^{\epsilon}-\Omega_t+1-\delta}, \tag{30}$$

From (26) and (30), the steady-state values of Φ_t and Ω_t, denoted as Φ and Ω, are determined by

$$\Phi = \frac{\sigma\beta}{\theta\left[1+\mu-(1-\sigma)\,\beta\right]}, \tag{31}$$

$$(1+\eta\beta\sigma\Phi)\,\Omega = \left[(1-\hat{\tau})-\alpha\beta\,(1-\tau_k)\right] D\Omega^{\epsilon} + (1-\beta)\,(1-\delta)\,. \tag{32}$$

We first substitute (31) into (32) and then plot the left-hand side (LHS) and right-hand side (RHS) of (32) in Figure 3.

Figure 3a shows that when $\alpha > 1/2$ (i.e., $\epsilon < 0$), there is a unique steady-state equilibrium value of Ω. In this case, an increase in μ raises the steady-state equilibrium value of Ω if and only if capital intensity $\eta > 0$ in the DM. Intuitively, higher inflation increases the cost of consumption in the DM where money is used as a medium of exchange. Due to this lower demand for consumption and a positive capital intensity in the DM, there is less incentive to accumulate physical capital, which is a factor input for production in the DM. Furthermore, the lower demand for consumption in the DM shifts the demand for consumption to the CM. Both of these effects lead to an increase in the consumption-capital ratio Ω in the CM.

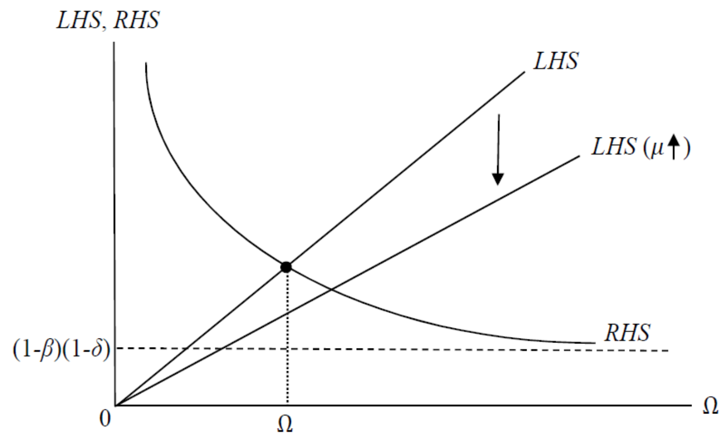

Figure 3a: Unique equilibrium under $\alpha > 1/2$

Figure 3b shows that when $\alpha < 1/2$ (i.e., $\epsilon > 1$) and σ is sufficiently large, there are two steady-state equilibrium values of Ω denoted as $\{\Omega^{low}, \Omega^{high}\}$. In this case, an increase in μ leads to an increase in Ω^{low} but a decrease in Ω^{high}. Given the two equilibria, we have global indeterminacy. The intuition can be understood as follows. Substituting $G_t = (\hat{\tau}A)^{1/\alpha} k_t h_t^{(1-\alpha)/\alpha}$ into (22) yields $Y_{x,t} = \hat{\tau}^{(1-\alpha)/\alpha} A^{1/\alpha} k_t h_t^{(1-\alpha)/\alpha}$, where $(1-\alpha)/\alpha > 1$ if and only if $\alpha < 1/2$ (i.e., labor intensity $1-\alpha > 1/2$ in the CM). When $(1-\alpha)/\alpha > 1$, the aggregate production function exhibits increasing returns to scale in labor, which in turn gives rise to an upward-sloping labor demand curve in the CM. Together with a horizontal labor supply curve from the quasi-linear preference, global indeterminacy arises. Finally, when $\alpha < 1/2$ (i.e., $\epsilon > 1$) and σ is sufficiently small, there is no equilibrium, and we rule out this parameter space by assumption.

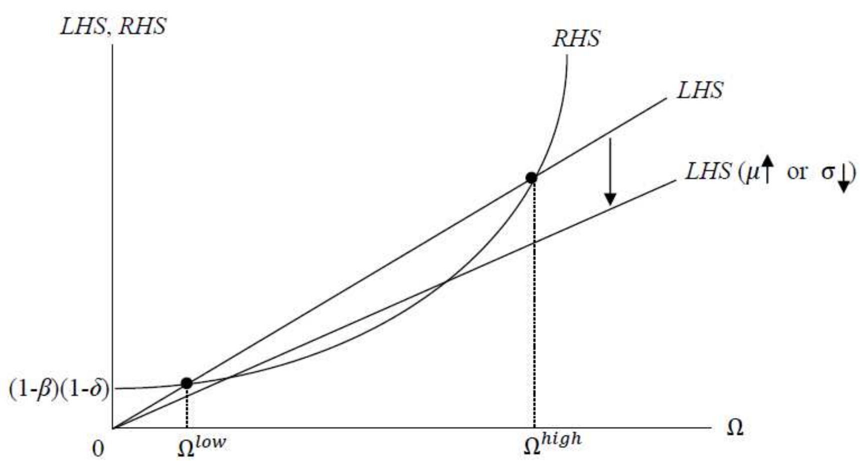

Figure 3b: Multiple equilibria under $\alpha < 1/2$

Figure 1 implies that Φ_{t+1} in (30) jumps to its unique steady-state value Φ given in (31). Therefore, the two-dimensional dynamic system degenerates to a one-dimensional dynamic system for Ω_t.[10] Taking a linear approximation around the steady-state equilibrium value Ω and using (32), we derive

$$\Omega_{t+1} = (1-\xi)\Omega + \xi\Omega_t \equiv F(\Omega_t), \tag{33}$$

where $\xi \equiv [(1-\delta) + (1-\epsilon)(1-\hat{\tau}) D\Omega^{\epsilon}] / \{\beta [(1-\delta) + \alpha (1-\epsilon)(1-\tau_k) D\Omega^{\epsilon}]\}$ is the characteristic root of the dynamical system. Figure 4 plots the phase diagram of the *local* dynamics of Ω_t under $\alpha > 1/2$. When $\alpha > 1/2$ (i.e., $\epsilon < 0$), the characteristic root ξ is greater than one. In this case, Figure 4 shows that the unique steady-state equilibrium exhibits saddle-point stability; therefore, Ω_t always jumps to the unique steady state. However, empirical values of labor income share suggest that labor intensity $1-\alpha$ is usually greater than 0.5. Therefore, $\alpha < 1/2$ is the more relevant parameter space, which we examine next.

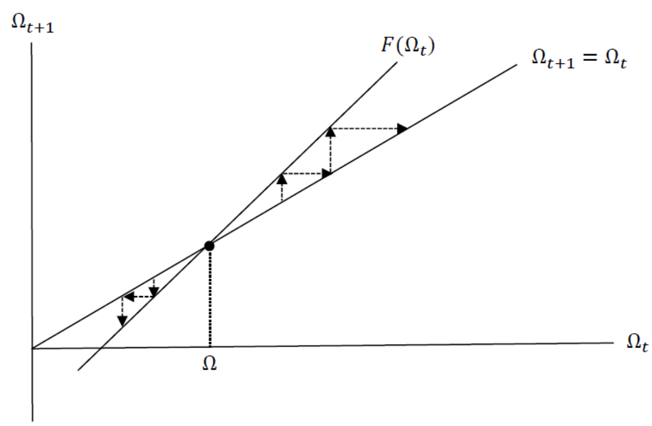

Figure 4: Phase diagram of Ω_t under $\alpha > 1/2$

For the case of $\alpha < 1/2$ (i.e., $\epsilon > 1$), it would be easier to understand the results if we first plot the relationship between the characteristic root ξ and the steady-state equilibrium value Ω. Also, it is useful to recall that $\xi \in (-1, 1)$ implies a dynamically stable (i.e., locally indeterminate) system and that a system is dynamically unstable (i.e., locally determinate) if $\xi < -1$ or $\xi > 1$. Figure 5 shows that the equilibrium Ω^{low} is always dynamically unstable because $\Omega^{low} < \Omega^*$ which implies $\xi > 1$, whereas the equilibrium Ω^{high} can be either dynamically unstable (when $\Omega^{high} > \Omega^{**}$ which implies $\xi < -1$ or $\xi > 1$) or dynamically stable (when $\Omega^{high} < \Omega^{**}$ which implies $\xi \in (-1, 1)$).[11]

[10] Exploring the dynamics of the two-dimensional system would yield the same results; see Appendix B.

