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Abstract

This study develops a Schumpeterian growth model with heterogeneous households and heterogeneous firms to explore the effects of monetary policy on innovation and income inequality. Household heterogeneity arises from an unequal distribution of wealth. Firm heterogeneity arises from random quality improvements and a cost of entry. We find that under endogenous firm entry, inflation has inverted-U effects on economic growth and income inequality. We also calibrate the model for a quantitative analysis and find that the model is able to match the growth-maximizing inflation rate and the inequality-maximizing inflation rate that we estimate using cross-country panel data.

JEL classification: O30, O40, D30, E41
Keywords: inflation, income inequality, economic growth, heterogeneity

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1 Introduction

The seminal study by Tobin (1965) initiated an influential literature in macroeconomics that explores the relationship between inflation and economic growth. Studies in this literature have focused on how inflation affects economic growth via the accumulation of physical capital and/or human capital.[1] However, an important insight from the seminal study by Solow (1956) is that economic growth is ultimately driven by technological progress. Therefore, it is important to also understand the effects of inflation in a growth model with endogenous technological progress. Marquis and Reffett (1994) explore the effects of inflation in the R&D-based growth model developed by Romer (1990). However, this early study by Marquis and Reffett (1994) and many subsequent studies in this branch of the literature have mostly focused on a representative-household setting with homogeneous firms. In this study, we address this issue and obtain novel results.

Specifically, we develop a monetary Schumpeterian growth model with heterogeneous firms and heterogenous households. We model firm heterogeneity in the Schumpeterian quality-ladder model by assuming that the step size of quality improvements is randomly drawn from a Pareto distribution. Then, to allow for endogenous firm entry, we assume that R&D entrepreneurs need to pay an entry cost to enter the market after observing the step size of their quality improvements. As a result, an entrepreneur would enter the market if and only if her quality improvement is sufficiently large, which in turn generates an endogenous distribution of quality improvements that are implemented. Motivated by the empirical evidence in Piketty (2014), we consider an unequal distribution of wealth as an important source of income inequality. Therefore, we model household heterogeneity in the Schumpeterian model by assuming that households have different levels of wealth in order to generate an endogenous income distribution. Within this growth-theoretic framework, we explore the effects of monetary policy on innovation and income inequality. In summary, we find that inflation has inverted-U effects on economic growth and income inequality under endogenous firm entry.

The inverted-U effect of inflation on economic growth under endogenous entry of heterogeneous firms is the same as in Chu *et al.* (2017). They showed that inflation increases the cost of R&D via the cash-in-advance (CIA) constraint on R&D and decreases the arrival rate of innovation, which is a negative effect of inflation on economic growth.[2] The lower rate of creative destruction however increases the expected value of future profits and the market value of inventions, which in turn lowers the entry threshold for quality improvements. With more inventions being implemented, inflation also has a positive effect on economic growth. These positive and negative effects together generate an inverted-U effect of inflation on economic growth so long as the entry cost is sufficiently large.

Interestingly, this inverted-U effect of inflation on economic growth leads to a novel inverted-U effect of inflation on income inequality in the Schumpeterian model. In our model, income inequality is increasing in the ratio of wealth income to wage income. Therefore, either an increase in the real interest rate or an increase in the value of financial assets would increase income inequality. Given the Euler equation under which the real interest rate is increasing in the growth rate of consumption, the abovementioned inverted-U effect of inflation on economic

[1] See for example Stockman (1981), Abel (1985), Dotsey and Ireland (1996) and Gillman and Kejak (2005).

[2] Marquis and Reffett (1994) also find a negative effect of inflation on R&D, which is supported by empirical evidence based on cross-country panel regressions in Chu *et al.* (2015).

growth causes an inverted-U effect on the real interest rate and hence also an inverted-U effect on income inequality. Furthermore, inflation has both positive and negative effects on the value of financial assets. On the one hand, by slowing down the rate of creative destruction, inflation increases the market value of monopolistic firms, which in turn increases the value of financial assets. On the other hand, by lowering the entry threshold for quality improvements, inflation reduces the average step size of quality improvements implemented in the market and decreases the average markup ratio, which in turn decreases the market values of monopolistic firms and financial assets. Combining all these effects yields an overall inverted-U effect of inflation on income inequality, which exists only under endogenous entry of heterogeneous firms. Finally, we calibrate the model to perform a quantitative analysis and find that our model is able to match a growth-maximizing inflation rate of 15% and an inequality-maximizing inflation rate of 10% that are estimated using cross-country panel data.

This study relates to the literature on innovation and economic growth. Romer (1990) develops the seminal R&D-based growth model in which economic growth is driven by the invention of new products. Segerstrom *et al.* (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992) develop the Schumpeterian quality-ladder model in which economic growth is driven by the innovation of higher-quality products. For tractability, these seminal studies and many subsequent studies assume a constant step size of quality improvement. Important exceptions include Klette and Kortum (2004) and Minniti *et al.* (2013). Minniti *et al.* (2013) develop a Schumpeterian growth model with random step sizes of quality improvements drawn from a Pareto distribution.[3] Chu *et al.* (2017) extend the elegant model in Minniti *et al.* (2013) by allowing for a Hopenhayn-Melitz-type entry cost to generate endogenous entry of heterogeneous firms.[4] This study further extends the representative-household model in Chu *et al.* (2017) by introducing heterogenous households with different asset holdings. In other words, this study contributes to the literature by developing a Schumpeterian growth model with two dimensions of heterogeneity among households and firms.

This study also relates to the literature on innovation and inflation. In this literature, the seminal study by Marquis and Reffett (1994) analyzes the effects of inflation on innovation in a variant of the Romer variety-expanding model. Subsequent studies analyze the effects of inflation in the Schumpeterian quality-ladder model; see for example Chu and Lai (2013), Chu and Cozzi (2014), Chu *et al.* (2015), He and Zou (2016), Huang *et al.* (2017) and Neto *et al.* (2017). However, all these studies feature a constant step size of quality improvement. As a result, these studies predict a monotonic relationship between inflation and economic growth, which is different from the inverted-U relationship between inflation and economic growth often found in empirical studies.[5] As a result, Chu *et al.* (2017) develop a monetary Schumpeterian growth model with endogenous entry of heterogeneous firms,[6] and they show that their model can generate an inverted-U relationship between inflation and economic growth and match

[3] A recent study by Iwaisako and Ohki (2017) develops a Schumpeterian growth model with random quality improvements drawn from a uniform distribution.

[4] See also Baldwin and Robert-Nicoud (2008), Haruyama and Zhao (2008) and Gustafsson and Segerstrom (2010) who adapt a similar entry cost into the R&D-based growth model with heterogeneous firms, but they do not consider random increments on the quality ladder.

[5] See for example Bick (2010) and Lopez-Villavicencio and Mignon (2011) for recent studies.

[6] See also Hori (2017) and Arawatari *et al.* (2018) who consider monetary policy in the Romer variety-expanding model with heterogeneity in the productivity of R&D entrepreneurs.

empirical estimates of the growth-maximizing inflation rate under plausible parameter values. However, all the abovementioned studies feature a representative household; therefore, they cannot be used to analyze the implications of monetary policy on the income distribution. To fill this important gap in the literature, this study introduces heterogeneous households into the model in Chu *et al.* (2017) and analyzes the effects of monetary policy on income inequality in addition to innovation and economic growth.

This study also relates to the literature on innovation and income inequality. Representative studies include Chou and Talmain (1996), Li (1998), Zweimuller (2000), Foellmi and Zweimuller (2006), Aghion *et al.* (2015), Kiedaisch (2016), Grossman and Helpman (2018) and Jones and Kim (2018). These studies focus on the relationship between income inequality and innovation. Our study complements these interesting studies by exploring the effects of monetary policy on innovation and income inequality. Chu and Cozzi (2018) explore the effects of R&D subsidies and patent policy on income inequality, but not monetary policy. More importantly, Chu and Cozzi (2018) focus on a Schumpeterian growth model with a constant step size of quality improvement. We show that endogenous entry of heterogeneous firms is necessary for the emergence of an inverted-U effect of inflation on income inequality that is consistent with our empirical finding.[7]

The rest of this study is organized as follows. Section 2 presents the model and solves the market equilibrium of the aggregate economy. Section 3 explores the distributions of wealth and income. Section 4 analyzes the effects of monetary policy. Section 5 provides a quantitative analysis. Section 6 concludes. Proofs are relegated to the appendix.

2 A Schumpeterian model with heterogeneous firms and heterogeneous households

The Schumpeterian quality-ladder model is based on Grossman and Helpman (1991).[8] We consider the monetary Schumpeterian growth model in Chu *et al.* (2017) featuring (a) a CIA constraint on R&D,[9] (b) lab-equipment specifications for innovation and entry that use final good as the input, (c) random quality improvements as in Minniti *et al.* (2013) and (d) a fixed entry cost that generates endogenous entry of heterogeneous firms as in Hopenhayn (1992) and Melitz (2003). Furthermore, we follow Chu and Cozzi (2018) to introduce heterogeneous households with different asset holdings into the monetary Schumpeterian model.

2.1 Households

There is a unit continuum of households, which are indexed by $h \in [0, 1]$. They have identical preferences over consumption $c_t(h)$ but own different levels of wealth. Household h has the

[7] See also Natob (2015) who finds an inverted-U effect of inflation on income inequality using dynamic panel regressions with cross-country data.

[8] See also Aghion and Howitt (1992).

[9] See Chu *et al.* (2015) for a discussion of empirical evidence for the presence of cash requirements on R&D and also Berentsen *et al.* (2012) for a discussion of theoretical justifications and microfoundations.

following utility function:

$$u(h) = \int_0^\infty e^{-\rho t} \ln c_t(h) dt, \tag{1}$$

where the parameter $\rho > 0$ is the subjective discount rate. Household h supplies one unit of labor to earn wage income w_t and maximizes utility $u(h)$ subject to

$$\dot{a}_t(h) + \dot{m}_t(h) = r_t a_t(h) - \pi_t m_t(h) + i_t b_t(h) + w_t + \tau_t - c_t(h). \tag{2}$$

$a_t(h)$ is the real value of financial assets (i.e., equity of monopolistic firms) owned by household h, and r_t is the real interest rate. $m_t(h)$ is the real value of cash holdings of household h, and π_t is the inflation rate. $b_t(h)$ is the amount of cash borrowed from household h by entrepreneurs for R&D, and i_t is the nominal interest rate.[10] The CIA constraint is given by $b_t(h) \leq m_t(h)$. Finally, the government provides a lump-sum transfer τ_t to each household.[11]

From standard dynamic optimization, household h's consumption path is given by

$$\frac{\dot{c}_t(h)}{c_t(h)} = r_t - \rho, \tag{3}$$

which shows that the growth rate of consumption is the same across households such that $\dot{c}_t(h)/c_t(h) = \dot{c}_t/c_t$ for all $h \in [0,1]$, where $c_t \equiv \int_0^1 c_t(h)dh$ denotes aggregate consumption. Therefore, the growth rate of aggregate consumption is also given by

$$\frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{4}$$

2.2 Final good

Final good y_t is produced by a unit continuum of competitive firms using the following Cobb-Douglas production function:

$$y_t = A_t L_t^\theta K_t^{1-\theta}, \tag{5}$$

where L_t is labor input and $\theta \in (0,1)$ measures labor intensity in production. K_t is a composite of a unit continuum of differentiated intermediate goods $k_t(j)$ given by

$$K_t = \exp\left(\int_0^1 \ln k_t(j) dj\right). \tag{6}$$

A_t captures a productive externality from $\overline{K}_t$ such that $A_t = \overline{K}_t^\varepsilon$,[12] where $\overline{K}_t$ is the aggregate level of K_t and the parameter $\varepsilon \in [0,\theta)$ measures the degree of this productive externality.[13] From profit maximization using (5), the conditional demand function for L_t is

$$w_t L_t = \theta y_t. \tag{7}$$

[10] It can be shown as a no-arbitrage condition that the rate of return on borrowing $b_t(h)$ must equal $r_t + \pi_t$.

[11] The transfer is financed by seigniorage. Alternatively, we can assume that seigniorage is used to finance a public good, in which case all our results would be the same so long as the (potentially utility-enhancing) public good is non-productive.

[12] See for example Ho *et al.* (2007) for a discussion of this type of productive externality. Here we assume that the externality arises from the production of intermediate goods.

