Measures of policy distance and inequality / disproportionality of votes and seats

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Measures of policy distance and inequality / disproportionality of votes and seats

Didactics and routines using *Mathematica*

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Abstract

Let v be a vector of votes for parties and s a vector of their seats gained in the House of Commons or the House of Representatives. We use a single zero for the lumped category of "Other", of the wasted vote, for parties that got votes but no seats. Let $V = 1'v$ be total turnout and $S = 1's$ the total number of seats, and let $w = v / V$ and $z = s / S$ be the perunages (or per ten or percentages). Let $d[w, z]$ be the inequality / disproportionality of votes and seats. This can be the angle between the vectors(AID) and the sine-diagonal (SDID) measure based upon this. Parties can also be scored with policy vector p, using a "left-to-right" policy scale [0, 10]. A common voter-legislative distance is the weighted average $a = p' (z - w)$. With AID $d[w, z]$ the present paper looks into the properties of $d[p\, w, p\, z]$. The latter term for variable w and z given p works as a disproportionality measure, and for variable p given w and z works as policy congruence. We can define an angular policy distance (APD) $pd[w, z, p]$ that employs this $d[p\, w, p\, z]$ properly. The APD is much more sensitive than the weighted average, but Sqrt[Abs[a]] has remarkably similar behaviour.

Keywords

Votes, Seats, Electoral System, Policy Distance, Disproportionality, Angular Distance, Sine-Diagonal Inequality / Disproportionality, Loosemore-Hanby, Gallagher, Descriptive Statistics, Education

MSC2010

00A69 General applied mathematics
28A75 Measure and integration. Length, area, volume, other geometric measure theory
62J20 Statistics. Diagnotics
97M70 Mathematics education. Behavioral and social sciences

JEL

A100 General Economics: General
D710 Social Choice; Clubs; Committees; Associations,
D720 Political Processes: Rent-seeking, Lobbying, Elections, Legislatures, and Voting Behavior
D630 Equity, Justice, Inequality, and Other Normative Criteria and Measurement

Cloud

This *Mathematica* notebook is also available at:

https://www.wolframcloud.com/objects/thomas-cool/Voting/2018-02-02-PolicyDistance.nb

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References

Start (evaluate this subsection for the initialisation packages)

1. Introduction

Votes and seats

Our target application concerns *votes* and *seats*. For example, the outcomes for Parties A and B might be *votes* = {4.9, 5.1} and *seats* = {5.1, 4.9}, both scores in the range [0, 10].

We use v for votes, s for seats. Their sum totals are $V = 1'v$ and $S = 1's$. The vectors $w = v / V$ and $z = s / S$ are called "unitised" to express that these vectors add up to 1 each. On occasion we will write $w = U[v]$ and $z = U[s]$ for this unitisation, e.g. when $U[c\, z]$ might be shorter than $c\, z / 1' (c\, z)$.

Normalisation to 10 rather than 1 or 100

For political science there is the choice between training students to grow sensitive to small numbers or create more sensitive measures that also allow direct communication with the larger public. For the Richter scale for earthquakes, the approach was to use logarithms, so that smaller values could be compared easier with larger ones. For our purposes the square root is adequate, with an easy change from 10 (grade) to 100 (percentage).

For this subject of votes and seats it is better to use the **grades** in [0, 10], like *votes* = {4.9, 5.1}. The reason is that variables in [0, 1] causes leading zeros, while percentages in the range [0, 100] suggest a degree of precision that is often overdone.

For the scale we will use the symbol Đ (capital eth). For example *w* = Đ *v* / *V*. The default value in this paper is Đ = 10.

```
? UnitĐ
```

UnitĐ[x, f:Đ] gives f x / (Plus @@ Abs[x])
UnitĐ[Clear] unprotects and clears Đ
UnitĐ[Set, x] sets Đ to the value x and protects
The default protected value of Đ is 10

```
? Đ
```

The default value is 10, but it might be reset, see UnitĐ.
Compare the use of % as having the value 1/100. Write [ESC]D–[ESC] or "\[CapitalEth]". Pronounce here as "deka" instead of "eth", and 1 / Đ as "decim"

```
UnitĐ[{49, 51}] // N
```

```
{4.9, 5.1}
```

```
1 / Đ == 10 Percent
```

$$\frac{1}{10} == 10\ \text{Percent}$$

A distance *d* of inequality / disproportionality (ID) for votes and seats

There are two measures that translate into each other:

(1) A measure of *association, proportionality* or *congruence* has Đ (default 10) for a perfect match and 0 for a total mismatch.

(2) A measure of *distance, disproportionality* or *incongruence* has Đ (default 10) for a total mismatch and 0 for a perfect match.

The common notion is *distance* but for votes and seats we would like to see *equal proportions*, whence we rather speak about an *inequality / disproportionality* (ID) measure.

The inequality / disproportionality measure *d* for votes and seats is independent of multiplication of the vectors by scalar multiples, with ***d*[*v*, *s*] = *d*[*w*, *z*]**.

Political science on electoral systems has provided various measures of distance between votes and seats, of which we will mention the two main ones just now: The *Loosemore-Hanby* measure is based upon the *absolute difference* (ALHID), and the *Gallagher* measure is based upon the *Euclidean distance* (EGID), both with corrections to get an outcome in [0, Đ]. For 2 parties these measures generate the same outcome. It appears that these measures have some drawbacks. My suggestion from econometrics and political economy is to use the *angle* between the vectors, which in particular gives the transformation into the *Sine-Diagonal Inequality / Disproportionality* (SDID), see Colignatus (2017ac and 2018).

For this present notebook it suffices to use a simpler indicator based upon the angle. Since the maximum angle between nonnegative vectors is 90 degrees, we can interprete the angle also as share of 90 degrees, scaled to Đ (default 10). This transform is called *AngularID* (AID).

For the following example, we have an inequality or disproportionality of votes and seats of 0.25 in [0, 10], or a quarter grade, or 2.5%.

```
? AngularID
```

AngularID[x, y] gives the angular inequality / disproportionality. This is the angle between real vectors x and y, divided by Pi/2 radian or 90 Degree, times Đ. Output is in [0, Đ]

```
AngularID[{4.9, 5.1}, {5.1, 4.9}]
0.254614
```

The angulare inequality / disproportionality is a bit more sensitive than the absolute difference.

```
? AbsLoosemoreHanbyID
```

AbsLoosemoreHanbyID[v, s] takes the absolute value of the differences of UnitĐ[v] and UnitĐ[s], divided by 2 to correct for double counting: Đ * Total[Abs[v / v'1 – s / s'1]] / 2. Outcomes are in the range [0, Đ]

```
AbsLoosemoreHanbyID[{4.9, 5.1}, {5.1, 4.9}]
0.2
```

PM. Whether AID is simpler than SDID is a matter of view. Both are based upon the cosine between the vectors, that can be found from the improduct in linear algebra. The AID then applies ArcCos to find the angle. The SDID uses the Pythagorean Theorem and sin^2 + cos^2 = 1. Some might find the latter simpler. For the present discussion the interpretation that an AID of 0.25 means 0.25 * 9 ≈ 2.5 degrees can be appreciated.

Equal / proportional representation (EPR) and district representation (DR)

There are various electoral systems. The key distinction concerns systems with equal / proportional representation (EPR) and systems with district representation (DR).

(1) With EPR we have $z \approx w$ and $d \approx 0$. For EPR, the distance d reaffirms that the system is EPR, and there our analysis essentially stops. For EPR, it is more relevant how far parties are apart for different topics, see the research by Stokman et al. (2013).

(2) The distance d is mainly interesting for systems with DR. In systems with district representation, there will be much strategic voting, because voters may fear that their vote is wasted on parties

that have no chance of winning the district seat. A common strategy is to vote for the party that might win against the worst alternative. Thus the votes v do not reflect the true first preferences. In that case $d[v, s]$ is biased and we should use $d[v^*, s]$, with the true first preferences v^* and their shares w^*. Political scientists are advised to have exit polls at (half) elections to get estimates for the true first preferences.

- Discussion of electoral systems: https://mpra.ub.uni-muenchen.de/82513/

A policy distance *pd*

Political parties have votes v and seats s, but can also be scored on a policy scale p, with its elements in [0, Đ]. In that case we get a policy distance ***pd*[*v, s, p*]**. A common term for such a scale is "left-to-right" even though those terms are rather out-dated. A basic reference is to Golder & Stramski (2010). A left-to-right scale has much less detail than the more complex methods as mentioned by Stokman et al. (2013).

A common policy distance measure is the **weighted average** $a = p'z - p'w = p' (z - w)$. Observe that a has the form $pd[w, z, p]$ too. Another notation is to take the matrix of policy differences $P = p - p'$, and then find $a = z' P w$. Under EPR $a \approx 0$. Under DR this is biased again, and we actually should use $a^* = p'(z - w^*) = z' P\ w^*$.

PM. The weighted average a must be calculated correctly to keep it in the [-Đ, Đ] range. When a is calculated with unitised w and z, then p in [0, Đ] will cause that a is in [0, Đ] too.

Apparently the literature has little discussion of using d for measuring such a policy distance. While the weighted average *adds* the components $p\ w$ and $p\ z$, we might consider to regard them *unadded.* Thus, we might consider using $d[p\ w, p\ z]$ and $d[p\ w^*, p\ z]$ or some related construct. Our terminology is:

(1) We call $d[p\ w, p\ z]$ the *angular policy term* (APT).

(2) We will use the APT to construct an *angular policy distance* (APD).

(3) An intermediate step between the APT and APD is the *angular policy measure* (APM). This APM will consist of $m = d[w, z] - d[p\ w, p\ z]$ on occasion but formally of $m = max - d[p\ w, p\ z]$, with the local maximum, while for symmetric cases the maximum is $d[w, z]$.

The comparison of *a, d,* APT, APM and APD is the major focus of this paper.

In the political science literature on electoral systems, the distance $d[w, z]$ on the inequality / disproportionality of the votes and seats consisted historically out of the *absolute difference* (Loosemore-Hanby) or the *Euclidean distance* (Gallagher). Given the drawbacks of these measures, we might imagine an effort amongst political scientists to overcome such drawbacks by also looking at a policy indicator p. Since 2017 there is a better measure $d[w, z]$ for inequality / disproportionality, based upon the angle between the vectors of votes and seats. Now the question arises what $pd[w, z, p]$ would still improve about the analysis.

A ruling coalition

Parties can form a ruling coalition that commands a majority of the seats. Let c contain elements 1 or 0, with 1 for a party in the ruling coalition and 0 for a party in opposition. In a *common sense* or *business minded* House, this c will depend upon topics for which such a majority can be bargained for, with different *c* for different packages of topics. In a *politically charged* House, the divisions may be along ideological lines, whatever package, with more permanence about ruling parties and parties in opposition.

In general there is the distinction:

(1) The *voter-legislative distance.* This is the case above, with use of *d*[*w, z*], *pd*[*w, z, p*] and *a*. Golder & Stramski (p95) w.r.t. a policy distance: “(...) substantively representative legislatures increase things like perceived levels of democratic legitimacy and responsiveness, satisfaction with democracy, political participation, or personal efficacy and trust in the political process.”

(2) The *voter-executive distance*. This might use *d*[*w, c z*] and *pd*[*w, c z, p*] as indicators. For *a* it is tempting to use *a* = *p’* (*c z* - *w*). However, we should keep in mind that the weighted average requires unitised variables. Thus the proper formula is *a* = *p’* (*U*[*c z*] - *w*). This is often called “voter-government distance” but the government consists of executive, legislative and judiciary branches.

The 2016 US House of Representatives as an example

The US (half) elections of 2016 for the House of Representatives generated the following distributions for the votes and seats, using the order {Democrats, Republicans, Other}:

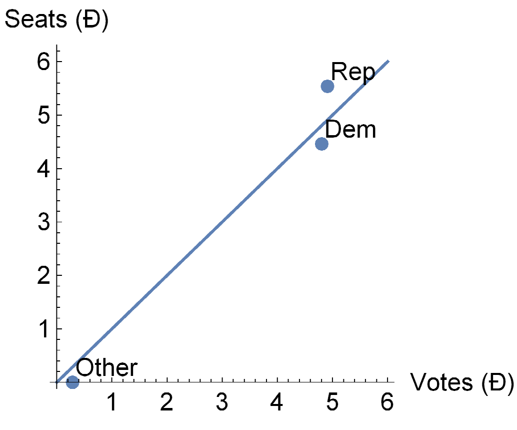
```
votes = {dem = 61776554, rep = 63173815, 128627010 - rep - dem};
seats = {194, 241, 0};
```

- https://en.wikipedia.org/wiki/United_States_House_of_Representatives_elections,_2016#Results_summary

Transforming these into grades and a plot:

```
{vts = UnitĐ[votes], sts = UnitĐ[seats]} // N
{{4.80277, 4.9114, 0.285837}, {4.45977, 5.54023, 0.}}
```

```
Show[ListLinePlot[{{0, 0}, 6 {1, 1}}],
 ListPlot[MapThread[Labeled, {Transpose[{vts, sts}], {Dem, Rep, Other}}],
  PlotStyle → PointSize[Large]], AxesLabel → {"Votes (Đ)", "Seats (Đ)"},
 AspectRatio → 1, BaseStyle → {FontSize → 13}]
```



The USA have *district representation* (DR) which differs from *equal or proportional representation* (EPR) (like in Holland). In a US district, a candidate is elected with the plurality rule. The votes for other candidates than the winner in the district are discarded. That the Democrats and Republicans are still relatively close to the 45 Degree line of equality or proportionality of votes and seats is the effect of geography, and perhaps the median voter theorem. The word “election” is over-used when it refers to such different meanings for either district (DR) or equal / proportional representation (EPR). Given the discarding of votes in the system of district representation, we rather should speak about “half elections”.

In district representation, there will also be more strategic voting, in which voters will not vote for a smaller party for fear that their vote will be discarded. This effect *cannot* be measured by the common inequality or disproportionality measures, since we only have the recorded votes and not the true first preferences.

A plot of the topic of this paper

Let us use arbitrary data on the left-to-right policy scale, and assume that the majority party forms the executive coalition.

```
policy = {4, 6, 5};
coalition = SingleMajority[seats]
```

```
{0, 1, 0}
```

The following plot gives the topic of this paper. The horizontal axis gives the *voter-legislative* distance, and the vertical axis gives the *voter-executive* distance. We have four distance measures:

(1) The angular inequality / disproportionality (AID), using *d*[*w, z*] and *d*[*w, c z*].

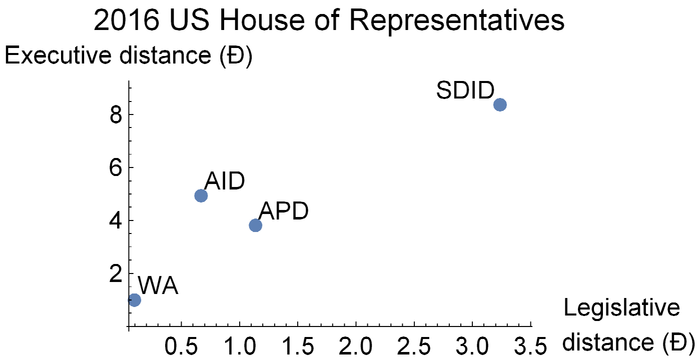
(2) The angular policy distance (APD), using *pd*[*w, z, p*] and *pd*[*w, c z, p*].

(3) The weighted average (WA), using *p’* (*z* - *w*) and *p’* (*U*[*c z*] - *w*).

(4) The sine-diagonal inequality / disproportionality (SDID): using *sdid*[*w, z*] and *sdid*[*w, c z*].

The SDID measure works as a magnifying glass for disproportionalities, and dominates the other outcomes. The USA has a SDID disproportionality in votes and seats of about 3.2 out of 10, or 32%. Observe that the US has DR. In EPR, we will see $z \approx w$, and thus the horizontal axis will be mostly 0. A question is what the measures contribute to the analysis, while it is already clear that DR is a suboptimal system. It would be better when the USA would switch to EPR and use the methods by Stokman et al. (2013).

```
AngularPolicyDistancePlot[votes, seats, policy,
 coalition, PlotLabel → "2016 US House of Representatives"]
```



The WA policy measure for the US is 0.1 on a [0, 10] scale, or 1%, which causus us to wonder whether the uncertainties in scoring this *p* would not dominate any analysis. Still, 0.1 is a significant figure for disproportionality in votes and seats (see the earlier discussion on 0.25).

```
{WAvPolicyDistance[votes, seats, policy],
  WAvPolicyDistance[{10, 0}, {0, 10}, {0, 10}]} // N
```

```
{0.0971831, 10.}
```

The WA voter-executive distance in this example is close to 1, meaning that the Republicans use their majority to implement policy at their position 6 rather than the country average. This measure only records the possibility, and does not suggest that it would be wise to do so.

An issue and an approach

This paper solves an issue on content and employs some didactics to do so.

On content: The issue is how d relates to $a = p' (z - w)$, with the question whether $d[w, z]$ can be transformed into a policy distance $pd[w, z, p]$, perhaps using $d[p\, w, p\, z]$ or a related construct.

On didactics: This issue requires three dimensions: v, s and p. Especially education is challenged on this. It appears that *Mathematica* allows us to better grasp the issues. *Mathematica* is a system for doing mathematics on the computer. It allows the combination of texts, symbols, numbers, graphs, patterns, motion, sound, etcetera. The following application again shows the power of using this integrated system.

Order of discussion

Historically, the political science on electoral systems before 2017 apparently was relatively unaware of the angular distance measure. For the policy distance the weighted average provided its services. Thus it is tempting to start the discussion below with this weighted average, as a common ground to start from. However, this would distract from the present news. (i) It is better to start with the construction of above new APD. (ii) Then we can return to the common ground on the weighted average. (iii) A combination of APD and WA is put into **Appendix C**. Before we start, though, it appears better to discuss: (0) the intended application and required properties.

Larger framework: 1W1V

This discussion forms part of a larger framework given by Colignatus (2017b): "*One woman, one vote. Though not in the USA, UK and France*" (1W1V). My diagnosis is that "political science on electoral systems" is still in the Humanities and pre-science. This branch of political science relies more upon common language than upon sharp definitions that are relevant for empirics. An example is the use of the word "election" for any system, while so-called "elections" under DR are actually only "half-elections" since the votes are discarded that are not for the district winner (and one is supposed to be "represented" by someone whom one explicitly did not vote for). On the other hand, there are also mathematicians who deal with their definitions abstractly, without a proper grounding in empirical research. My invitation to empirical researchers is to help make a difference, notably in re-engineering the theory on electoral systems.

- https://mpra.ub.uni-muenchen.de/82513/

Routines in this notebook

```
? Cool`Voting`Angular`*
```

▼ Cool`Voting`Angular`

AbsLoosemoreHanbyID	AngularPolicyTerm	SineDiagonalID
AllZeroButOneQ	CombinedPolicyDistance	SingleMajority
AngleProxy	Đ	UnitĐ
AngPolMax	EuclidGallagherID	WAvAngPolicyMeasure
AngularID	IndepAngPolicyDistance	WAvPlot
AngularPolicyDistance	MaxAngularPolicies	WAvPolicyDistance
AngularPolicyDistancePlot	PlusAngPolicyDistance	WAvSqrtPolicyDistance
AngularPolicyMeasure	PolicyNorm	WebsterSainteLagueID

2. Inequality / disproportionality measures of votes and seats

Disproportionality measures

The *Absolute Difference* (ALHID) and *Euclidean Distance* (EGID) measures are already used in the literature on votes and seats, and the sine-diagonal measure is a new suggestion from 2017. For this paper we use the simpler *Angular Distance.*

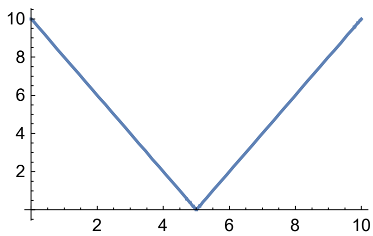
The literature uses values of ALHID and EGID in the [0, 100] range. We use the [0, 10] range here.