[11] We will show that $\Omega^{high} > \Omega^*$ and also derive Ω^* and Ω^{**} in Appendix A.

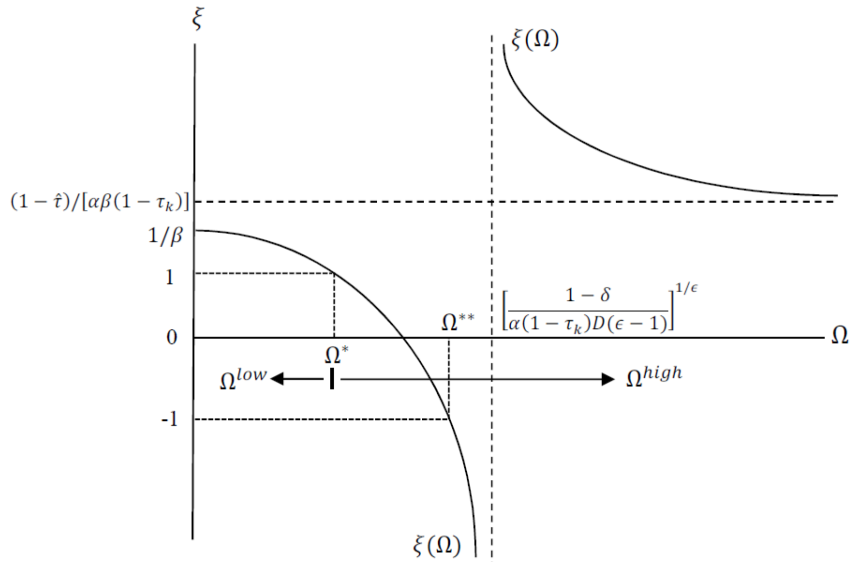

Figure 5: Relationship between ξ and Ω under $\alpha < 1/2$

Recall from Figure 3b that Ω^{high} is increasing in the value of the matching parameter σ in the DM. Then, Figure 6a shows that when $\alpha < 1/2$ and σ is not too large,[12] the equilibrium Ω^{high} is locally indeterminate (i.e., dynamically stable) because $\Omega^* < \Omega^{high} < \Omega^{**}$ whereas the equilibrium Ω^{low} is always locally determinate (i.e., dynamically unstable) because $\Omega^{low} < \Omega^*$. When Ω^{low} is unstable and Ω^{high} is stable, Ω_t reaching the unstable equilibrium Ω^{low} is a measure-zero event. In this case, the economy is subject to sunspot fluctuations around the stable equilibrium Ω^{high}.

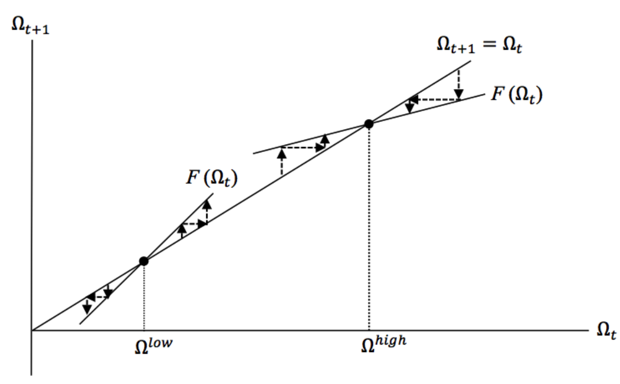

Figure 6a: Phase diagram of Ω_t under
$\alpha < 1/2$ and a small σ

[12] Here we assume that σ is sufficiently large for the presence of equilibria but not excessively large. In the proof of Proposition 1, we explicitly derive these threshold values; see Appendix A.

Figure 6b[13] shows that when $\alpha < 1/2$ and σ is sufficiently large, the two equilibria are both locally determinate (i.e., dynamically unstable) because $\Omega^{high} > \Omega^{**}$ and $\Omega^{low} < \Omega^{*}$. In this case, it is possible for Ω_t to jump to either equilibrium. Therefore, unlike the case with a small σ, we cannot rule out the steady-state equilibrium Ω^{low} under a sufficiently large σ. We summarize these results in Proposition 1.

Proposition 1 *If $\alpha > 1/2$, then there exists a unique steady-state equilibrium value of Ω_t, which exhibits saddle-point stability. If $\alpha < 1/2$, then there exist two equilibria. One is locally determinate and the other one is locally indeterminate under a sufficiently small σ whereas they are both locally determinate under a sufficiently large σ.*

Proof. See Appendix A. ■

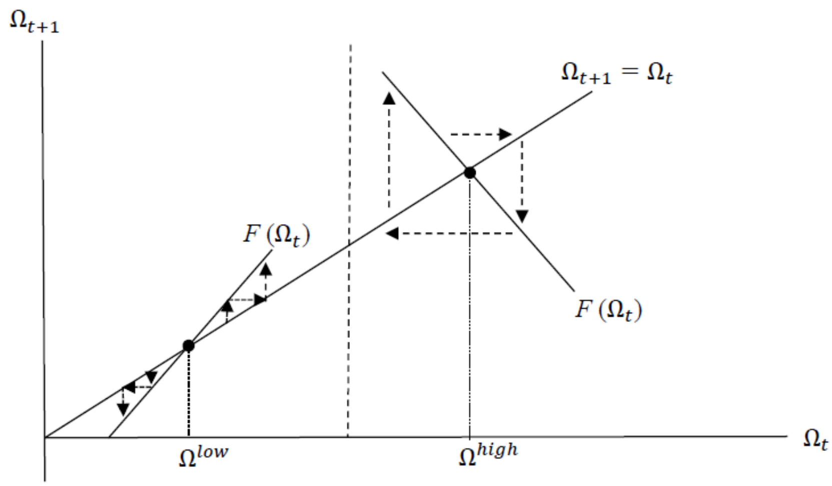

Figure 6b: Phase diagram of Ω_t under $\alpha < 1/2$ and a large σ

4 Inflation and economic growth

In this section, we examine the relationship between inflation and economic growth. Given that in our analysis we treat the growth rate of money supply m_t as an exogenous policy parameter μ, we first need to discuss the relationship between μ and the endogenous inflation rate π. Along a balanced-growth path, aggregate variables, such as output, consumption, capital and real money balance, grow at the same long-run growth rate g. In other words, the growth rate of m_t/p_t is equal to g, which in turn implies that $(1+g) = (1+\mu)/(1+\pi)$

[13] In this figure, we draw the case in which the characteristic root at the steady-state equilibrium Ω^{high} is $\xi < -1$. One can also draw the case of $\xi > 1$.

because the growth rates of m_t and p_t are respectively μ and π. From the approximation $\ln(1+X) \approx X$, the relationship $(1+\pi) = (1+\mu)/(1+g)$ becomes $\pi = \mu - g(\mu)$, where the long-run growth rate $g(\mu)$ is a function of μ as we will show below. Taking the derivative yields $\partial\pi/\partial\mu = 1 - g'(\mu)$. Therefore, if money growth μ has a negative effect on economic growth g, then it must have a positive effect on inflation π implying also a negative relationship between inflation and economic growth. Even if money growth μ has a positive effect on economic growth g, it would still have a positive effect on inflation π so long as its effect on economic growth is not excessively large (i.e., $g'(\mu) < 1$). In this case, the positive relationship between money growth and economic growth implies also a positive relationship between inflation and economic growth. Therefore, the relationship between money growth and economic growth generally carries over to inflation and economic growth.

Using (29), we obtain the following expression for the long-run growth rate of the economy:

$$g \equiv \frac{k_{t+1}}{k_t} - 1 = (1 - \hat{\tau}) D\Omega_t^{\epsilon} - \Omega_t - \delta. \tag{34}$$

In the case of a unique equilibrium (i.e., $\alpha > 1/2$ and $\epsilon < 0$), we have $\partial g/\partial\Omega < 0$. Furthermore, Figure 3a shows that $\partial\Omega/\partial\mu > 0$ given $\eta > 0$. Therefore, the overall effect of μ on g is negative. Intuitively, an increase in inflation leads to a higher cost of money holding, which in turn increases the cost of consumption and reduces the level of consumption in the DM. As a result, there are less incentives to accumulate capital for production in the DM, and the lower rate of capital accumulation leads to a lower growth rate of the economy. We summarize this result in Proposition 2. This result is similar to the one in Chu *et al.* (2014), except that we have generalized the model to asymmetric degrees of capital intensity in the CM and the DM and shown that it is the degree of capital intensity η in the DM that determines whether inflation affects economic growth.