[13] Our analytical result is robust to the absence of this productive externality (i.e., $\varepsilon = 0$ and $A_t = 1$). We allow for this parameter to improve the ability of the model to match data in the quantitative analysis.

From profit maximization using (5) and (6), the conditional demand function for $k_t(j)$ is

$$p_t(j)k_t(j) = (1-\theta)\, y_t, \tag{8}$$

where $p_t(j)$ is the price of $k_t(j)$.[14]

2.3 Intermediate goods

There is a unit continuum of industries indexed by $j \in [0, 1]$. In each industry j, there is a monopolistic industry leader, who holds a patent on the latest technology and dominates the market until the arrival of the next innovation.[15] The production function of the leader in industry j is

$$k_t(j) = q_t(j, \omega_j)x_t(j), \tag{9}$$

where $q_t(j, \omega_j)$ is the quality-level of the leader in industry j and ω_j is an integer denoting the quality vintage of the intermediate goods produced by the leader in industry j. $x_t(j)$ is the quantity of input j produced using final good with an one-to-one technology (i.e., $x_t(j)$ units of final good produce $x_t(j)$ units of input j). From Bertrand competition, the equilibrium price of $k_t(j)$ is a markup over the marginal cost $1/q_t(j, \omega_j)$ given by

$$p_t(j) = \frac{\lambda_t(j)}{q_t(j, \omega_j)}, \tag{10}$$

where the markup ratio $\lambda_t(j) \equiv q_t(j, \omega_j)/q_t(j, \omega_j - 1)$ is determined by the size of the quality improvement by the leader in industry j. The equilibrium level of monopolistic profit is

$$\Pi_t(j) = \left[\frac{\lambda_t(j) - 1}{\lambda_t(j)}\right] p_t(j)k_t(j) = \left[\frac{\lambda_t(j) - 1}{\lambda_t(j)}\right](1-\theta)y_t \equiv \Pi_t(\lambda), \tag{11}$$

where the second equality uses (8).

2.4 R&D and entry

In this section, we present the three steps of innovation. First, an entrepreneur invents a higher quality product. Then, the size of the quality improvement is randomly drawn from a Pareto distribution. Finally, if and only if the quality improvement is sufficiently large, then the entrepreneur would pay a fixed entry cost to enter the market.

[14] Here $p_t(j)$ is denominated in units of the final good.

[15] See Cozzi (2007) for a discussion of this Arrow replacement effect.

2.4.1 Invention

R&D is performed by competitive entrepreneurs. If an entrepreneur employs $R_t(j)$ units of final good to engage in innovation in industry j, then she would succeed in inventing the next higher-quality product in the industry with an instantaneous probability $\phi_t(j)$ given by

$$\phi_t(j) = R_t(j)/\alpha_t. \tag{12}$$

For convenience, we define a composite parameter $\psi \equiv \theta - \varepsilon$, where $\psi \in (0, \theta]$. To ensure balanced growth, $\alpha_t \equiv \alpha Q_t^{(1-\psi)/\psi}$ measuring the difficulty of R&D is increasing the aggregate technology level Q_t,[16] which is defined as

$$Q_t \equiv \exp\left(\int_0^1 \ln q_t(j, \omega_j) dj\right). \tag{13}$$

To facilitate the payment of $R_t(j)$, the entrepreneur needs to borrow the amount $\zeta R_t(j)$ of cash from households, where $\zeta \in (0, 1]$ is the CIA parameter. The borrowing cost is determined by the nominal interest rate i_t. Therefore, the total cost of R&D is $(1 + \zeta i_t)R_t(j)$. Let's use $v_t^e(j, \omega_j + 1)$ to denote the expected value of an invention before the realization of the size of its quality improvement. The R&D condition is given by

$$\phi_t(j) v_t^e(j, \omega_j + 1) = (1 + \zeta i_t) R_t(j) \Leftrightarrow v_t^e(j, \omega_j + 1) = (1 + \zeta i_t) \alpha Q_t^{(1-\psi)/\psi}. \tag{14}$$

2.4.2 Random quality improvements

We follow Minniti *et al.* (2013) to assume that when an R&D entrepreneur invents a higher-quality product in industry j, the quality step size $\lambda_t(j) > 1$ is randomly drawn from a stationary Pareto distribution with the following probability density function:

$$f(\lambda) = \frac{1}{\kappa} \lambda^{-\frac{1+\kappa}{\kappa}}, \tag{15}$$

where the parameter $\kappa \in (0, 1)$ determines the shape of the Pareto distribution. Given that the expected value of $\lambda_t(j)$ is equal across industries, (11) implies that the expected value $\Pi_t^e(j)$ of monopolistic profit $\Pi_t(j)$ is also the same across industries such that $\Pi_t^e(j) = \Pi_t^e$ for $j \in [0, 1]$. Therefore, we follow the standard treatment to focus on the symmetric equilibrium in which the arrival rate of innovation is equal across industries, such that $\phi_t(j) = \phi_t$ for $j \in [0, 1]$.[17] As a result, the expected value of an invention does not depend on j such that $v_t^e(j, \omega_j + 1) = v_t^e$ for $j \in [0, 1]$.

[16] Venturini (2012) provides empirical evidence for the presence of increasing R&D difficulty.

[17] Cozzi *et al.* (2007) provide a theoretical justification for the symmetric equilibrium to be the unique rational-expectation equilibrium in the Schumpeterian model.

2.4.3 Endogenous firm entry

Following Hopenhayn (1992) and Melitz (2003), we consider a fixed entry cost to generate an endogenous entry of heterogeneous firms. Let's denote $v_t(\lambda)$ as the *ex post* value of an invention (i.e., after the realization of the quality step size λ). In this case, the entry condition is given by

$$v_t(\lambda) \geq \beta_t, \tag{16}$$

where the entry cost $\beta_t = \beta Q_t^{(1-\psi)/\psi}$ is proportional to $Q_t^{(1-\psi)/\psi}$ to ensure balanced growth. Given that $\Pi_t(\lambda)$ is increasing in λ, there exists a threshold quality level $\tilde{\lambda}_t$ above which $v_t(\lambda) \geq \beta_t$ for all $\lambda \geq \tilde{\lambda}_t$. Also, it can be shown that $v_t(\lambda)/Q_t^{(1-\psi)/\psi}$ is stationary in equilibrium. Then, Lemma 1 shows that the threshold quality level $\tilde{\lambda}$ in $v_t(\tilde{\lambda}) = \beta_t$ is stationary.

Lemma 1 *There exists a unique and stationary threshold quality level $\tilde{\lambda}_t = \tilde{\lambda}$ for all t.*

Proof. See Appendix A. ■

Given the stationary threshold $\tilde{\lambda}$, Lemma 2 derives the no-arbitrage condition for the expected value v_t^e of an invention. In (17), $\Pr(\lambda \geq \tilde{\lambda}) = \tilde{\lambda}^{-1/\kappa}$ is the probability that a randomly drawn quality step size is larger than the threshold $\tilde{\lambda}$.

Lemma 2 *The no-arbitrage condition for the expected value v_t^e of an invention is*

$$r_t = \frac{\Pi_t^e + \dot{v}_t^e + \Pr(\lambda \geq \tilde{\lambda})\dot{\beta}_t - \Pr(\lambda \geq \tilde{\lambda})\phi_t[v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t]}{v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t}. \tag{17}$$

Proof. See Appendix A. ■

2.5 Monetary authority

We consider the nominal interest rate i_t as the policy instrument, which is exogenously set by the monetary authority. The Fisher equation is given by $i_t = \pi_t + r_t$, where $\pi_t \equiv \dot{P}_t/P_t$ is the inflation rate and P_t is the price level of final good. Given the aggregate nominal money balance $M_t \equiv P_t m_t$, the growth rate of the aggregate nominal money balance is

$$\mu_t \equiv \frac{\dot{M}_t}{M_t} = \pi_t + \frac{\dot{m}_t}{m_t} = i_t - r_t + \frac{\dot{m}_t}{m_t} = i_t - \rho - \frac{\dot{c}_t}{c_t} + \frac{\dot{m}_t}{m_t}, \tag{18}$$

where the last equality uses the aggregate consumption path in (4). It can be shown that given a stationary nominal interest rate i, aggregate consumption c_t and aggregate real money balance m_t grow at the same rate on the balanced growth path. Therefore, on the balanced growth path, the growth rate of the nominal money balance is determined by the nominal interest rate such that $\mu = i - \rho$. The government uses the seigniorage revenue $\dot{M}_t$ to finance a lump-sum transfer τ_t that has a real value given by

$$\tau_t = \frac{\dot{M}_t}{P_t} = \mu_t \frac{M_t}{P_t}, \tag{19}$$

which yields $\tau_t = (i - \rho) m_t$ on the balanced growth path.

2.6 Decentralized equilibrium

The equilibrium is a time path of allocations $\{c_t(h), a_t(h), m_t(h), b_t(h), y_t, L_t, k_t(j), x_t(j), R_t(j)\}$ and a time path of prices $\{w_t, r_t, p_t(j), v_t(\lambda)\}$. Also, at each instance of time, the following conditions hold:

- household $h \in [0, 1]$ maximizes utility taking $\{w_t, r_t\}$ as given;
- competitive firms produce final good y_t to maximize profit taking prices as given;
- monopolistic firm $j \in [0, 1]$ produces intermediate good $k_t(j)$ and chooses $\{x_t(i), p_t(j)\}$ to maximize profit;
- competitive R&D entrepreneurs choose $R_t(j)$ to maximize expected profit taking $v_t(\lambda)$ as given;
- the market-clearing condition for labor holds such that $L_t = 1$;
- the market-clearing condition for final good holds such that $\int_0^1 c_t(h)dh + \int_0^1 x_t(j)dj + \int_0^1 R_t(j)dj + \tilde{\lambda}^{-1/\kappa}\phi_t\beta_t = y_t$;
- the total amount of cash owned by households equals the amount of cash borrowed by entrepreneurs such that $\int_0^1 m_t(h)dh = \int_0^1 b_t(h)dh = \zeta \int_0^1 R_t(j)dj$;
- the total value of assets owned by households equals the value of all monopolistic firms such that $\int_0^1 a_t(h)dh = \int_0^1 v_t(j)dj \equiv v_t$;
- the monetary authority uses seigniorage to finance a lump-sum transfer $\tau_t = \dot{M}_t/P_t$.

2.7 Aggregate economy

First, we derive the growth rate of aggregate technology Q_t by differentiating the log of (13) with respect to time and using the law of large numbers:

$$\frac{\dot{Q}_t}{Q_t} = \left[\int_0^1 \ln \lambda_t(j)dj\right] \Pr(\lambda \geq \tilde{\lambda})\phi_t = \left[\int_{\tilde{\lambda}}^\infty (\ln \lambda)\, \tilde{f}(\lambda)d\lambda\right] \tilde{\lambda}^{-1/\kappa}\phi_t = (\ln \tilde{\lambda} + \kappa)\tilde{\lambda}^{-1/\kappa}\phi_t, \tag{20}$$

where the truncated density function $\tilde{f}(\lambda)$ as a result of the threshold $\tilde{\lambda}$ is defined as

$$\tilde{f}(\lambda) \equiv \frac{f(\lambda)}{\int_{\tilde{\lambda}}^{\infty} f(\lambda) d\lambda} = \tilde{\lambda}^{\frac{1}{\kappa}} f(\lambda). \tag{21}$$

In (20), $\tilde{\lambda}^{-1/\kappa}\phi_t$ is the *composite* arrival rate of *implementable* quality improvements and $\kappa + \ln\tilde{\lambda}$ is the average step size of implemented quality improvements. Then, we derive the aggregate production function for y_t in the following lemma:

Lemma 3 *The aggregate production function for y_t is given by*

$$y_t = \left(\frac{1-\theta}{\tilde{\lambda}e^{\kappa}} Q_t\right)^{\frac{1-\psi}{\psi}}. \tag{22}$$

Proof. See Appendix A. ■

The aggregate production function in (22) implies that the growth rate of aggregate output y_t is given by

$$g_t \equiv \frac{\dot{y}_t}{y_t} = \frac{1-\psi}{\psi}\frac{\dot{Q}_t}{Q_t} = \frac{1-\psi}{\psi}(\ln\tilde{\lambda} + \kappa)\tilde{\lambda}^{-1/\kappa}\phi_t, \tag{23}$$

where the last equality uses (20). Lemma 4 shows that given a stationary nominal interest rate i, the aggregate economy jumps to a unique and stable balanced growth path along which ϕ and g are also stationary.

