Markets and games: a simple equivalence among the core, equilibrium and limited arbitrage

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MARKETS AND GAMES: A SIMPLE EQUIVALENCE AMONG THE CORE, EQUILIBRIUM AND LIMITED ARBITRAGE

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ABSTRACT

This note provides simple proofs of the equivalence among the core, equilibrium and limited arbitrage in markets with short sales, and with uniform strictly convex preferences.

1. INTRODUCTION

This note provides very short and simple proofs of the equivalence among the core, competitive equilibrium and *limited arbitrage*. Limited arbitrage is a condition defined on the endowments and preferences of the traders in an Arrow Debreu economy; it was introduced in Chichilnisky (1991) and it has an antecedent in Chichilnisky and Heal (1984).

The expression *limited arbitrage* is used to describe economies where only bounded, or limited, gains are available to the traders at their initial endowments. This means that there exists one price—the same for all traders—at which affordable trades can only increase their utilities by limited, or bounded, amounts. Limited arbitrage is related to, but nonetheless different from, the concept of no-arbitrage used in finance, see Chichilnisky (1992, 1995a).

The results in this note provide a very succinct and simple exposition of part of a much larger and more complex area. I draw from earlier

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results establishing for the first time a condition—limited arbitrage—which is simultaneously necessary and sufficient for the existence of a competitive equilibrium and for the non-emptiness of the core. (Chichilnisky 1991, 1992, 1993a, 1994b).[1] The simple proofs provided here work only for economies with short sales (so that trades are in $\mathbb{R}^N$), and when the preferences are convex, uniform and contain no half lines in their indifferences, such as for example those which are representable by strictly concave utilities.[2] It has been established that the equivalence between limited arbitrage and the existence of an equilibrium holds in great generality: it is true in markets with short sales, i.e. when $X = \mathbb{R}^N$ and also in the classic Arrow Debreu case when no short sales are allowed[3] $X = \mathbb{R}^{N+}$; furthermore it holds also with finitely and with infinitely many commodities.[4] Limited arbitrage is also equivalent to the non-emptiness of the core with short sales, and either finitely or infinitely many markets, see also Chichilnisky and Heal (1991). In general, complex proofs are required. However by specializing on a simpler case, when $X = \mathbb{R}^{N+}$ and uniform preferences with indifference surfaces which contain no half lines, I can use known results in the literature[5] Chichilnisky and Heal (1984, 1993), and offer proofs which are very short and simple. Just a few lines suffice.

Section 4 shows that the no-arbitrage Condition C, introduced in 1984 by Chichilnisky and Heal (1984, 1993) in a paper which provided the first proof of existence of a competitive equilibrium in Walrasian economies with short sales, is also simultaneously necessary and sufficient for the existence of a competitive equilibrium and for the nonemptiness of the core in the special cases considered in this paper. The relationship between the results of Chichilnisky and Heal (1984, 1993) and the

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[1] Other literature is discussed in Chichilnisky (1992) and also below.

[2] This excludes the classic Arrow Debreu model in which the trading space is $\mathbb{R}^{N+}$, and preferences which are linear or have half lines in their indifferences. The results in Chichilnisky (1991, 1992, 1993a, 1993b, 1994a, 1994b, 1995a, 1995b) include $\mathbb{R}^{N+}$ and all the preferences mentioned above.

[3] See Chichilnisky (1991, 1992, 1993a, 1994b, 1995a).

[4] Chichilnisky and Heal (1984, 1993).

[5] E.g. the 1984 results in Chichilnisky and Heal (1984, 1993) which introduced a no-arbitrage condition C and proved it is sufficient for the existence of an Arrow Debreu equilibrium in markets with short sales (finite or infinite dimensional). Subsequently, existence results which are special cases of the above were given in Werner (1987), Nielsen (1989), Page (1987), and Chichilnisky (1992). The latter provides necessary as well as sufficient conditions. Theorem 1 of Chichilnisky and Heal (1984, 1993) establishes the existence of an equilibrium under this paper's conditions. See Proposition 4 of Section 4 below.

subsequent results in Werner (1987), Nielsen (1989) and Page (1987) is also discussed in Section 4.

In addition to market equilibrium and the core, the concept of limited arbitrage is central to other forms of resource allocation: it is necessary and sufficient for the existence of continuous anonymous social choice rules which respect unanimity on preferences similar to those of the traders (Chichilnisky 1993c, 1991) and for the elimination of Condorcet cycles and a resolution of Arrow's impossibility theorem on choices of large utility values (Chichilnisky 1994b, 1995a). Limited arbitrage is therefore a unifying condition which is crucial for the most frequently used forms of resource allocation: by markets, games and social choice.