Proposition 2 *If $\alpha > 1/2$, then there exists a unique balanced growth equilibrium in which a higher money growth rate μ reduces economic growth.*

Proof. See Appendix A. ■

In the more plausible case of $\alpha < 1/2$, we have multiple equilibria, and it would be more transparent if we use (28) to express the long-run growth rate of the economy as

$$g \equiv \frac{x_{t+1}}{x_t} - 1 = \beta\left[1 - \delta + \alpha(1 - \tau_k) D\Omega^{\epsilon} + \eta\sigma\Phi\Omega\right] - 1, \tag{35}$$

where Φ is the steady-state ratio of real money balance to consumption in the CM as shown in (31). The ratio of real money balance to consumption in the DM is decreasing in the growth rate of money supply, and this result can be shown as follows:

$$\frac{\partial\Phi}{\partial\mu} = -\frac{\sigma\beta}{\theta\left[(1+\mu) - \beta(1-\sigma)\right]^2} < 0. \tag{36}$$

Intuitively, a higher money growth rate increases inflation, which in turn raises the cost of money holding. Equation (35) also shows that a larger Ω corresponds to a higher growth rate for a given Φ because ϵ is positive (recall that $\alpha < 1/2$ implies $\epsilon > 1$). Therefore, Ω^{high} corresponds to the high-growth equilibrium g^{high} whereas Ω^{low} corresponds to the low-growth equilibrium g^{low}.

Figure 3b shows that Ω^{high} is decreasing in μ given $\eta > 0$. Together with the result that Φ is also decreasing in μ, we find that the high-growth equilibrium growth rate g^{high} is decreasing in the money growth rate μ. Therefore, the effect of inflation on growth in the high-growth equilibrium is the same as in the unique equilibrium. However, the intuition behind these results is different. In the case of the high-growth equilibrium, an increase in inflation reduces the consumption-capital ratio Ω^{high} in the CM, and this counterintuitive result is due to the presence of global indeterminacy. From (34), we see that Ω has a positive effect on g via $D\Omega^{\epsilon}$ (when ϵ is positive) and a negative effect on g via $-\Omega$. The overall relationship between g and Ω in (34) is a U-shaped function[14] as we show in Figure 7.[15] Because Ω^{high} is always on the upward-sloping side of the U-shape, the increase in μ leads to a decrease in both Ω^{high} and g^{high}. In this case, when inflation decreases consumption in the CM, it causes the individuals to also want to consume less leisure and raise their supply of labor in the CM. Given that the labor demand curve is upward sloping due to productive government spending, this increase in labor supply (i.e., a *downward* shift in the horizontal labor supply curve) leads to a surprising decrease in equilibrium labor input, which in turn reduces the levels of output and capital investment.

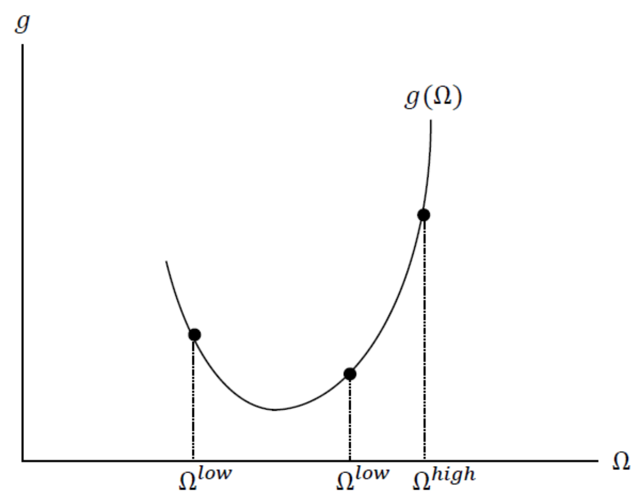

Figure 7: Relationship between g and Ω in (34) when $\epsilon > 1$

As for Ω^{low}, it is increasing in μ given $\eta > 0$ as shown in Figure 3b. However, g^{low} can be either increasing or decreasing in μ. Recall from (34) that g is a U-shaped function in Ω when

[14] Recall that $\epsilon > 1$ when $\alpha < 1/2$.

[15] In Figure 7, the equilibria $\{\Omega^{low}, \Omega^{high}\}$ are determined by the intersection of $g(\Omega)$ in (34) and $g(\Omega)$ in (35), where the latter is a monotonically increasing function in Ω when ϵ is positive. We do not draw (35) in Figure 7 to simplify the diagram.

$\epsilon > 1$. Therefore, when Ω^{low} is sufficiently small, the increase in Ω^{low} caused by an increase in μ reduces the growth rate g^{low}. Intuitively, higher inflation reduces both consumption and the incentives to accumulate capital for production in the DM. This lower rate of capital accumulation causes the lower growth rate. This is the negative capital-accumulation effect of inflation. In contrast, when Ω^{low} is sufficiently large, the increase in Ω^{low} caused by an increase in μ raises the growth rate g^{low}. Intuitively, when inflation increases consumption in the CM, it causes the individuals to also want to consume more leisure and reduce their supply of labor in the CM. Given that the labor demand curve is upward sloping due to productive government spending, this decrease in labor supply (i.e., an upward shift in the horizontal labor supply curve) leads to a surprising increase in equilibrium labor input, which in turn increases the levels of output and capital investment. This is the novel positive labor-market effect of inflation in the presence of productive government spending. Therefore, the overall effect of μ on the low-growth equilibrium growth rate g^{low} is generally a U-shaped function. However, as we will show in Proposition 3, it is also possible for the labor-market effect to always dominate the capital-accumulation effect (i.e., when Ω^{low} is always on the upward-sloping side of the U-shape in Figure 7) or for the capital-accumulation effect to always dominate the labor-market effect (i.e., when Ω^{low} is always on the downward-sloping side of the U-shape). We summarize these results in Proposition 3.

Proposition 3 *If $\alpha < 1/2$, then a higher money growth rate μ has the following effects on economic growth: the high-growth equilibrium g^{high} is decreasing in μ whereas the low-growth equilibrium g^{low} can be an increasing, a decreasing or a U-shaped function in μ.*

Proof. See Appendix A. ■

5 Taxation and economic growth

In the original Barro model, the relationship between labor income tax and economic growth is monotonically positive, which is rather unrealistic. Empirical studies tend to find ambiguous relationships between taxation and growth.[16] Our extension of the Barro model with matching frictions and equilibrium indeterminacy indeed predicts ambiguous relationships between labor income tax and growth.[17] The intuition can be explained as follows. On the one hand, increasing the labor income tax rate generates more tax revenue for productive government spending, which causes a positive effect on economic growth. On the other hand, increasing the labor income tax rate affects labor supply. When the model features a unique and determinate equilibrium, the reduction in labor supply causes a negative effect on economic growth. When the model features indeterminacy and multiple equilibria, how the labor income tax rate affects equilibrium labor becomes ambiguous and differs across equilibria.

[16] See for example Huang and Frentz (2014) for a concise survey that summarizes the contrasting empirical findings in the literature.

[17] It is useful to note that the relationship between capital income tax and economic growth in our model can also be positive or negative, which is also the case in the original Barro model.