AbsLoosemoreHanbyID (ALHID)

```
? AbsLoosemoreHanbyID
```

AbsLoosemoreHanbyID[v, s] takes the absolute value of the differences of UnitĐ[v] and UnitĐ[s], divided by 2 to correct for double counting: Đ * Total[Abs[v / v'1 – s / s'1]] / 2. Outcomes are in the range [0, Đ]

```
Plot[AbsLoosemoreHanbyID[{Đ - t, t}, {t, Đ - t}], {t, 0, Đ}]
```



The LHID measure has the useful interpretation that it gives the share of displaced seats between parties for a House of Đ seats (corrected for double counting). In the US, 0.6 seats of a House of 10 seats are displaced, more than 5%.

```
AbsLoosemoreHanbyID[votes, seats] // N
```

```
0.628834
```

The ALHID distance is insensitive to the location of a 1 grade difference, say at {4, 6} or {9, 1}.

```
{ad1 = AbsLoosemoreHanbyID[{4, 6}, {5, 5}],
  ad2 = AbsLoosemoreHanbyID[{9, 1}, {10, 0}], ad1 / ad2} // N
```

```
{1., 1., 1.}
```

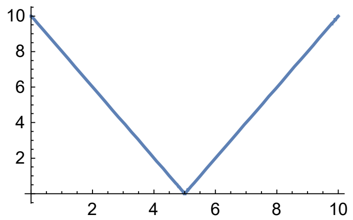
- https://en.wikipedia.org/wiki/Loosemore%E2%80%93Hanby_index

EuclidGallagherID (EGID)

```
? EuclidGallagherID
```

EuclidGallagherID[v, s] takes the Euclided distance of UnitĐ[v] and UnitĐ[s], divided by Sqrt[2] to correct for double counting. Outcomes are in the range [0, Đ]. For length 2 the outcome is the same as AbsLoosemoreHanbyID

```
Plot[EuclidGallagherID[{Đ - t, t}, {t, Đ - t}], {t, 0, Đ}]
```



The 2016 US House data have 3 parties, and thus the EGID can have a different outcome than the ALHID. The US EGID is a bit larger than a half grade or 5%.

```
EuclidGallagherID[votes, seats] // N
```

```
0.545336
```

For two parties, the insensitivity is the same for EGID as ALHID.

```
{ad1 = EuclidGallagherID[{4, 6}, {5, 5}],
  ad2 = EuclidGallagherID[{9, 1}, {10, 0}], ad1 / ad2} // N
```

```
{1., 1., 1.}
```

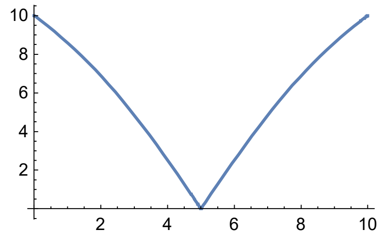
- https://en.wikipedia.org/wiki/Gallagher_index

Angular inequality / disproportionality (AID)

```
? AngularID
```

AngularID[x, y] gives the angular inequality / disproportionality. This is the angle between real vectors x and y, divided by Pi/2 radian or 90 Degree, times Đ. Output is in [0, Đ]

```
Plot[AngularID[{Đ - t, t}, {t, Đ - t}], {t, 0, Đ}]
```



The 2016 US AID would suggest a disproportionality in the USA of 0.7 grades or 7%.

```
AngularID[votes, seats] // N
```

```
0.66848
```

The angular distance is sensitive to the location of 1 grade difference. It shows a halving or doubling of the angular distance, depending where one starts.

```
{ad1 = AngularID[{4, 6}, {5, 5}],
  ad2 = AngularID[{9, 1}, {10, 0}], ad1 / ad2} // N
```

```
{1.25666, 0.704466, 1.78385}
```

Because of this greater sensitivity the angular distance is a better measure for disproportionality than ALHID and EGID.

- https://en.wikipedia.org/wiki/Angular_distance

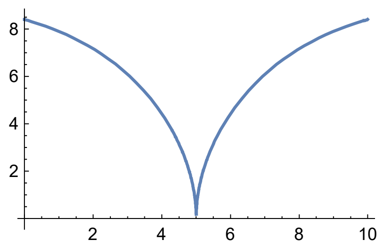
Sine-Diagonal (SDID)

```
? SineDiagonalID
```

SineDiagonalID[v, s] gives the Sqrt (greater sensitivity) of the Sine of the angle between the vectors, multiplied by 10 to get rid of leading zeros, and with the sign of Covariance[v, s] < –1 for majority switches. Thus it ranges in [–10, 10]

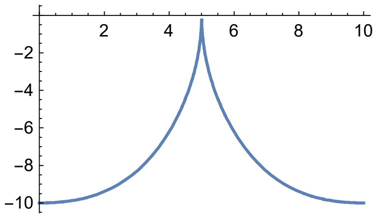
If we do not look at input of opposites.

```
Plot[SineDiagonalID[{50, 50}, {t, Đ - t}], {t, 0, Đ}]
```



The input of opposites generates negative outcomes, indicating majority switches between votes and seats.

```
Plot[SineDiagonalID[{Đ - t, t}, {t, Đ - t}], {t, 0, Đ}]
```



The SDID measure is designed to be sensitive to disproportionality. For the US it gives 3.2 on a scale of 10, thus 32%. The SDID uses a magnifying glass like the scale of Richter on earthquakes does. (Smaller earthquakes are made comparable with the bigger ones.)

```
SineDiagonalID[votes, seats] // N
```

```
3.23746
```

The SDID is steeper than the AID, which makes the ratio of inner versus outer range values for the displacement a bit lower.

```
{ad1 = SineDiagonalID[{4, 6}, {5, 5}],
  ad2 = SineDiagonalID[{9, 1}, {10, 0}], ad1 / ad2} // N
```

```
{4.4285, 3.32312, 1.33263}
```

- https://mpra.ub.uni-muenchen.de/81389/

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3. Before and after (half) elections

Before the elections *p* and after the elections *q*

Suppose that parties announce their policy stance p before the (half) election and that the parties might change their policy stance to q after the (half) election. The relevant weighted average then is $a = q'z - p'w$ under EPR and $a^* = q'z - p'w^*$ under DR. In this case the policy difference matrix is $P = q - p'$, and $a = z' P q$.

Using d we might consider using $d[p\ w, q\ z]$ and $d[p\ w^*, q\ z]$ or some related construct.

Caveat

As a political economist, I regard it as “revealed preferences” (a term coined by Paul Samuelson) what parties people vote for and what policy those enact. When researchers try to determine what the “policy stance” p of the parties is on a scale of [0, 10] then it seems to me that this is operationally dubious and needlessly complicated, since we already have the essential distinctions between parties with their votes and seats, including the distance $d[v, s]$ between them. Remarkably, some researchers who prefer a policy distance tend to regard the distance between votes and seats $d[v, s]$ as devoid of much meaning, and they thus do not value the interpretation in terms of revealed preference. The present discussion thus only occurs because there is a section in the literature that tries to look into such a policy distance.

Perhaps there might be an argument for district representation (DR) that the policy measure p is useful as an *alternative* to trying to recover the true first votes v^*. However, $a = p'(z - w)$ still is based upon the actual votes. It would still be better to find w^*. In that case $d[w^*, z]$ has only one uncertainty w^*, and this could still be a better measure than $a^* = p'(z - w^*)$ with the two uncertainties of both w^* and p.

That said, the after-elections change from p to q forms a challenge to the use of d. We might wonder whether we would still have the same parties, once they have changed position. While we do not need to join parties who have the same policy positions, since they might still be different in other aspects (like candidates), perhaps we must split parties when there is a tiny difference between p and q. Then, however, d quickly becomes 0. Thus, we reject the option to split parties (unless they do so themselves). Thus $d[p\ w, q\ z]$ and $d[p\ w^*, q\ z]$ or some related construct might still be a relevant approach to look into.

Combining coalition, views and changes of views

The following table combines above distinctions. We should not overlook the possibility of using $d[w, z]$ and $d[w, c\ z]$, thus not using p at all. When we include a policy indicator, then we distinguish between either using only p or using both p and q. Keep in mind that some formulas require unitisation U.

Before & after	Only c	p, c	p, q, c
Legislative	d[w, z]	p	p, q
Exec. Coalition	d[w, c z]	p, c p	p, q c

This article assumes the left column and will tend to look at the voter-legislative distinction $q = p$, which concerns the middle column, with the use of c as a corollary. This paper is already rather involved, and thus we skip looking at $q \neq p$. Having good foundations on p however clarifies where changes must be made to allow for q.

Coalition, using $d[w, c\ z]$

Let c contain elements 1 or 0, with 1 for a party in a coalition and 0 for a party in opposition.

When $d[w, z]$ is the *voter-legislative distance* then $d[w, c\ z]$ is a useful *voter-executive distance.*

Take above case of the US 2016 and let us assume that the Republicans form the ruling coalition. The voter-legislative distance, of almost 0.7 grades or 7%, changes into a voter-executive distance of 4.9 grades or almost 50%. This datapoint was already plotted in the graph in the *Introduction.*

```
{AngularID[vts, sts], AngularID[vts, coalition sts]} // N
```

```
{0.66848, 4.93444}
```

The outcomes will tend to be around 5 off 10 or 50% in general. When votes and seats are in deadlock, then the random toss of a coin still produces this 5 off 10 or 50%.

```
deadlock = {5, 5};
coalition = {0, 1};
{AngularID[deadlock, deadlock], AngularID[deadlock, coalition deadlock]} // N
```

```
{0., 5.}
```

The total outcome can still be 10 in extreme cases. When the first party gets 100% of the votes but doesn't manage to get a majority in seats, then the voter-executive outcome is 10.

```
{AngularID[{10, 0}, {4.9, 5.1}], AngularID[{10, 0}, {0, 1} {4.9, 5.1}]} // N
{5.12731, 10.}
```

The message may be that the distinction between 50% and 100% might be accurate but less informative than the description of the actual situation. However, we would use such measures in *Comparative Political Systems,* with regressions over nations, and then such numerical indicators could be of use.

4. Properties and first distinctions

Poperties for a policy distance

At this stage we can already identify properties that a policy distance $pd[w, z, p]$ must satisfy:

(P.1) When $w = z$ then $pd[w, z, p] = d[w, z] = 0$. This happens for $a = p'(z - w)$ and $d[p\, w, p\, z]$.

(P.2) When all parties have the same policy $p = \lambda\, 1$ then there is zero policy incongruence too.

(i) This happens for $a = p'(z - w)$, since $\lambda\, 1'(z - w) = \lambda\,(1'z - 1'w) = 0$.

(ii) For $p = \lambda\, 1$ then $d[p\, w, p\, z] = d[w, z] \geqslant 0$. Thus the latter does not satisfy this criterion for a policy distance. The *term* (APT) is not a *distance* (APD). Then we might consider $m = d[w, z] - d[p\, w, p\, z]$, and $p = \lambda\, 1$ causes that $m = 0$. It appears that m might turn negative for asymmetric input, so that m by itself cannot be a distance.

(P.3) Extreme positions would be dubious to consider, like when two parties are opposed like $p = \{0, 10\}$, with one party fully on the left and another fully on the right. The weighted average then would give $a = z[[2]] - w[[2]]$, with the meaning of the difference in shares of seats over votes for the rightist party, which is a dubious statistic. For the weighted average, such an extreme p would be a *selector* of parties rather than *weighing* them (and using the left-to-right scale as the selection criterion). (The binary case would be symmetric though.) Also in such an extreme case $d[p\, w, p\, z] = 0$, which makes some sense from its own logic, but which is still dubious, notably when we have extremes $w = \{0, 10\}$ and $z = \{10, 0\}$. When $p\, z = \{0, 10\}\, \{10, 0\} = \{0, 0\}$, then we have the zero vector, and then there is no angular measure.

(P.4) When we consider symmetric cases, notably $z = 10 - w$ in the plots below, then $pd[w, z, p]$ and $d[w, z]$ might show a regularity, like $pd \leqslant d$, yet there might be other outcomes for cases without symmetry.

First distinctions on content

On content, we may already introduce some first findings:

(A.1) We find that $f[w, z\,; p] = d[p\, w, p\, z]$ *given* p works like $d[w, z]$, thus as disproportionality of votes and seats. Property P.1 is of key relevance here. The exception though is P.3 that blocks sensitivity to w and z. One option is to disregard the case, another option is to consider 0 as a local maximum.

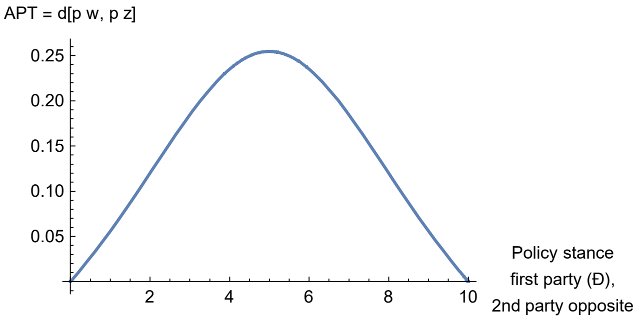
(A.2) We also find that $g[p\,; w, z] = d[p\, w, p\, z]$ *given* w *and* z works like a policy congruence measure, and thus into the other direction. The congruence shows by moving from P.3 to P.2, when the congruence and the measure both rise.

In an earlier text (2017b, p45 and its Appendix J) I suggested that the APT *d*[*p v, p s*] might be a policy distance itself. I retract this suggestion. This is only true for the perspective of *f*[*v, s* ; *p*] = *d*[*p v, p s*] or assuming a given *p.* There is another perspective for *g*[*p* ; *v, s*] = *d*[*p v, p s*]. A widening policy gap between parties does not *increase* but *reduce* the angle of vectors *p v* and *p s,* since the components actually come closer to each other. A clear example compares *p* = {0, 10} and *p* = {5, 5}. Thus *d*[*p v, p s*] is a measure of disproportionality given *p,* and a measure of congruence given *v* and *s.* However, this very phenomenon provides the suggestion of a useful transform of APT for the creation of a proper APD.

The following plots the perspective how the APT given *v* and *s* is a measure of policy congruence.

```
votes3 = {4.9, 5.1};
seats3 = {5.1, 4.9};
```

```
Plot[policy = {r, Đ - r};
 AngularID[policy votes3, policy seats3] , {r, 0, Đ}, AxesLabel →
  {"Policy stance\nfirst party (Đ),\n2nd party opposite", "APT = d[p w, p z]"}]
```



These are numerical outcomes for various policy positions.

```
{policy = {0, 10}; AngularID[policy votes3, policy seats3],
  policy = {2, 7}; AngularID[policy votes3, policy seats3] ,
  policy = {6, 6}; AngularID[policy votes3, policy seats3] }  // N
```

```
{0., 0.134565, 0.254614}
```

The *angular policy term d*[*p w, p z*] thus shows both *policy incongruence* and *disproportionality for votes and seats*. The design of an overall *policy distance* using this *term* thus is more involved.

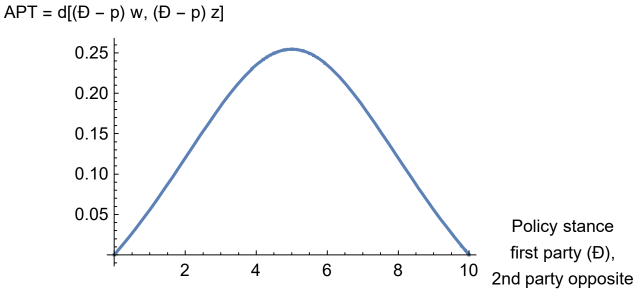
Some technical issues

The graphical plots below tend to use opposite formulas, notably *z* = {*t,* Đ - *t*}, with *t* for the seats of the first party, and *w* = Đ - *z.* Yet we must be aware that *w* and *z* actually have some independence.

We will tend to use *p* = {*r*, Đ - r} with *r* the "right position of the first party". When *r* ranges over [0, Đ] then *p* runs from *opposite policy positions* to *equal policy positions*, to *opposed policy positions* again. Yet we should keep in mind that values like *p* = {7, 6} would be possible too.

When *d*[*p w, p z*] *given w and z* works as policy congruence, then perhaps *d*[(Đ - *p*) *w,* (Đ - *p*) *z*] might work as policy incongruence ? Indeed input Đ - *p* would run in the opposite direction of *p.* Or use *d*[*w* / *p, z* / *p*]. However, when *p* goes from *opposite* to *equal* to *opposite*, then Đ - *p* and Đ / *p* do the same. Using Abs[Đ/2 - *p*] would eliminate the left-to-right scale.

```
Plot[policy = {r, Đ - r}; AngularID[(Đ - policy) votes3, (Đ - policy) seats3] ,
 {r, 0, Đ}, AxesLabel → {"Policy stance\nfirst party (Đ),\n2nd party opposite",
   "APT = d[(Đ - p) w, (Đ - p) z]"}]
```



5. Basics of the Angular Policy Distance (APD)

Geometric mean of independent factors

The above mentioned the matrix of policy differences *P* = *p* - *p'*. A *policy distance* on *P* is the Euclidean norm, corrected for double counting (factor 2) and the number of parties (above 2). I write || *p* - *p'* || though I haven't checked all conditions of a formal norm. The following norm has basically the same scale as *p,* though there might be outliers.

```
? PolicyNorm
```

PolicyNorm[p] scores the differences in policy positions p. Let m = Outer[Plus, p, – p] and n = Length[p], then output is Sqrt[m^2 / (2 (n–1)) // Flatten // Total]. This corrects for double counting and the number of parties. Output has basically the same scale as p

For example, when there are four parties, and the first is fully rightist and the others are fully leftist, then the matrix of differences is as follows, and the policy norm is 10.

```
policy = {10, 0, 0, 0}; Outer[Plus, policy, -policy] // MatrixForm
```

$$\begin{pmatrix} 0 & 10 & 10 & 10 \\ -10 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 \end{pmatrix}$$

```
PolicyNorm[policy]
```

10

For example, when *p* = {*r*, 1 - *r*} with *r* the position on a left-to-right scale [0, 1] for the first party:

```
PolicyNorm[{r, 1 - r}] // FullSimplify
```

$$\sqrt{(1-2\,r)^2}$$

For example, when *p* = {*r*, Đ - *r*} with *r* the position on a left-to-right scale [0, 10] for the first party:

```
PolicyNorm[{r, Đ - r}] // FullSimplify
```

$$2\sqrt{(-5+r)^2}$$

This policy norm might violate the [0, Đ] range but such outliers can be neglected.

```
PolicyNorm[{10, 10, 0, 0}] // N
```

```
11.547
```

Appendix D discusses putting the policy norm on top of *d*[*w, z*].

