Sion's minimax theorem and Nash equilibrium of symmetric multi-person zero-sum game

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Abstract

About a symmetric multi-person zero-sum game we will show the following results.

(1) Sion's minimax theorem plus the coincidence of the maximin strategy and the minimax strategy are proved by the existence of a symmetric Nash equilibrium.

(2) The existence of a symmetric Nash equilibrium is proved by Sion's minimax theorem plus the coincidence of the maximin strategy and the minimax strategy.

Thus, they are equivalent. If a zero-sum game is asymmetric, maximin strategies and minimax strategies of players do not correspond to Nash equilibrium strategies. If it is symmetric, the maximin strategies and the minimax strategies constitute a Nash equilibrium. However, with only the minimax theorem there may exist an asymmetric equilibrium in a symmetric multi-person zero-sum game.

Keywords: multi-person zero-sum game, Nash equilibrium,Sion's minimax theorem.

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1 Introduction

We consider the relation between Sion's minimax theorem for a continuous function and the existence of Nash equilibrium in a symmetric multi-person zero-sum game. We will show the following results.

(1) Sion's minimax theorem plus the coincidence of the maximin strategy and the minimax strategy are proved by the existence of a symmetric Nash equilibrium.

(2) The existence of a symmetric Nash equilibrium is proved by Sion's minimax theorem plus the coincidence of the maximin strategy and the minimax strategy.

Thus, they are equivalent. An example of such a game is a relative profit maximization game in a Cournot oligopoly. Suppose that there are $n \geq 3$ firms in an oligopolistic industry. Let $\bar{\pi}_i$ be the absolute profit of the i-th firm. Then, its relative profit is

$$\pi_i = \bar{\pi}_i - \frac{1}{n-1}\sum_{j=1, j\neq i}^{n} \bar{\pi}_j.$$

We see

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \bar{\pi}_i - \frac{1}{n-1}(n-1)\sum_{j=1}^{n} \bar{\pi}_j = 0.$$

Thus, the relative profit maximization game in a Cournot oligopoly is a zero-sum game[1]. If the oligopoly is asymmetric because the demand function is not symmetric (in a case of differentiated goods) or firms have different cost functions (in both homogeneous and differentiated goods cases), maximin strategies and minimax strategies of firms do not correspond to Nash equilibrium strategies. However, if the demand function is symmetric and the firms have the same cost function, the maximin strategies and the minimax strategies constitute a Nash equilibrium. With only the minimax theorem there may exist an asymmetric equilibrium in a symmetric multi-person zero-sum game.

In Section 3 we will show the main results, and in Section 4 we present an example of an asymmetric n-person zero-sum game.

2 The model and Sion's minimax theorem

Consider a symmetric n-person zero-sum game with $n \geq 3$ as follows. There are n players, 1, 2, …, n. The set of players is denoted by N. A vector of strategic variables is $(s_1, s_2, \dots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$. S_i is a convex and compact set in a linear topological space for each $i \in N$. The payoff functions of the players are $u_i(s_1, s_2, \dots, s_n)$ for $i \in N$. We assume

> u_i for each $i \in N$ is continuous on $S_1 \times S_2 \times \cdots \times S_n$, quasi-concave on S_i for each $s_j \in S_j$, $j \in N$, $j \neq i$, and quasi-convex on S_j for $j \in N$, $j \neq i$ for each $s_i \in S_i$.

[1] About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

Symmetry of a game means that the payoff functions of the players are symmetric, and in the payoff function of each Player i, Players j and k, $j,k \neq i$, are interchangeable. If the game is symmetric and zero-sum, we have

$$\sum_{i=1}^{n} u_i(s_1, s_2, \dots, s_n) = 0, \tag{1}$$

for given $(s_1, s_2, \dots, s_n)$. Also all S_i's are identical. Denote them by S.

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \max_{x\in X} f(x,y).$$

We follow the description of this theorem in Kindler (2005).

Suppose that $s_k \in S_k$ for all $k \in N$ other than i and j, $j \neq i$, are given. Denote a vector of such s_k's by $s_{-i,j}$. Then, $u_i(s_1, s_2, \dots, s_n)$ is written as $u_i(s_i, s_j, s_{-i,j})$, and it is a function of s_i and s_j. We can apply Lemma 1 to such a situation, and get the following lemma.