Lemma 4 *The aggregate economy jumps to a unique and stable balanced growth path.*

Proof. See Appendix A. ■

The no-arbitrage condition for the *ex-post* value of an invention with $\lambda \geq \tilde{\lambda}$ is given by

$$\frac{\Pi_t(\lambda)}{v_t(\lambda)} = r + \tilde{\lambda}^{-1/\kappa}\phi - \frac{\dot{v}_t(\lambda)}{v_t(\lambda)} = \rho + \tilde{\lambda}^{-1/\kappa}\phi, \tag{24}$$

where the last equality uses the aggregate consumption path in (4) and the property that c_t and v_t both grow at the steady-state equilibrium growth rate in (23). Then, substituting (11) and (24) into the entry condition $v_t(\tilde{\lambda}) = \beta Q_t^{(1-\psi)/\psi}$, we obtain

$$\left(\frac{\tilde{\lambda}-1}{\tilde{\lambda}}\right)\frac{1}{\beta} = \frac{(\rho + \tilde{\lambda}^{-1/\kappa}\phi)Q_t^{(1-\psi)/\psi}}{(1-\theta)y_t}. \tag{25}$$

From (17), the equilibrium value of v_t^e on the balanced growth path is determined by

$$\frac{\Pi_t^e}{v_t^e + \tilde{\lambda}^{-1/\kappa}\beta_t} = r_t + \tilde{\lambda}^{-1/\kappa}\phi - \frac{\dot{v}_t^e + \tilde{\lambda}^{-1/\kappa}\dot{\beta}_t}{v_t^e + \tilde{\lambda}^{-1/\kappa}\beta_t} = \rho + \tilde{\lambda}^{-1/\kappa}\phi, \tag{26}$$

where the last equality uses the aggregate consumption path in (4) and the property that c_t, v_t^e and β_t all grow at the steady-state equilibrium growth rate in (23). The expected value of monopolistic profit is given by

$$\Pi_t^e = \left[\int_{\tilde{\lambda}}^{\infty}\left(\frac{\lambda-1}{\lambda}\right)f(\lambda)d\lambda\right](1-\theta)y_t = \left[\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}^{\frac{1+\kappa}{\kappa}}}\right](1-\theta)y_t. \tag{27}$$

Substituting (26) and (27) into the R&D condition in (14) yields

$$\left[\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}^{\frac{1+\kappa}{\kappa}}}\right]\frac{1}{(1+\zeta i)\,\alpha+\tilde{\lambda}^{-1/\kappa}\beta} = \frac{(\rho+\tilde{\lambda}^{-1/\kappa}\phi)Q_t^{(1-\psi)/\psi}}{(1-\theta)y_t}. \tag{28}$$

Combining (25) and (28), we obtain the following condition:

$$(\tilde{\lambda}-1)\tilde{\lambda}^{1/\kappa} = \frac{\beta}{\alpha}\frac{\kappa}{1+\kappa}\frac{1}{1+\zeta i}, \tag{29}$$

where the left-hand side is monotonically increasing in $\tilde{\lambda}$. Therefore, (29) implicitly determines the unique equilibrium value of $\tilde{\lambda}$ as a decreasing function in the nominal interest rate i. Using (22), (25) and (29), we obtain the following condition:

$$\tilde{\lambda}^{-1/\kappa}\phi = \frac{\tilde{\lambda}^{-(1/\kappa+1/\psi)}}{1+\zeta i}\frac{\kappa}{1+\kappa}\frac{(1-\theta)^{1/\psi}}{\alpha e^{\kappa(1-\psi)/\psi}} - \rho, \tag{30}$$

which determines the unique equilibrium value of the composite innovation rate $\tilde{\lambda}^{-1/\kappa}\phi$. The right-hand side of (30) is decreasing in the nominal interest rate i for a given value of $\tilde{\lambda}$; however, $\tilde{\lambda}$ is also decreasing in i. Therefore, the overall effect of i on $\tilde{\lambda}^{-1/\kappa}\phi$ is ambiguous. The following proposition from Chu *et al.* (2017) summarizes the overall effects of i on $\tilde{\lambda}^{-1/\kappa}\phi$ and g.

Proposition 1 *If the entry cost parameter β is sufficiently large (small), then an increase in the nominal interest rate i has an inverted-U (a negative) effect on the composite innovation rate $\tilde{\lambda}^{-1/\kappa}\phi$ and the equilibrium growth rate g.*

Proof. See Appendix A. ■

Intuitively, when the entry cost β is zero, the nominal interest rate i has no effect on the distribution of quality improvements that are implemented because all firms enter the market. In this case, the entry threshold becomes $\tilde{\lambda} = 1$, and the equilibrium growth rate $g = \frac{1-\psi}{\psi}\kappa\phi$ is monotonically decreasing in the nominal interest rate i via the innovation arrival rate ϕ. However, when the entry cost β is positive, the nominal interest rate i affects the entry threshold $\tilde{\lambda}$ in addition to the innovation arrival rate ϕ. In this case, $\Pr(\lambda \geq \tilde{\lambda}) = \tilde{\lambda}^{-1/\kappa}$ is increasing in the nominal interest rate i because an increase in the nominal interest rate i reduces the entry threshold $\tilde{\lambda}$ and leads to more quality improvements being implemented. Therefore, the overall effects of the nominal interest rate i on the composite innovation rate $\tilde{\lambda}^{-1/\kappa}\phi$ and the equilibrium growth rate $g = \frac{1-\psi}{\psi}(\ln\tilde{\lambda}+\kappa)\tilde{\lambda}^{-1/\kappa}\phi$ become ambiguous and follow an inverted-U pattern when the entry cost β is sufficiently large.

3 Wealth and income distributions

In this section, we show that the wealth distribution is stationary and exogenously determined by its initial distribution. Then, we show that the income distribution is also stationary but endogenously affected by the nominal interest rate.

3.1 Wealth distribution

In equilibrium, household $h \in [0, 1]$ lends all its cash to entrepreneurs such that $m_t(h) = b_t(h)$. Substituting this condition into (2) yields

$$\dot{a}_t(h) + \dot{b}_t(h) = r_t[a_t(h) + b_t(h)] + w_t + \tau_t - c_t(h), \tag{31}$$

where we have also used the Fisher equation $r_t = i_t - \pi_t$. Aggregating (31) for all h, we have

$$\dot{a}_t + \dot{b}_t = r_t(a_t + b_t) + w_t + \tau_t - c_t. \tag{32}$$

Let's denote $z_t(h) \equiv a_t(h) + b_t(h)$ as household h's wealth, which consists of financial assets and bond holdings. Then, we define $s_{z,0}(h) \equiv z_0(h)/z_0$ as the initial share of wealth owned by household h, and $s_{z,0}(h)$ is exogenously given at time 0. We consider a general distribution function of initial wealth share with a mean of one and a standard deviation of $\sigma_z > 0$.

Taking the log of wealth share $s_{z,t}(h) \equiv z_t(h)/z_t$ at time t and differentiating the resulting expression with respect to time yield

$$\frac{\dot{s}_{z,t}(h)}{s_{z,t}(h)} = \frac{\dot{z}_t(h)}{z_t(h)} - \frac{\dot{z}_t}{z_t} = \frac{c_t - w_t - \tau_t}{z_t} - \frac{c_t(h) - w_t - \tau_t}{z_t(h)}. \tag{33}$$

Then, (33) can be re-expressed as

$$\dot{s}_{z,t}(h) = \frac{c_t - w_t - \tau_t}{z_t} s_{z,t}(h) - \frac{s_{c,t}(h)c_t - w_t - \tau_t}{z_t}, \tag{34}$$

where $s_{c,t}(h) \equiv c_t(h)/c_t$ is the share of consumption by household h at time t. Taking the log of $s_{c,t}(h)$ and differentiating the resulting expression with respect to time yield

$$\frac{\dot{s}_{c,t}(h)}{s_{c,t}(h)} = \frac{\dot{c}_t(h)}{c_t(h)} - \frac{\dot{c}_t}{c_t}. \tag{35}$$

Given that $\dot{c}_t(h)/c_t(h) = \dot{c}_t/c_t$ from (3) and (4), (35) becomes $\dot{s}_{c,t}(h) = 0$ for all t, which in turn implies $s_{c,t}(h) = s_{c,0}(h)$ for all t.[18] Given that $\{a_t, b_t, z_t, c_t, w_t, \tau_t\}$ all grow at the same rate g in equilibrium, (34) represents a one-dimensional differential equation, which describes the potential evolution of $s_{z,t}(h)$ given an initial $s_{z,0}(h)$. In Appendix A, we show that the coefficient on $s_{z,t}(h)$ in (34) is positive and equal to ρ. Together with the fact that $s_{z,t}(h)$ is a state variable, the only solution consistent with long-run stability is $\dot{s}_{z,t}(h) = 0$ for all t, which is achieved by consumption share $s_{c,t}(h)$ jumping to its steady-state value shown in Appendix A. Lemma 5 shows that as an equilibrium outcome, the wealth distribution is stationary and remains the same as the initial distribution, which is exogenously given at time 0.

[18] $s_{c,0}(h)$ is an endogenous variable to be determined in Appendix A.

Lemma 5 *The wealth share of household $h \in [0,1]$ is given by $s_{z,t}(h) = s_{z,0}(h)$ for all t.*

Proof. See Appendix A. ■

3.2 Income distribution

From (31), before-transfer income earned by household h is given by[19]

$$I_t(h) = r_t z_t(h) + w_t. \tag{36}$$

Aggregating (36) yields total income earned by all households given by

$$I_t = r_t z_t + w_t. \tag{37}$$

Combining (36) and (37) yields the share of income earned by household h given by

$$s_{I,t}(h) \equiv \frac{I_t(h)}{I_t} = \frac{s_{z,0}(h)\ r_t z_t + w_t}{r_t z_t + w_t}, \tag{38}$$

which also uses $z_t(h) = s_{z,t}(h) z_t = s_{z,0}(h) z_t$ from Lemma 5. The distribution function of income share $s_{I,t}(h)$ has a mean of one and the following standard deviation:

$$\sigma_{I,t} \equiv \sqrt{\int_0^1 [s_{I,t}(h) - 1]^2 dh} = \frac{r_t z_t}{r_t z_t + w_t} \sqrt{\int_0^1 [s_{z,0}(h) - 1]^2 dh} = \frac{r_t z_t / w_t}{1 + r_t z_t / w_t} \sigma_z, \tag{39}$$

which is also the coefficient of variation of income and is increasing in $r_t z_t / w_t$. As discussed in Chu and Cozzi (2018), income inequality $\sigma_{I,t}$ is increasing in $r_t z_t / w_t$ because an unequal distribution of wealth is the source of income inequality in the model. Therefore, whenever interest income $r_t z_t$ increases relative to wage income w_t, the degree of income inequality increases.

Lemma 6 *Income inequality is increasing in the ratio of interest income to wage income.*

Proof. Equation (39) shows that $\sigma_{I,t}$ is increasing in $r_t z_t / w_t$. ■

Recall that total wealth is given by $z_t = a_t + b_t$. The amount of financial assets a_t in the economy is given by

$$a_t = v_t = \int_{\tilde{\lambda}}^{\infty} v_t(\lambda) \tilde{f}(\lambda) d\lambda = \left[\tilde{\lambda}^{1/\kappa} (1 + \zeta i)\, \alpha + \beta \right] Q_t^{(1-\psi)/\psi}, \tag{40}$$

which uses $\int_{\tilde{\lambda}_t}^{\infty} v_t(\lambda) f(\lambda) d\lambda = (1 + \zeta i)\, \alpha_t + \tilde{\lambda}_t^{-1/\kappa} \beta_t$. Using (7), (22) and (40), we derive

$$\frac{a}{w} = \left[\tilde{\lambda}^{1/\kappa} (1 + \zeta i)\, \alpha + \beta \right] \frac{(\tilde{\lambda} e^{\kappa})^{(1-\psi)/\psi}}{\theta (1-\theta)^{(1-\psi)/\psi}}. \tag{41}$$

[19] We consider after-transfer income in Section 5.3.