This PolicyNorm neglects the weights of votes and seats. We can multiply the norm on policy with the angular distance on votes and seats. Multiplying values in [0, 1] with values in [0, 1] generates small numbers, while multiplying values in [0, 10] and [0, 10] generates high numbers. Thus we take the geometric mean as an operation that is neutral to the scale. See the following subsection for a plot.

```
? IndepAngPolicyDistance
```

IndepAngPolicyDistance[v, s, p] gives Sqrt[AngularID[v, s] * PolicyNorm[p]]
IndepAngPolicyDistance[Plot3D] gives a 3D plot for s = {t, Đ – t}, v = Đ – s, and p = {r, Đ – r}

```
IndepAngPolicyDistance[{4.9, 5.1}, {5.1, 4.9}, {r, Đ - r}] // FullSimplify //
  PowerExpand // N
```

$0.713602 \sqrt{-5. + r}$

This independent combination neglects interaction though.

A proposed angular policy distance (APD)

To include interaction between *w, z* and *p* we combine the measure *i* of independent factors with (Đ - *i*) times a measure *m* = *d*[*v, s*] - *d*[*p v, p s*].

(1) When *d*[*v, s*] ⩾ *d*[*p v, p s*] then this still works as a distance measure for *v* and *s,* given *p.*

(2) When *d*[*p v, p s*] is a policy congruence term given *v* and *s,* and when *x* is independent of *p,* then *x* - *d*[*p v, p s*] gives incongruence.

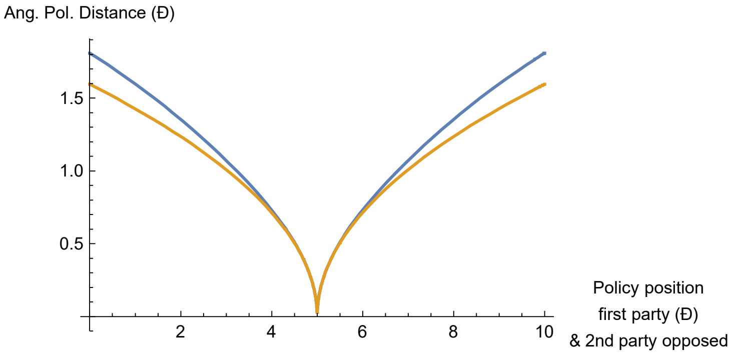
Henceforth the proposed angular policy distance: APD = *i* + (Đ - *i*) *m* / Đ. The interaction is limited, yet this will be larger when the independent *i* is close to 0. It appears that *m* may be negative, but this value falls away against *i.* We first scale to 1 and then to Đ for output.

```
? AngularPolicyDistance
```

AngularPolicyDistance[v, s, p] combines i = IndepAngPolicyDistance[v, s, p] / Đ, d = AngularID[v, s] / Đ, t = AngularPolicyTerm[v, s, p] / Đ. Output is Đ (i + (1 – i) (d – t)). PM. It may be that d – t < 0 but still 0 <= APD <= Đ

The following plots the proposed *angular policy distance* (APD) (blue) and the underlying measure of independent factors (yellow). Given that the votes and seats are relatively close together in this example, the maximal distance can be only about 1.5 grades or 15% instead of 100% (of the 90 degree angle). It appears that the interaction is only relevant for the larger differences in policy positions.

```
Plot[{AngularPolicyDistance[{4.9, 5.1}, {5.1, 4.9}, {r, Đ - r}],
  IndepAngPolicyDistance[{4.9, 5.1}, {5.1, 4.9}, {r, Đ - r}]
 }, {r, 0, Đ},
 AxesLabel → {"Policy position\nfirst party (Đ)\n& 2nd party opposed",
   "Ang. Pol. Distance (Đ)"}]
```



The above plot assumes symmetry. Let us consider non-symmetry. Let *Party A* lose the (half) elections in terms of the popular count but still win the House with a remarkable majority. We expect a sizable difference, unless the policy positions are the same with *Party B*. Let us assume that *B* has a policy position at 7. The following plots the value of the measure for various positions of *Party A*. We find the value 0 exactly at 7.

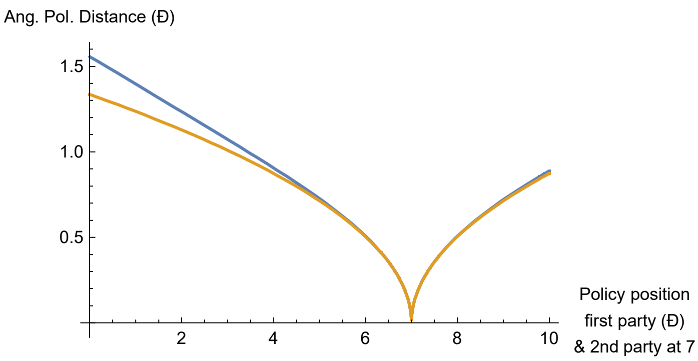
```
FindMinimum[IndepAngPolicyDistance[{4.9, 5.1}, {6, 4}, {polpos, 7}], {polpos, 4}]
```

••• FindMinimum: The line search decreased the step size to within the tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the function. You may need more than MachinePrecision digits of working precision to meet these tolerances.

$\{4.95824 \times 10^{-8}, \{polpos \to 7.\}\}$

The minimum that we found holds for both functions of course.

```
Plot[{AngularPolicyDistance[{4.9, 5.1}, {5.1, 4.9}, {r, 7}],
  IndepAngPolicyDistance[{4.9, 5.1}, {5.1, 4.9}, {r, 7}]
 }, {r, 0, Đ}, AxesLabel →
  {"Policy position\nfirst party (Đ)\n& 2nd party at 7", "Ang. Pol. Distance (Đ)"}]
```



6. Properties and application of the APD

The US House 2016

The following two applications of the APD have already been shown in the graph in the *Introduction.*

Assume the policy scores of {4, 6, 5} for the Dem, Rep and Other parties on a left-to-right scale. Then the *voter-legislative* distance would be around 1.1 on a scale of [0, 10], or 11% of the full 90 degree angle.

```
AngularPolicyDistance[votes, seats, {4, 6, 5}] // N
```

```
1.13657
```

Assuming that the Rep have the ruling coalition, the *voter-executive* distance *dp*[*w, c z, p*] generates outcome 3.8 on a scale of [0, 10], or 38%.

```
? SingleMajority
```

SingleMajority[s] uses z = s/Total[s] and returns
a vector, with 1 at a position i when z[[i]] > 1/2, and 0 elsewhere

```
coalition = SingleMajority[seats]
```

```
{0, 1, 0}
```

```
AngularPolicyDistance[votes, {0, 1, 0} seats, {4, 6, 5}] // N
```

```
3.80697
```

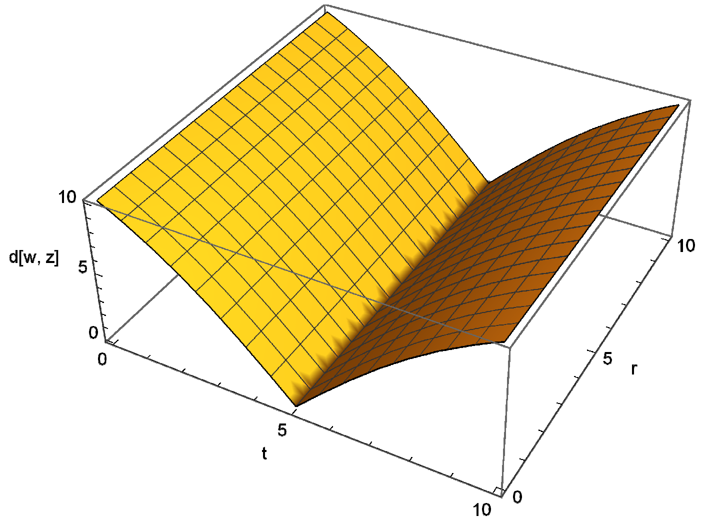
The former section already introduced the *angular policy distance* (APD), but let us look closer at its construction.

Plot of *d*[*w, z*]

The following plots get us acquainted with the relevant 3D space.

The above gave the 2D plot of the angular distance *d*[*w, z*]. In 3D, it is independent of *p.*

```
Plot3D[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r};
  AngularID[w, z]], {t, 0, Ð}, {r, 0, Ð}, AxesLabel → {"t", "r", " d[w, z]"}]
```



Plot of Sqrt[*d*[*w, z*] * ||*p* - *p'*||]

We first look into the policy distance created by independent factors.

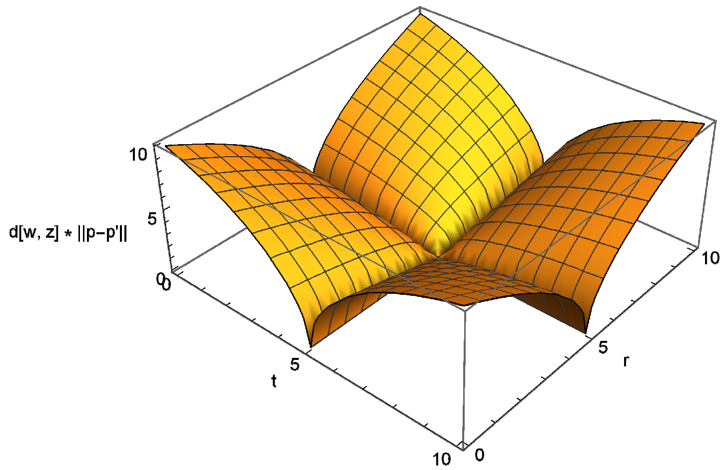
The following plots Sqrt[*d*[*w, z*] * ||*p* - *p'*||], or the geometric mean of *d*[*w, z*] and the norm of the policy differences between the parties ||*p* - *p'*||. This geometric mean provides a measure that combines *disproportionality in votes and seats* with *policy differences*. There are no interaction terms.

```
? IndepAngPolicyDistance
```

IndepAngPolicyDistance[v, s, p] gives Sqrt[AngularID[v, s] ⋆ PolicyNorm[p]]

IndepAngPolicyDistance[Plot3D] gives a 3D plot for s = {t, Ð – t}, v = Ð – s, and p = {r, Ð – r}

```
Plot3D[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r};
  IndepAngPolicyDistance[w, z, p]], {t, 0, Ð},
 {r, 0, Ð}, AxesLabel → {"t", "r", " d[w, z] ⋆ ||p-p'|| "}]
```



Two properties are:

(ad P.1) When $w = z$ then we get $d[w, z] = 0$, which fits P.1.

(ad P.2) When $p = \lambda\, 1$ then we get 0 too, which fits P.2.

These properties also apply to nonsymmetric cases, see the former section (with the 2nd party at position 7).

Plot of *dp*[*w, z, p*] or APD

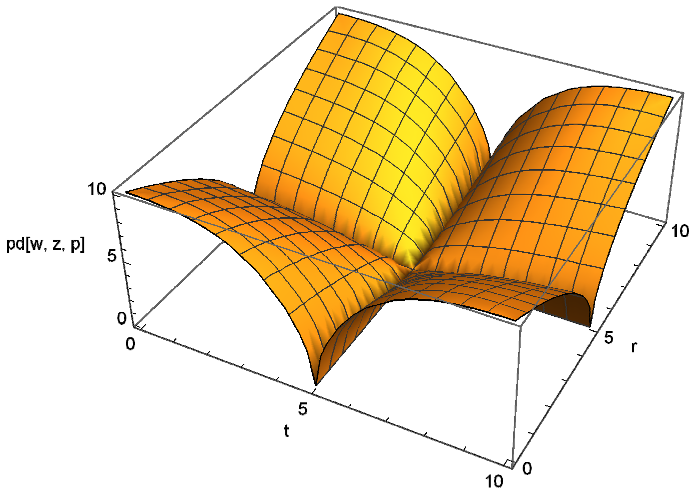
The proposed angular policy distance APD combines the independent measure *i* with a dependent measure *m* checked by (1 - *i*), i.e. APD = *i* + (1 - *i*) *m.* The latter formula uses unitised variables, and thus must be corrected for the scale [0, Đ].

Here we only plot APD and then the next section will discuss the construction of *m.*

```
? AngularPolicyDistance
```

AngularPolicyDistance[v, s, p] combines i = IndepAngPolicyDistance[v, s, p] / Đ, d = AngularID[v, s] / Đ, t = AngularPolicyTerm[v, s, p] / Đ. Output is Đ (i + (1 – i) (d – t)). PM. It may be that d – t < 0 but still 0 <= APD <= Đ

```
Plot3D[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
 AngularPolicyDistance[w, z, p], {t, 0, Đ},
 {r, 0, Đ}, AxesLabel → {"t", "r", "pd[w, z, p] "}]
```



Relating Δ*s* to Δ*p*

We want to be able to say that Δ*s* is equivalent to some Δ*p,* with equivalence now judged by having the same value of *pd*[*v, s, p*]. A statement like this is rather artificial, since its meaning depends upon our measures. However, once these notions are commonly accepted then this allows for such communication. Below we will look at different values of *s* and the resulting outcome of the angular policy distance.

Let us use the US House of Representatives (half) election results of 2016. Let us assume a policy stance of {4, 6, 5}.

```
votes = {dem = 61 776 554, rep = 63 173 815, 128 627 010 - rep - dem};
seats = {194, 241, 0};
{w = UnitĐ[votes], z = UnitĐ[seats]} // N
```

```
{{4.80277, 4.9114, 0.285837}, {4.45977, 5.54023, 0.}}
```

```
policy = {4, 6, 5};
{AngularID[w, z], PolicyNorm[policy],
  pdorg = AngularPolicyDistance[w, z, policy]} // N
```

```
{0.66848, 1.73205, 1.13657}
```

The (angular) disproportionality of votes and seats is 0.67 grade points. The policy norm is 1.7 (approximately comparable to grade points). The angular policy distance is the geometric mean of these, 1.1 grade points on a scale of [0, 10].

Let us assume Δ*s* = {10, -10, 0} shift in *seats* in favour of the Democrats.

```
newseats = seats + {10, -10, 0}
```

```
{204, 231, 0}
```

```
{pdnew = AngularPolicyDistance[votes, newseats , {4, 6, 5}], pdnew - pdorg} // N
```

```
{0.877463, -0.259108}
```

```
eq = AngularPolicyDistance[votes, seats, {4, 6 + dp, 5}] ==
   AngularPolicyDistance[votes, newseats, {4, 6, 5}] ;
```

```
sol = FindRoot[eq, {dp, 0}]
```

```
{dp → -0.832026}
```

```
6 + dp /. sol
```

```
5.16797
```

Thus the equivalent of 10 seats more for the Democrats would be a shift in the policy stance of the Republicans of almost 1 grade to the left, from 6 to 5.2.

It would be stronger to argue for EPR though, when the distances would disappear.

```
AngularID[w, w] == AngularPolicyDistance[w, w, policy]
```

```
True
```

Of course this only considers the *voter-legislative* distance. Also in EPR there still is a *voter-executive* distance.

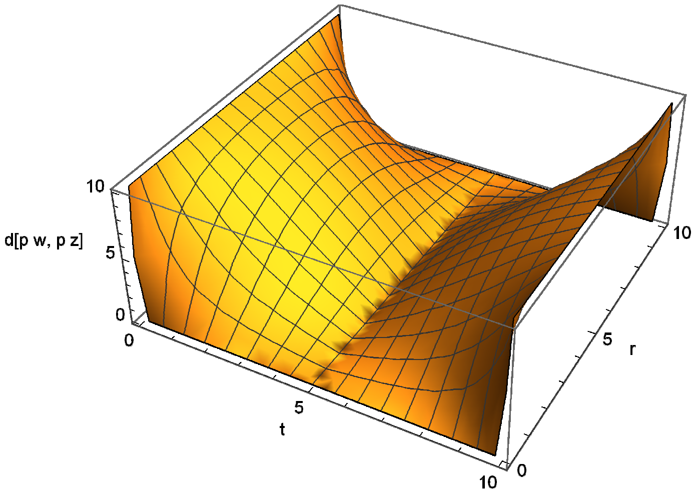
7. Focusing on *d*[*p w, p z*] (APT)

Comparing the APD with APT *d*[*p w, p z*]

Having the APD *pd*[*w, z, p*], we can look now at the problematic APT *d*[*p w, p z*], using the APD as a yardstick, or as a proper distance measure that is sensitive to both policy and votes and seats.

The plot of *d*[*p w, p z*] provides a challenge.

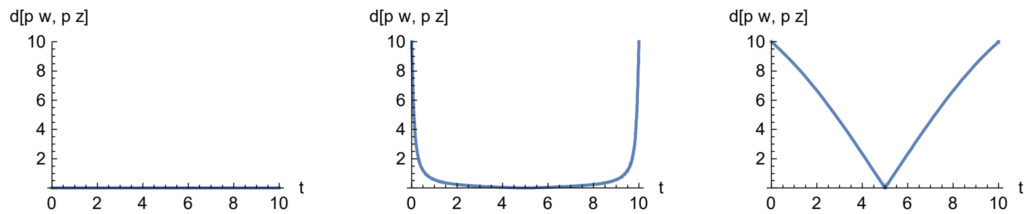
```
Plot3D[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r};
  AngularID[p w, p z]], {t, 0, Ð}, {r, 0, Ð}, AxesLabel → {"t", "r", " d[p w, p z] "}]
```



The 3D plot can be supported by some sections at different values of *r*.

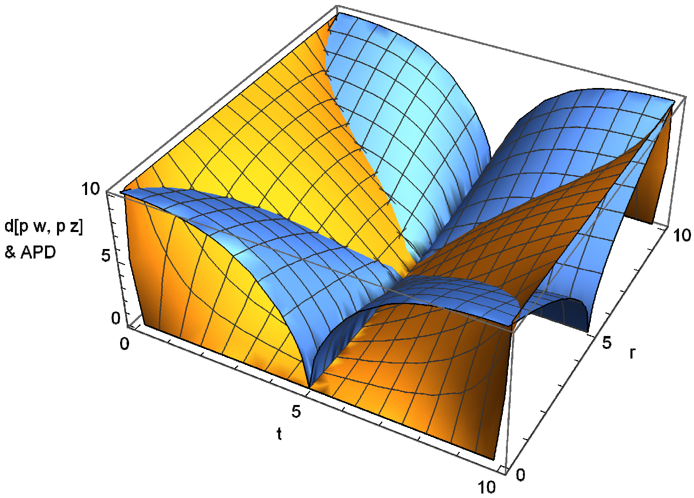
```
plaid[rval_] :=
 Plot[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r} /. r → rval;
   AngularID[p w, p z]], {t, 0, Ð},
  AxesLabel → {"t", "d[p w, p z] "}, PlotRange → {0, Ð}]
```

```
GraphicsGrid[{ plaid /@ {0, .1, 4} }]
```



Let us confront *d*[*p w*, *p z*] (yellow) with the APD yardstick (blue), and remember that the APD is 0 for $w = z$ and $p = \lambda\, 1$.

```
Plot3D[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r};
  {AngularID[p w, p z],
   AngularPolicyDistance[w, z, p]}
 ], {t, 0, Ð}, {r, 0, Ð}, AxesLabel → {"t", "r", " d[p w, p z]\n & APD "}]
```



We evaluate the properties of the APT as follows:

(ad P.1) When $w = z$ then the value becomes 0 for any *p.* This is as desired.