Lemma 2. *Let $j \neq i$, and S_i and S_j be non-void convex and compact subsets of two linear topological spaces, and let $u_i : S_i \times S_j \to \mathbb{R}$ given $s_{-i,j}$ be a function that is continuous on $S_1 \times S_2 \times \cdots \times S_n$, quasi-concave on S_i and quasi-convex on S_j. Then*

$$\max_{s_i\in S_i} \min_{s_j\in S_j} u_i(s_i, s_j, s_{-i,j}) = \min_{s_j\in S_j} \max_{s_i\in S_i} u_i(s_i, s_j, s_{-i,j}).$$

We assume that $\arg\max_{s_i\in S_i} \min_{s_j\in S_j} u_i(s_i, s_j, s_{-i,j})$ and $\arg\min_{s_j\in S_j} \max_{s_i\in S_i} u_i(s_i, s_j, s_{-i,j})$ are unique, that is, single-valued for any pair of i and j. By the maximum theorem they are continuous in $s_{-i,j}$.

Also, throughout this paper we assume that the maximin strategy and the minimax strategy of players in any situation are unique, and the best response of players in any situation is unique.

Since we consider a symmetric game, by Lemma 2 we can assume that when $s_{-i,j} = s_{-k,l}$,

$$\begin{aligned}\max_{s_i\in S} \min_{s_j\in S} u_i(s_i, s_j, s_{-i,j}) &= \max_{s_k\in S} \min_{s_l\in S} u_k(s_k, s_l, s_{-k,l}) \\ &= \min_{s_l\in S} \max_{s_k\in S} u_k(s_k, s_l, s_{-k,l}) = \min_{s_j\in S} \max_{s_i\in S} u_i(s_i, s_j, s_{-i,j}),\end{aligned}$$

$$\arg\max_{s_i\in S} \min_{s_j\in S} u_i(s_i, s_j, s_{-i,j}) = \arg\max_{s_k\in S} \min_{s_l\in S} u_k(s_k, s_l, s_{-k,l}),$$

and

$$\arg\min_{s_l\in S}\max_{s_k\in S} u_k(s_k, s_l, s_{-k,l}) = \arg\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, s_{-i,j}) \text{ for } i, j, k, l \in N.$$

They mean

$$\begin{aligned}\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, s_{-i,j}) &= \max_{s_j\in S}\min_{s_i\in S} u_j(s_i, s_j, s_{-i,j})\\ = \min_{s_i\in S}\max_{s_j\in S} u_j(s_i, s_j, s_{-i,j}) &= \min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, s_{-i,j}),\end{aligned}$$

$$\arg\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, s_{-i,j}) = \arg\max_{s_j\in S}\min_{s_i\in S} u_j(s_i, s_j, s_{-i,j}),$$

and

$$\arg\min_{s_i\in S}\max_{s_j\in S} u_j(s_i, s_j, s_{-i,j}) = \arg\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, s_{-i,j}) \text{ for any } i, j.$$

Assume $(s_1, s_2, \dots, s_n) = (s, s, \dots, s)$, and let $\mathbf{s}_{-i,j}$ be a vector of $s_k,\ k \in N,\ k \neq i, j$ such that $s_k = s$. Then, for a symmetric game Lemma 2 is rewritten as follows.

Lemma 3. *Let $j \neq i$, and S_i and S_j be non-void convex and compact subsets of two linear topological spaces, let $u_i : S_i \times S_j \to \mathbb{R}$ given $\mathbf{s}_{-i,j}$ be a function that is continuous on $S_1 \times S_2 \times \cdots \times S_n$, quasi-concave on S_i and quasi-convex on S_j, and assume $S_i = S_j = S$. Then*

$$\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) \text{ for any } i, j.$$

3 The main results

Consider a Nash equilibrium of a symmetric multi-person zero-sum game. Let $s_i^*,\ i \in N$, be the values of s_i's which, respectively, maximize $u_i,\ i \in N$, given $s_j^*, j \neq i$, in a neighborhood around $(s_1^*, s_2^*, \dots, s_n^*)$ in $S_1 \times S_2 \times \cdots \times S_n = S^n$. Then,

$$u_i(s_1^*, \dots, s_i^*, \dots, s_n^*) \geq u_i(s_1^*, \dots, s_i, \dots, s_n^*) \text{ for all } s_i \neq s_i^*,\ i \in N.$$

If the Nash equilibrium is symmetric, all s_i^*'s are equal at equilibria. Then, $u_i(s_1^*, \dots, s_i^*, \dots, s_n^*)$'s for all i are equal, and by the property of zero-sum game they are zero.