The amount of borrowing b_t in the economy is given by

$$b_t = \zeta R_t = \zeta \alpha Q_t^{(1-\psi)/\psi} \phi_t, \tag{42}$$

where the last equality uses (12). Using (7), (22) and (42), we derive

$$\frac{b}{w} = \zeta \alpha \phi \frac{(\tilde{\lambda} e^{\kappa})^{(1-\psi)/\psi}}{\theta (1-\theta)^{(1-\psi)/\psi}}. \tag{43}$$

Using (4), (41) and (43), we derive the ratio of total interest income to wage income as

$$\frac{rz}{w} = \frac{r(a+b)}{w} = (\rho + g) \left[\tilde{\lambda}^{1/\kappa} (1 + \zeta i) \alpha + \beta + \zeta \alpha \phi \right] \frac{(\tilde{\lambda} e^{\kappa})^{(1-\psi)/\psi}}{\theta (1-\theta)^{(1-\psi)/\psi}}, \tag{44}$$

where the growth rate g, the quality threshold $\tilde{\lambda}$ and the innovation arrival rate ϕ are determined by (23), (29) and (30), respectively.

4 Monetary policy on growth and inequality

In this section, we explore the effects of monetary policy on economic growth and income inequality. We begin by exploring the relationship between the inflation rate and the nominal interest rate. From the Fisher equation, the inflation rate is given by

$$\pi = i - r = i - g(i) - \rho, \tag{45}$$

where the last equality uses (4). Differentiating the steady-state equilibrium inflation rate π in (45) with respect to the nominal interest rate i yields

$$\frac{\partial \pi}{\partial i} = 1 - \frac{\partial g(i)}{\partial i}. \tag{46}$$

Therefore, so long as $\partial g(i)/\partial i < 1$, the relationship between the steady-state equilibrium inflation rate and the nominal interest rate is positive.[20] This positive *long-run* relationship between the inflation rate and the nominal interest rate is supported by empirical studies such as Mishkin (1992) and Booth and Ciner (2001). In the following sections, we explore the effects of the nominal interest rate on economic growth and income inequality. It is useful to note that any relationship between the nominal interest rate and growth/inequality would also apply to inflation and growth/inequality given the positive relationship between inflation and the nominal interest rate.

[20] Under our calibrated parameter values, the equilibrium inflation rate is indeed increasing in the nominal interest rate.

4.1 Monetary policy under a zero entry cost

We first consider the case of a zero entry cost $\beta = 0$. In this case, the threshold quality level becomes $\tilde{\lambda} = 1$. Then, the equilibrium growth rate in (23) becomes $g = \frac{1-\psi}{\psi}\kappa\phi$, where the innovation arrival rate ϕ in (30) simplifies to

$$\phi = \frac{1}{1+\zeta i}\frac{\kappa}{1+\kappa}\frac{(1-\theta)^{1/\psi}}{\alpha e^{\kappa(1-\psi)/\psi}} - \rho, \tag{47}$$

which is decreasing in the nominal interest rate i. As for the effect of the nominal interest rate i on income inequality, we know from Lemma 6 that we simply have to examine how i affects the ratio of total interest income to wage income in (44). We begin by examining separately the effects of i on ra/w and rb/w.

Under a zero entry cost, the ratio of asset interest income to wage income simplifies to

$$\frac{ra}{w} = \left(\rho + \frac{1-\psi}{\psi}\kappa\phi\right)\frac{1}{\rho+\phi}\frac{\kappa}{1+\kappa}\frac{1-\theta}{\theta}, \tag{48}$$

which uses (4), (41) and (47). Recall that the innovation arrival rate ϕ is decreasing in the nominal interest rate i. Therefore, (48) shows that the nominal interest rate i has two opposing effects on the ratio ra/w of asset interest income to wage income. First, an increase in i reduces the real interest rate $r = \rho + \frac{1-\psi}{\psi}\kappa\phi$ by decreasing innovation and the equilibrium growth rate. This corresponds to the *interest-rate* effect of innovation on income inequality identified in Chu and Cozzi (2018), who consider R&D subsidies instead of monetary policy. Second, an increase in i reduces the rate of creative destruction and raises the asset-wage ratio a/w. This corresponds to the *asset-value* effect of innovation on income inequality in Chu and Cozzi (2018). Equation (48) shows that as $\rho \to 0$, the two effects cancel each other. For the more general case with $\rho > 0$, differentiating ra/w in (48) with respect to i yields the following result:

$$\frac{\partial ra/w}{\partial i} > 0 \Leftrightarrow \kappa < \frac{\psi}{1-\psi}. \tag{49}$$

Therefore, the positive *asset-value* effect of i on income inequality dominates the negative *interest-rate* effect of i on income inequality if and only if $\kappa < \psi/(1-\psi)$. This result generalizes the one in Chu and Cozzi (2018), who consider a symmetric quality step size and find that the asset-value effect of R&D subsidies dominates the interest-rate effect of R&D subsidies if and only if the quality step size is sufficiently small. In the case of asymmetric quality step sizes, the average quality step size is increasing in κ. Therefore, a small value of κ implies a small average quality step size, under which the asset-value effect dominates the interest-rate effect of monetary policy on income inequality.

Under a zero entry cost, the ratio of bond interest income to wage income simplifies to

$$\frac{rb}{w} = \left(\rho + \frac{1-\psi}{\psi}\kappa\phi\right)\zeta\phi\frac{\alpha(e^{\kappa})^{(1-\psi)/\psi}}{\theta(1-\theta)^{(1-\psi)/\psi}}, \tag{50}$$

which uses (4) and (43). Equation (50) shows that the ratio rb/w of bond interest income to wage income is increasing in the innovation arrival rate ϕ, which in turn is decreasing in the nominal interest rate i. The first negative effect is that an increase in i reduces the real interest rate $r = \rho + \frac{1-\theta}{\theta}\kappa\phi$ by decreasing innovation and the equilibrium growth rate. The second negative effect is that an increase in i decreases R&D and the amount of borrowing, which in turn decreases the bond-wage ratio b/w.

Combining (48) and (50) yields the ratio of total interest income to wage income given by

$$\frac{rz}{w} = \left(\rho + \frac{1-\psi}{\psi}\kappa\phi\right)\left[\frac{1}{\rho+\phi}\frac{\kappa}{1+\kappa}\frac{1-\theta}{\theta} + \zeta\phi\frac{\alpha(e^{\kappa})^{(1-\psi)/\psi}}{\theta(1-\theta)^{(1-\psi)/\psi}}\right]. \tag{51}$$

As $\rho \to 0$, the two effects of the nominal interest rate i on the ratio ra/w of asset interest income to wage income cancel each other. In this case, we are left with the negative effects of i on the ratio rb/w of bond interest income to wage income. For the more general case in which $\rho > 0$, the overall effect of i on rz/w depends on the relative value of κ and $\psi/(1-\psi)$. If $\kappa > \psi/(1-\psi)$, then the effects of i on ra/w and rb/w are both negative. In this case, the overall effect of the nominal interest rate on income inequality is negative. If $\kappa < \psi/(1-\psi)$, then the effect of i on ra/w is positive whereas the effect of i on rb/w is negative. In this case, the overall effect of the nominal interest rate on income inequality can be positive, negative or U-shaped. Proposition 2 summarizes these results.

Proposition 2 *Given a zero entry cost parameter β, an increase in the nominal interest rate has the following effects: (a) it has a negative effect on income inequality if $\kappa > \psi/(1-\psi)$ and (b) it may have a positive, negative or U-shaped effect on income inequality if $\kappa < \psi/(1-\psi)$.*

Proof. See Appendix A. ■

4.2 Monetary policy under a positive entry cost

We now consider the general case of a positive entry cost $\beta > 0$. Recall that the effects of the nominal interest rate on income inequality depend on how it affects the ratio of total interest income to wage income, which depends on r, a/w and b/w. Proposition 1 shows that the nominal interest rate has an inverted-U effect on the equilibrium growth rate under a sufficiently large entry cost β. Therefore, the nominal interest rate also has an inverted-U effect on the real interest rate $r = \rho + g$ under a sufficiently large entry cost β. It is useful to note that this *interest-rate* effect works through the quality threshold $\tilde{\lambda}$ in addition to the innovation arrival rate ϕ in the previous section and in Chu and Cozzi (2018).

We now consider how the nominal interest rate affects the asset-wage ratio a/w. Substituting (29) and (30) into (41) yields

$$\frac{a}{w} = \frac{1}{\rho + \tilde{\lambda}^{-1/\kappa}\phi}\frac{\tilde{\lambda} - 1/(1+\kappa)}{\tilde{\lambda}}\frac{1-\theta}{\theta}, \tag{52}$$

where the quality threshold $\tilde{\lambda} > 1$ is determined by (29) and decreasing in the nominal interest rate i. In the previous section with $\beta = 0$, the quality threshold is simply $\tilde{\lambda} = 1$. In this special case, the positive *asset-value* effect works through the innovation arrival rate ϕ, which in turn is decreasing in i. However, in the more general case with $\beta > 0$, the asset-value effect also works through the quality threshold $\tilde{\lambda}$ via two channels. First, it decreases the threshold quality level $\tilde{\lambda}$, which in turn speeds up the arrivals of implementable innovations and increases the rate of creative destruction. This effect works to decrease a/w. Second, the lower average quality step size also reduces the average markup ratio and the average value of monopolistic firms, which in turn decreases a/w. It is useful to note that these negative *asset-value* effects have the opposite sign as the one in the previous section by working through a different channel that is the quality threshold $\tilde{\lambda}$, which is absent in Chu and Cozzi (2018).

We now consider how the nominal interest rate affects the bond-wage ratio b/w. Substituting (29) and (30) into (43) yields

$$\frac{b}{w} = \zeta\alpha\tilde{\lambda}^{1/\kappa}\left[\frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\psi}}\frac{(1-\theta)^{1/\psi}}{\beta e^{\kappa(1-\psi)/\psi}} - \rho\right]\frac{(\tilde{\lambda}e^{\kappa})^{(1-\psi)/\psi}}{\theta(1-\theta)^{(1-\psi)/\psi}}. \tag{53}$$

Equation (53) shows that the nominal interest rate i affects b/w through $\tilde{\lambda}$ via multiple channels. The main effect is similar to and complements the one in the previous section but once again works through a different channel that the nominal interest rate reduces the quality threshold and the average quality step size, which in turn decreases the average markup ratio and the expected value of monopolistic profits. This general-equilibrium effect in turn reinforces the direct negative direct of i on R&D and the amount of borrowing as well as the bond-wage ratio b/w.

In Section 4.1, we find that in the case of a zero entry cost $\beta = 0$ and a positive discount rate $\rho > 0$, an increase in the nominal interest rate has both positive and negative effects on income inequality. In this section, we find that in the case of a positive entry cost $\beta > 0$, an increase in the nominal interest rate has additional effects on income inequality via endogenous firm entry. Therefore, when the entry cost β and the discount rate ρ are both positive and the CIA parameter ζ, which determines the effects of the nominal interest rate, is sufficiently large, we find that an increase in the nominal interest rate has a potentially inverted-U effect on income inequality. Proposition 3 shows that the effect of the nominal interest rate on income inequality is firstly increasing and eventually decreasing.

Proposition 3 *If the product of the discount rate and the entry cost (i.e., $\rho\beta$) is positive and the CIA parameter ζ is sufficiently large, then the effect of the nominal interest rate i on income inequality is firstly increasing and eventually decreasing.*

Proof. See Appendix A. ∎

It is important to note that this non-monotonic and potentially inverted-U effect of the nominal interest rate on income inequality is different from the U-shaped effect under a zero entry cost $\beta = 0$ in Proposition 2. The reason is that as Proposition 1 shows, the nominal interest rate has an inverted-U effect on the equilibrium growth rate if and only if the entry cost β is sufficiently large. In other words, endogenous firm entry is necessary for the emergence of an inverted-U effect of the nominal interest rate on economic growth, which in turn generates an inverted-U effect on income inequality that is otherwise absent without endogenous firm entry.

The main mechanisms behind this inverted-U effect on income inequality can be summarized as follows. Given that the real interest rate is increasing in the growth rate of consumption, the inverted-U effect of the nominal interest rate on economic growth causes an inverted-U effect on the real interest rate. Furthermore, the nominal interest rate has both positive and negative effects on the value of assets. On the one hand, by slowing down the innovation arrival rate, the nominal interest rate increases the market value of monopolistic firms, which in turn increases the value of assets. On the other hand, by lowering the entry threshold for quality improvements, the nominal interest rate reduces the average step size of implemented quality improvements and decreases the average markup ratio, which in turn decreases the market values of monopolistic firms and assets. Combining all these effects yields an overall inverted-U effect of the nominal interest rate on income inequality, which exists only under endogenous entry of heterogeneous firms.