(ad P.2) When all parties have the same position, then $p = \lambda\, 1$, and $d[p\ w, p\ z] = d[w, z] \geqslant 0$. The latter will only be 0 when $w = z$.

(ad P.3) When *p* is extreme like {0, 10} for two parties, then the value becomes 0. Since only one party is selected, this tends to follow the logic of disproportionality *d* for any *w* and *z.*

```
votes3 = {4.9, 5.1};
seats3 = {5.1, 4.9};
```

```
policy = {0, 10}; AngularID[policy votes3, policy seats3] (* APT *)
```

```
0.
```

However, the latter runs counter to the yardstick of the angular policy distance APD.

```
AngularPolicyDistance[votes3, seats3, {0, 10}] // N
```

```
1.80965
```

In sum, the APT $d[p\ w, p\ z]$ satisfies the property of being 0 on $w = z$, but it fails on the other conditions for a policy distance measure.

A plot of $m = d[w, z] - d[p\ w, p\ z]$ and its square root, for symmetric cases

There is the difference $m = d[w, z] - d[p\ w, p\ z]$. The following is still in the range [0, 10], but see below for negative outcomes for asymmetric cases.

Measure *m* is interesting because:

(i) Looking at $m[w, z\,;\, p]$, we see a difference between two disproportionality measures for *w* and *z,* given *p,* and we may still have disproportionality on balance, especially when $m \geqslant 0$.

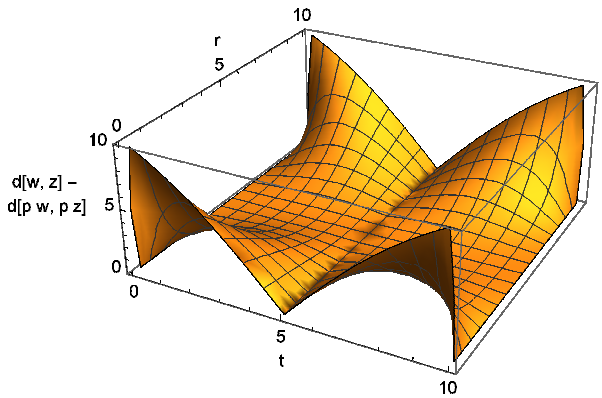
(ii) Looking at $m[p\,;\, w, z]$, the negative sign of $- d[p\ w, p\ z]$ turns the policy congruence, given *w* and *z,* into policy incongruence.

For symmetric input, *m* satisfies P.1 (*w* = *z* implies 0) and P.2 (*p* = λ 1 implies 0). For P.3 at the extreme positions like *z* = {0, 10} and {10, 0} we find outcome 0 for any *p*, which runs counter to nonzero values for the APD.

Thus we see a graph that is more in accordance with the APD. What spoils the situation, however, are: (iii) the extreme values of *w* and *z* (P.3), and (iv) asymmetric cases (P.4) (see below).

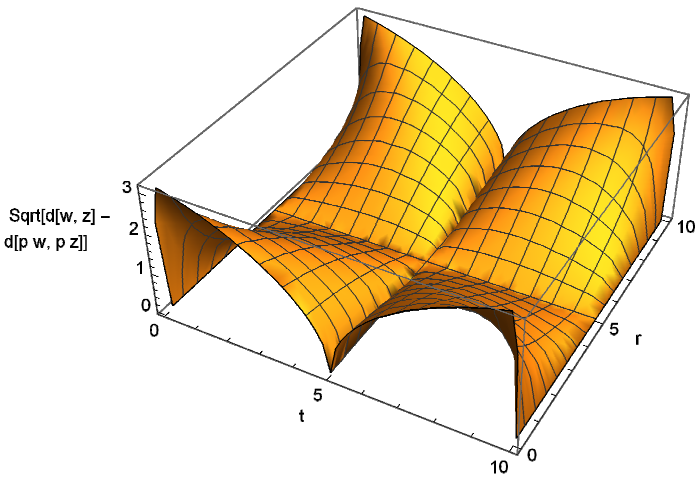
Because of the latter drawbacks, we would tend to discard this *m,* but there is a potential application in **Appendix C**, and hence we continue looking at its properties.

```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  AngularID[w, z] - AngularID[p w, p z]], {t, 0, Đ},
 {r, 0, Đ}, AxesLabel → {"t", "r", " d[w, z] - \nd[p w, p z] "}]
```



We get a more pronounced profile by using Sqrt[*m*] = Sqrt[*d*[*w, z*] - *d*[*p w, p z*]]. PM. There is an effect of scaling, and we actually should use Đ Sqrt[*m* / Đ]. The following plot doesn't have this correction, and we find a maximal value around 3 only.

```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  Sqrt[AngularID[w, z] - AngularID[p w, p z]]], {t, 0, Đ},
 {r, 0, Đ}, AxesLabel → {"t", "r", " Sqrt[d[w, z] - \nd[p w, p z]] "}]
```



PM. At the extremes *z* = {0, 10} or {10, 0} both *d*[*w, z*] and *d*[*p w, p z*] are 10, and thus their difference is 0, except when *p* is extreme at {0, 10} of {0, 10} when the angular measure is undefined, when {0, 10} {10, 0} generates input {0, 0}.

```
policy = {0, 10}; AngularID[policy {0, 10}, policy {10, 0}]
```

```
Indeterminate
```

For that reason, the following routine checks upon such extreme cases, and returns a proper 1.

```
? AngularPolicyTerm
```

AngularPolicyTerm[v, s, p] gives AngularID[p v, p s], controlling for the effects (in this order): (1) if v == s, return 0, (2) when p v or p s is {0, ..., 0}, return Đ

```
AngularPolicyTerm[{0, 10}, {10, 0}, policy]
```

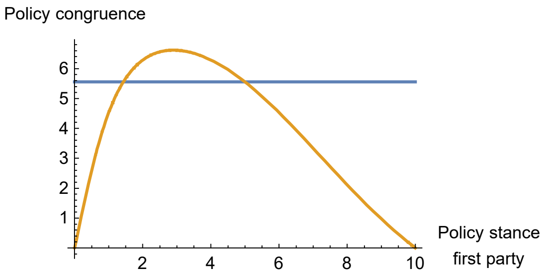
```
10
```

Asymmetric cases for *m* = *d*[*w*, *z*] - *d*[*p w*, *p z*]

The above assumes symmetry. For asymmetric input *m* = *d*[*w*, *z*] - *d*[*p w*, *p z*] may become negative. We may show this by plotting *d* for the two elements and then observe that *d*[*w*, *z*] does not touch *d*[*p w*, *p z*] at its maximum. Remember that the APT given *w* and *z* represents policy congruence.

```
votes5 = {9, 1} ; seats5 = {4, 6};
```

```
Plot[Evaluate[p = {r, Đ - r};
  {AngularID[votes5, seats5],
   AngularID[p votes5, p seats5]
  }], {r, 0, Đ}, AxesLabel → {"Policy stance\nfirst party", "Policy congruence"}]
```



A general routine for *M* = *max* - *d*[*p w*, *p z*]

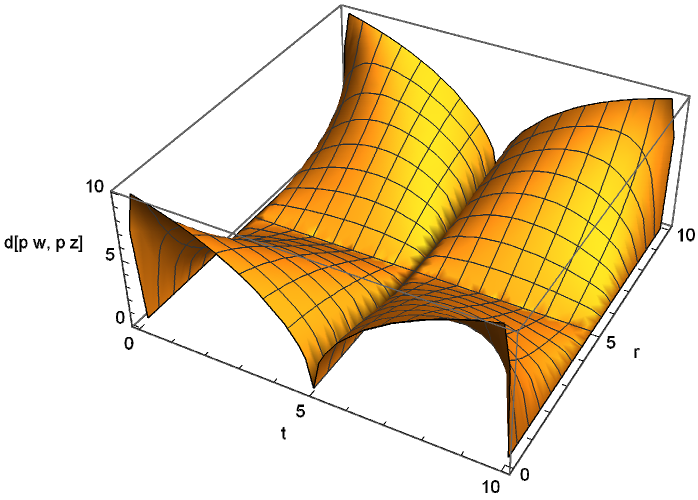
Given *v* and *s*, we can determine the local maximum of *d*[*q v*, *q s*] with *q* in the unit simplex. Subsequently we can define a *level* and a *relative* measure *M*, that uses this maximum, to be distinguished from *m*. Let us also introduce an exponent *f* with default *f* = 1/2 (for the square root).

(i) APM[*v*, *s*, *p*] = (*max* - *d*[*p v*, *p s*])^*f*, with *max* the maximal value of *d*[*q v*, *q s*] with *q* in the unit simplex. For symmetric cases *max* = *d*[*v*, *s*], but this will be different for asymmetric cases.

(ii) RPM[*v*, *s*, *p*] = (1 - *d*[*p v*, *p s*] / *max*)^*f*. **Appendix C** discusses how this RPM can be combined with the weighted average *a*.

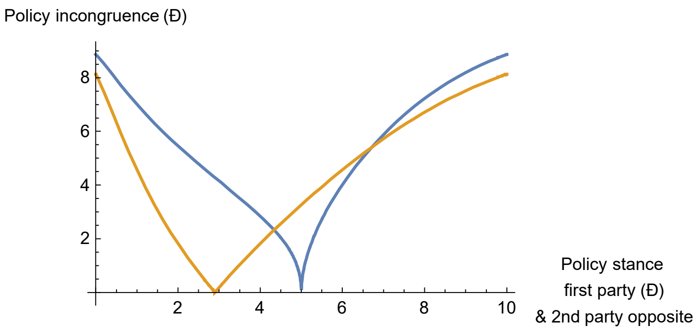
The following plot uses the routine AngularPolicyMeasaure, and generates Sqrt[*d*[*w*, *z*] - *d*[*p w*, *p z*]] as in the above, though now corrected for (1) the use of a maximum, (2) the proper scale Đ.

```
Plot3D[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r} ;
 AngularPolicyMeasure[ w, z, p], {t, 0, Đ},
 {r, 0, Đ}, AxesLabel → {"t", "r", "d[p w, p z] "}]
```



The following comparison of APD (blue) and APM (yellow) shows the asymmetry. APD is asymmetric itself and APM is fully shifted. Because of the maximum condition, the APM no longer satisfies the rule that $p = \lambda\, 1$ implies 0.

```
Plot[Evaluate[p = {r, Đ - r};
  {AngularPolicyDistance[votes5, seats5, p], (*blue*)
   AngularPolicyMeasure[votes5, seats5, p]  (*yellow*)
  }], {r, 0, Đ},
 AxesLabel → {"Policy stance\n first party (Đ)\n & 2nd party opposite",
   "Policy incongruence (Đ)"}]
```



```
? AngularPolicyMeasure
```

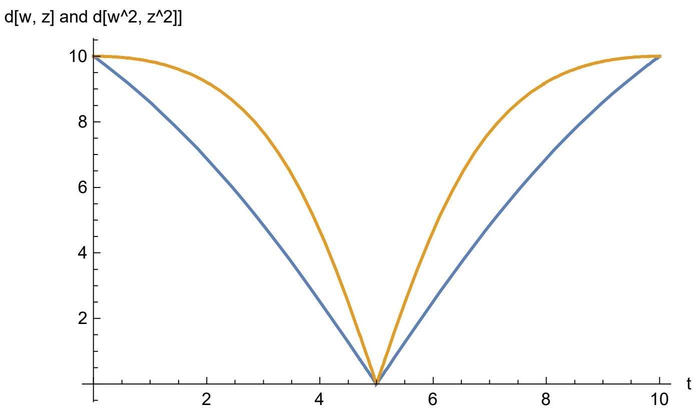
AngularPolicyMeasure[v, s, p] for vectors of votes, seats and policy positions. As f[v, s ; given p] this is disproportionality of votes and seats, and as g[p ; given v and s] this is policy incongruence. For symmetric cases the APM = 0 when v = labda s and/or p = mu 1. The option Level –> True | False gives the level outcome, otherwise the relative (default True). Let APT[q] = AngularID[q v, q s]. Let apt = APT[p]. Let mx = Max[APT[q]] with q over the unit simplex, given v and s. With symmetry mx = AngularID[v, s]. The level outcome is Đ ((mx – apt)/Đ)^fct. The relative output is (1 – apt/mx)^fct, with fct taken from the Exponent option (default fct = 1/2). See AngularPolicyMeasure["Results"]
Option Symbol –> True | False controls how the maximum is determined. If False: application of FindMinimum on APT[q]. If True (default): Option Max –> mx allows the choice: A numeric mx is used directly, mx = Automatic uses mx = AngularID[v, s], and another value calls FindMinimum which works for numeric v and s. A numerical value Max –> Đ would turn the graph upside down. PM. See AngularPolicyDistance (APD) that always satisfies APD = 0 for v = labda s and/or p = mu 1

Potential relation between *d*[*p w*, *p z*] and *d*[*w*, *p*] and *d*[*z*, *p*]

We might get some information from the angles between the weights and the policy positions, namely *d*[*w*, *p*] and *d*[*z*, *p*]. **Appendix A** looks closer into the potential use of *d*[*w*, *p*] and *d*[*z*, *p*].

It is not quite possible to argue that when *p* is similar to *w* and *z*, or *d*[*w*, *p*] ≈ 0 ≈ *d*[*z*, *p*], then *d*[*p w*, *p z*] ≈ *d*[*w*, *z*] too. The reason is that there might be quite some difference between *d*[*w*, *z*] and *d*[*w*^2, *z*^2].

```
Plot[Evaluate[z = {t, Đ - t}; w = {Đ - t, t};
  {AngularID[w, z], AngularID[w^2, z^2]}],
 {t, 0, Đ}, AxesLabel → {"t", " d[w, z] and d[w^2, z^2]] "}]
```



8. The weighted average policy distance (WA)

The weighted average as an indicator

A measure of policy distance between the House and the electorate is the weighted average:

$a = q'z - p'w = q \,.\, z - p \,.\, w$, using different notations of the improduct, assuming *unitised* variables.

This may also be read as $a = z \,.\, P \,.\, w$, in which $P = q - p'$ is the matrix of policy differences.

Assuming $p = q$ for our case, gives $a = p'z - p'w = p'(z - w) = p \,.\, (z - w)$.

The weighted average can have negative outcomes. It becomes a distance for Abs[a] = Abs[$p \,.\, (z - w)$]. The outcome $a = 0$ means high congruence and zero incongruence. Thus Abs[a] is a *measure of incongruence* and 1 - Abs[a] is a *measure of policy congruence*. We use Đ Abs[a] to compare with APD, or we use w and z in [0, 1] and p in [0, Đ].

The weighted average is zero, or $a = 0$:

(ad P.1) when $z = w$,

(ad P.2) when all parties have the same policy position, since $p = \lambda\, 1$ gives $\lambda\, 1'(z - w) = \lambda\,(1'z - 1'w) = 0$.

The discussion on the policy distance a thus may also be less relevant when the electoral system aspires at equality / proportionality (EPR), or $z \approx w$. This holds more in general for any distance measure, e.g. when (angular) distance $d[p\, v, p\, s]$ is used, since $d[p\, w, p\, w] = 0$ too.

The issue gets more perspective when $q \neq p$. Then: (ad P.1) if $z = w$ then $a = q'z - p'w = (q' - p')\, w$ and (ad P.2) if $p = \lambda\, 1$ and $q = \mu\, 1$, then $a = q'z - p'w = \mu - \lambda$. In the case of EPR these are relatively simple measures, and the complexity in DR is to find the w^*.

The use of $d[p\, v, p\, s] = d[p\, w, p\, z]$ fits the weighted average, as we may write $a = 1'(p\, z - p\, w)$. A distinction however is that the weighted average uses the levels of $p\, z$ and $p\, w$, while those levels are immaterial to the angular inequaly / disproportionality distance.

Numerical example and the slope of the line

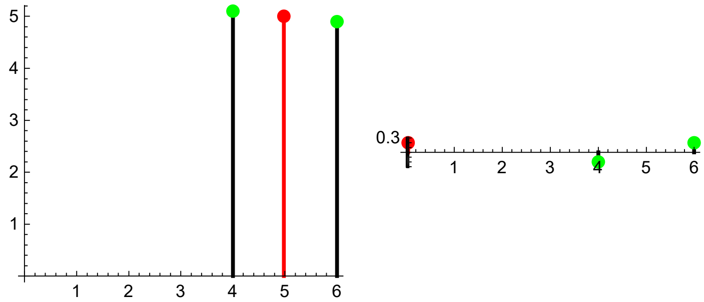
Suppose that *Party A* (“Democrats”) scores 4 on the policy scale and *Party B* (“Republicans”) scores 6, and that the parties do no change position after the (half) election ($p = q$).

```
policy = {4, 6};
votes3 = {5.1, 4.9}; (* unitise for a *)
seats3 = {4.9, 5.1}; (* unitise for a *)
{electorate = policy . votes3 / Đ, house = policy . seats3 / Đ, house - electorate}
```

```
{4.98, 5.02, 0.04}
```

The following is a visualisation of the weighted average. The LHS graph gives the votes, the RHS graph gives the difference between the votes and seats. The horizontal axis gives the policy positions of the parties. The heights of lines at the policy positions (green dots) gives the votes for the parties. The weighted average is in red. Primarily important is its horizontal position. Its height is given as the mean of the absolute values of the weights.

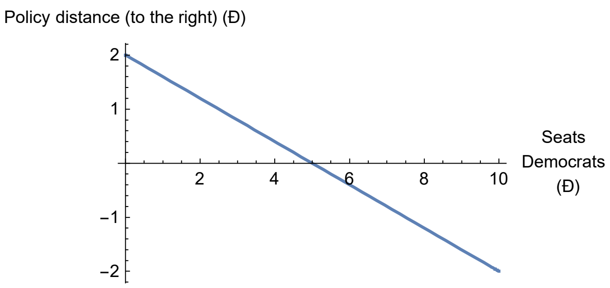
```
GraphicsGrid[{{WAvPlot[votes3, policy], WAvPlot[seats3 - votes3, policy]}}]
```



In this arbitrary example, the electorate takes a position with a weighted average 4.98 slightly below the middle of the scale. The House takes a position 5.02 on average slightly above the middle of the scale. The change from votes to seats causes a shift of the overall policy position to the right. The latter is a dubious statistic, when the Republicans have a majority of 5.1 grades in seats, and thus might opt for a full-out policy at their level of preference on 6 of the "left-to-right" policy scale. This is actually the distinction between the *voter-legislative* and the *voter-executive* distances, mentioned above. Supposedly political competition might force the parties to pursue policy around the middle, see also the Golder & Stramski quote above. In that sense the weighted average is still a useful statistic.