We show the following theorem.

Theorem 1. *The existence of Nash equilibrium in a symmetric multi-person zero-sum game implies Sion's minimax theorem, and implies that the maximin strategy and the minimax strategy for each pair of players coincide at the symmetric Nash equilibrium.*

Proof. (1) Let $(s^*, s^*, \dots, s^*)$ be a symmetric Nash equilibrium of an n-person zero-sum game. Then,

$$u_i(s^*, s^*, \mathbf{s}^*_{-i,j}) = \max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) \geq u_i(s_i, s^*, \mathbf{s}^*_{-i,j}). \tag{2}$$

$\mathbf{s}^*_{-i,j}$ is a vector of $s_k,\ k \in N,\ k \neq i, j$ such that $s_k = s^*$. Since the game is zero-sum,

$$u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) + (n-1)u_j(s_i, s^*, \mathbf{s}^*_{-i,j}) = 0,\ j \neq i$$

imply

$$u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = -(n-1)u_j(s_i, s^*, \mathbf{s}^*_{-i,j}).$$

This equation holds for any s_i. Thus,

$$\arg\max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = \arg\min_{s_i \in S} u_j(s_i, s^*, \mathbf{s}^*_{-i,j}).$$

By the assumption of the uniqueness of the best responses, they are unique.

By the symmetry of the game,

$$\arg\max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = \arg\min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) = s^*.$$

Therefore,

$$u_i(s^*, s^*, \mathbf{s}^*_{-i,j}) = \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) \leq u_i(s^*, s_j, \mathbf{s}^*_{-i,j}).$$

With (2), we get

$$\max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = u_i(s^*, s^*, \mathbf{s}^*_{-i,j}) = \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}).$$

This means

$$\begin{aligned}\min_{s_j \in S}\max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) &\leq \max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) \\ &= \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) \leq \max_{s_i \in S}\min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}).\end{aligned} \tag{3}$$

On the other hand, since

$$\min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \leq u_i(s_i, s_j, \mathbf{s}^*_{-i,j}),$$

we have

$$\max_{s_i \in S}\min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \leq \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}).$$

This inequality holds for any s_j. Thus,

$$\max_{s_i \in S}\min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \leq \min_{s_j \in S}\max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}).$$

With (3), we obtain the following minimax theorem (Lemma 3).

$$\max_{s_i \in S}\min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \min_{s_j \in S}\max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}).$$

(3) implies

$$\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \max_{s_i\in S} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}),$$

and

$$\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \min_{s_j\in S} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}).$$

From

$$\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \le u_i(s_i, s^*, \mathbf{s}^*_{-i,j}),$$

and

$$\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \max_{s_i\in S} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}),$$

we have

$$\arg\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \arg\max_{s_i\in S} u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) = s^*.$$

Also, from

$$\max_{s_j\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \ge u_i(s^*, s_j, \mathbf{s}^*_{-i,j}),$$

and

$$\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \min_{s_j\in S} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}),$$

we get

$$\arg\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \arg\min_{s_j\in S} u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) = s^*.$$

Therefore,

$$\arg\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \arg\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}).$$

□

Next we show the following theorem.

Theorem 2. *Sion's minimax theorem plus the coincidence of the maximin strategy and the minimax strategy imply the existence of a symmetric Nash equilibrium.*

Proof. Let $\mathbf{s} = (s, s, \dots, s)$. By the minimax theorem

$$\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}).$$

Assume

$$\arg\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \arg\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}). \tag{4}$$

Consider the following function;

$$s \to \arg\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}).$$

Since u_i is continuous and S is compact, this function is also continuous. Thus, by the Glicksberg fixed point theorem there exists a fixed point. Denote it by $\tilde{s}$. Let $\tilde{\mathbf{s}} = (\tilde{s}, \tilde{s}, \dots, \tilde{s})$. Then, from the minimax theorem and

$$\tilde{s} = \arg\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \arg\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

we have

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \max_{s_i \in S} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}).$$