5 Quantitative analysis

In this section, we provide a quantitative analysis. In Section 4.1, we use cross-country data to estimate the empirical effects of inflation on economic growth and income inequality. In Section 4.2, we calibrate the model to data as well as regression estimates and then simulate the effects of inflation on growth and inequality in the model. Section 4.3 considers after-transfer income inequality to examine the robustness of our results.

5.1 Empirical estimation

To facilitate the subsequent calibration, we first provide an empirical estimation of the effects of inflation on economic growth and income inequality. Here we use cross-country panel data to estimate the following regressions:

$$\begin{aligned} g_{it} &= \gamma_1 \pi_{it} + \gamma_2 \pi_{it}^2 + \Gamma X_{it} + \delta_i + \epsilon_{it}, \\ \sigma_{it} &= \omega_1 \pi_{it} + \omega_2 \pi_{it}^2 + \Omega X_{it} + \delta_i + \eta_{it}, \end{aligned}$$

where g_{it} denotes the growth rate of real GDP in country i at time t, π_{it} denotes the inflation rate from the GDP deflator in country i at time t, and σ_{it} denotes income inequality in country i at time t. Income inequality is collected from the World Income Inequality Database (WIID) version 3.4. This database provides information on the income share of each decile group; i.e., the share of total income going to each tenth of the population ordered according to the income level of each group. To be more specific, the first decile group includes the poorest 10% of the population, while the tenth decile group includes the richest 10%. We use the data to calculate the standard deviation of income share and the ratio of income between groups. This database also provides the Gini index, which is another conventional measure of income inequality. X_{it} is a vector of the following control variables: a constant, the degree of openness, the unemployment rate, and investment risks. We follow Fan and Gao (2017) to use the investment profile index and the corruption index from the International Country Risk Guide to measure investment risks.[21] δ_i is the country fixed effect.[22]

To be consistent with our innovation-driven growth model, we focus on high-income countries. We consider data from 1995 to 2014. In the first four columns, we define high-income countries according to the definition given by the WIID. In the last four columns, we define high-income countries according to the classification given by the World Bank (WB). Table 1 shows that the overall effects of inflation on economic growth and income inequality follow an inverted-U pattern. The growth-maximizing inflation rate is about 15%,[23] whereas the inequality-maximizing inflation rate is about 10%.[24]

Table 1: Effects of inflation on economic growth and income inequality[25]

	WIID				WB			
	growth	income inequality			growth	income inequality		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
π_{it}	1.071***	0.026**	0.103**	0.104*	1.057***	0.027**	0.108**	0.109*
	(0.283)	(0.013)	(0.042)	(0.054)	(0.283)	(0.012)	(0.043)	(0.055)
π_{it}^2	-0.034**	-0.001**	-0.006***	-0.005*	-0.033**	-0.001**	-0.006***	-0.005*
	(0.014)	(0.001)	(0.002)	(0.003)	(0.014)	(0.001)	(0.002)	(0.003)
country fixed effect	YES	YES	YES	YES	YES	YES	YES	YES
control variables	YES	YES	YES	YES	YES	YES	YES	YES
observations	571	463	463	463	570	459	459	459
R-squared	0.178	0.920	0.897	0.926	0.182	0.916	0.892	0.920

5.2 Calibration and simulation

We now calibrate the model to perform a quantitative analysis on the relationship between inflation and economic growth/income inequality. The model features the following structural parameters $\{\zeta, \rho, \theta, \kappa, \alpha, \beta, \varepsilon\}$ and the policy instrument i. We normalize the CIA parameter ζ to unity. Then, we set the discount rate ρ to a conventional value of 0.05. As for labor intensity θ, we set it to a value of 0.58 in the US; see for example Elsby *et al.* (2013). As for the Pareto distribution parameter κ, we calibrate its value by matching an average real GDP per capita growth rate of 0.014 in the US. As for the R&D cost parameter α and the entry cost parameter β, we calibrate their values by matching the growth-maximizing inflation rate and

[21] We rescale these two indexes into a number between zero and ten.

[22] If we controlled year fixed effects instead, our results (available upon request) would still hold.

[23] This estimate is between the two estimates of 12% and 19% reported in Bick (2010), who uses a panel threshold model. Our estimate of 15% is also similar to the whole-sample estimate in Lopez-Villavicencio and Mignon (2011). Although Lopez-Villavicencio and Mignon (2011) find a lower threshold for advanced economies, their high-income countries include only a subset of the OECD countries.

[24] This estimate is much lower than the estimate in Natob (2015), who however focuses on developing countries.

[25] *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Robust standard errors are corrected by clustering at the country level in parentheses. Columns 1 and 5 correspond to the GDP growth rate. Columns 2-4 and 6-8 use different measures of income inequality. Specifically, columns 2 and 6 correspond to the standard deviation of income share. Columns 3 and 7 correspond to the income difference between the top 10% and the bottom 10% of the population. Columns 4 and 8 correspond to the Gini coefficient. In the first (last) four columns, high-income countries follow the classification by the WIID (WB).

the inequality-maximizing inflation rate estimated in the previous section. We calibrate the monetary policy instrument i by matching the average inflation rate in the US, which is about 0.025 in the past two decades. As for the productive externality parameter ε, we consider a range of values from 0.43 to 0.55.[26] The calibrated parameter values are summarized in Table 2. We find that if the productive externality parameter ε were too small (large), then the implied value of the Pareto distribution parameter κ would approach one (zero), under which the average value of λ approaches infinity (one).

Table 2: Calibration

ε	ρ	θ	κ	α	β	i
0.430	0.050	0.580	0.938	$3.249*10^{-5}$	$1.727*10^{-5}$	0.089
0.470	0.050	0.580	0.670	$3.022*10^{-6}$	$1.358*10^{-6}$	0.089
0.510	0.050	0.580	0.416	$2.246*10^{-8}$	$8.391*10^{-9}$	0.089
0.550	0.050	0.580	0.174	$6.950*10^{-16}$	$2.113*10^{-16}$	0.089

Figure 1 simulates the relationship between inflation and economic growth. We find that the relationship between inflation and economic growth follows an inverted-U pattern. When the inflation rate increases from the benchmark value of 0.025 to 0.150, the equilibrium rate of economic growth increases from the benchmark value of 0.0140 to a maximum value of about 0.0154 in all four cases. After that, any further increase in inflation is associated with a decline in economic growth.

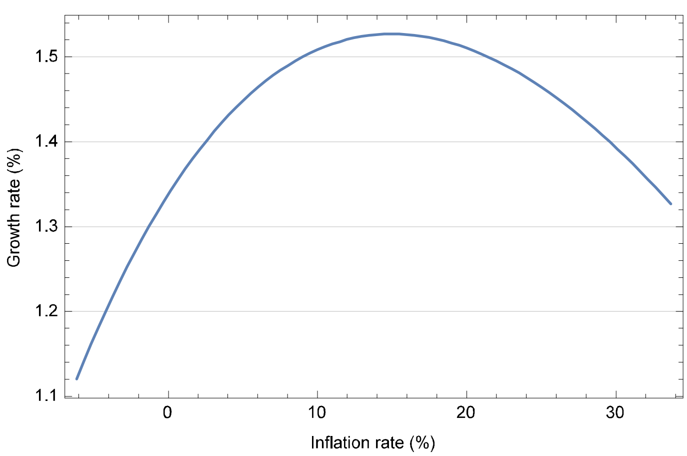

Figure 1a: Inflation and economic growth ($\varepsilon = 0.43$)

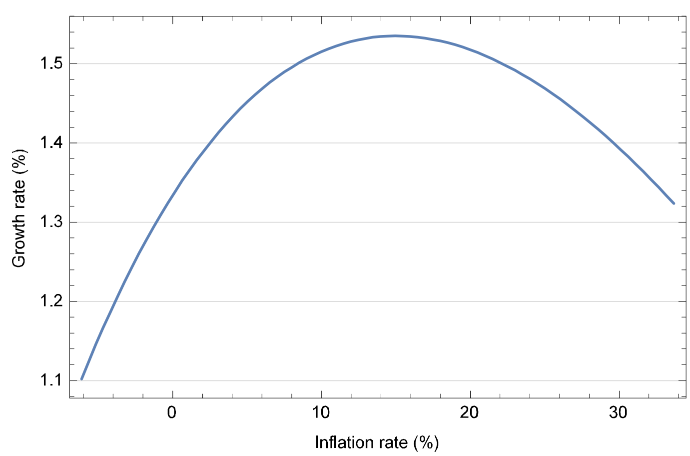

Figure 1b: Inflation and economic growth ($\varepsilon = 0.47$)

[26] This range of values is consistent with Ho *et al.* (2007). Soerensen and Whitta-Jacobsen (2010, p. 219) survey the empirical literature and conclude that the degree of externality from capital goods is between 0.45 and 0.75.

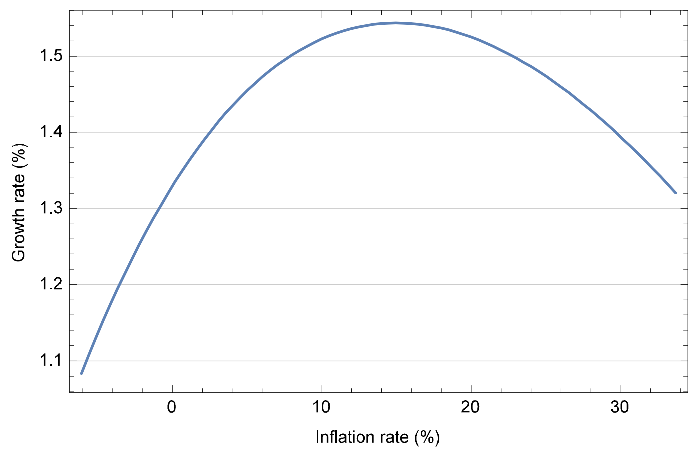

Figure 1c: Inflation and economic growth ($\varepsilon = 0.51$)

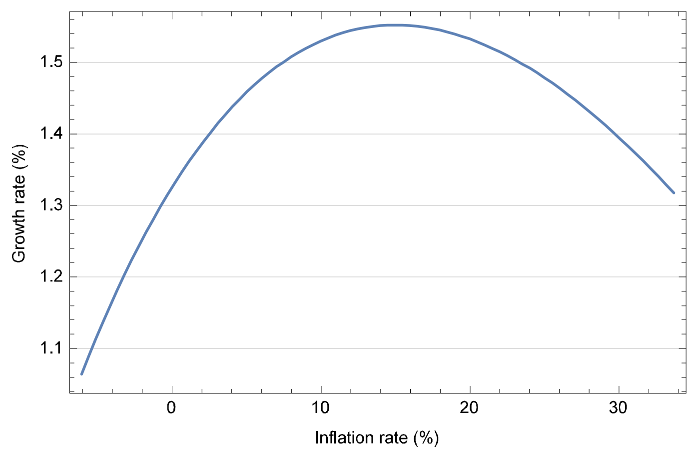

Figure 1d: Inflation and economic growth ($\varepsilon = 0.55$)

Figure 2 simulates the relationship between inflation and income inequality. We find that the relationship between inflation and income inequality also follows an inverted-U pattern. When the inflation rate increases from the benchmark value of 0.025 to 0.100, the increase in the coefficient of variation of income ranges from 0.4% (in the case of $\varepsilon = 0.43$) to 0.7% (in the case of $\varepsilon = 0.55$).[27] Therefore, larger productive externality amplifies the effect of inflation on income inequality. When the inflation rate is above 0.1, any further increase in inflation is associated with a decline in income inequality.