The policy distance between House and electorate, or the weighted average, is a line, of which the slope depends upon the initial policy difference 6 - 4 = 2. When the Democrats begin with all votes and zero seats then the Republicans with all seats can shift the electoral choice at 4 to the policy average at 6. Observe the unitisation of *w* and *z* by Đ.

```
Plot[z = {t, Đ - t}; w = {Đ - t, t}; {4, 6} . (z - w) / Đ, {t, 0, Đ},
 AxesLabel → {"Seats\nDemocrats\n  (Đ)", "Policy distance (to the right)  (Đ)"}]
```

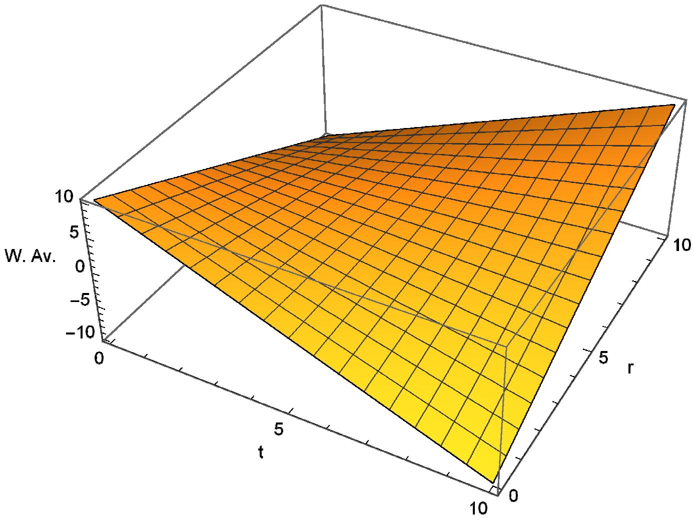


The WA Saddle

For opposing policy stances $p = \{r, Đ - r\}$, with r the position to the right for the first party, we have a line for the weighted average a, of which the slope is determined from the difference $2r$ - Đ.

Assuming opposing scores for votes and seats, the weighted average forms a saddle.

```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  p . (z - w) / Đ], {t, 0, Đ}, {r, 0, Đ}, AxesLabel → {"t", "r", " W. Av."}]
```



Thus US 2016 example

We had these votes and seats for the US House 2016 on a scale of [0, 10].

```
{vts, sts} // N
{{4.80277, 4.9114, 0.285837}, {4.45977, 5.54023, 0.}}
```

The *voter-legislative* distance is only 0.1 grade or 1%.

```
policy = {4, 6, 5};
{electorate = policy . vts / Đ, house = policy . sts / Đ, house - electorate} // N
{5.01086, 5.10805, 0.0971831}
```

The *voter-executive* distance $a = p' (U[c\, z] - w)$ in terms of unitised variables.

```
coalition = SingleMajority[sts]
{0, 1, 0}

majority = UnitĐ[coalition sts]
{0, 10, 0}
```

The voter-executive distance is almost 1 point, or 0.99 grade, or 9.9%. While the electoral average position is at 5, the Republican majority may execute their view at 6.

```
{executive = policy . majority / Đ, executive - electorate} // N
{6., 0.989137}
```

The maximal shift in policy positions from left-to-right is 6 - 4 = 2, and thus the 1 point of the voter-executive distance is quite a distance, almost 50% of the maximum. However, a figure like 50% is not informative itself, since the true meaning depends upon the slope of the line.

Partial differences Δ*p* and Δ*s*

We can compare a slope along one dimension with a slope along another dimension.

Consider the situation $\Delta a = \Delta p \,.\, (z - w) = p \,.\Delta(z - w)$, neglecting crossterm cases. We want to say that Δp is equivalent to some $\Delta(z - w)$. This is not necessarily a very relevant topic for political science but it helps us to understand some of the properties of the weighted average and perhaps policy distance measures in general.

We return to the binary case votes3 and seats3.

For example: if the Democrats would get $ds = 0.1$ gradepoints more seats at the cost of the Republicans, so that the prospective seats would be {5, 5}, then the policy distance would reduce to 0.02. This policy distance would also have been achieved if the Republicans had become less conservative at $5 = 6 + dp$ instead of 6 at the old seats. Thus the change in seats can be seen as quite a change in policy position (given the same reduction of the weighted average). Eliminating Đ on both sides:

```
sol = Solve[{4 , 6 + dp} . (seats3 - votes3) ==
    {4, 6} . ( seats3 + ds {1, -1} - votes3), dp] // Simplify
```

```
{{dp → 0. - 10. ds}}
```

```
% /. ds → 0.1
```

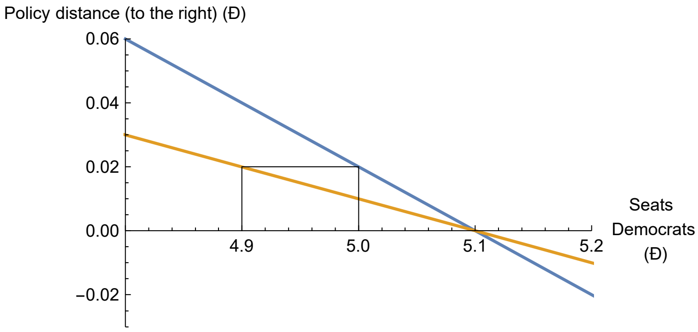
```
{{dp → -1.}}
```

The shift from 6 to 5 would reduce the slope of the weighted average line. The Dems and Reps would be closer together on their positions, and thus the difference in votes and seats would have less of an impact. The policy distance in the new line at the old seats would be the same as the old line at the new seats.

To plot this, we need explicit functions, with 1 the old line (blue) and 2 the new line (yellow).

```
wavlin1[t_] = (z = {t, Đ - t}; w = {5.1, 4.9}; {4, 6} . (z - w) / Đ ) // Simplify;
wavlin2[t_, dp_] =
  (z = {t, Đ - t}; w = {5.1, 4.9}; {4 , 6 + dp} . (z - w) / Đ) // Simplify;
(* the above was: sol = Solve[wavlin2[4.9, dp] ==  wavlin1[4.9 + ds], dp] *)
```

```
dif = -10 ds /. ds → 0.1;
new = 4.9 + ds /. ds → 0.1;
p1 = Plot[{ wavlin1[t], wavlin2[t, dif]}, {t, 0, Đ},
   AxesLabel → {"Seats\n Democrats\n  (Đ)", "Policy distance (to the right) (Đ)"},
   PlotRange → {{4.8, 5.2}, {-.03, .06}}];
p2 = Graphics[{ Line[ {{new, 0}, {new, wavlin1[new]} ,
      {4.9, wavlin2[4.9, dif]}, {4.9, 0} }] }];
Show[
 p1,
 p2]
```



9. Relation of the WA to ALHID and APD

The absolute distance measure

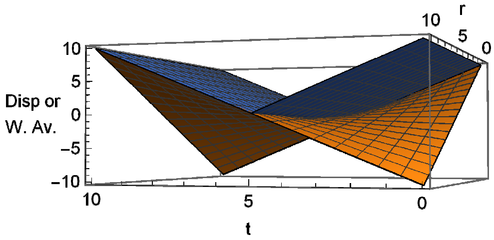
If the weighted average $a = p \,.\, (z - w)$ is a sound measure for the policy distance between electorate and House, then Abs[z - w] might be seen as a good measure for disproportionality for the votes and seats. We are reminded of the Loosemore-Hanby (ALHID) measure. The measure gives the share of displaced seats, say the grades in a House with 10 seats, corrected for double counting.

Observe the distinction between **Abs[*p* . (*z* - *w*)]** and ***p* . Abs[*z* - *w*]** (not shown) and **ALHID[*p w*, *p z*]** (shown in **Appendix B**, and with internal unitisation of *p w* and *p z*). We will only plot the first here.

Comparing the WA Saddle with ALHID[*w*, *z*]

The combination of the plots for the Loosemore-Hanby measure (going from 2D to 3D) and the policy distance saddle shows that Loosemore-Hanby is always larger than or equal to the weighted average saddle. This holds for this arrangement of two parties with opposite policies, and we may infer that this holds in general. ALHID already plots at range [0, Đ] and we correct a for it too.

```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
   {p . (z - w) / Đ, AbsLoosemoreHanbyID[w, z]}],
  {t, 0, Đ}, {r, 0, Đ}, AxesLabel → {"t", "r", "Disp or\nW. Av."}]
```



The Loosemore-Hanby distance measure is less attractive as an indicator of disproportionality, see the discussion above. Whatever that be, we still regard the weighted average for the policy stance as a useful indicator. Above properties for the weighted average remain true whether we call Abs[*z* - *w*] disproportionality or not.

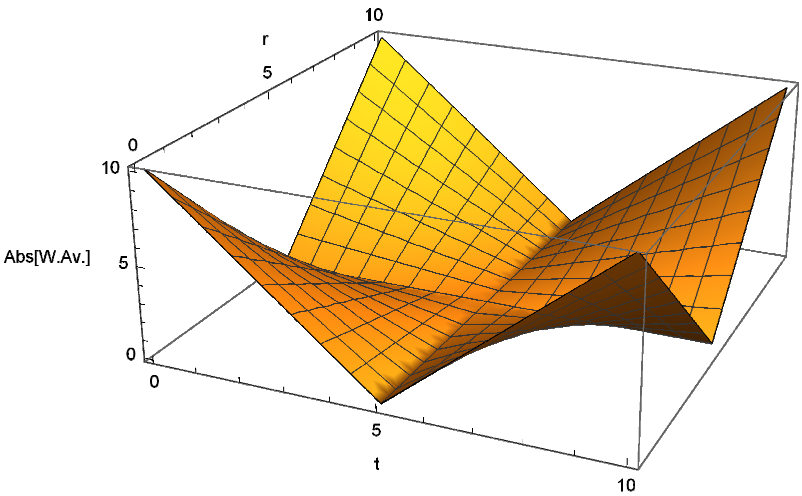
Abs[*p* . (*z* - *w*)]

The absolute value of the weighted average Abs[*a*] = Abs[*p* . (*z* - *w*)] shows incongruence for policy and disproportionality for votes and seats. The measure satisfies:

(ad P.1) When $w = z$ then Abs[*a*] = 0.

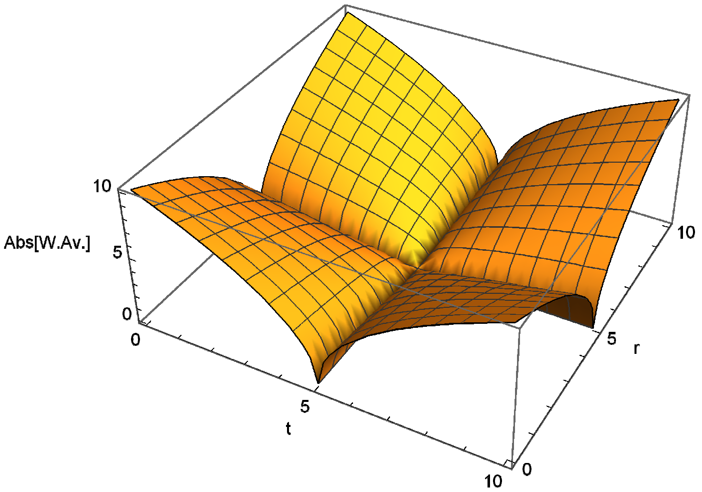
(ad P.2) When $p = \lambda\, 1$ then Abs[*a*] = 0.

```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r} ;
  Abs[p . (z - w) / Đ]], {t, 0, Đ}, {r, 0, Đ}, AxesLabel → {"t", "r", "Abs[W.Av.] "}]
```



The measure can be made more sensitive to smaller values by taking the square root. Observe that the correction for Đ now causes that it is mentioned in the variables themselves.

```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r} ;
  Sqrt[Abs[p . (z - w)]]], {t, 0, Đ},
 {r, 0, Đ}, AxesLabel → {"t", "r", "Abs[W.Av.] "}]
```



The following repeats the two perspectives that we had for the angular distance, though specifying the formulas that use Abs.

(ad P.1) We may regard *f*[*w, z* ; *p*] = Abs[*p* . (*z* - *w*)] as a function of *w* and *z,* given some *p.* The greater the distance between *w* and *z* the more spread out *f*[*w, z* ; *p*]. Also, *w* = *z* will cause an outcome of zero, Abs[*a*] = 0. This means that *f*[*w, z*] given *p* behaves like a disproportionality measure *d*[*w, z*].

(ad P.2) We may regard *g*[*p* ; *w, z*] = Abs[*p* . (*z* - *w*)] as a function of *p,* given some *w* and *z.* When *p* = *λ* 1 then Abs[*p* . (*z* - *w*)] = *λ* Abs[1 . (*z* - *w*)] = 0. The more extreme *p* (with opposite values *r* and Đ - r here, with ultimately *p* = {0, 10} or {10, 0}), the higher the maximum of *g*[*p*]. This means that *g*[*p*] is a measure of policy incongruence.

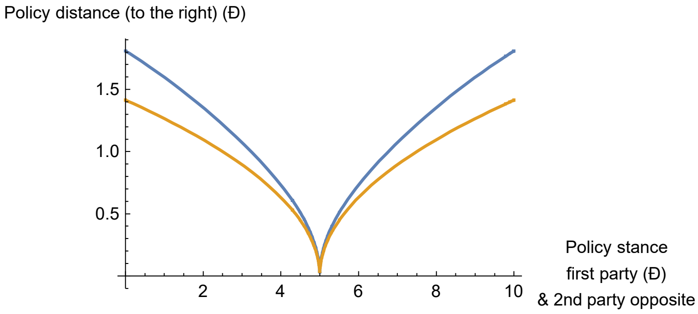
Thus from one perspective (votes and seats) we have *disproportionality* and from another perspective (policy stances) we have *incongruence*.

For all practical purposes, it seems that Abs[*a*] seems a useful measure with a clear interpretation. The normal value *a* also includes the direction of change.

Comparing Sqrt[Abs[*a*]] and the APD

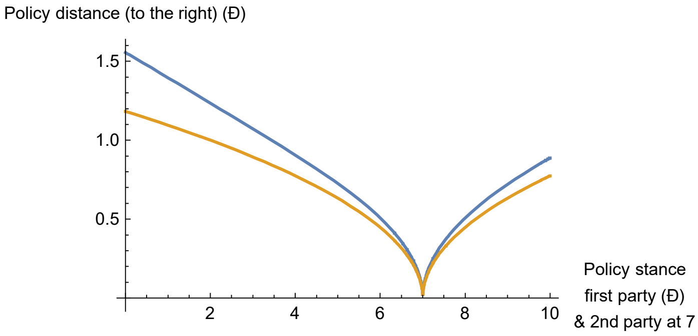
The following is a comparison between APD (blue) and Sqrt[Abs[*a*]] (yellow).

```
Plot[Evaluate[p = {r, Đ - r};
  {AngularPolicyDistance[votes3, seats3, p], Sqrt[Abs[p . (seats3 - votes3)]]
  }], {r, 0, Đ},
 AxesLabel → {"Policy stance\nfirst party (Đ)\n& 2nd party opposite",
   "Policy distance (to the right) (Đ)"}]
```



The following fixes the second party at 7, which allows less variation and generates a slightly lower outcome.

```
Plot[Evaluate[p = {r, 7};
  {AngularPolicyDistance[votes3, seats3, p], Sqrt[Abs[p . (seats3 - votes3)]]
  }], {r, 0, Đ}, AxesLabel → {"Policy stance\nfirst party (Đ)\n& 2nd party at 7",
   "Policy distance (to the right) (Đ)"}]
```



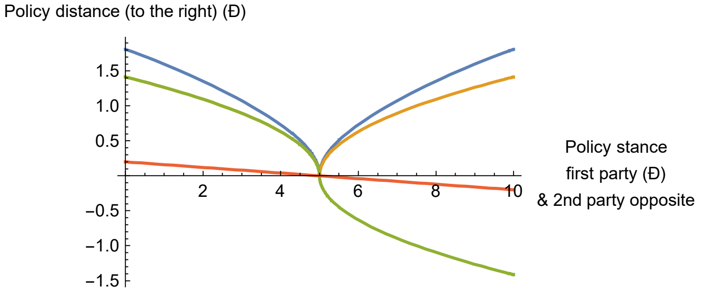
Retaining the sign of *a* for Sqrt[Abs[*a*]]

It is an idea to retain the sign of *a*. The following plots *a* (red), the square root of its absolute value (yellow), the combination with the sign (green), and the APD for comparison (blue).

```
? WAvSqrtPolicyDistance
```

WAvSqrtPolicyDistance[v, s, p] first calculates the weighted average wa = WAvPolicyDistance[v, s, p] and then puts out Sign[wa] Đ Sqrt[Abs[wa] / Đ], assuming that p is in [0, Đ]

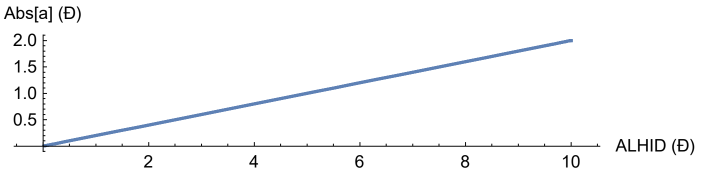
```
Plot[Evaluate[p = {r, Đ - r}; wa = p . (seats3 - votes3) / Đ;
  {AngularPolicyDistance[votes3, seats3, p], Đ Sqrt[Abs[wa] / Đ],
   WAvSqrtPolicyDistance[votes3, seats3, p], wa
  }], {r, 0, Đ},
 AxesLabel → {"Policy stance\nfirst party (Đ)\n& 2nd party opposite",
   "Policy distance (to the right) (Đ)"}]
```



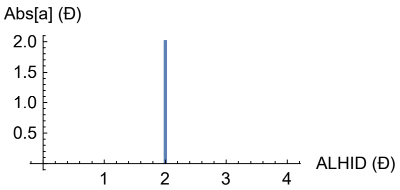
Parametric plot of Abs[*p* . (*z* - *w*)] and ALHID

The above 3D plot already showed how Abs[*a*[*w, z, p*]] and ALHID[*w, z*] relate. Perhaps in some cases a parametric plot might be helpful too.

```
ParametricPlot[Evaluate[
  z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  {AbsLoosemoreHanbyID[w, z], Abs[p . (z - w) / Đ]} /. r → 4],
 {t, 0, Đ}, AxesLabel → {"ALHID (Đ)", "Abs[a] (Đ)"}]
```




```
ParametricPlot[Evaluate[
  z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  {AbsLoosemoreHanbyID[w, z], Abs[p . (z - w) / Đ]} /. t → 4],
 {r, 0, Đ}, AxesLabel → {"ALHID (Đ)", "Abs[a] (Đ)"}]
```



10. Conclusion

This paper belongs to the framework of Colignatus (2017b) (1W1V) of “re-engineering the political science on electoral systems”. To evaluate (half) elections, we can look at votes *v* and seats *s* but also at the policy stances *p* of the parties. We can distinguish the voter-legislative and the voter-executive distances. The latter refers to the ruling coalition in the legislative.

The key distinction is between systems with equal or proportional representation (EPR) and those with district representation (DR). In the latter we only have half-elections. What might be seen as a single seat election for a district actually is inadequate for the multiple-seats election for the legislative. In a district, the votes are discarded that are not for the winner of the district, whence those voters are not represented in the legislative. Saying that those voters are represented by the winner whom they did not vote for, is changing the meaning of representation.

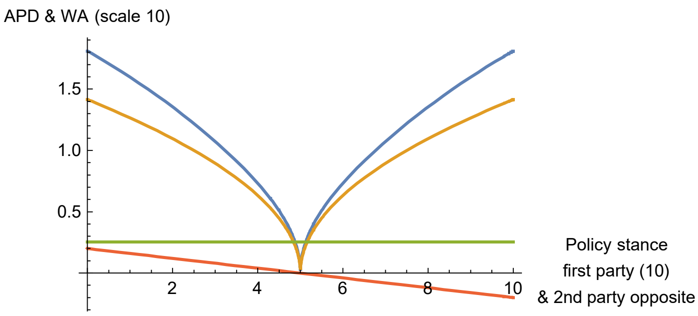
The traditional measures for inequality or disproportionality in votes and seats, as used in the political science on electoral systems, are the absolute difference (Loosemore-Hanby) and the Euclidean distance (Gallagher). This article uses the angle between the vectors, that is more sensitive. This gives the angular inequality / disproportionality (AID). See Colignatus (2017a) for the use of the sine of the angle. For this present paper the use of the sine distracts from the main purpose.