2. DEFINITIONS

An economy E has $H \geqslant 2$ traders who trade $n \geqslant 2$ commodities or assets; the trading space X is $\mathbb{R}^{N+}$. A trader i is described by an initial endowment or property rights vector Ω_i in $\mathbb{R}^N$, and by a convex preference represented by a utility function $u_i\colon \mathbb{R}^N \to \mathbb{R}$, which is continuous and monotonically increasing. Without loss of generality we may choose a quasiconcave utility representation satisfying $u_i(0) = 0$, and $\sup_{x \in \mathbb{R}^N} (u(x)) = \infty$. A market economy is $E = \{X, \Omega_i \in \mathbb{R}^{N+}, u_i\colon X \to \mathbb{R}, i = 1, \ldots, H\}$. The trading opportunities which could yield unbounded utility increases for the ith trader are described by net trades in the **global cone** defined as[6] $A_i(\Omega_i) = \{y \in \mathbb{R}^N\colon \forall x \in \mathbb{R}^N, \exists \lambda > 0\colon u_i(\Omega_i + \lambda y) > u_i(x)\}$: this is a concept originally introduced in Chichilnisky (1991) and (1992), and contains global information about the trader. I make three assumptions on preferences, $\forall i$:

(1) **non-satiation:** $A_i(\Omega_i) \neq \phi$;
(2) **uniformity:** the smallest closed set containing $A_i(\Omega_i)$, denoted $\bar{A}_i(\Omega_i)$, is the set of directions of net trades from Ω_i along which u_i does not decrease and is the same $\forall \Omega_i \in \mathbb{R}^N$. This condition is automatically satisfied by preferences which admit a representation by

[6] The global cone A_i and its closure $\bar{A}_i$ (i.e. the smallest closed set containing A_i) are different in general from the "recession cone" used by Rockafellar (1970) and others, e.g. Werner (1987). For example, with Leontief-type preferences the recession cone is the closed upper contour of the preference while the global cone A_i is an open set. Furthermore when a preference in $\mathbb{R}^N$ has several directions to which the indifferences asymptote, then the recession cone is different also from the closed set $\bar{A}_i$, see e.g. Proposition 2 in Chichilnisky (1995a). For examples see Proposition 2 in Chichilnisky (1995a).

concave utilities,[7] or by smooth utilities with gradients bounded away from zero.[8] It is used to prove the necessity of limited arbitrage for the existence of an equilibrium and the core;

(3) **no flats:** preferences contain no half-lines in their indifference surfaces, a condition which is automatically satisfied by strictly convex preferences.[9]

Under assumptions (1) (2) and (3), the boundary of the global cone, ∂A_i, is non-empty and it consists of those directions along which the utility increases strictly and asymptotically approaches but never reaches a maximum. A competitive equilibrium is a price $p^* \in \mathbb{R}^N$ and an allocation $(x_1^*, \ldots, x_H^*) \in X^H$: $\sum_i (x_i^* - \Omega_i) = 0$ and $\forall\, i\; u_i(x_i^*) = \mathrm{Max}\,(u_i(x_i))$ over the set $\{x_i\colon\ (p^*,\ x_i - \Omega_i) = 0\}$. An allocation $(x_1^*, \ldots, x_H^*) \in X^H$ is in the core if $\sum_i (x_i^* - \Omega_i) = 0$ and $\sim \exists\, N \subset 1, \ldots, H$ and $y_{i,\, i \in N}$: $\sum_{i \in N} (y_i - \Omega_i) = 0$, $\forall\, i \in N$, $u_i(y_i) \geqslant u_i(x_i^*)$ and $\exists\, h \in N$ s.t. $u_h(y_h) > u_h(x_h^*)$. The trader s market cone D_i is the set of all those prices at which all trading opportunities in $\bar{A}_i$ are unaffordable: $D_i = \{p \in \mathbb{R}^N\colon\ \forall\, y \in \bar{A}_i,\ (p,\ y) > 0\}$.

Definition 1. *The market economy E has limited arbitrage when all its market cones intersect:*

$$\bigcap_{i=1}^{H} D_i \neq \phi.$$

This means that there exists one price, the same for all traders, at which the trades they can afford only increase their utilities by limited, or bounded, amounts. The geometry of limited arbitrage is simple: it means that the traders global cones cannot contain net trades which add up to zero: With two traders: $\sim \exists\, x,\, y$ such that $x + y = 0$, $x \in \bar{A}_1$ and $y \in \bar{A}_2$. In other words: the cones $\bar{A}_i$ must lie on one side of a given price hyperplane. Figure 1 illustrates an economy E_1 with two traders and two assets which has limited arbitrage. Its cones are A_1 and A_2 and the price line p leaves both cones on one side. Therefore net trades in directions which lead to unbounded utility gains are unaffordable by all traders from their initial endowments at price p. The economy of Figure 2 does not

[7] For a proof see Chichilnisky (1995b).