$\tilde{\mathbf{s}}_{-i,j}$ is a vector of $s_k,\ k \in N,\ k \neq i, j$ such that $s_k = \tilde{s}$. Since

$$u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) \leq \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

and

$$\min_{s_j \in S} u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

we get

$$\arg\min_{s_j \in S} u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) = \arg\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \tilde{s}.$$

Also, since

$$u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) \geq \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

and

$$\max_{s_i \in S} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) = \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

we obtain

$$\arg\max_{s_i \in S} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) = \arg\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \tilde{s}.$$

Therefore,

$$u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) \geq u_i(\tilde{s}, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) \geq u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}),$$

and so $(\tilde{s}, \tilde{s}, \dots, \tilde{s})$ is a symmetric Nash equilibrium of an n-person zero-sum game. □

4 Note on the case where (4) is not assumed.

Let $\mathbf{s} = (s, s, \dots, s)$, and define

$$s^1 = \arg\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}),$$

$$s^2 = \arg\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}).$$

Let $\bar{s}$ be the fixed point of the following function;

$$s \to s^1(s).$$

Then, by the minimax theorem

$$\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \min_{s_j\in S} u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = \min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}).$$

$\bar{\mathbf{s}}_{-i,j}$ is a vector of $s_k,\ k \in N,\ k \neq i$ such that $s_k = \bar{s}$. Since

$$\max_{s_i\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) \geq u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}),$$

and

$$\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \min_{s_j\in S} u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}),$$

we have

$$\arg\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \arg\min_{s_j\in S} u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = s^2.$$

Then,

$$\min_{s_j\in S}\max_{s_i\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \max_{s_i\in S} u_i(s_i, s^2, \bar{\mathbf{s}}_{-i,j}).$$

Since

$$\min_{s_j\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) \leq u_i(s_i, s^2, \bar{\mathbf{s}}_{-i,j}),$$

and

$$\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \max_{s_i\in S} u_i(s_i, s^2, \bar{\mathbf{s}}_{-i,j}),$$

we have

$$\arg\max_{s_i\in S}\min_{s_j\in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \arg\max_{s_i\in S} u_i(s_i, s^2, \bar{\mathbf{s}}_{-i,j}) = \bar{s}. \tag{5}$$

Because the game is symmetric and zero-sum,

$$(n-1)u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) + u_j(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = 0.$$

Thus,

$$u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = -(n-1)u_j(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}).$$

This means

$$\arg\min_{s_j\in S} u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = \arg\max_{s_j\in S} u_j(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = s^2. \tag{6}$$

(5) is applicable to each player other than one player denoted by j in (6). Therefore, if $s^2 \neq \bar{s}$, there may exist an asymmetric Nash equilibrium denoted as follows.

$$(\bar{s}, \dots, \bar{s}, s^2, \bar{s}, \dots, \bar{s})$$

In which only $s_j = s^2$. Of course, Theorem 1 means that there always exists a symmetric Nash equilibrium. Thus, in this case we have multiple equilibria.

5 Example of asymmetric multi-person zero-sum game

Consider a three-person game. Suppose that the payoff functions of players are

$$\pi_1 = (a - s_1 - s_2 - s_3)s_1 - c_1 s_1 - \frac{1}{2}[(a - s_2 - s_1 - s_3)s_2 - c_2 s_2 + (a - s_3 - s_2 - s_1)s_3 - c_3 s_3],$$

$$\pi_2 = (a - s_2 - s_1 - s_3)s_2 - c_2 s_2 - \frac{1}{2}[(a - s_1 - s_2 - s_3)s_1 - c_1 s_1 + (a - s_3 - s_2 - s_1)s_3 - c_3 s_3],$$

and

$$\pi_3 = (a - s_3 - s_2 - s_1)s_3 - c_3 s_3 - \frac{1}{2}[(a - s_1 - s_2 - s_3)s_1 - c_1 s_1 + (a - s_2 - s_1 - s_3)s_2 - c_2 s_2].$$