To show these results, the rest of this section is devoted to investigate the relationship between the labor income tax rate τ_h and economic growth. For simplicity, we set the tax rate of capital income to zero $\tau_k = 0$. From (32), the marginal effect of tax rate τ_h on the consumption-capital ratio Ω in CM is given by

$$\frac{\partial \Omega}{\partial \tau_h} = -\frac{\frac{D\Omega^{\epsilon+1}}{[\tau_h(1-\tau_h)]}\left(\frac{1-\alpha}{2\alpha-1}\right)\underbrace{\left[\tau_h\left(\tau_h-\alpha\right)-\left(1-\alpha\beta\right)\left(2\tau_h-1\right)\right]}_{?}}{\left[1-\left(1-\alpha\right)\tau_h-\alpha\beta\right]\left(\epsilon-1\right)D\Omega^{\epsilon}-\left(1-\beta\right)\left(1-\delta\right)}. \tag{37}$$

For convenience, we define a threshold value τ_h^* given by

$$\tau_h^* \equiv \frac{1}{2}\left\{\left[\alpha+2\left(1-\alpha\beta\right)\right]-\sqrt{\left[\alpha+2\left(1-\alpha\beta\right)\right]^2-4\left(1-\alpha\beta\right)}\right\}.$$

It is useful to note that $\tau_h^* \in (0, 1/2)$ when $\alpha > 1/2$ whereas $\tau_h^* \in (1/2, 1)$ when $\alpha < 1/2$. As a result, equation (37) shows that if $\alpha > 1/2$ (i.e., $\epsilon < 0$), then the equilibrium Ω is an inverted U-shaped function in τ_h. For the case of $\alpha < 1/2$ (i.e., $\epsilon > 1$), the equilibrium Ω^{low} is an U-shaped function in τ_h whereas the equilibrium Ω^{high} is an inverted U-shaped function in τ_h.

Finally, we explore the growth effect of taxation. Differentiating (35) with respect to τ_h yields

$$\frac{\partial g}{\partial \tau_h} = \frac{\alpha\beta D\Omega^{\epsilon}\left(1-2\tau_h\right)}{\left[\tau_h\left(1-\tau_h\right)\right]}\left(\frac{1-\alpha}{2\alpha-1}\right)+\beta\left\{\alpha D\epsilon\Omega^{\epsilon-1}+\eta\sigma\Phi\right\}\left(\frac{\partial\Omega}{\partial\tau_h}\right). \tag{38}$$

Equation (38) shows that an increase in the labor income tax rate τ_h has ambiguous effects on economic growth as summarized in the following two propositions.

Proposition 4 *If $\alpha > 1/2$, then an increase in the labor income tax rate has an inverted-U effect on the equilibrium growth rate.*

Proof. See Appendix A. ■

Proposition 5 *If $\alpha < 1/2$, then an increase in the labor income tax rate has a U-shaped effect on growth in the low-growth equilibrium whereas it has an inverted-U effect on growth in the high-growth equilibrium.*

Proof. See Appendix A. ■

6 Conclusion

In this study, we have explored the effects of inflation in a monetary search-and-matching model. A novelty of our analysis is productive government expenditure that generates endogenous growth and sunspot-driven indeterminacy in the model. We find that when labor intensity in the CM is below a threshold, the model features a unique equilibrium in which inflation has a negative effect on growth so long as capital intensity is positive in the DM.

When labor intensity in the CM is above the threshold which is empirically the more likely scenario, the model may feature two equilibria, in which the two equilibria display different comparative statics of growth with respect to inflation. Specifically, the high-growth equilibrium features a negative effect of inflation on growth whereas the low-growth equilibrium may feature a negative, a positive or a non-monotonic effect of inflation on growth. Furthermore, under a sufficiently high matching probability in the DM, both equilibria are locally determinate; therefore, either equilibrium may emerge in the economy.

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Appendix A

Proof of Proposition 1. Equation (30) shows that the variable Φ_t jumps to its unique steady state Φ given in (31). We substitute Φ into (30) to obtain the following autonomous one-dimensional dynamical system for Ω_t:

$$\frac{\Omega_{t+1}}{\Omega_t} = \frac{\beta\left[1-\delta+\alpha\left(1-\tau_k\right)D\Omega_{t+1}^{\epsilon}+\eta\sigma\Phi\Omega_{t+1}\right]}{D\Omega_t^{\epsilon}-\Omega_t+1-\delta}. \tag{A1}$$

Taking a linear approximation around the steady-state equilibrium Ω yields

$$\Omega_{t+1} = \Omega + \frac{(1-\delta)+(1-\epsilon)\left(1-\hat{\tau}\right)D\Omega^{\epsilon}}{\beta\left[(1-\delta)+\alpha\left(1-\epsilon\right)\left(1-\tau_k\right)D\Omega^{\epsilon}\right]}\left(\Omega_t-\Omega\right), \tag{A2}$$

where we have used (32). Based on (A2), the characteristic root ξ of the dynamical system can be expressed as

$$\xi \equiv \frac{(1-\delta)+(1-\epsilon)\left(1-\hat{\tau}\right)D\Omega^{\epsilon}}{\beta\left[(1-\delta)+\alpha\left(1-\epsilon\right)\left(1-\tau_k\right)D\Omega^{\epsilon}\right]}. \tag{A3}$$

The local stability properties of the steady state are determined by comparing the number of the stable root with the number of predetermined variables in the dynamical system. In (A2), there is no predetermined variable because Ω_t is a jump variable. As a result, the steady-state equilibrium Ω is locally determinate when the characteristic root is unstable (i.e., $|\xi| > 1$) whereas it is locally indeterminate when the characteristic root is stable (i.e., $|\xi| < 1$). Given these properties, we have the following results. First, if $\alpha > 1/2$ (i.e., $\epsilon < 0$), then the dynamical system exists a unstable root. This result implies that Ω_t displays saddle-point stability and equilibrium uniqueness as shown in Figures 3a and 4.

Second, if $\alpha < 1/2$ (i.e., $\epsilon > 1$), then whether the root is unstable or stable is determined by the steady-state equilibrium value of Ω. The result implies that multiple equilibria may emerge as shown in Figure 3b. To derive a range for the steady-state equilibrium value of Ω, we first make use of (32) to obtain

$$\frac{\partial LHS}{\partial \Omega} = \frac{\partial RHS}{\partial \Omega} \Rightarrow \Omega^* \equiv \left[\frac{(1-\beta)(1-\delta)}{\left(1-\hat{\tau}-\alpha\beta\left(1-\tau_k\right)\right)(\epsilon-1)D}\right]^{1/\epsilon}, \tag{A4}$$

where Ω^* is a threshold value under which $\Omega^{low} < \Omega^*$ and $\Omega^{high} > \Omega^*$ as shown in Figure 8. Notice that for any values of $\tau_h, \tau_k \in (0,1)$, we have

$$\begin{aligned}
1-\hat{\tau}-\alpha\beta\left(1-\tau_k\right) &= 1-(1-\alpha)\tau_h-\alpha\tau_k-\alpha\beta\left(1-\tau_k\right) \\
&= 1-\alpha\beta-(1-\alpha)\tau_h-\alpha(1-\beta)\tau_k \\
&> 1-\alpha\beta-(1-\alpha)-\alpha(1-\beta)=0.
\end{aligned}$$

This implies $\Omega^* > 0$.

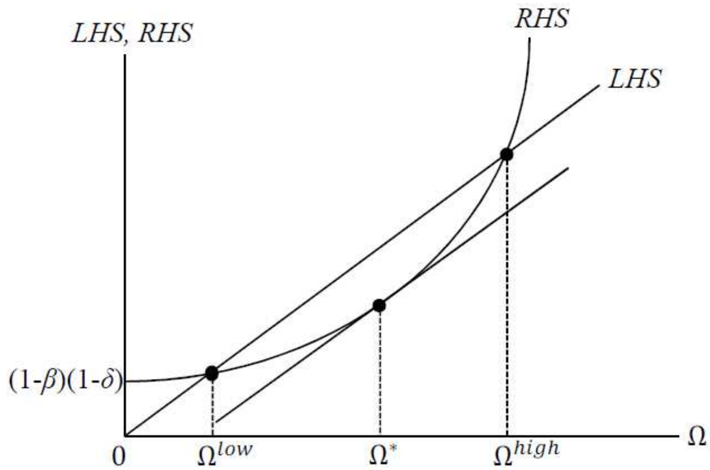

Figure 8

A steady-state equilibrium Ω is dynamically stable if $\xi \in (-1, 1)$. One can manipulate (A3) to show that $\xi \in (-1, 1)$ is equivalent to

$$\Omega^* < \Omega < \Omega^{**}, \tag{A5}$$

where $\Omega^{**} \equiv \{[(1+\beta)(1-\delta)] / [(1-\hat{\tau}+\alpha\beta(1-\tau_k))(\epsilon-1)D]\}^{1/\epsilon}$. Therefore, a steady-state equilibrium Ω is locally indeterminate if $\Omega \in (\Omega^*, \Omega^{**})$ whereas it is locally determinate if $\Omega < \Omega^*$ or $\Omega > \Omega^{**}$. We can now conclude that Ω^{low} is locally determinate because $\Omega^{low} < \Omega^*$. However, Ω^{high} can be either locally indeterminate when $\Omega^* < \Omega^{high} < \Omega^{**}$ or it can be locally determinate when $\Omega^{high} > \Omega^{**}$.