The purpose of this present article was to see whether the angle might also be used for the policy stances *p*. When we have unitised vectors of votes *w* and seats *z* and AID *d*[*w, z*] then this generates the possibility to also use the *angular policy term* (APT) *d*[*p w, p z*], and this can be compared to the weighted average (WA) *a* = *p'z* - *p'w.*

This paper suggested the *angular policy distance* (APD) *pd*[*w, z, p*]. The APD combines (1) the independent factors of the policy distances ||*p* - *p'*|| and the inequality / disproportionality of votes and seats AID *d*[*w, z*] and (2) the interaction given by the APT *d*[*p w, p z*].

The following plot compares the angular distance (AID) (green), APD (blue) and the WA (red) and the WA square root (yellow), using votes *v* and seats *s* that are fairly close but that imply a majority switch, and policy positions *p* = {*r,* 10 - *r*}, with the policy stance of the first party *r* ranging in [0, 10].

```
votes3 = {5.1, 4.9}; seats3 = {4.9, 5.1};
Plot[Evaluate[p = {r, 10 - r}; wa = p . (seats3 - votes3) / 10;
  {AngularPolicyDistance[votes3, seats3, p],
   Sqrt[Abs[wa / 10]] 10, AngularID[votes3, seats3], wa
  }], {r, 0, 10}, AxesLabel →
  {"Policy stance\nfirst party (10)\n& 2nd party opposite", "APD & WA (scale 10)"}]
```



Some points are:

(1) The angular distance *d*[*w, z*] (green) does not depend upon policy, and indicates that votes and seats are disproportional. The outcome of 0.25 on a scale of 10 or a quarter grade is too large, and this political community is advised to switch to a better system of EPR.

(2) When the votes and seats differ only 0.2 points in absolute terms (corrected for double counting), then the WA (red) can only range between [-0.2, 0.2], which means that the WA is not a sensitive measure. Yet it has a sign and perhaps a more familiar interpretation as a policy distance.

(3) The APD (blue) is much more sensitive than the WA but without the sign. We might borrow the sign from the WA though, and flip the RHS.

(4) The latter causes the question: why not boost the WA, like taking its square root (yellow) ? Here we could flip the RHS too. (This is the routine WAvSqrtPolicyDistance, above.) Perhaps WA contains all that we need, and the sqrt might also deal with the insensitivity of the ALHID.

The numerical values of the APD are new. Their meaning translates to the angle between the vectors. Their meaning is more "if the *angular distance* is a better measure for disproportionality for votes and seats, then the *angular policy distance* might be a better measure than the weighted average for policy stances". One might get used to the numbers and see how they work out in practice, like one also has to get used to the meaning of the numbers in the Richter scale on earthquakes.

The *Introduction* contained a plot with values for weighted average and APD for the 2016 US House of Representatives, with the voter-legislative and voter-executive distances, for a guesstimate of the policy positions. The APD inserts another dot into the plot, compared to WA on policy and AID and SDID on disproportionality. Comparing AID and APD, the voter-legislative distance increases and the voter-executive distance decreases. It is too early to attach much meaning to these figures. One would have to look at results over more years to see whether this generates additional information or has relevance for forecasts.

Overall, all these distances have little relevance for EPR when $z \approx w$. Nations in the world are advised to switch to EPR as in Holland. The findings above then might still have some use for comparing policy positions before and after (true) elections - but that is another topic. And we should not forget about the multidimensional analysis as in Stokman et al. (2013).

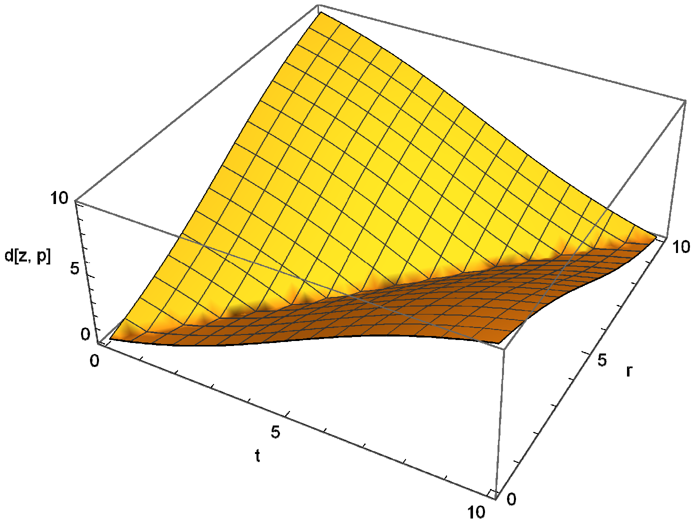
Appendix A. Using $d[w, p]$ and $d[z, p]$

Plots $d[w, p]$ and $d[z, p]$ and their maximum Max[$d[w, p]$, $d[z, p]$]

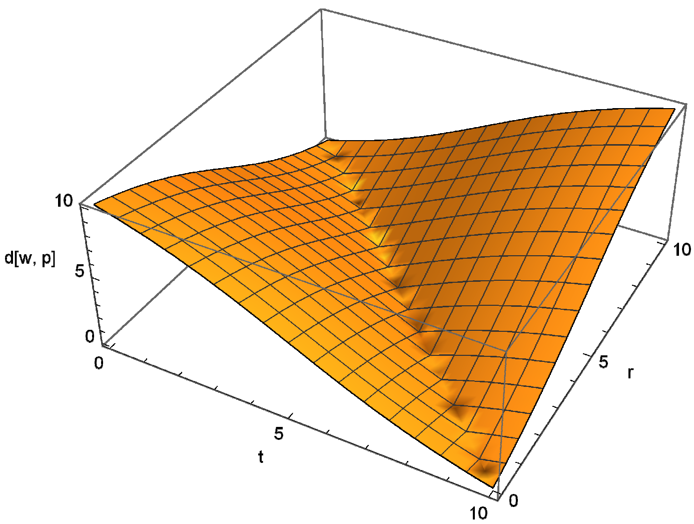
Since we have both the APD and the measure with independent factors (IndepAngPolicyDistance) itself, it is doubtful whether we should try to continue looking for another measure that involves interaction. However, the following was looked into in the past, and may be discussed, for it may provide another insight in non-independent interaction.

We may also consider $d[w, p]$ and $d[z, p]$. These are based upon special angles, namelijk between the weights and the policy positions. In the weighted average a they are multiplied and added, and here we consider their angular distance. A high value in p might compensate for a low value in w, not in terms of politics but purely in terms of the numbers. For the following plots, we should remain aware of the presumed symmetries in these plots. We first plot for z and then for w, and then take their maximum value.

```
Plot3D[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r};
  AngularID[z, p]], {t, 0, Ð}, {r, 0, Ð}, AxesLabel → {"t", "r", " d[z, p]"}]
```




```
Plot3D[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r};
  AngularID[w, p]], {t, 0, Ð}, {r, 0, Ð}, AxesLabel → {"t", "r", " d[w, p]"}]
```



It appears that *pd*[*w, z, p*] = Max[*d*[*w, p*], *d*[*z, p*]] behaves *somewhat* as disproportionality for *w* and *z* given *p,* and behaves *somewhat* as policy incongruence for *p* given *w* and *z.*

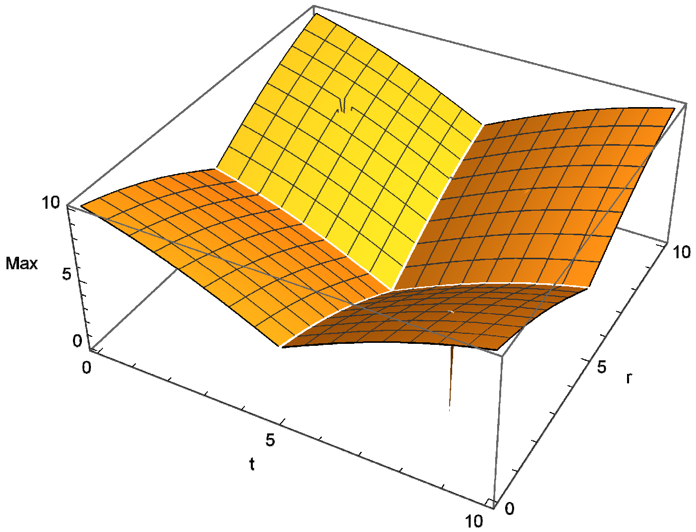
```
Plot3D[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r};
  Max[AngularID[z, p], AngularID[w, p]]],
 {t, 0, Ð}, {r, 0, Ð}, AxesLabel → {"t", "r", " Max "}]
```

Max: Invalid comparison with 0. + 1.34158 × 10^{-7} i attempted.

Max: Invalid comparison with 0. + 1.34158 × 10^{-7} i attempted.

Max: Invalid comparison with 0. + 1.34158 × 10^{-7} i attempted.

General: Further output of Max::nord will be suppressed during this calculation.



Let us consider the meanings of the angles between *w* and *p* on the one hand and *z* and *p* on the other hand. The “sink” shape can be explained as follows:

(1) If *d*[*w, p*] is small, then *d*[*z, p*] could be larger if *d*[*w, z*] is disproportional, and otherwise a smaller value is acceptable on both counts.

(2) If *d*[*w, p*] is large, then *d*[*z, p*] could be smaller if *d*[*w, z*] is disproportional, and otherwise a larger value is acceptable on both counts.

The following show that the maximum has drawbacks:

(ad P.1) When $w = z$ then we need not expect the same outcome as $d[w, z] = 0$.

(ad P.2) When $p = \lambda\, 1$ then we need not expect 0 either.

This causes us to be wary of this as a potential measure of distance.

```
? MaxAngularPolicies
```

MaxAngularPolicies[v, s, p] gives Max[AngularID[v,
p]], AngularID[s, p]], and note that this is not a proper distance
MaxAngularPolicies[Plot3D] gives a 3D plot for s = {t, Ð – t}, v = Ð – s, and p = {r, Ð – r}

The above 3D plot assumes symmetry. Let is consider non-symmetry. In this case, *Party A* loses the (half) elections in terms of the popular count with {4.9, 5.1} but wins the House with a remarkable majority with {4, 6}. We expect a sizable difference, unless the policy positions are the same with *Party B*. Let us assume that *B* has a position at 6 on the left-to-right scale. The following plots the value of the measure for various positions of *Party A*. Remarkably, the minimum is not at 6 but at around 7, and the minimum is not 0.

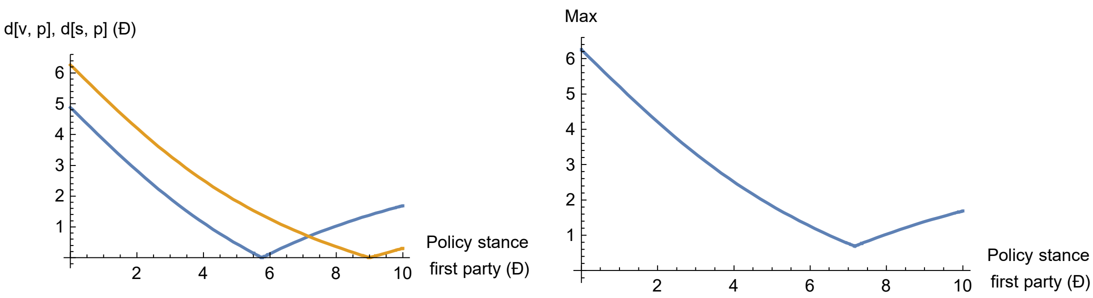
```
FindMinimum[MaxAngularPolicies[{4.9, 5.1}, {6, 4}, {polpos, 6}], {polpos, 5}]
```

FindMinimum: The line search decreased the step size to within the tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the function. You may need more than MachinePrecision digits of working precision to meet these tolerances.

```
{0.691983, {polpos → 7.17136}}
```

In the following plot, policy positions *polpos* to the right of 7 for *Party A* do not contribute to a much higher policy incongruence, presumably because of the interaction between *z* = {6, 4} and *p* = {*polpos,* 6}. We first plot the two angular distances *d*[*v, p*] and *d*[*s, p*], and then their maximum.

```
GraphicsGrid[{{ Plot[{AngularID[{4.9, 5.1}, {polpos, 6}],
      AngularID[ {6, 4}, {polpos, 6}]},
    {polpos, 0, Ð},
    AxesLabel → {"Policy stance\n first party (Ð)", "d[v, p], d[s, p] (Ð)"}],
   Plot[MaxAngularPolicies[{4.9, 5.1}, {6, 4}, {polpos, 6}],
    {polpos, 0, Ð}, AxesOrigin → {0, 0},
    AxesLabel → {"Policy stance\n first party (Ð)", "Max"}]
     }}]
```



Combined: geometric mean of independence and maximum

If we take the geometric mean of above two measures, the one using independence and the other using the maximum, then we make sure that there are zero outcomes at *w* = *z* and *p* = *λ* 1. This combination generates a measure that meets our current specifications P.1 and P.2.

```
? CombinedPolicyDistance
```

CombinedPolicyDistance[v, s, p] gives the geometric
mean (sqrt) of IndepAngPolicyDistance and MaxAngularPolicies
CombinedPolicyDistance[Plot3D] gives a 3D plot for s = {t, Ð – t}, v = Ð – s, and p = {r, Ð – r}

The 3D plot is not perfect due to the Maximum condition. However, the zero's are respected when *w* = *z,* even though the 3D plot does not show this.

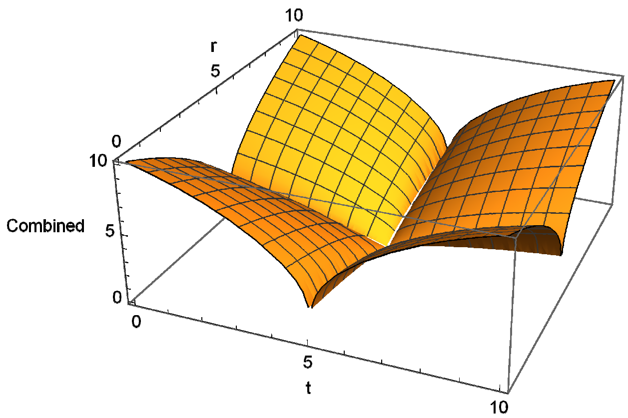
```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  CombinedPolicyDistance[w, z, p]], {t, 0, Đ},
 {r, 0, Đ}, AxesLabel → {"t", "r", " Combined "}]
```

Max: Invalid comparison with 0. + 2.10734 × 10^{-8} ⅈ attempted.

Max: Invalid comparison with 0. + 2.10734 × 10^{-8} ⅈ attempted.

Max: Invalid comparison with 0. + 2.10734 × 10^{-8} ⅈ attempted.

General: Further output of Max::nord will be suppressed during this calculation.




```
CombinedPolicyDistance[{a, b}, {a, b}, {r, Đ - r}]
0
```

Non-symmetry

The above 3D plot assumes symmetry. Let is consider non-symmetry. In this case, *Party A* loses the (half) elections in terms of the popular count but wins the House with a remarkable majority. We expect a sizable difference, unless the policy positions are the same with *Party B.* Let us assume that *B* has a position at 6. The following plots the value of the measure for various positions of *Party A.* We find the value 0 exactly at *polpos* = 6. This works as expected, though the 0 is only approximate due to numerical routines.

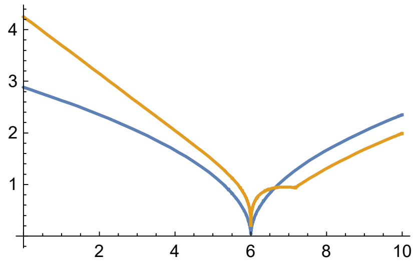
```
FindMinimum[CombinedPolicyDistance[{4.9, 5.1}, {6, 4}, {polpos, 6}], {polpos, 5}]
```

FindMinimum: The line search decreased the step size to within the tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the function. You may need more than MachinePrecision digits of working precision to meet these tolerances.

```
{0.0149279, {polpos → 6.}}
```

A drawback seems that some properties of the Max function might carry over into the combination. However, the main drawbacks w.r.t. P.1 and P.2 have been eliminated. The outcome reflects the interaction between votes and seats and the policy positions. In the following, we plot both the independent measure and the combination. For the latter, the policy positions to the right of 6 for *Party A* do not contribute to a much higher policy incongruence, presumably because of the interaction between *z* = {6, 4} and *p* = {*polpos,* 6}. The small curve towards point 7 has been explained above, with the same effect.

```
Plot[{IndepAngPolicyDistance[{4.9, 5.1}, {6, 4}, {polpos, 6}],
  CombinedPolicyDistance[{4.9, 5.1}, {6, 4}, {polpos, 6}]}, {polpos, 0, Ð}]
```



The *combined measure* satisfies the conditions P.1 and P.2 for both *disproportionality of votes and seats* and *policy incongruence.* The behaviour has been shown in the plots with symmetry and asymmetry. The numerical outcomes could be accepted *as is,* even for the remarkable behavior at points like 7 above. When widely different combinations still have the same outcome of the measure, then this is just the outcome.

However, the proposed *angular policy distance* APD in the main body of the text has a simpler structure without the numerically troubling Max condition, and such remarkable points like 7 above. Thus the combination that has been discussed here finds only mentioning in this appendix.

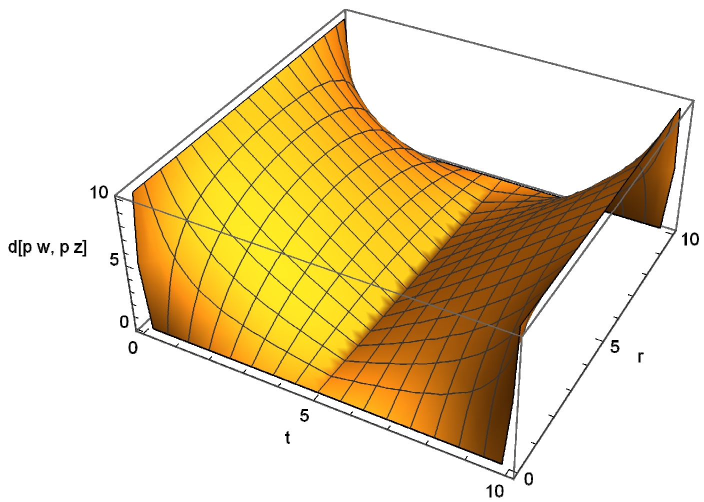
Appendix B. *d*[*p w, p z*] for ALHID

An indicator *d*[*p w, p z*] for (i) congruence on policy and (ii) disproportionality on votes and seats

The following repeats the discussion for the APD, that uses the angular distance, now for the ALHID.

With *d*[*w, z*] as a distance measure, like the Loosemore-Hanby (ALHID) measure above, then *d*[*p w, p z*] may say something about the policy congruence or incongruence too. The plot for ALHID is quite similar to the plot for the APT.