This is a model of relative profit maximization in a three firms Cournot oligopoly with constant marginal costs and zero fixed cost producing a homogeneous good. $s_i, i = 1, 2, 3$, are the outputs of the firms. The conditions for maximization of $\pi_i,\ i = 1, 2, 3$, are

$$\frac{\partial \pi_1}{\partial s_1} = a - 2s_1 - (s_2 + s_3) - c_1 + \frac{1}{2}(s_2 + s_3) = 0,$$

$$\frac{\partial \pi_2}{\partial s_2} = a - 2s_2 - (s_1 + s_3) - c_2 + \frac{1}{2}(s_1 + s_3) = 0,$$

and

$$\frac{\partial \pi_3}{\partial s_3} = a - 2s_3 - (s_2 + s_1) - c_3 + \frac{1}{2}(s_2 + s_1) = 0.$$

The Nash equilibrium strategies are

$$s_1 = \frac{3a - 5c_1 + c_2 + c_3}{9},\ s_2 = \frac{3a - 5c_2 + c_1 + c_3}{9},\ s_3 = \frac{3a - 5c_3 + c_2 + c_1}{9}. \tag{7}$$

We consider maximin and minimax strategy about Player 1 and 2. The condition for minimization of π_1 with respect to s_2 is $\frac{\partial \pi_1}{\partial s_2} = 0$. Denote s_2 which satisfies this condition by $s_2(s_1, s_3)$, and substitute it into π_1. Then, the condition for maximization of π_1 with respect to s_1 given $s_2(s_1, s_3)$ and s_3 is

$$\frac{\partial \pi_1}{\partial s_1} + \frac{\partial \pi_1}{\partial s_2}\frac{\partial s_2}{\partial s_1} = 0.$$

We call the strategy of Player 1 obtained from these conditions the maximin strategy of Player 1 to Player 2. It is denoted by $\arg\max_{s_1} \min_{s_2} \pi_1$. The condition for maximization of π_1 with respect to s_1 is $\frac{\partial \pi_1}{\partial s_1} = 0$. Denote s_1 which satisfies this condition by $s_1(s_2, s_3)$, and substitute it into π_1. Then, the condition for minimization of π_1 with respect to s_2 given $s_1(s_2, s_3)$ is

$$\frac{\partial \pi_1}{\partial s_2} + \frac{\partial \pi_1}{\partial s_1}\frac{\partial s_1}{\partial s_2} = 0.$$

We call the strategy of Player 2 obtained from these conditions the minimax strategy of Player 2 to Player 1. It is denoted by $\arg\min_{s_2}\max_{s_1}\pi_1$. In our example we obtain

$$\arg\max_{s_1}\min_{s_2}\pi_1=\frac{3a-4c_1+c_2}{9},$$

$$\arg\min_{s_2}\max_{s_1}\pi_1=\frac{6a-9s_3-2c_1-4c_2}{9}.$$

Similarly, we get the following results.

$$\arg\max_{s_2}\min_{s_1}\pi_2=\frac{3a-4c_2+c_1}{9},$$

$$\arg\min_{s_1}\max_{s_2}\pi_2=\frac{6a-9s_3-2c_2-4c_1}{9},$$

$$\arg\max_{s_1}\min_{s_3}\pi_1=\frac{3a-4c_1+c_3}{9},$$

$$\arg\min_{s_3}\max_{s_1}\pi_1=\frac{6a-9s_2-2c_1-4c_3}{9},$$

$$\arg\max_{s_3}\min_{s_1}\pi_3=\frac{3a-4c_3+c_1}{9},$$

$$\arg\min_{s_1}\max_{s_3}\pi_3=\frac{6a-9s_2-2c_3-4c_1}{9},$$

$$\arg\max_{s_2}\min_{s_3}\pi_2=\frac{3a-4c_2+c_3}{9},$$

$$\arg\min_{s_3}\max_{s_2}\pi_2=\frac{6a-9s_1-2c_2-4c_3}{9},$$

$$\arg\max_{s_3}\min_{s_2}\pi_3=\frac{3a-4c_3+c_2}{9},$$

$$\arg\min_{s_2}\max_{s_3}\pi_3=\frac{6a-9s_1-2c_3-4c_2}{9}.$$

If the game is asymmetric, for example, $c_2\neq c_3$, $\arg\max_{s_1}\min_{s_2}\pi_1\neq\arg\max_{s_1}\min_{s_3}\pi_1$, $\arg\max_{s_2}\min_{s_3}\pi_2\neq\arg\max_{s_3}\min_{s_2}\pi_3$, $\arg\min_{s_3}\max_{s_2}\pi_2\neq\arg\min_{s_2}\max_{s_3}\pi_3$, and so on. However, if the game is symmetric, we have $c_2=c_3=c_1$ and