Next, we examine how the matching probability σ affects the steady-state equilibrium values of $\{\Phi, \Omega\}$, which in turn affect the dynamical properties of Ω_t. Differentiating (31) and (32) with respect to σ yields

$$\frac{\partial \Phi}{\partial \sigma} = \frac{\beta(1+\mu-\beta)}{\theta[(1+\mu)-\beta(1-\sigma)]^2} > 0. \tag{A6}$$

$$\frac{\partial \Omega}{\partial \sigma} = \underbrace{\frac{\eta\beta\Omega^2}{(1-\hat{\tau}-\alpha\beta(1-\tau_k))(\epsilon-1)D\Omega^\epsilon-(1-\beta)(1-\delta)}}_{?}\underbrace{\left(\Phi+\sigma\frac{\partial \Phi}{\partial \sigma}\right)}_{+}. \tag{A7}$$

Equation (A6) indicates that increasing σ has a positive effect on Φ. Equation (A7) shows that increasing σ has an ambiguous effect on Ω. Specifically, if and only if $\Omega > \Omega^*$, then Ω is increasing in σ. The result implies that an increase in σ may cause Ω^{high} to change from being locally indeterminate (i.e., $\Omega^{high} < \Omega^{**}$) to being locally determinate (i.e., $\Omega^{high} > \Omega^{**}$). Finally, it can be shown that when σ is sufficiently large (small), we must obtain $\Omega^{high} > \Omega^{**}$ ($\Omega^{high} < \Omega^{**}$). To prove this statement, we make use of (32) to obtain

$$(1+\eta\beta\sigma\Phi) = [1-\hat{\tau}-\alpha\beta(1-\tau_k)]\,D\Omega^{\epsilon-1} + \frac{(1-\beta)(1-\delta)}{\Omega}. \tag{A8}$$

The right-hand side (RHS) of (A8) is increasing in Ω, and this result can be shown as follows:

$$\frac{\partial RHS}{\partial \Omega} = \frac{1}{\Omega^2}\underbrace{\{[1-\hat{\tau}-\alpha\beta(1-\tau_k)](\epsilon-1)D\Omega^{\epsilon}-(1-\beta)(1-\delta)\}}_{+} > 0. \tag{A9}$$

Suppose we have an equilibrium $\Omega^{high} < \Omega^{**}$ under a certain value of σ. In this case, substituting Ω^{**} into the RHS of (A8) yields $(1+\eta\beta\sigma\Phi) < (RHS)_{\Omega=\Omega^{**}}$. Given $\partial\Phi/\partial\sigma > 0$, we know that there exists a larger value of σ denoted as σ^{**} such that $(1+\eta\beta\sigma\Phi) = (RHS)_{\Omega=\Omega^{**}}$ at $\sigma = \sigma^{**}$, where

$$\sigma^{**} \equiv \frac{1}{2\eta\beta^2}\left\{\theta\beta(\Theta-1)+\sqrt[2]{[\theta\beta(\Theta-1)]^2+4\eta\theta\beta^2(\Theta-1)[1+\mu-\beta]}\right\} > 0, \tag{A10}$$

$$\Theta \equiv [1-\hat{\tau}-\alpha\beta(1-\tau_k)]D(\Omega^{**})^{\epsilon-1}+\frac{(1-\beta)(1-\delta)}{\Omega^{**}} > 1. \tag{A11}$$

By analogous inference, we substitute Ω^{*} into (A8) to derive

$$\sigma^{*} \equiv \frac{1}{2\eta\beta^2}\left\{\theta\beta(\Psi-1)+\sqrt[2]{[\theta\beta(\Psi-1)]^2+4\eta\theta\beta^2(\Psi-1)[1+\mu-\beta]}\right\} > 0, \tag{A12}$$

$$\Psi \equiv [1-\hat{\tau}-\alpha\beta(1-\tau_k)]D(\Omega^{*})^{\epsilon-1}+\frac{(1-\beta)(1-\delta)}{\Omega^{*}} > 1. \tag{A13}$$

As a result, if σ is sufficiently large (i.e., $\sigma > \sigma^{**}$), then Ω^{high} changes from being locally indeterminate to being locally determinate. On the contrary, if σ is sufficiently small (i.e., $\sigma \in (\sigma^{*}, \sigma^{**})$), then Ω^{high} exists and is locally indeterminate. ■

Proof of Proposition 2. Differentiating (32) with respect to μ and using (36) yield

$$\frac{\partial \Omega}{\partial \mu} = \underbrace{\frac{\eta\beta\sigma\Omega^2}{[1-\hat{\tau}-\alpha\beta(1-\tau_k)](\epsilon-1)D\Omega^{\epsilon}-(1-\beta)(1-\delta)}}_{?} \times \underbrace{\frac{\partial\Phi}{\partial\mu}}_{-}. \tag{A14}$$

Given $\alpha > 1/2$ and $\epsilon < 0$, we have the following results. First, there is a unique steady-state equilibrium value of Ω which is increasing in μ, given $\eta > 0$, as reported in Figure 3a. Second, based on (34), the growth rate is monotonically decreasing in the consumption-capital ratio in the CM (i.e., $\partial g/\partial\Omega = \epsilon(1-\hat{\tau})D\Omega^{\epsilon-1}-1 < 0$). We make use of these results and take the differentials of (34) with respect to μ to obtain

$$\frac{\partial g}{\partial \mu} = \underbrace{\frac{\partial g}{\partial \Omega}}_{-} \times \underbrace{\frac{\partial \Omega}{\partial \mu}}_{+} < 0. \tag{A15}$$

Equation (A15) shows that if $\alpha > 1/2$, there exists a unique balanced-growth equilibrium in which an increase in μ reduces g. ■

Proof of Proposition 3. Given $\alpha < 1/2$ and $\epsilon > 1$, (A14) shows that given $\eta > 0$, an increase in μ leads to a decrease in Ω when $\Omega > \Omega^*$ whereas it leads to an increase in Ω when $\Omega < \Omega^*$. In other words, when $\alpha < 1/2$ and $\epsilon > 1$, an increase in μ increases Ω^{low} and decreases Ω^{high} as shown in Figure 3b. We take the differentials of (35) with respect to μ to obtain

$$\frac{\partial g}{\partial \mu} = \underbrace{\left\{ \eta\sigma\beta\Omega + \frac{\eta\sigma\beta^2\Omega^2 \left[\alpha\epsilon\left(1-\tau_k\right) D\Omega^{\epsilon-1} + \eta\sigma\Phi\right]}{\left[1-\hat{\tau}-\alpha\beta\left(1-\tau_k\right)\right]\left(\epsilon-1\right) D\Omega^{\epsilon} - \left(1-\beta\right)\left(1-\delta\right)} \right\}}_{?} \times \underbrace{\frac{\partial \Phi}{\partial \mu}}_{-}, \tag{A16}$$

where we have used (A14). Equations (A14) and (A16) show that when $\Omega > \Omega^*$, g is decreasing in μ. In other words, the high-growth equilibrium g^{high} is decreasing in μ.