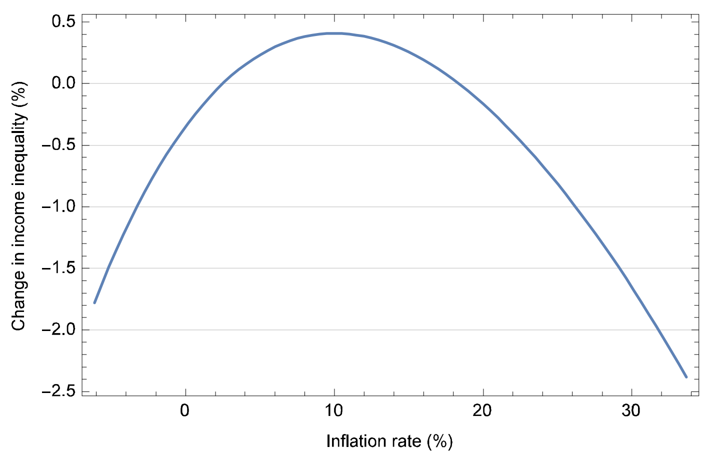

Figure 2a: Inflation and income inequality ($\varepsilon = 0.43$)

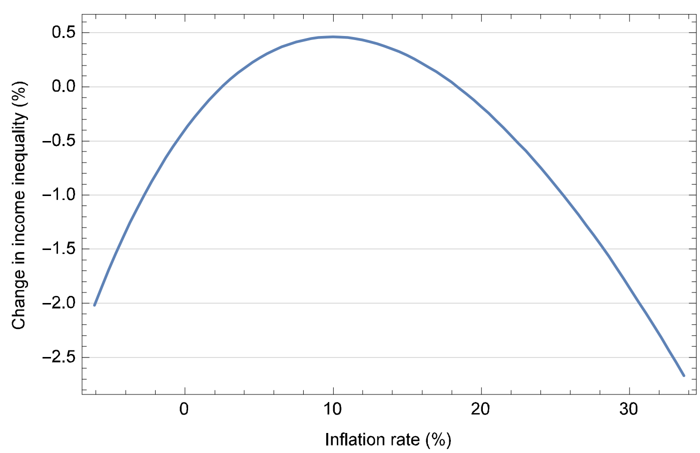

Figure 2b: Inflation and income inequality ($\varepsilon = 0.47$)

[27] These numbers refer to percent changes from the benchmark value.

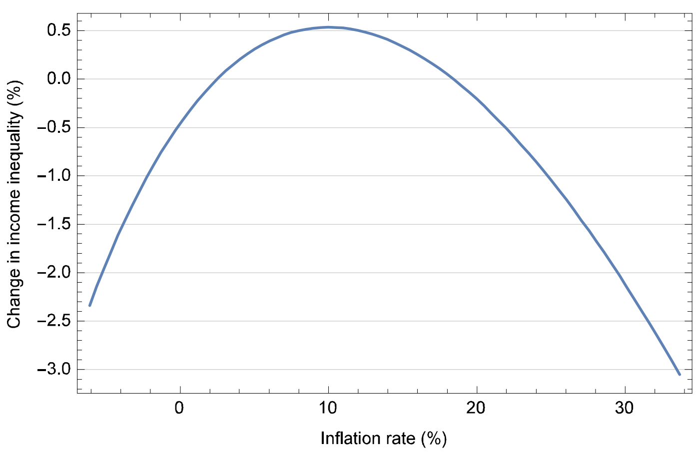

Figure 2c: Inflation and income inequality ($\varepsilon = 0.51$)

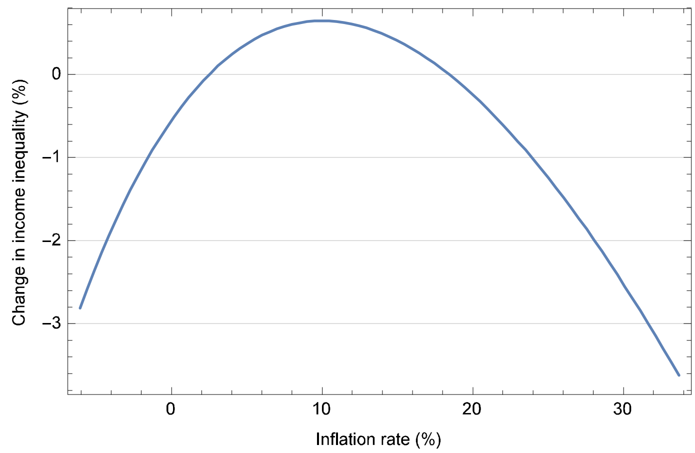

Figure 2d: Inflation and income inequality ($\varepsilon = 0.55$)

5.3 After-transfer income inequality

In this section, we consider after-transfer income inequality. After-transfer income of household $h \in [0, 1]$ is

$$I_t^a(h) = r_t z_t(h) + w_t + \tau_t, \tag{54}$$

where $\tau_t = \dot{M}_t / P_t = (i - \rho) m_t$. Then, after-transfer income share of household h is

$$s_{I,t}^a(h) = \frac{s_{z,0}(h)\ r_t z_t + w_t + \tau_t}{r_t z_t + w_t + \tau_t}, \tag{55}$$

and the standard deviation of after-transfer income share is

$$\sigma_I^a = \frac{rz/(w + \tau)}{1 + rz/(w + \tau)} \sigma_z, \tag{56}$$

which is increasing in $rz/(w+\tau)$. In the rest of this section, we use the same parameter values as in the previous section and simulate the effects of inflation on after-transfer income inequality. Figure 3 shows that the relationship between inflation and after-transfer income inequality is very similar to the relationship between inflation and before-transfer income inequality in Figure 2. Therefore, adding seigniorage as a lump-sum transfer to households does not change our results.

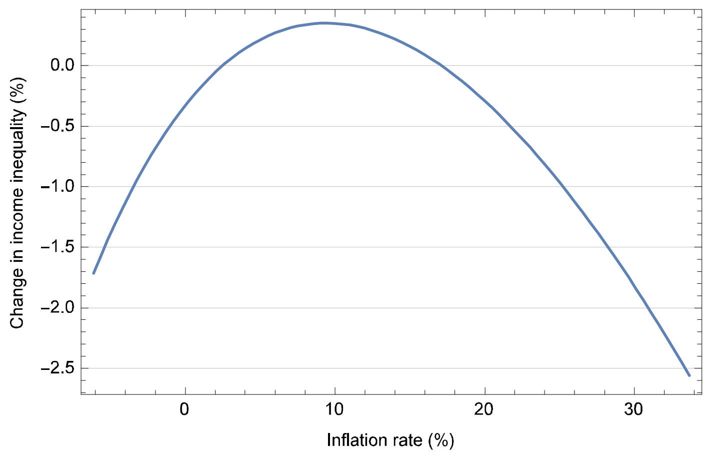

Figure 3a: Inflation and income inequality ($\varepsilon = 0.43$)

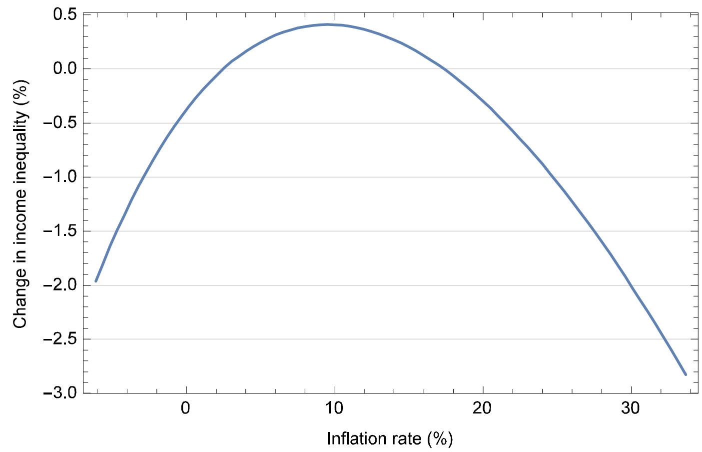

Figure 3b: Inflation and income inequality ($\varepsilon = 0.47$)

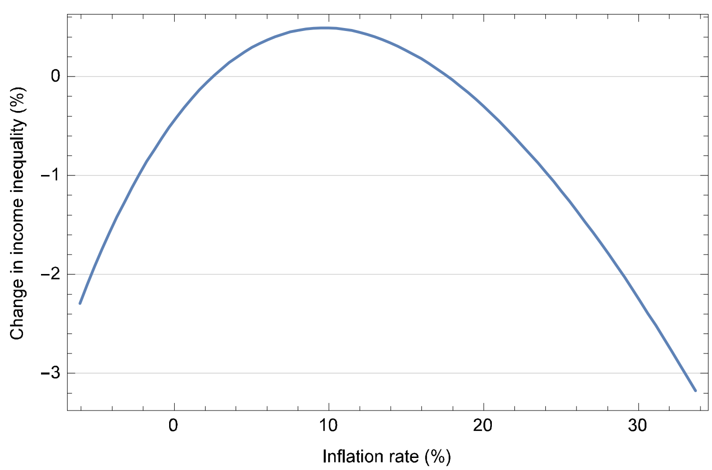

Figure 3c: Inflation and income inequality ($\varepsilon = 0.51$)

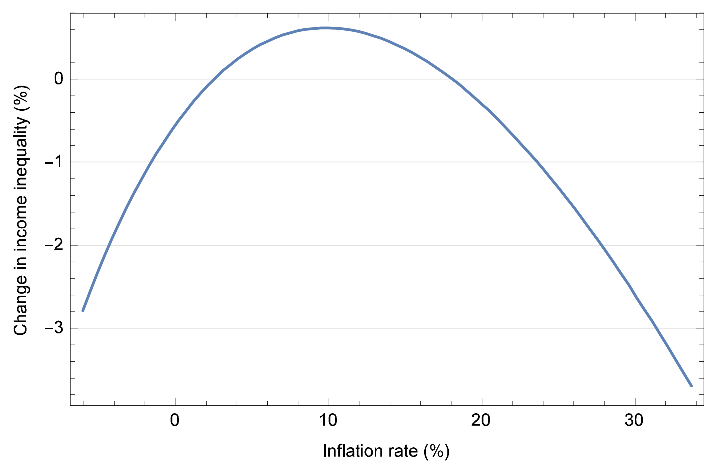

Figure 3d: Inflation and income inequality ($\varepsilon = 0.55$)

6 Conclusion

In this study, we have developed a Schumpeterian growth model with two dimensions of heterogeneity among households and firms. We model household heterogeneity by assuming that households own different levels of wealth, which in turn generate an endogenous distribution of income. We model firm heterogeneity by assuming random quality improvements and a cost of entering a market, which together generate an endogenous distribution of implemented quality improvements. Both the income distribution and the implemented quality distribution are affected by monetary policy. Within this monetary growth-theoretic framework, we find that inflation has inverted-U effects on both economic growth and income inequality. Furthermore, we calibrate our model to match the growth-maximizing and inequality-maximizing inflation rates that are estimated using cross-country panel data.

Finally, it is useful to note that our model could feature scale effects as in the first-generation R&D-based growth model in seminal studies by Romer (1990), Segerstrom *et al.* (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992).[28] We sidestep this issue by normalizing the supply of labor to unity. Alternatively, one can remove scale effects in the Schumpeterian growth model by considering the semi-endogenous-growth approach in Segerstrom (1998) or the second-generation approach in Peretto (1998, 2007). We leave this potentially interesting extension to future research.