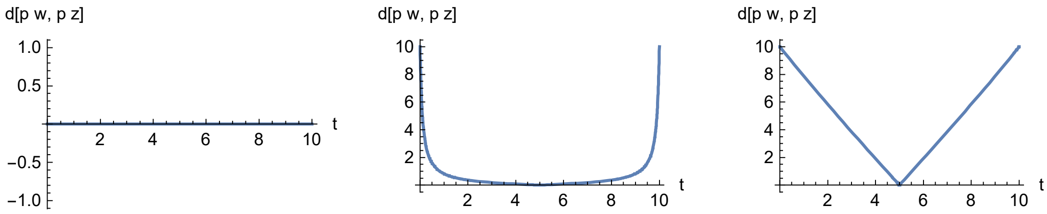
```
Plot3D[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r};
  AbsLoosemoreHanbyID[p w, p z]], {t, 0, Ð},
 {r, 0, Ð}, AxesLabel → {"t", "r", "d[p w, p z] "}]
```



The 3D plot can be supported by some sections at different values of *r*.

```
plalhid[rval_] :=
 Plot[Evaluate[z = {t, Ð - t}; w = {Ð - t, t}; p = {r, Ð - r} /. r → rval ;
   AbsLoosemoreHanbyID[p w, p z]], {t, 0, Ð},
  AxesLabel → {"t", "d[p w, p z] "}, PlotRange → All]
```

```
GraphicsGrid[{ plalhid /@ {0, .1, 4} }]
```



We again look at the two perspectives. We repeat the analysis so that this Appendix can be a bit more independent from the main body of the text.

(i) We may regard *f*[*w, z* ; *p*] = *d*[*p w, p z*] as a function of *w* and *z,* given some *p.* The greater *d*[*w, z*] the more spread out *d*[*p w, p z*]. Also *w* = *z* (here *t* = 1/2) or *z* = *μ w* will cause an outcome of zero, since *d*[*p w, p w*] = 0. This means that *f*[*w, z*] (given *p*) behaves like the disproportionality measure *d*[*w, z*]. See below for the comment on the extreme values, given *p* on {0, 10} or {10, 0}.

(ii) We may regard *g*[*p* ; *w, z*] = *d*[*p w, p z*] as a function of *p,* given some *w* and *z.* When *p* = *λ* 1 then *d*[*p w, p z*] = *d*[*w, z*]. (Here *r* = 5 gives *p* = 5 {1, 1}.) This need not be its maximal value, and the maximum need not be 10 either, though this happens to be the case in above (symmetric) plot, except for outcome 0 at *p* on {0, 10} or {10, 0}. The more extreme *p* (with opposite values *r* and Ð - r here), the lower *g*[*p*], with ultimately 0 for *p* = {0, 10} or {10, 0}. This means that *g*[*p*] is a measure of policy congruence.

Thus from one perspective (votes and seats) we have *disproportionality* and from another perspective (policy stances) we have *congruence*.

The exceptional aspects are: (a) The disproportionality levels might not be entirely comparable, since they depend upon *p,* (b) At the extreme cases of *p* like {0, 10} or {10, 0} the measure plunges to 0, for any value of *w* or *z*. The reason is that at the extremes only nonzero *p*[*i*] *w*[*i*] and *p*[*i*] *z*[*i*] are compared, but also unitised, whence the compared vectors are the same. We might say that we actually do not have a measure here. The outcome of zero congruence for (ii) is okay however. Then the outcome for (i) may also be interpreted as that the disproportionality is 0 *given* that extreme value of *p* with zero congruence.

The ALHID measure

For the ALHID we also have a *measure m* = *d*[*w, z*] - *d*[*p w, p z*].

Let us use two properties now:

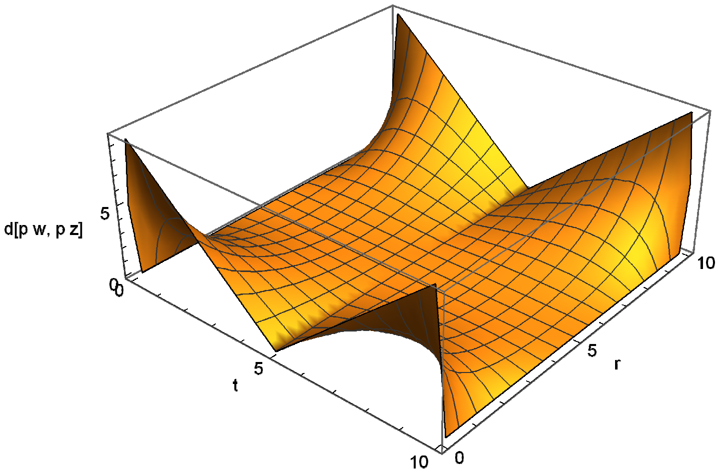
(1) For symmetric cases *z* = Ð - *w,* we (tend to) have *d*[*w, z*] ≥ *d*[*p w, p z*]. Thus *d*[*w, z*] - *d*[*p w, p z*] ≥ 0 is still a disproportionality measure, *given p.*

(2) With *g*[*p*] policy congruence and some value *x* that does not depend upon *p,* it follows that *x* - *g*[*p*] *given w and z* is a measure for policy incongruence.

Thus *d*[*w, z*] - *d*[*p w, p z*] ⩾ 0 is both a measure of *disproportionality* for *w* and *z* and a measure of *incongruence* for *p,* in all their combinations. When we plot this, then we should see the same pattern as for Abs[*a*] = Abs[*p'* (*z* - *w*)] but with more nonlinearity. The exception could be the development of the values to the boundaries.

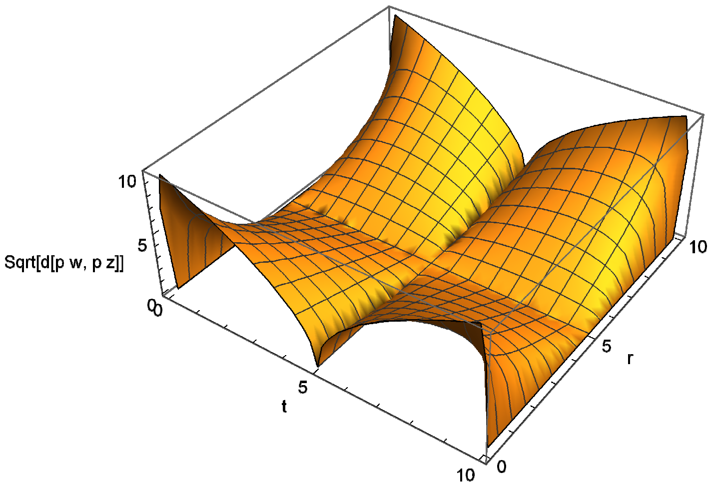
For the Loosemore-Hanby measure we find the following. The outcome is rather flat in the middle ranges, since *d*[*w, z*] and *d*[*p w, p z*] apparently are close together.

```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  AbsLoosemoreHanbyID[w, z] - AbsLoosemoreHanbyID[p w, p z]], {t, 0, Đ},
 {r, 0, Đ}, AxesLabel → {"t", "r", "d[p w, p z]"}, PlotRange → All]
```



We get more sensitivity by taking the square root - but we need to keep track of the scale Đ.

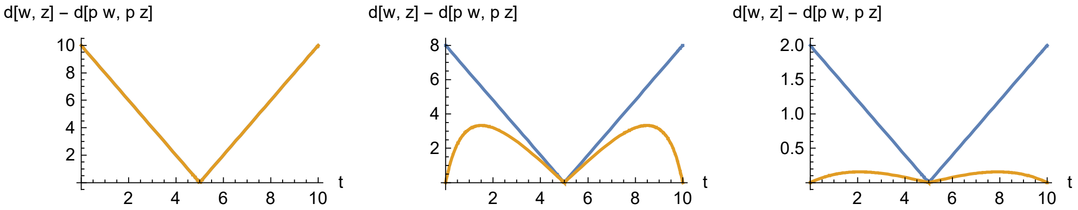
```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  Đ Sqrt[(AbsLoosemoreHanbyID[w, z] - AbsLoosemoreHanbyID[p w, p z]) / Đ]], {t, 0, Đ},
 {r, 0, Đ}, AxesLabel → {"t", "r", "Sqrt[d[p w, p z]]"}, PlotRange → All]
```



This may be supported by some 2D plots. For extreme values of *r,* the outcome reduces to the ALHID measure itself. When *r* get closer to 1/2, then also Abs[*a*] gets a lower slope and lower maximal value (check the vertical axes), but ALHID[*w, z*] - ALHID[*p w, p z*] drops much faster. Also, the outcome becomes zero for opposite values of votes and seats, when ALHID[*w, z*] actually is Đ.

```
plaid[rval_] := Plot[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
   {Abs[p . (z - w) / Đ], AbsLoosemoreHanbyID[w, z] - AbsLoosemoreHanbyID[p w, p z]} /.
    r → rval], {t, 0, Đ}, AxesLabel → {"t", "d[w, z] - d[p w, p z] "}]
```

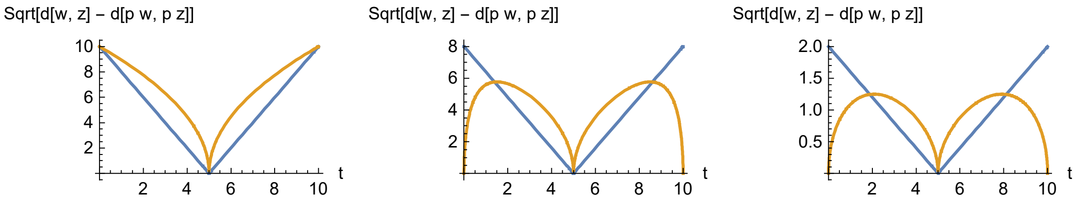
```
GraphicsGrid[{ plaid /@ {0, 1, 4} }]
```



Now using the Sqrt.

```
plaid[rval_] := Plot[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
   {Abs[p . (z - w) / Đ],
     Đ Sqrt[(AbsLoosemoreHanbyID[w, z] - AbsLoosemoreHanbyID[p w, p z]) / Đ]} /.
    r → rval], {t, 0, Đ}, AxesLabel → {"t", "Sqrt[d[w, z] - d[p w, p z]]"}]
```

```
GraphicsGrid[{ plaid /@ {0, 1, 4} }]
```



Investigating a maximum

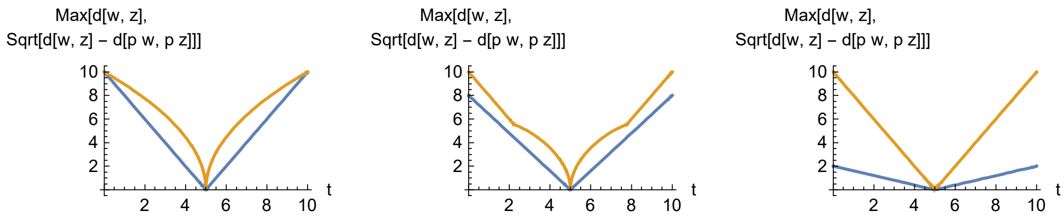
The above still uses symmetry. For asymmetry, we may have to look for a local maximum.

The following plots use the weighted average for reference (blue line, chech the vertical axis).

For a disproportionality index, the values at the extremes that curve back to the horizontal axis are problematic. This is just a possible answer: Max[*d,* Sqrt[*d* - *dp*]], avoiding a dependence like Max[*a,* Sqrt[*d* - *dp*]].

```
plaid[rval_] := Plot[Evaluate[z = {t, Đ - t};
   w = {Đ - t, t}; p = {r, Đ - r}; alhid = AbsLoosemoreHanbyID[w, z];
   {Abs[p . (z - w) / Đ], Max[alhid,
      Đ Sqrt[(alhid - AbsLoosemoreHanbyID[p w, p z]) / Đ]]} /. r → rval],
  {t, 0, Đ}, AxesLabel → {"t", "Max[d[w, z], \nSqrt[d[w, z] - d[p w, p z]]]"}]

GraphicsGrid[{ plaid /@ {0, 1, 4} }]
```



At this point, we would start repeating the analysis for ALHID which we already did for the AngularID. Since the latter has the better property of being more sensitive, there is no need to further develop the ALHID.

Appendix C. Combining WA with the RPM

Combining properties

In Section 7 we defined the measures APM and RPM, to be distinguished from the distance APD.

The following combines the weighted average *a* with the RPM.

wavapm[*v, s, p*] = *a* + *a* (1 - *rpm*) = *a* (2 - *rpm*), with *a* = *p* . (*z* - *w*) for unitised *z* and *w*. When *p* is in [0, Đ] then *a* will be in [-Đ, Đ] and *wavapm* will tend to be so too.

The *wavapm* is a theoretical construct, and it is not clear whether it can be of practical use, whence it is put in this Appendix. The last subsection of this appendix discusses potential application.

Underlying properties

The following starts with numerical examples. The APT *d*[*p v, p s*] for given *v* and *s* becomes smaller for more extreme values of *p,* whence it is a measure of congruence and whence a *suitable* transform APD can be used for an angular policy distance.

Though we still have to construct the *wavapm,* the following table can already print its values.

Let us consider the outcomes of the APD, *a, wavapm,* APM and the APT. This next case assumes that the parties are still fairly close and that the angular distance is small.

```
votes3 = {5.1, 4.9};
seats3 = {4.9, 5.1};
differenceAndAngle[p_] := {AngularPolicyDistance[votes3, seats3, p], (*APD*)
    p . (seats3 - votes3) / Đ, (* WA or a *)
    WAvAngPolicyMeasure[votes3, seats3, p], (* wavapm *)
    AngularPolicyMeasure[votes3, seats3, p, Exponent → 1], (* APM *)
    AngularID[p votes3, p seats3] (* APT *)
    } // N // Chop

policies = {{7, 2}, {6, 4}, {5, 5}, {7, 7}, {7, 8}, {4, 6}, {2, 7}, {0, 10}};
TableForm[differenceAndAngle /@ policies, TableHeadings →
  {policies, {"APD", "WA", "wavapm[v, s, p]", "apm[v, s, p]", "d[p v, p s]"}}]
```

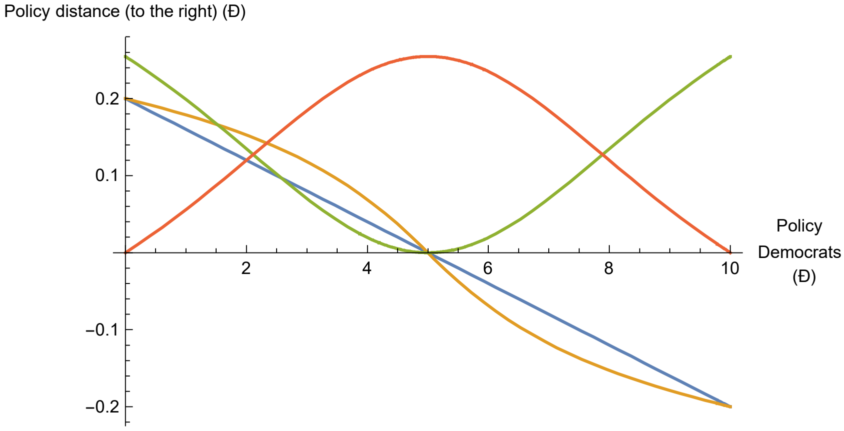
	APD	WA	wavapm[v, s, p]	apm[v, s, p]	d[p v, p s]
{7, 2}	1.23481	-0.1	-0.131335	0.120049	0.134565
{6, 4}	0.731773	-0.04	-0.0689112	0.0195672	0.235047
{5, 5}	0	0	0	0	0.254614
{7, 7}	0	0	0	0	0.254614
{7, 8}	0.50673	0.02	0.0381195	0.00225085	0.252363
{4, 6}	0.731773	0.04	0.0689112	0.0195672	0.235047
{2, 7}	1.23481	0.1	0.131335	0.120049	0.134565
{0, 10}	1.80965	0.2	0.2	0.254614	0

The table shows: (1) APT is low for high policy differences, and APT has a relative maximum when policies are equal, whence the APT is a measure of policy congruence. (2) The APM is a transform Max - APT. (3) The APD and WA are measures of policy distance (taking the absolute values). (4) The *wavapm* behaves like the distance measures (taking the absolute values).

The measure *wavapm* combines the weighted average with the angular disproportionality of votes, seats and policy.

The following graph shows how the Democrats move from the left to the right, and the Republicans from the right to the left. The weighted average *a* generates a (blue) line and has a sign, given that the Republicans have 2% more seats. The (red) hill shape is *d*[*p v, p s*], thus actually congruence (given *v* and *s*), and it is not so high since the votes and seats are fairly close. Its oppositely curved (green) line is the level angular policy measure APM[*v, s, p*]. The combination *wavapm* (yellow) of the weighted average (blue) with the angular disproportionality (red or green) gives a stronger reaction of the policy distance (yellow compared to blue) to the disproportionality in votes and seats.

```
Plot[Evaluate[p = {r, Đ - r}; Rest[differenceAndAngle[p]]], {r, 0, Đ},
 AxesLabel → {"Policy\nDemocrats\n  (Đ)", "Policy distance (to the right)  (Đ)"}]
```



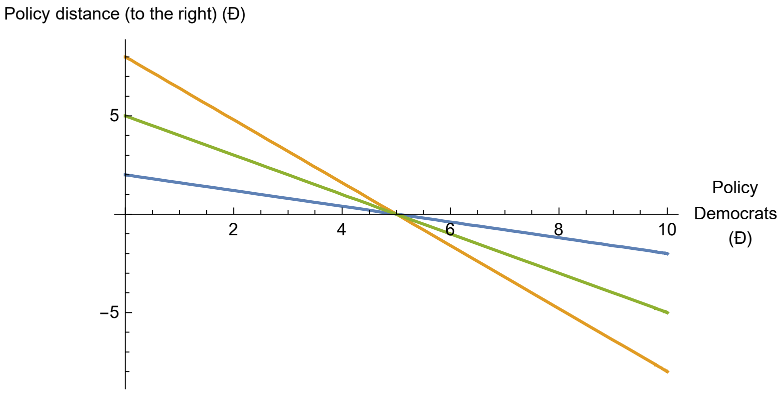
A more involved discussion with three examples

The disproportionality between votes and seats affects the policy distance. Let us take three cases, and for reference plot the weighted average policy distance between House and electorate. The first case (blue) is almost zero and thus we plot with a factor 10. Check the positions at the extremes.

```
votes3 = {5.1, 4.9}; seats3 = {4.9, 5.1}; (*blue, use a factor 10*)
votes4 = {9, 1} ; seats4 =  {1, 9}; (*yellow*)
votes5 = {9, 1} ; seats5 = {4, 6}; (*green*)
```

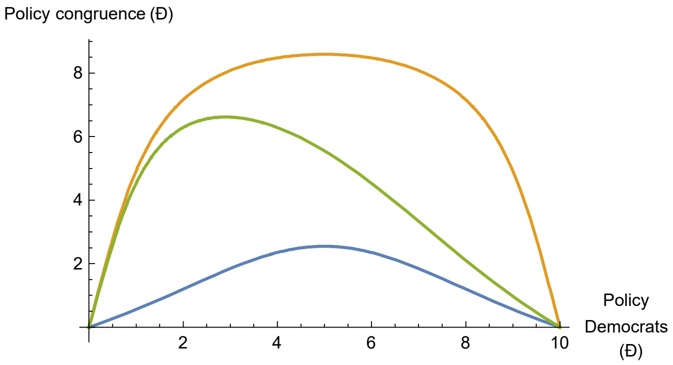
In all cases the Republicans have the House. When the Democrats would move to the left (r < Đ - r) then the Republicans move the House position to the right of the electorate. When the Democrats would move to the right (r > Đ - r) then the Republicans move the House to the left of the electorate. (One might say that this happened in the abolition of slavery, with Abraham Lincoln a Republican.)

```
pav = Plot[Evaluate[p = {r, Đ - r};
   {10 p . (seats3 - votes3) / Đ,  p . (seats4 - votes4) / Đ, p . (seats5 - votes5) / Đ
   }], {r, 0, Đ},
  AxesLabel → {"Policy\nDemocrats\n  (Đ)", "Policy distance (to the right)  (Đ)"}]
```



For the nonlinear plots, first consider $d[p\ v, p\ s]$. For ease we assume that the parties have opposite policy stances. The first case (blue) again is plotted with a factor of 10. Let us repeat some of the properties above, and remember that each curve assumes given v and s (though different per curve).

```
Plot[Evaluate[p = {r, Đ - r};
  {10 AngularID[p votes3, p seats3], (*blue*)
   AngularID[p votes4, p seats4],  (*yellow*)
   AngularID[p votes5, p seats5]  (*green*)
  }], {r, 0, Đ},
 AxesLabel → {"Policy\nDemocrats\n  (Đ)", "Policy congruence (Đ)"}]
```



Remarkably, the maximal values of $d[p\ v, p\ s]$ are close to the maximal values of the weighted average a.