As for the case of $\Omega < \Omega^*$, we substitute (32) into (A16) to derive

$$\frac{\partial g}{\partial \mu} = \underbrace{\left[\epsilon\left(1-\hat{\tau}\right) D\Omega^{\epsilon-1} - 1\right]}_{?} \times \underbrace{\frac{\partial \Omega}{\partial \mu}}_{+}, \tag{A17}$$

where we have used (A14). Equation (A17) shows that when $\Omega < \Omega^*$, an increase in μ has an ambiguous effect on g. This result implies that g^{low} may be decreasing in μ, increasing in μ or a U-shaped function in μ. To prove this statement, we define a threshold value $\overline{\Omega} \equiv \left[1/\left(\epsilon\left(1-\hat{\tau}\right) D\right)\right]^{1/(\epsilon-1)}$ and make use of (A8). Based on $\Omega < \Omega^*$, the right-hand side (RHS) of (A8) is decreasing in Ω, and this result can be shown as follows:

$$\frac{\partial RHS}{\partial \Omega} = \frac{1}{\Omega^2} \underbrace{\left\{\left[1-\hat{\tau}-\alpha\beta\left(1-\tau_k\right)\right]\left(\epsilon-1\right) D\Omega^{\epsilon} - (1-\beta)(1-\delta)\right\}}_{-} < 0. \tag{A18}$$

We first examine the case in which g^{low} is always decreasing in μ. Suppose we have an equilibrium $\Omega^{low} < \overline{\Omega}$ under a certain value of μ. In this case, substituting $\overline{\Omega}$ into the RHS of (A8) yields $(1+\eta\beta\sigma\Phi) > (RHS)_{\Omega=\overline{\Omega}}$. Although $\partial\Phi/\partial\mu < 0$, it is possible for $1 > (RHS)_{\Omega=\overline{\Omega}}$ even as $\mu \to \infty$. This is the case when the following condition holds: $(1-\beta)(1-\delta) < (1-1/\epsilon)\overline{\Omega} + \alpha\beta\left(1-\tau_k\right) D\overline{\Omega}^{\epsilon}$. This result shows that an increase in μ does not cause Ω^{low} to change from $\Omega^{low} < \overline{\Omega}$ to $\Omega^{low} > \overline{\Omega}$. Therefore, under $(1-\beta)(1-\delta) < (1-1/\epsilon)\overline{\Omega} + \alpha\beta\left(1-\tau_k\right) D\overline{\Omega}^{\epsilon}$, Ω^{low} is always on the downward-sloping side of $g(\Omega)$ as shown in Figure 7, which in turn implies that g^{low} is decreasing in μ.

We now examine the case in which g^{low} is a U-shaped function in μ. Once again, suppose that we have an equilibrium $\Omega^{low} < \overline{\Omega}$ under a certain value of μ and that the following condition holds: $(1-\beta)(1-\delta) > (1-1/\epsilon)\overline{\Omega} + \alpha\beta\left(1-\tau_k\right) D\overline{\Omega}^{\epsilon}$. In this case, given $\partial\Phi/\partial\mu < 0$, we know that there exists a larger value of μ denoted as $\overline{\mu}$ such that $(1+\eta\beta\sigma\Phi) = (RHS)_{\Omega=\overline{\Omega}}$ at $\mu = \overline{\mu}$, where

$$\overline{\mu} \equiv \frac{\eta\beta^2\sigma^2}{\theta\left\{\left[1-\hat{\tau}-\alpha\beta\left(1-\tau_k\right)\right]/\left[\epsilon\left(1-\hat{\tau}\right)\right] + (1-\beta)(1-\delta)\left[\epsilon\left(1-\hat{\tau}\right) D\right]^{1/(\epsilon-1)} - 1\right\}} + \beta(1-\sigma) - 1. \tag{A19}$$

This result shows that an increase in μ may cause Ω^{low} to change from $\Omega^{low} < \overline{\Omega}$ to $\Omega^{low} > \overline{\Omega}$. Specifically, under $(1-\beta)(1-\delta) > (1-1/\epsilon)\overline{\Omega} + \alpha\beta(1-\tau_k) D\overline{\Omega}^{\epsilon}$,[18] Ω^{low} is on the downward-sloping side of $g(\Omega)$ when $\mu < \overline{\mu}$, whereas Ω^{low} is on the upward-sloping side of $g(\Omega)$ when $\mu > \overline{\mu}$ as shown in Figure 7. Therefore, the overall effect of μ on g^{low} follows a U-shaped function.

Finally, we examine the case in which g^{low} is always increasing in μ. Suppose we have an equilibrium $\Omega^{low} > \overline{\Omega}$. In this case, substituting $\overline{\Omega}$ into the RHS of (A8) yields $(1 + \eta\beta\sigma\Phi) < (RHS)_{\Omega=\overline{\Omega}}$. Although $\partial\Phi/\partial\mu < 0$, it is possible for $(1 + \eta\beta\sigma/\theta) < (RHS)_{\Omega=\overline{\Omega}}$ even at $\mu = \underline{\mu}$, where $\underline{\mu} = \beta - 1$ is the lower bound on μ (i.e., the zero lower bound on the nominal interest rate). This is the case when the following condition holds: $(1-\beta)(1-\delta) > (1 - 1/\epsilon + \eta\beta\sigma/\theta)\overline{\Omega} + \alpha\beta(1-\tau_k) D\overline{\Omega}^{\epsilon}$. In this case, a decrease in μ does not cause Ω^{low} to change from $\Omega^{low} > \overline{\Omega}$ to $\Omega^{low} < \overline{\Omega}$ for $\mu \geq \underline{\mu}$. Therefore, under $(1-\beta)(1-\delta) > (1-1/\epsilon+\eta\beta\sigma/\theta)\overline{\Omega} + \alpha\beta(1-\tau_k) D\overline{\Omega}^{\epsilon}$, Ω^{low} is always on the upward-sloping side of $g(\Omega)$ as shown in Figure 7, which in turn implies that g^{low} is increasing in μ.

To sum up, the overall relationship between the low-growth equilibrium g^{low} and the money growth rate μ can be positive, negative or U-shaped. ■

Proof of Proposition 4. Substituting (37) into (38), we obtain

$$\frac{\partial g}{\partial \tau_h} = \frac{\frac{\beta D\Omega^{\epsilon+1}}{[\tau_h(1-\tau_h)]}\left(\frac{1-\alpha}{2\alpha-1}\right)[\phi(\tau_h) - \varpi(\tau_h)]}{[1-(1-\alpha)\tau_h - \alpha\beta](\epsilon-1)D\Omega^{\epsilon} - (1-\beta)(1-\delta)}, \tag{A20}$$

where

$$\phi(\tau_h) \equiv \eta\sigma\Phi\tau_h(\alpha-\tau_h) - (\eta\sigma\Phi+\alpha)(1-2\tau_h), \tag{A21}$$

$$\varpi(\tau_h) \equiv \alpha(1-\alpha) D[\tau_h(1-\tau_h)]\Omega^{\epsilon-1}. \tag{A22}$$

For any values of α, we obtain that $\phi(\tau_h)$ holds the following properties: (a) $\partial\phi(\tau_h)/\partial\tau_h > 0$ and $\partial^2\phi(\tau_h)/(\partial\tau_h)^2 < 0$, (b) $\phi(0) = -(\eta\sigma\Phi+\alpha) < 0$, (c) $\phi(1) = \alpha(1+\eta\sigma\Phi) > 0$. We define a threshold value $\tilde{\tau}_h \in (0,1)$ given by $\phi(\tilde{\tau}_h) = 0$. It is obvious that $\tilde{\tau}_h$ is unique and $\phi(\tau_h) < 0\,(>0)$ for any $\tau_h < \tilde{\tau}_h\,(>\tilde{\tau}_h)$. As for $\varpi(\tau_h)$, we first take differentials of it with respect to τ_h to obtain

$$\frac{\partial\varpi(\tau_h)}{\partial\tau_h} = \alpha(1-\alpha)D\Omega^{\epsilon-2}\left\{\left(\frac{\alpha}{2\alpha-1}\right)(1-2\tau_h)\Omega + (\epsilon-1)[\tau_h(1-\tau_h)]\frac{\partial\Omega}{\partial\tau_h}\right\}. \tag{A23}$$

Substituting (37) into (A23) yields

$$\frac{\partial\varpi(\tau_h)}{\partial\tau_h} = \frac{\alpha D\Omega^{\epsilon-1}\left(\frac{1-\alpha}{2\alpha-1}\right)[(2\alpha-1)(\epsilon-1)D\Omega^{\epsilon}\Delta + \alpha(2\tau_h-1)(1-\beta)(1-\delta)]}{[1-(1-\alpha)\tau_h-\alpha\beta](\epsilon-1)D\Omega^{\epsilon} - (1-\beta)(1-\delta)}, \tag{A24}$$

where $\Delta(\tau_h) \equiv (1-\alpha)\tau_h^2 + (1-\alpha\beta)(1-2\tau_h)$. Based on this result, we define a threshold value

$$\tau_h^{**} = \frac{2(1-\alpha\beta) - \sqrt{4(1-\alpha\beta)^2 - 4(1-\alpha)(1-\alpha\beta)}}{2(1-\alpha)},$$