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Appendix A

Proof of Lemma 1. It follows from (14) that $v_t^e + \tilde{\lambda}_t^{-1/\kappa}\beta_t = (1+\zeta i)\,\alpha_t + \tilde{\lambda}_t^{-1/\kappa}\beta_t$. Differentiating both sides of this equation with respect to time t yields

$$\dot{v}_t^e + \tilde{\lambda}_t^{-1/\kappa}\dot{\beta}_t = (1+\zeta i)\,\alpha_t\frac{\dot{\alpha}_t}{\alpha_t} + \tilde{\lambda}_t^{-1/\kappa}\beta_t\frac{\dot{\beta}_t}{\beta_t} \Leftrightarrow \frac{\dot{v}_t^e + \tilde{\lambda}_t^{-1/\kappa}\dot{\beta}_t}{v_t^e + \tilde{\lambda}_t^{-1/\kappa}\beta_t} = \frac{1-\psi}{\psi}\frac{\dot{Q}_t}{Q_t}, \tag{A1}$$

where the first equality cancels $\beta_t d(\tilde{\lambda}_t^{-1/\kappa})/dt$ from both sides and the second equality uses $\alpha_t = \alpha Q_t^{(1-\psi)/\psi}$ and $\beta_t = \beta Q_t^{(1-\psi)/\psi}$. Using (A1) and $\Pr(\lambda \geq \tilde{\lambda}_t) = \tilde{\lambda}_t^{-1/\kappa}$, we modify (17) as

$$r_t = \frac{\Pi_t^e}{v_t^e + \tilde{\lambda}_t^{-1/\kappa}\beta_t} + \frac{1-\psi}{\psi}\frac{\dot{Q}_t}{Q_t} - \tilde{\lambda}_t^{-1/\kappa}\phi_t. \tag{A2}$$

Similarly, we modify (24) for $\lambda = \tilde{\lambda}_t$ as

$$r_t = \frac{\Pi(\tilde{\lambda}_t)}{v_t(\tilde{\lambda}_t)} + \frac{1-\psi}{\psi}\frac{\dot{Q}_t}{Q_t} - \tilde{\lambda}_t^{-1/\kappa}\phi_t, \tag{A3}$$

which uses the entry condition $v_t(\tilde{\lambda}_t) = \beta_t = \beta Q_t^{(1-\psi)/\psi}$ and $\Pr(\lambda \geq \tilde{\lambda}_t) = \tilde{\lambda}_t^{-1/\kappa}$. From (A2) and (A3), we have

$$\frac{\Pi_t^e}{(1+\zeta i)\,\alpha + \tilde{\lambda}_t^{-1/\kappa}\beta} = \frac{\Pi_t(\tilde{\lambda}_t)}{\beta}, \tag{A4}$$

where

$$\Pi_t^e \equiv \int_{\tilde{\lambda}_t}^{\infty}\Pi_t(\lambda)f(\lambda)d\lambda = \left(\frac{\tilde{\lambda} - 1/(1+\kappa)}{\tilde{\lambda}^{\frac{1+\kappa}{\kappa}}}\right)(1-\theta)y_t \tag{A5}$$

and

$$\Pi_t(\tilde{\lambda}_t) = \frac{\tilde{\lambda}_t - 1}{\tilde{\lambda}_t}(1-\theta)y_t \tag{A6}$$

from (11). Using (A4)-(A6), we also have

$$\tilde{\lambda}_t^{\frac{1}{\kappa}}(\tilde{\lambda}_t - 1) = \frac{\kappa}{1+\kappa}\frac{1}{1+\zeta i}\frac{\beta}{\alpha}, \tag{A7}$$

which uniquely determines $\tilde{\lambda} > 1$ independent of t because the left-hand side of (A7) is increasing in $\tilde{\lambda}_t > 1$ and the right-hand side is independent of t. ■

Proof of Lemma 2. In the symmetric equilibrium, we have $v_t^e(i, \omega_i + 1) = v_t^e$, which can be expressed as

$$v_t^e \equiv \int_1^{\tilde{\lambda}} 0\ f(\lambda)d\lambda + \int_{\tilde{\lambda}}^{\infty}[v_t(\lambda) - \beta_t]\ f(\lambda)d\lambda = \int_{\tilde{\lambda}}^{\infty} v_t(\lambda)f(\lambda)d\lambda - \Pr(\lambda \geq \tilde{\lambda})\beta_t. \tag{A8}$$

Substituting the no-arbitrage condition for the value of an implemented innovation $v_t(\lambda) = [\Pi_t(\lambda) + \dot{v}_t(\lambda) - \Pr(\lambda \geq \tilde{\lambda})\phi_t v_t(\lambda)]/r_t$ into (A8) yields

$$r_t[v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t] = \Pi_t^e + \int_{\tilde{\lambda}}^{\infty} \dot{v}_t(\lambda) f(\lambda) d\lambda - \Pr(\lambda \geq \tilde{\lambda})\phi_t \int_{\tilde{\lambda}}^{\infty} v_t(\lambda) f(\lambda) d\lambda, \tag{A9}$$

which uses (A8) and $\Pi_t^e \equiv \int_{\tilde{\lambda}}^{\infty} \Pi_t(\lambda) f(\lambda) d\lambda$. Then, we use the R&D condition $v_t^e = (1+\zeta i)\,\alpha_t$ to derive

$$\int_{\tilde{\lambda}}^{\infty} v_t(\lambda) f(\lambda) d\lambda = (1+\zeta i)\,\alpha_t + \Pr(\lambda \geq \tilde{\lambda})\beta_t. \tag{A10}$$

Differentiating both sides in (A10) with respect to t yields

$$\int_{\tilde{\lambda}}^{\infty} \dot{v}_t(\lambda) f(\lambda) d\lambda = (1+\zeta i)\,\dot{\alpha}_t + \Pr(\lambda \geq \tilde{\lambda})\dot{\beta}_t. \tag{A11}$$

By substituting (A11) into (A9), with $v_t^e = (1+\zeta i_t)\,\alpha_t$, we can obtain

$$r_t[v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t] = \Pi_t^e + \dot{v}_t^e + \Pr(\lambda \geq \tilde{\lambda})\dot{\beta}_t - \Pr(\lambda \geq \tilde{\lambda})\phi_t[v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t], \tag{A12}$$

which is equivalent to (17). ■

Proof of Lemma 3. Substituting (8) and (10) into (6) yields

$$K_t = (1-\theta)\, y_t Q_t \exp\left(-\int_0^1 \ln \lambda_t(j) dj\right), \tag{A13}$$

which uses (13) for Q_t. Given that $\lambda_t(j) > \tilde{\lambda}$ for implemented innovations, the truncated distribution function for implemented innovations is as follows:

$$\tilde{f}(\lambda) \equiv \frac{f(\lambda)}{\int_{\tilde{\lambda}}^{\infty} f(\lambda) d\lambda} = \tilde{\lambda}^{\frac{1}{\kappa}} f(\lambda). \tag{A14}$$

By this,

$$\exp\left(-\int_0^1 \ln \lambda_t(j) dj\right) = \frac{1}{\tilde{\lambda} e^{\kappa}} \tag{A15}$$

holds. Substituting (A13)-(A15) into $y_t = K_t^{1-\theta+\varepsilon} = K_t^{1-\psi}$ from (5) yields (22). ■

Proof of Lemma 4. Define $\tilde{c}_t \equiv c_t / Q_t^{(1-\psi)/\psi}$. Then, from (4), it holds that

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = r_t - \rho - \frac{1-\psi}{\psi}\frac{\dot{Q}_t}{Q_t}. \tag{A16}$$

From (A3) and (A6), with $v_t(\tilde{\lambda}) = \beta_t = Q_t^{(1-\psi)/\psi}\beta$, the real interest rate can be expressed as

$$r_t = \frac{\tilde{\lambda}-1}{\tilde{\lambda}}\frac{1-\theta}{\beta}\frac{y_t}{Q_t^{(1-\psi)/\psi}} + \frac{1-\psi}{\psi}\frac{\dot{Q}_t}{Q_t} - \tilde{\lambda}^{-1/\kappa}\phi_t. \tag{A17}$$

Substituting (22) and (A17) into (A16) yields

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{(1-\theta)^{1/\psi}}{\beta e^{\kappa(1-\psi)/\psi}} \frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\psi}} - \tilde{\lambda}^{-1/\kappa}\phi_t - \rho. \tag{A18}$$

To obtain the equilibrium expression of ϕ_t off the balanced growth path, we will derive the total demand for final goods. First, we use (8)-(10), and to have

$$\int_0^1 x_t(j)dj = \int_0^1 \frac{(1-\theta)\, y_t}{\lambda_t(j)} dj = (1-\theta)\, y_t \int_{\tilde{\lambda}}^{\infty} \frac{\tilde{f}(\lambda)}{\lambda} d\lambda = \frac{(1-\theta) y_t}{(1+\kappa)\, \tilde{\lambda}}. \tag{A19}$$

Then from (12), we have

$$\int_0^1 R_t(j)dj = \alpha Q_t^{(1-\psi)/\psi} \phi_t. \tag{A20}$$

Combining (A19) and (A20) with the final good market condition yields

$$\phi_t = \frac{1}{\alpha + \beta \tilde{\lambda}^{-1/\kappa}} \left[\left(1 - \frac{1-\theta}{(1+\kappa)\, \tilde{\lambda}}\right) \left(\frac{1-\theta}{\tilde{\lambda} e^{\kappa}}\right)^{(1-\psi)/\psi} - \tilde{c}_t \right] \equiv \phi(\underset{-}{\tilde{c}_t}). \tag{A21}$$

which also uses (22) for y_t. Finally, by substituting (A21) into (A18), we have a one-dimensional differential equation in $\tilde{c}_t$. Given that ϕ_t decreases with $\tilde{c}_t$, the right-hand side of (A18) is increasing in $\tilde{c}_t$. The dynamics of $\tilde{c}_t$ is saddle-point stable; i.e., $\tilde{c}_t$ jumps to the unique steady-state $\tilde{c}$ at $t = 0$. Accordingly, (A18) determines the stationary equilibrium value of $\tilde{\lambda}^{-1/\kappa}\phi_t$ as in (30). Then, (A21) determines the steady-state value of $\tilde{c}_t$ as

$$\tilde{c} = \left(1 - \frac{1-\theta}{(1+\kappa)\, \tilde{\lambda}}\right) \left(\frac{1-\theta}{\tilde{\lambda} e^{\kappa}}\right)^{(1-\psi)/\psi} - \phi \left(\alpha + \beta \tilde{\lambda}^{-1/\kappa}\right). \tag{A22}$$

■

Proof of Proposition 1. From (29) and (30), we have

$$\tilde{\lambda}^{-1/\kappa}\phi = \frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\psi}} \frac{(1-\theta)^{1/\psi}}{\beta e^{\kappa(1-\psi)/\psi}} - \rho, \tag{A23}$$

which is an inverted-U shaped function in $\tilde{\lambda}$ that is maximized at $\tilde{\lambda} = 1/(1-\psi)$. We naturally focus on a non-trivial case where $\tilde{\lambda}^{-1/\kappa}\phi > 0$. There are, thus, lower and upper bounds of $\tilde{\lambda}$, say λ_- and λ_+, such that $\tilde{\lambda}^{-1/\kappa}\phi > 0$ holds if and only if $\tilde{\lambda} \in (\lambda_-, \lambda_+)$. By (29), $\tilde{\lambda}$ is decreasing in i, thereby having an upper bound, denoted as $\overline{\lambda}$, due to $i \geq 0$. It is easy to verify that $\overline{\lambda}$ increases from 1 to ∞ as β increases from 0. When β is such large that $\overline{\lambda} > 1/(1-\psi)$ holds, there is an inverted-U shaped relationship between $i \geq 0$ and $\tilde{\lambda}^{-1/\kappa}\phi$, noting $\tilde{\lambda}$ monotonically decreases with $i \geq 0$. Then, when β is small such that $\overline{\lambda} < 1/(1-\psi)$, the relationship is monotonically negative for any $i \geq 0$.

Differentiating (23) with respect to $\tilde{\lambda}$ yields

$$\left(\frac{dg}{d\tilde{\lambda}}\right)\frac{\beta\psi\tilde{\lambda}^{1+1/\psi}}{1-\psi} \tag{A24}$$
$$= \underbrace{\left(\frac{(1-\theta)^{1/\psi}}{e^{\kappa(1-\psi)/\psi}}\left(\tilde{\lambda}-1\right)-\beta\rho\tilde{\lambda}^{1/\psi}\right)}_{\equiv\Lambda_1(\tilde{\lambda})}-\underbrace{\frac{(1-\theta)^{1/\psi}}{\psi e^{\kappa(1-\psi)/\psi}}\left(\ln\tilde{\lambda}+\kappa\right)\left((1-\psi)\tilde{\lambda}-1\right)}_{\equiv\Lambda_2(\tilde{\lambda})},$$

where we have used (23). It is easy to prove the following properties of Λ_1 and Λ_2. On the one hand, $\Lambda_1(\tilde{\lambda})$ is maximized at $\tilde{\lambda} = 1/(1-\psi)$ and strictly concave, with $\Lambda_1(\tilde{\lambda}) > 0$ for $\tilde{\lambda} \in (\lambda_-, \lambda_+)$. On the other hand, $\Lambda_2(\tilde{\lambda})$ increases from $\Lambda_2(1) < 0$ to ∞ as $\tilde{\lambda}$ increases from 1, with $\Lambda_2(1/(1-\psi) = 0$. Taking into account these facts, Figure 4 illustrates the graphs of the two functions. As this shows, there must uniquely exist a threshold level of $\tilde{\lambda} \in (1/(1-\theta), \lambda_+)$, denoted as Λ^* in Figure 4, below (above) which $\Lambda_1(\tilde{\lambda}) > (<)\Lambda_2(\tilde{\lambda})$, that is, $dg/d\tilde{\lambda} > (<)0$. Recalling that $\tilde{\lambda}$ increases with β and then $\tilde{\lambda}$ decreases with i, we can show that the relationship between i and g is also inverted-U shaped (negative) if β is large (small). ■