The following repeats the properties that we already saw above, now in a different context.

(i) The graphs get lower, when moving from yellow, to green to blue. When $w = z$ then the APT will be zero, since $d[p\ w, p\ w] = 0$. This zero outcome might still be seen as high policy congruence, since also $d[v, s] = 0$ and $a = 0$. The greater $d[v, s]$ the more spread out $d[p\ v, p\ s]$ too.

(ii) In a plot of $d[p\ v, p\ s]$ as a function of p, the level of $d[v, s]$ shows up as a horizontal line (shown above). When $p = \lambda\ 1$ then $d[p\ v, p\ s] = d[v, s]$. This means that $d[p\ v, p\ s]$ is not a mere measure of congruence but that its value must be related to $d[v, s]$. However, notice the effect of asymmetry, see below.

(iii) The wider the policy differences, the closer $d[p\ v, p\ s]$ is to zero. When p has extreme values like {0, 10} then only those components will be selected, and the angle becomes zero. (Though we are looking at the binary case here.)

(iv) When v and s have opposite formulas then the graph may be symmetric, but more common would be asymmetry or skewness, whence $d[v, s]$ and $d[p\ v, p\ s]$ need not always have the same maximum (check green at $p = 1/2$).

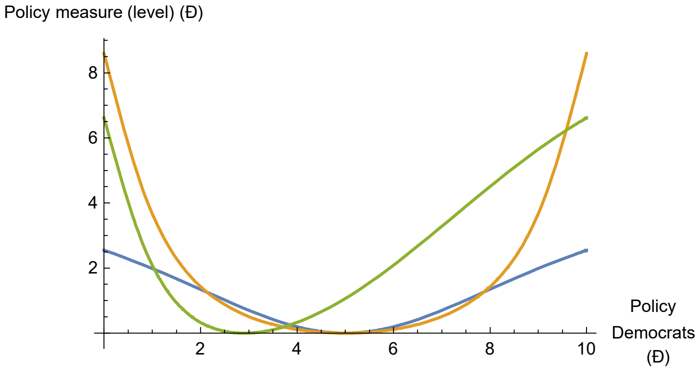
(v) Thus $d[p\ v, p\ s]$ *given v and s* is a congruence measure and not an incongruence measure. Its interpretation is somewhat difficult, however, given that $d[p\ v, p\ s]$ *given p* is a measure of disproportionality. A high outcome still relates to a high value in disproportionality, and a zero might be caused by either high congruence ($a = 0$ because either $w = z$ or $p = \lambda\ 1$) or high incongruence (extreme policy differences like $p = \{0, 10\}$).

Inverting above plot

It follows that APM[v, s, p] = $max - d[p\ v, p\ s]$ inverts above plot, where the maximum differs per case, as an *level* measure *given v and s*. It is an important design matter to allow for different maxima. Just like the weighted averages have different slopes, the different disproportionalities between votes and seats generate different shapes and such maxima.

```
Plot[Evaluate[p = {r, Đ - r};
  {10 AngularPolicyMeasure[votes3, seats3, p, Exponent → 1],
   AngularPolicyMeasure[votes4, seats4, p, Exponent → 1],
   AngularPolicyMeasure[votes5, seats5, p, Exponent → 1]
  }], {r, 0, Đ},
 AxesLabel → {"Policy\nDemocrats\n  (Đ)", "Policy measure (level) (Đ)"}]
```

FindMinimum: Encountered a gradient that is effectively zero. The result returned may not be a minimum; it may be a maximum or a saddle point.

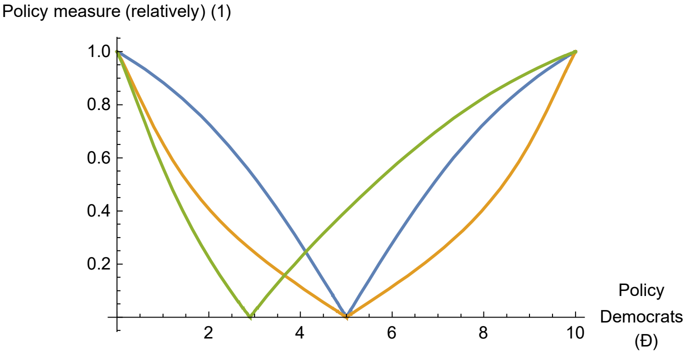


While the weighted average gives a line, this angular policy measure gives curves, that also depend nonlinearly upon the disproportionality of votes and seats. The curves are less straightforward to interprete than the weighted average. The *measure* APM cannot be used as the *distance* APD, since it does not satisfy properties P.1 and P.2 though.

The relative meaure

The *relative* measure RPM[*v, s, p*] = (1 - *d*[*p v, p s*] / *max*)^*f,* is made more sensitive by using a power, by default the square root (*f* = 1/2). For plotting we remove the factor 10 for the first case (blue). The relative measure is paradoxical. We would like to see that the score is higher when the disproportionalities are greater (yellow), yet here we see that the more proportional case (blue) has a higher score. The cause is that we have eliminated the impact of the maximum on the level score. Because of this paradox we would generally not use this particular relative measure.

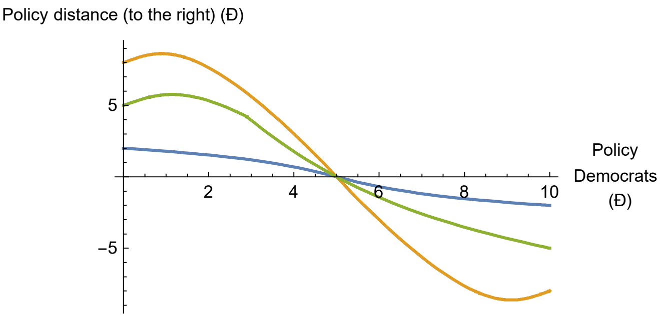
```
SetOptions[AngularPolicyMeasure, Level → False];
Plot[Evaluate[p = {r, Đ - r};
  {AngularPolicyMeasure[ votes3, seats3, p],  (* no factor 10 *)
   AngularPolicyMeasure[ votes4, seats4, p],
   AngularPolicyMeasure[ votes5, seats5, p]
  }], {r, 0, Đ},
 AxesLabel → {"Policy\nDemocrats\n  (Đ)", "Policy measure (relatively) (1)"}]
SetOptions[AngularPolicyMeasure, Level → True];
```



Combining the weighted average and the RPM

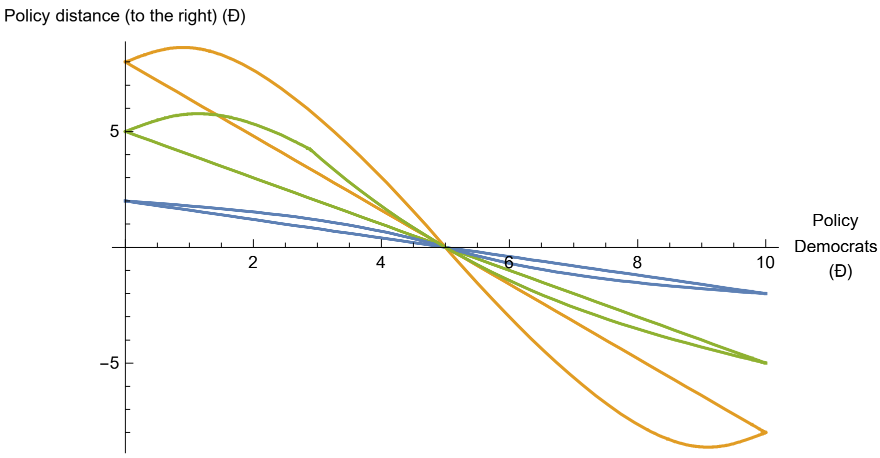
However, the relative measure can be used for the combination with the weighted average. Let b = (a * rpm) be the basic combination. We might consider looking at $a + b$, but the present reasoning is otherwise. The difference of b with a is given as $dif = a - b = a$ (1 - rpm), and we want to upscale a with this difference, as $a + dif$. Thus final output for the combination is $a + a$ (1 - rpm) = a * (2 - rpm). In this case, we scale again for blue. Given that we use a as a policy *distance*, the present combination may be seen as a policy *distance* too (taking the absolute value).

```
pavang = Plot[Evaluate[p = {r, Đ - r};
   {10 WAvAngPolicyMeasure[votes3, seats3, p],
    WAvAngPolicyMeasure[votes4, seats4, p],
    WAvAngPolicyMeasure[votes5, seats5, p]
   }], {r, 0, Đ},
  AxesLabel → {"Policy\nDemocrats\n  (Đ)", "Policy distance (to the right) (Đ)"}]
```



The following combines the plots for the weighted average and the combination. (i) Since the relative angular measure is 1 at the extremes, the curves join up with the lines of the weighted average. (ii) They all pass through r = 1/2 since then the weighted average is 0. (iii) They all assume *given v and s,* and the different assumptions for v and s cause lines with different slopes. (iv) The interpretation is that the greater disproportionality (moving from blue to green to yellow) may increase the policy distance *above* the value given by the weighted average, or *wavapm* ≥ *a*.

```
Show[pav, pavang,
 AxesLabel → {"Policy\nDemocrats\n  (Ð)", "Policy distance (to the right) (Ð)"}]
```



Potential application

The *wavapm* is a theoretical construct, and it is not clear whether it can be of practical use.

Thus we now have these possible measures in terms of the weighted average:

(1) The *voter-legislative* distance is $a = p'(z - w)$. The weighted average assumes that parties will not take advantage of their majority in the House, and will heed the median voter.

(2) The *voter-executive* distance is $a = p'(U[c\ z] - w)$. This allows that the majority neglects the median voter (though its policy position might be at 5).

(3) Perhaps this *wavapm* provides some intermediate ground. The majority or even the disproportionality in seats might seduce the parties to take more risk w.r.t. the median voter. If so, then the exponent *f* may also be used to describe the degree by which this happens.

Let us consider the 2016 US House again.

```
votes = {dem = 61776554, rep = 63173815, 128627010 - rep - dem};
seats = {194, 241, 0};
```

The assumed policy stances generate the following weighted average (*voter-legislative* distance). Since *p* is in [0, Ð], the weighted average *a* is so too. The outcome of *a* is rather low, but this derives from the slope of the line.

```
policy = {4, 6, 5};
policy . (UnitÐ[seats, 1] - UnitÐ[votes, 1]) // N
0.0971831
```

The *wavapm* does not terribly raise the value over *a*. The weighted average is rather insensitive to the disproportionality in votes and seats, and the *wavapm* inherits that property.

```
WAvAngPolicyMeasure[votes, seats, {4, 6, 5}]
0.100147
```

The outcome of the *wavapm* may also be interpreted as an increase of seats for the Rep at the cost of the Dem, and then again calculating the new weighted average *a*. It turns out that it would make only 1 seat difference.

```
eq = policy . (UnitĐ[seats + ds {-1, 1, 0}, 1] - UnitĐ[votes, 1]) == % /.
   Abs → Identity // FullSimplify
```

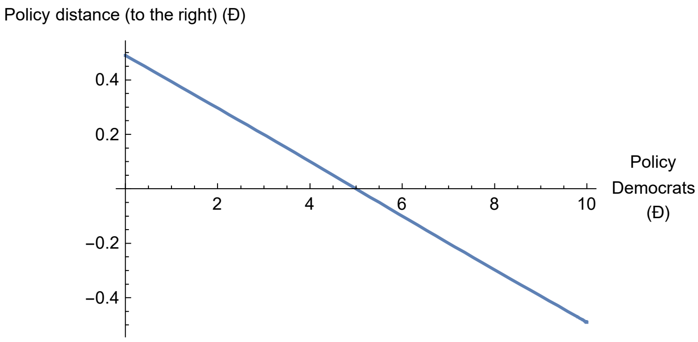
```
1. ds == 0.644625
```

```
Solve[eq, ds]
```

```
{{ds → 0.644625}}
```

The plot shows that the *wavapm* is almost a straight line, and rather indistinguishable from *a.* Most plots in this Appendix use dramatic values for votes and seats and policy stances {2, 8}, and in this practical example we are more around the middle, where events are less dramatic.

```
Plot[Evaluate[p = {r, Đ - r, 5};
  {WAvAngPolicyMeasure[votes, seats, p]
  }], {r, 0, Đ},
 AxesLabel → {"Policy\nDemocrats\n  (Đ)", "Policy distance (to the right) (Đ)"}]
```



A weighted average outcome of 0.12 would be tantamount to an increase of 5 seats for the Rep and a decrease of 5 for the Dem.

```
eq = policy . (UnitĐ[seats + ds {-1, 1, 0}, 1] - UnitĐ[votes, 1]) == 0.12 /.
   Abs → Identity // FullSimplify
```

```
1. ds == 4.96268
```

```
Solve[eq, ds]
```

```
{{ds → 4.96268}}
```

```
eq = WAvAngPolicyMeasure[votes, seats, {4, 6, 5}, Exponent → f] == 0.12
```

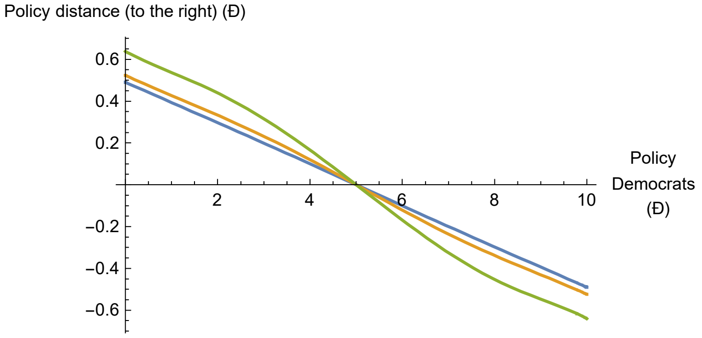
$$\frac{362\,510\,729\,\left(2 - 0.939936^{f}\right)}{3\,730\,183\,290} == 0.12$$

```
fr = FindRoot[eq, {f, 1 / 2}]
```

```
{f → 4.31999}
```

The following plots the default value *f* = 1/2 (square root) (blue), this found value *f* = 4.32 (yellow) and *f* = 20 (green) for comparison. If this kind of exercise would be useful for practice (say predicting future behaviour) then it might be wiser to use a formal specification that is more sensitive around the middle and less sensitive at the extremes.

```
Plot[Evaluate[p = {r, Đ - r, 5};
  {WAvAngPolicyMeasure[votes, seats, p],
   WAvAngPolicyMeasure[votes, seats, p, Exponent → fr[[1, 2]]],
   WAvAngPolicyMeasure[votes, seats, p, Exponent → 20]
  }], {r, 0, Đ},
 AxesLabel → {"Policy\nDemocrats\n  (Đ)", "Policy distance (to the right) (Đ)"}]
```



Appendix D. ||*p* - *p*’|| on top of *d*[*w*, *z*]

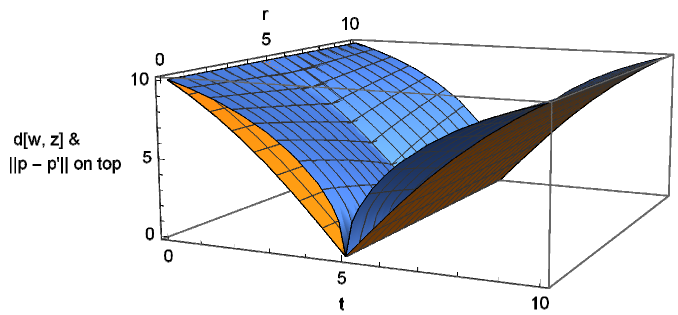
The following is only to show the outcome when we would put the policy norm on top of the angular distance.

```
? PlusAngPolicyDistance
```

PlusAngPolicyDistance[v, s, p] puts an independent dp = PolicyNorm[p / Đ] distance on top of
the d = AngularID[v, s] / Đ, keeping the range [0, Đ], giving Đ Sqrt[d (1 – (1–d)(1–dp))]
PlusAngPolicyDistance[Plot3D] plots 3D both d and the result, for s = {t, Đ – t}, v = Đ – s, and p = {r, Đ – r}

When $p = \lambda$ 1, then this measure is equal to *d*[*w*, *z*], and then it doesn’t satisfy P.2.

```
Plot3D[Evaluate[z = {t, Đ - t}; w = {Đ - t, t}; p = {r, Đ - r};
  {AngularID[w, z], PlusAngPolicyDistance[w, z, p]}
 ], {t, 0, Đ}, {r, 0, Đ}, AxesLabel → {"t", "r", " d[w, z] &\n||p - p'|| on top "}]
```



References

Colignatus is the name in science of Thomas Cool, econometrician and teacher of mathematics, Scheveningen, Holland.

Colignatus, Th. (2017a), “Comparing votes and seats with cosine, sine and sign, with attention for the slope and enhanced sensitivity to disproportionality", https://mpra.ub.uni-muenchen.de/81389/

Colignatus, Th. (2017b), "One woman, one vote. Though not in the USA, UK and France", https://mpra.ub.uni-muenchen.de/82513/

Colignatus, Th. (2017c), "Statistics, slope, cosine, sine, sign, significance and R-squared", https://boycottholland.wordpress.com/2017/10/21/statistics-slope-cosine-sine-sign-significance-and-r-squared/

Colignatus, Th. (2018), "Comparing the Aitchison distance and the angular distance for use as inequality or disproportionality measures for votes and seats", https://www.wolframcloud.com/objects/thomas-cool/Voting/2018-01-18-Aitchison.nb

Golder, M. & Stramski, J. (2010), "Ideological congruence and electoral institutions", American Journal of Political Science, 54(1), 90-106, http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.606.8475&rep=rep1&type=pdf

Stokman, F., Knoop, J. van der & Oosten, R. C. H. van (2013), "Modeling collective decision making", in Nee, V., Snijders, T. A. B. & Wittek, R. (eds.) (2013), "Handbook of Rational Choice Social Research", Stanford University Press, p. 151-182. See also https://www.rug.nl/staff/f.n.stokman, while Dutch readers may look at http://decide.nl