[18] To be more precise, we also need $(1-\beta)(1-\delta) < (1-1/\epsilon+\eta\beta\sigma/\theta)\overline{\Omega} + \alpha\beta(1-\tau_k) D\overline{\Omega}^{\epsilon}$ as we will show in the next part of the proof.

where $\tau_h^{**} \in (1/2, 1)$. Using this threshold value τ_h^{**}, we derive $\Delta(\tau_h^{**}) = 0$ and $1 - \alpha\beta = \frac{(1-\alpha)\left(\tau_h^{**}\right)^2}{2\tau_h^{**}-1}$. Moreover, substituting (32) into (A22) yields

$$\varpi(\tau_h) = \frac{\alpha(1-\alpha)\left[\tau_h(1-\tau_h)\right]}{1-(1-\alpha)\tau_h - \alpha\beta}\left[(1+\eta\beta\sigma\Phi) - \frac{(1-\beta)(1-\delta)}{\Omega}\right]. \tag{A25}$$

By the definition of τ_h^{**}, we obtain that for any $\tau_h \in (\tau_h^{**}, 1)$, there is

$$\frac{d}{d\tau_h}\left[\frac{\tau_h(1-\tau_h)}{1-(1-\alpha)\tau_h-\alpha\beta}\right] = \frac{\Delta(\tau_h)}{\left[1-(1-\alpha)\tau_h-\alpha\beta\right]^2} < 0.$$

Hence for any $\tau_h \in [\tau_h^{**}, 1]$, we can get

$$\begin{aligned}
\phi(\tau_h) - \varpi(\tau_h) &\geq \phi(\tau_h^{**}) - \frac{\alpha(1-\alpha)\left[\tau_h^{**}(1-\tau_h^{**})\right]}{1-(1-\alpha)\tau_h^{**}-\alpha\beta}\left[(1+\eta\beta\sigma\Phi) - \frac{(1-\beta)(1-\delta)}{\Omega}\right] \\
&= \phi(\tau_h^{**}) - \alpha(2\tau_h^{**}-1)\left[(1+\eta\beta\sigma\Phi) - \frac{(1-\beta)(1-\delta)}{\Omega}\right] \\
&= \frac{\alpha(1-\beta)(1-\delta)}{\Omega}(2\tau_h^{**}-1) + \eta\sigma\Phi\left[\tau_h^{**}(\alpha-\tau_h^{**}) + (1-\alpha\beta)(2\tau_h^{**}-1)\right] \\
&= \frac{\alpha(1-\beta)(1-\delta)}{\Omega}(2\tau_h^{**}-1) + \alpha\eta\sigma\Phi\tau_h^{**}(1-\tau_h^{**}) > 0,
\end{aligned}$$

where in the second and third line we use $1-\alpha\beta = \frac{(1-\alpha)\left(\tau_h^{**}\right)^2}{2\tau_h^{**}-1}$, and Ω is the value given by (32) at τ_h. Combining with the fact that $\varpi(\tau_h) > 0$ and the definition of $\tilde{\tau}_h$, we can get $\phi(\tau_h) < \varpi(\tau_h)$ on $\tau_h \leq \tilde{\tau}_h$ and $\phi(\tau_h) > \varpi(\tau_h)$ on $\tau_h \geq \tau_h^{**}$. The continuity of $\phi(\tau_h)$ and $\varpi(\tau_h)$ implies that there exists at least one zero point of $\phi(\tau_h) - \varpi(\tau_h)$, and the zero points only lie in $(\tilde{\tau}_h, \tau_h^{**})$.

Next we prove that the zero point of $\phi(\tau_h) - \varpi(\tau_h)$ is unique on $(\tilde{\tau}_h, \tau_h^{**})$ when $\alpha > 1/2$ and $\epsilon < 0$. For such τ_h, the corresponding Ω must satisfy (32) and $\phi(\tau_h) = \varpi(\tau_h)$ simultaneously. Combining these two equations and eliminating $D\Omega^{\epsilon}$ yield

$$\frac{\phi(\tau_h)\left[1-\alpha\beta-(1-\alpha)\tau_h\right]}{\eta\sigma\Phi\tau_h(1-\tau_h)} = \frac{\alpha(1-\alpha)}{\eta\sigma\Phi}\left[1+\eta\beta\sigma\Phi - \frac{(1-\beta)(1-\delta)}{\Omega}\right]. \tag{A26}$$

It is obvious that the zero point τ_h and corresponding Ω also satisfy (32) and (A26) simultaneously. Since $\tau_h > \tilde{\tau}_h$, the left-hand side (LHS) of (A26) is larger than zero, and its derivative satisfies

$$
\begin{aligned}
\frac{\partial LHS}{\partial \tau_h} &= \frac{\left[(1-\tau_h)^2+\alpha+\frac{\alpha}{\eta\sigma\Phi}\right]\left[1-\alpha\beta-(1-\alpha)\tau_h\right]}{(1-\tau_h)^2\tau_h} \\
&\quad -(1-\alpha\beta)\frac{\left(\alpha+2+\frac{2\alpha}{\eta\sigma\Phi}\right)\tau_h-\tau_h^2-\left(1+\frac{\alpha}{\eta\sigma\Phi}\right)}{(1-\tau_h)\tau_h^2} \\
&\propto \left[(1-\tau_h)^2+\alpha+\frac{\alpha}{\eta\sigma\Phi}\right]\frac{1-\alpha\beta-(1-\alpha)\tau_h}{1-\alpha\beta} \\
&\quad -\frac{\left(\alpha+2+\frac{2\alpha}{\eta\sigma\Phi}\right)\tau_h-\tau_h^2-\left(1+\frac{\alpha}{\eta\sigma\Phi}\right)}{\tau_h}(1-\tau_h) \\
&= \frac{1+\frac{\alpha}{\eta\sigma\Phi}-\tau_h^2}{\tau_h}(1-\tau_h)^2+\left[(1-\tau_h)^2+\alpha+\frac{\alpha}{\eta\sigma\Phi}\right]\frac{\alpha(1-\beta)\tau_h}{1-\alpha\beta} \\
&> 0.
\end{aligned}
$$

Since the right-hand side of (A26) is increasing in Ω, then this implies that the Ω given by (A26) is increasing in τ_h on $(\tilde{\tau}_h, \tau_h^{**})$. Given $\alpha > 1/2$ and $\epsilon < 0$, we have $\tilde{\tau}_h > \tau_h^*$, then the equilibrium Ω given by (32) is decreasing in τ_h on $(\tilde{\tau}_h, \tau_h^{**})$. Because the zero point always exists, then there is a unique solution to equations (32) and (A26), i.e. the zero point of $\phi(\tau_h) - \varpi(\tau_h)$ is unique, below (above) which $\partial g/\partial \tau_h > 0 (< 0)$. The figure below shows this relationship. This result implies an inverted-U relationship between g and τ_h.

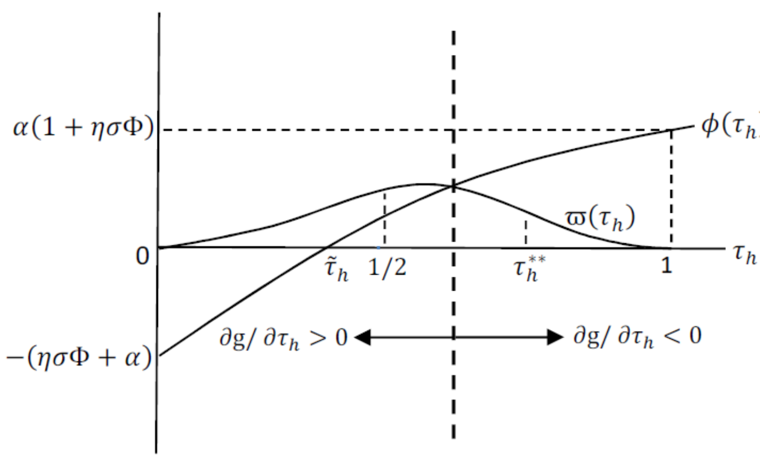

Figure 9

■

Proof of Proposition 5. For the case of $\alpha < 1/2$ and $\epsilon > 1$, the proof of Proposition 4 shows that zero points of $\phi(\tau_h) - \varpi(\tau_h)$ also exist in both high equilibrium and low equilibrium, and the zero points τ_h take values in $(\tilde{\tau}_h, \tau_h^{**})$. Since $\Omega^{low} < \Omega^* < \Omega^{high}$ for any values of τ_h, then the zero points of τ_h in high equilibrium and low equilibrium must be different according to the properties of (A26), which implies that there are at least two pairs of (τ_h, Ω) that satisfy (32) and $\phi(\tau_h) = \varpi(\tau_h)$ simultaneously. Notice that equations (32) and $\phi(\tau_h) = \varpi(\tau_h)$ imply (A26), and $\phi(\tau_h) = \varpi(\tau_h)$ implies