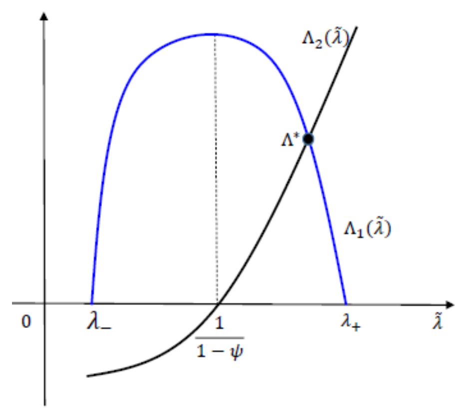

Figure 4: Proof of Proposition 1

Proof of Lemma 5. From (3), (4), and (35), we can show that $s_{c,t}(h) = s_{c,0}(h)$ holds for all t. Substituting this condition into (34) yields

$$\dot{s}_{z,t}(h) = \frac{c_t - w_t - \tau_t}{z_t}s_{z,t}(h) - \frac{s_{c,0}(h)c_t - w_t - \tau_t}{z_t}. \tag{A25}$$

According to Lemma 4, $\{c_t, w_t, \tau_t, z_t, m_t\}$ all grow at the same rate g in equilibrium. Using (4) and (32), it is easy to obtain

$$\frac{c_t - w_t - \tau_t}{z_t} = r_t - \frac{\dot{z}_t}{z_t} = \rho > 0. \tag{A26}$$

Therefore, the coefficient on $s_{z,t}(h)$ in (A25) is always positive, which in turn implies that $\dot{s}_{z,t}(h) = 0$ for all t is the only solution of (A25) consistent with long-run stability. Finally, imposing $\dot{s}_{z,t}(h) = 0$ on (A25) yields the steady-state value of $s_{c,t}(h)$ given by

$$s_{c,0}(h) = 1 - \frac{\rho\left[1 - s_{z,0}(h)\right]}{c/z}, \tag{A27}$$

where we can make use of (40) and (42) to derive

$$\frac{c}{z} = \frac{\tilde{c}}{\tilde{\lambda}^{1/\kappa}\left(1 + \zeta i\right)\alpha + \beta + \zeta\alpha\phi}. \tag{A28}$$

Note that $\tilde{c}$ is given by (A22). ■

Proof of Proposition 2. Differentiating rz/w in (51) with respect to ϕ yields

$$\frac{d(rz/w)}{d\phi} = \kappa\frac{\alpha(e^{\kappa})^{(1-\psi)/\psi}}{\theta(1-\theta)^{(1-\psi)/\psi}}\left\{\frac{\rho}{(\rho+\phi)^2}\left(\frac{1-\psi}{\psi}\kappa - 1\right)\frac{1}{1+\kappa}\frac{(1-\theta)^{1/\psi}}{\alpha e^{\kappa(1-\psi)/\psi}} + \zeta\left(\frac{\rho}{\kappa} + 2\frac{1-\psi}{\psi}\phi\right)\right\}. \tag{A29}$$

Given $\kappa > \psi/(1-\psi)$, (A29) shows that $d(rz/w)/d\phi > 0$. From (47), we know $d\phi/di < 0$. As a result, there is a negative effect of i and rz/w.

As for $\kappa < \psi/(1-\psi)$, we will show that there are three possibilities: for a feasible range of ϕ, (a) $d(rz/w)/d\phi < 0$, (b) $d(rz/w)/d\phi > 0$, or (c) $d(rz/w)/d\phi < (>)0$ if ϕ is smaller (larger). Before proceeding, it is useful to note that there is an upper bound of ϕ since $i \geq 0$ with (47), given by

$$\phi_{+} \equiv \underbrace{\frac{\kappa}{1+\kappa}\frac{(1-\theta)^{1/\psi}}{\alpha e^{\kappa(1-\psi)/\psi}}}_{\equiv\vartheta} - \rho.$$

We will derive a sufficient conditions for each case, by focusing on both ends of $\phi \in (0, \phi_{+}]$.

First, by substituting $\phi \to 0$ (i.e., the lower bound) into (A29), we can show that $d(rz/w)/d\phi > 0$ holds at $\phi \to 0$ if

$$\left(1 - \frac{\kappa}{\psi/(1-\psi)}\right) < \frac{\zeta\rho^2}{\vartheta}. \tag{A30}$$

Moreover, it is easy to derive $d^2(rz_t/w_t)/d\phi^2 > 0$ when $\kappa < \psi/(1-\psi)$. As a result, $d(rz/w)/d\phi > 0$ holds for any $\phi \in (0, \phi_{+}]$. Given $d\phi/di < 0$, in this case, there is a negative effect of i on rz/w.

Second, it is straightforward to verify that $d(rz/w)/d\phi < 0$ holds at $\phi \to 0$ if (A30) is violated. In this case, by substituting $\phi = \phi_{+}$ into (A29), we can show that $d(rz/w)/d\phi < 0$ also holds at the upper bound, $\phi = \phi_{+}$, if and only if

$$\left(1 - \frac{\kappa}{\psi/(1-\psi)}\right) > \zeta\vartheta\left[\frac{\vartheta}{\rho}\left(\frac{2\kappa}{\psi/\left(1-\psi\right)}\right) + \left(1 - \frac{2\kappa}{\psi/\left(1-\psi\right)}\right)\right]. \tag{A31}$$

We know $d^2(rz_t/w_t)/d\phi^2 > 0$ when $\kappa < \psi/(1-\psi)$. As a result, $d(rz/w)/d\phi < 0$ holds for any $\phi \in (0, \phi_{+}]$. Given $d\phi/di < 0$, in this case, there is a positive effect of i on rz/w.

Finally, if (A31) does not hold, there is a threshold value of ϕ below (above) which $d(rz/w)/d\phi <$ $(>)0$; i.e., there is a U-shaped relationship between i and rz/w. Therefore, the effect of i on rz/w can be negative, positive, or U-shaped. ■

Proof of Proposition 3. As in the proof of Proposition 1, we focus on the non-trivial case where $\tilde{\lambda}^{-1/\kappa}\phi > 0$, implying $\tilde{\lambda} \in (\lambda_-, \lambda_+)$ unless $\beta\rho = 0$. Recall that $1 < \lambda_- < 1/(1-\psi) < \lambda_+$. By (29) and (44), we have

$$\frac{rz}{w} = (\rho + g)\tilde{\lambda}^{(1-\psi)/\psi}\left[\frac{\tilde{\lambda} - 1/(1+\kappa)}{\tilde{\lambda} - 1} + \frac{\zeta\alpha}{\beta}\phi\right]\frac{\beta(e^{\kappa})^{(1-\psi)/\psi}}{\theta(1-\theta)^{(1-\psi)/\psi}}. \tag{A32}$$

By differentiating this with respect to $\tilde{\lambda}$,

$$\Xi\frac{d(rz/w)}{d\tilde{\lambda}} = \frac{\tilde{\lambda}(\rho+g)'}{(\rho+g)} + \frac{1-\psi}{\psi} + \tilde{\lambda}\frac{\left(\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1}\right)' + \frac{\zeta\alpha}{\beta}\phi'}{\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1} + \frac{\zeta\alpha}{\beta}\phi} \equiv \Psi(\tilde{\lambda}), \tag{A33}$$

where $\Xi > 0$ is a composite variable that is strictly positive.[29] By evaluating $\Psi(\tilde{\lambda})$ in (A33) at $\tilde{\lambda} \in \{\lambda_-, \lambda_+\}$, we can obtain

$$\Psi(\tilde{\lambda}) = \underbrace{\frac{1-(1-\psi)\tilde{\lambda}}{\tilde{\lambda}-1}\frac{1}{\psi}\left[\frac{1-\psi}{\psi}(\ln\tilde{\lambda} + \kappa) + (\alpha\zeta)\frac{(1+\kappa)\rho}{\beta}\frac{\tilde{\lambda}^{1/\kappa}(\lambda-1)}{(1+\kappa)\tilde{\lambda}-1}\right]}_{\equiv\Psi_1(\tilde{\lambda})} + \underbrace{\frac{1-\psi}{\psi} - \frac{\kappa}{(1+\kappa)\tilde{\lambda}-1}\frac{\tilde{\lambda}}{\tilde{\lambda}-1}}_{\equiv\Psi_2(\tilde{\lambda})}, \tag{A34}$$

which reflects $\tilde{\lambda}^{-1/\kappa}\phi = 0$ for $\tilde{\lambda} \in \{\lambda_-, \lambda_+\}$ with (30). For $\tilde{\lambda} = \lambda_-$, $\Psi_1(\tilde{\lambda}) > 0$ always holds due to $\lambda_- < 1/(1-\psi)$, but $\Psi_2(\tilde{\lambda}) \lesseqgtr 0$. For $\tilde{\lambda} = \lambda_+$, both $\Psi_1(\tilde{\lambda}) < 0$ and $\Psi_2(\tilde{\lambda}) > 0$ hold due to $\lambda_+ > 1/(1-\psi)$.

Given that $\tilde{\lambda}^{-1/\kappa}\phi$ is independent of ζ, λ_- and λ_+ are also independent of ζ. Thus, changes in ζ affect (A34) only through the second term of Ψ_1. Keeping $\tilde{\lambda} = \{\lambda_-, \lambda_+\}$ unchanged, it is possible to make $\Psi(\tilde{\lambda})$ larger (smaller) as one needs by increasing ζ, since the coefficient of Ψ_1, $\frac{1-(1-\psi)\tilde{\lambda}}{\tilde{\lambda}-1}$, is positive (negative) for $\tilde{\lambda} = \lambda_-$ ($\tilde{\lambda} = \lambda_+$). Therefore, for a sufficiently large ζ,[30] $\Psi(\lambda_-) > 0$ and $\Psi(\lambda_+) < 0$ hold; rz/w is first increasing and eventually decreasing in $\tilde{\lambda}$ on the feasible domain of $\tilde{\lambda}$. As we already mentioned, by (29), $\tilde{\lambda}$ has another upper bound, $\overline{\lambda}$, due to $i \geq 0$. Since, by (29), $\overline{\lambda}$ is decreasing in α and satisfies $\lim_{\alpha\to 0}\overline{\lambda} = \infty$, we can also prove that rz/w first increases and eventually decreases with i on the feasible domain of i, by taking an appropriately small value of α so that $\overline{\lambda} > 1/(1-\psi)$. ■

[29] Here $\Xi \equiv \frac{\tilde{\lambda}w}{rz}$. We can derive from (23), (29), and (30)

$$\frac{\tilde{\lambda}(\rho+g)'}{(\rho+g)} = \frac{1-\psi}{\psi}\;\frac{\frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\psi}}\frac{(1-\theta)^{1/\psi}}{\beta e^{\kappa(1-\psi)/\psi}} - \rho + (\ln\tilde{\lambda}+\kappa)\frac{(1-\theta)^{1/\psi}}{\beta e^{\kappa(1-\psi)/\psi}}\frac{1}{\psi}\frac{1-(1-\psi)\tilde{\lambda}}{\tilde{\lambda}^{1/\psi}}}{\rho + \frac{1-\psi}{\psi}(\ln\tilde{\lambda}+\kappa)\left(\frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\psi}}\frac{(1-\theta)^{1/\psi}}{\beta e^{\kappa(1-\psi)/\psi}} - \rho\right)}$$

and

$$\tilde{\lambda}\frac{\left(\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1}\right)' + \frac{\zeta\alpha}{\beta}\phi'}{\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1} + \frac{\zeta\alpha}{\beta}\phi} = \frac{-\frac{\kappa}{1+\kappa}\frac{\tilde{\lambda}}{(\tilde{\lambda}-1)^2} + \frac{\zeta\alpha}{\beta}\frac{\tilde{\lambda}^{1/\kappa}}{\kappa}\left\{\frac{1}{\tilde{\lambda}^{1/\psi}}\frac{(1-\theta)^{1/\psi}}{\beta e^{\kappa(1-\psi)/\psi}}\left[\left(1-\frac{\kappa}{\psi/(1-\psi)}\right)\tilde{\lambda} + \frac{\kappa-\psi}{\psi}\right] - \rho\right\}}{\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1} + \frac{\zeta\alpha}{\beta}\tilde{\lambda}^{1/\kappa}\left(\frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\psi}}\frac{(1-\theta)^{1/\psi}}{\beta e^{\kappa(1-\psi)/\psi}} - \rho\right)}.$$

[30] It is worth noting that there exists a sufficient condition for the lower bound of ζ to be less than 1.