$$\arg\max_{s_1}\min_{s_2}\pi_1=\arg\max_{s_2}\min_{s_1}\pi_2=\arg\max_{s_1}\min_{s_3}\pi_1=\arg\max_{s_3}\min_{s_1}\pi_3$$
$$=\arg\max_{s_2}\min_{s_3}\pi_2=\arg\max_{s_3}\min_{s_2}\pi_3=\frac{a-c_1}{3}.$$

All of the Nash equilibrium strategies of the players in (7) are also equal to $\frac{a-c_1}{3}$. Assume $s_2 = s_3 = s_1$ as well as $c_2 = c_3 = c_1$. Then,

$$\arg\min_{s_2}\max_{s_1}\pi_1 = \arg\min_{s_1}\max_{s_2}\pi_2 = \arg\min_{s_3}\max_{s_1}\pi_1 = \arg\min_{s_1}\max_{s_3}\pi_3$$
$$= \arg\min_{s_3}\max_{s_2}\pi_2 = \arg\min_{s_2}\max_{s_3}\pi_3 = \frac{2a - 3s_1 - 2c_1}{3}.$$

Further, if

$$s_1 = \arg\min_{s_1}\max_{s_2}\pi_2 = \arg\min_{s_1}\max_{s_3}\pi_3,$$

we obtain

$$\arg\min_{s_2}\max_{s_1}\pi_1 = \arg\min_{s_1}\max_{s_2}\pi_2 = \arg\min_{s_3}\max_{s_1}\pi_1 = \arg\min_{s_1}\max_{s_3}\pi_3$$
$$= \arg\min_{s_3}\max_{s_2}\pi_2 = \arg\min_{s_2}\max_{s_3}\pi_3 = \frac{a - c_1}{3}.$$

Therefore, the maximin strategy, the minimax strategy and the Nash equilibrium strategy for all players are equal.

6 Concluding Remark

In this paper we have shown that Sion's minimax theorem plus coincidence of the maximin strategy and the minimax strategy is equivalent to the existence of a symmetric Nash equilibrium in a symmetric multi-person zero-sum game. As we have shown in Section 4, if a game is asymmetric, the equivalence result does not hold.

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References

Glicksberg, I.L. (1952) "A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points." *Proceedings of the American Mathematical Society*, **3**, pp.170-174.

Kindler, J. (2005), "A simple proof of Sion's minimax theorem," *American Mathematical Monthly*, **112**, pp. 356-358.

Komiya, H. (1988), "Elementary proof for Sion's minimax theorem," *Kodai Mathematical Journal*, **11**, pp. 5-7.

Matsumura, T., N. Matsushima and S. Cato (2013) “Competitiveness and R&D competition revisited,” *Economic Modelling*, **31**, pp. 541-547.

Satoh, A. and Y. Tanaka (2013) “Relative profit maximization and Bertrand equilibrium with quadratic cost functions,” *Economics and Business Letters*, **2**, pp. 134-139, 2013.

Satoh, A. and Y. Tanaka (2014a) “Relative profit maximization and equivalence of Cournot and Bertrand equilibria in asymmetric duopoly,” *Economics Bulletin*, **34**, pp. 819-827, 2014.

Satoh, A. and Y. Tanaka (2014b), “Relative profit maximization in asymmetric oligopoly,” *Economics Bulletin*, **34**, pp. 1653-1664.

Sion, M. (1958), “On general minimax theorems,” *Pacific Journal of Mathematics*, **8**, pp. 171-176.

Tanaka, Y. (2013a) “Equivalence of Cournot and Bertrand equilibria in differentiated duopoly under relative profit maximization with linear demand,” *Economics Bulletin*, **33**, pp. 1479-1486.

Tanaka, Y. (2013b) “Irrelevance of the choice of strategic variables in duopoly under relative profit maximization,” *Economics and Business Letters*, **2**, pp. 75-83, 2013.

Vega-Redondo, F. (1997) “The evolution of Walrasian behavior,”, *Econometrica*, **65**, pp. 375-384.