$$\left(\frac{\gamma}{\theta}\right)^{(1-\alpha)/\alpha} \alpha^{(1-2\alpha)/\alpha} \left[(1-\alpha) A\right]^{-1/\alpha} \Omega = \phi(\tau_h)^{(1-2\alpha)/\alpha} \tau_h (1-\tau_h), \tag{A27}$$

then we can combine (A26) and (A27) to eliminate Ω and obtain

$$\begin{aligned} & \alpha(1-\alpha)(1-\beta)(1-\delta)\left(\frac{\gamma}{\theta}\right)^{(1-\alpha)/\alpha} \alpha^{(1-2\alpha)/\alpha}\left[(1-\alpha) A\right]^{-1/\alpha} \\ = \; & \alpha(1-\alpha)(1+\eta\beta\sigma\Phi)\tau_h(1-\tau_h)\phi(\tau_h)^{(1-2\alpha)/\alpha} - \left[1-\alpha\beta-(1-\alpha)\tau_h\right]\phi(\tau_h)^{(1-\alpha)/\alpha} \end{aligned} \tag{A28}$$

Next we prove that the zero point is unique in both high-growth and low-growth equilibrium. It suffices to show that (A28) has at most two solutions. The left-hand side of (A28) is a constant, and the derivative of right-hand side (RHS) of (A28) satisfies

$$\frac{\partial RHS}{\partial \tau_h} \propto \chi(\tau_h) + \psi(\tau_h),$$

where

$$\chi(\tau_h) = (1+\eta\beta\sigma\Phi)\frac{(1-2\alpha)\tau_h(1-\tau_h)\left[\alpha(\eta\sigma\Phi+2)+2\eta\sigma\Phi(1-\tau_h)\right]}{\phi(\tau_h)},$$

and

$$\psi(\tau_h) = \alpha(1+\eta\beta\sigma\Phi)(1-2\tau_h)+\phi(\tau_h)-\left[1-\alpha\beta-(1-\alpha)\tau_h\right]\left[\eta\sigma\Phi+2+\frac{2\eta\sigma\Phi}{\alpha}(1-\tau_h)\right].$$

$\psi(\tau_h)$ has the following properties: (a) $\partial\psi(\tau_h)/\partial\tau_h|_{\tau_h=1} = 2(1-\alpha)+\eta\sigma\Phi(3-2\beta-2\alpha\beta) > 0$ and $\partial^2\psi(\tau_h)/\partial\tau_h^2 < 0$, (b) $\psi(1) = -2\alpha(1-\beta) < 0$ and $\psi(0) > -\infty$. This implies that $\psi(\tau_h) < 0$ and $\partial\psi(\tau_h)/\partial\tau_h > 0$ on $\tau_h \in [0,1]$. Notice that when $\alpha < 1/2$ and $\epsilon > 1$, $\phi(1/2) = \eta\sigma\Phi(\alpha-1/2)/2 < 0$, which implies $\tilde{\tau}_h \in (1/2, 1)$. Then $\chi(\tau_h)$ has the following properties: (a) $\partial\chi(\tau_h)/\partial\tau_h < 0$ and $\partial\chi^2(\tau_h)/\partial\tau_h^2 > 0$[19] for any $\tau_h \in (\tilde{\tau}_h, 1]$, (b) $\chi(1) = 0$ and $\chi(\tilde{\tau}_h) = +\infty$. These results imply that $\chi(\tilde{\tau}_h) + \psi(\tilde{\tau}_h) > 0$ and $\chi(1)+\psi(1) < 0$. By continuity, there exists at least one $\tau_h \in (\tilde{\tau}_h, 1)$ such that $\frac{\partial RHS}{\partial \tau_h} = 0$.

Moreover, notice that $RHS|_{\tau_h=\tilde{\tau}_h} = RHS|_{\tau_h=1} = 0$, then there exists at most two solutions of τ_h to (A28). As we have discussed above, there are at least two τ_h's that satisfy (A28), hence there exist exactly two solutions of τ_h to (A28) with one in high-growth equilibrium and the other in low-growth equilibrium. Therefore, in both high and low equilibrium, the zero point of $\phi(\tau_h) - \varpi(\tau_h)$ is unique. Then making use of (A20) and the properties of Ω^{low} and Ω^{high}, there is a U-shaped relationship between g and τ_h in low-growth equilibrium, and an inverted-U relationship between g and τ_h in high-growth equilibrium. ■

[19] Derivations are available upon request.

Appendix B

In the main text, we have the following 2×2 dynamic system:

$$\Phi_{t+1} = \frac{1+\mu}{\beta(1-\sigma)}\Phi_t - \frac{\sigma}{\theta(1-\sigma)}, \tag{B1}$$

$$\frac{\Omega_{t+1}}{\Omega_t} = \frac{\beta\left[1-\delta+\alpha(1-\tau_k)D\Omega_{t+1}^{\epsilon}+\eta\sigma\Phi_{t+1}\Omega_{t+1}\right]}{(1-\hat{\tau})D\Omega_t^{\epsilon}-\Omega_t+1-\delta}. \tag{B2}$$

The steady-state values of Φ_t and Ω_t, denoted as Φ and Ω, are determined by

$$\Phi = \frac{\sigma\beta}{\theta[1+\mu-(1-\sigma)\beta]}, \tag{B3}$$

$$(1+\eta\beta\sigma\Phi)\Omega = [(1-\hat{\tau})-\alpha\beta(1-\tau_k)]D\Omega^{\epsilon}+(1-\beta)(1-\delta). \tag{B4}$$

Taking a linear approximation around the steady-state equilibrium values of Φ and Ω over the dynamic system and using (B3) and (B4), we derive

$$\begin{bmatrix} \Phi_{t+1}-\Phi \\ \Omega_{t+1}-\Omega \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \Phi_t-\Phi \\ \Omega_t-\Omega \end{bmatrix}, \tag{B5}$$

where

$$a_{11} = \frac{1+\mu}{\beta(1-\sigma)} > 1, \quad a_{12} = 0,$$

$$a_{21} = \frac{\beta\eta\sigma\Omega^2}{\beta[(1-\delta)+\alpha(1-\epsilon)(1-\tau_k)D\Omega^{\epsilon}]}\left[\frac{1+\mu}{\beta(1-\sigma)}\right],$$

$$a_{22} = \frac{(1-\delta)+(1-\epsilon)(1-\hat{\tau})D\Omega^{\epsilon}}{\beta[(1-\delta)+\alpha(1-\epsilon)(1-\tau_k)D\Omega^{\epsilon}]}.$$

Given $a_{12}=0$, the eigenvalues of the Jacobian matrix are a_{11} and a_{22}. Note that Φ_t and Ω_t are both non-predetermined variables. Blanchard and Kahn (1980) show that in a linear difference model with rational expectation, the dynamic system is (a) determinate if the number of eigenvalues of Jacobian outside the unit circle is equal to the number of non-predetermined variables and (b) indeterminate if the number of eigenvalues outside the unit circle is less than the number of non-predetermined variables. Given that $a_{11} > 1$,[20] the Blanchard-Kahn conditions imply that the dynamic system (B5) is (a) determinate if $|a_{22}| > 1$ and (b) indeterminate if $|a_{22}| < 1$. Finally, note that the eigenvalue a_{22} is equal to ξ in the main text; therefore, our analysis of indeterminacy in the main text is consistent with the Blanchard-Kahn conditions.

[20] Together with $a_{12}=0$, this implies that Φ_t jumps to its unique steady state Φ as Figure 1 shows.