[8] For a proof see Chichilnisky (1992, 1995a).

[9] Without the condition that preferences contain no half lines the arguments provided in this paper do not hold. This condition includes all strictly concave preferences, but it excludes linear preferences, preferences which are partly linear and many preferences which are convex but not strictly so.

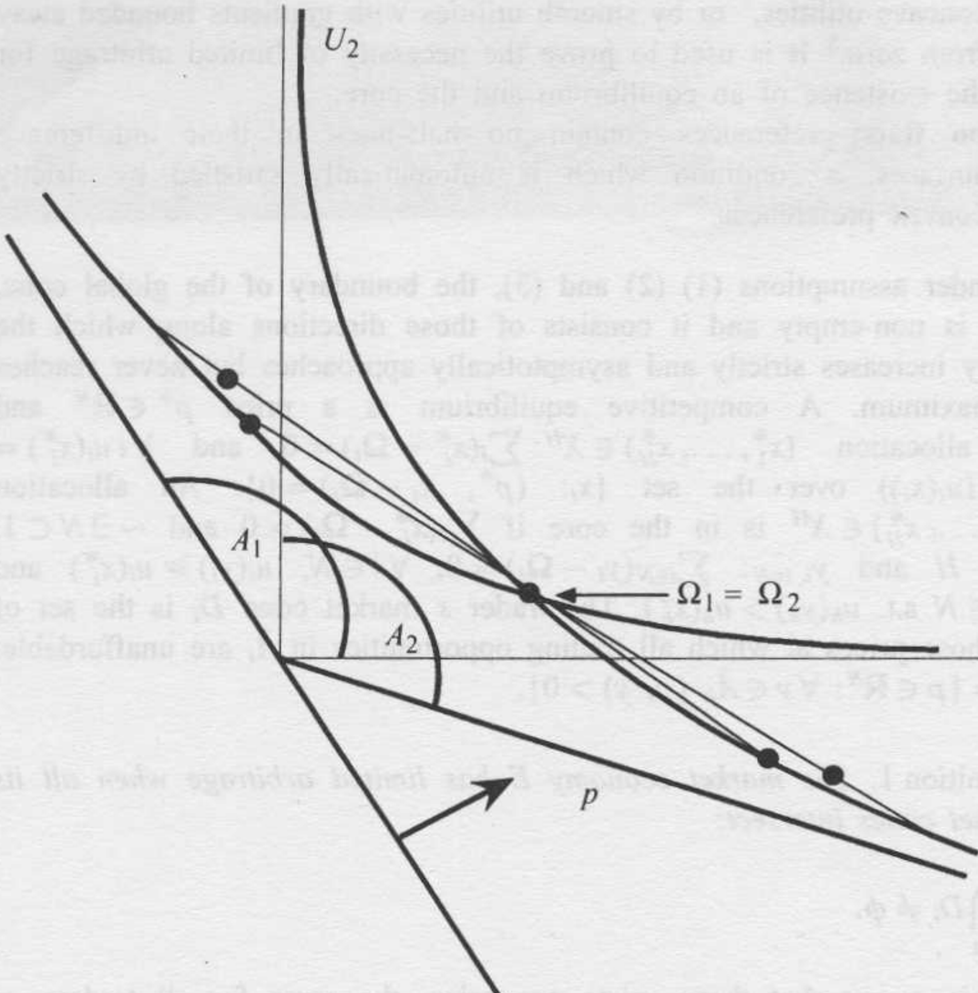

Figure 1: Limited arbitrage is satisfied. The two global cones lie in the halfspace defined by p. There are no feasible trades that increase utilities without limit: these would consist of pairs of points symmetrically placed about the common initial endowment, and such pairs of points lead to utility values below those of the endowments at a bounded distance from the initial endowments.

satisfy limited arbitrage: there are two directions of net trades $w_1 \in A_1$ and $w_1 \in A_2$, which yield unbounded increases in utility and which sum up to zero. Therefore, there is no price p at which all net trades in $\bar{A}_1$ and in $\bar{A}_2$ are unaffordable from initial endowments.

3. LIMITED ARBITRAGE, COMPETITIVE EQUILIBRIUM AND THE CORE

Well defined economies may have no competitive equilibrium. Examples of such failure in Arrow Debreu economies with convex and monotone

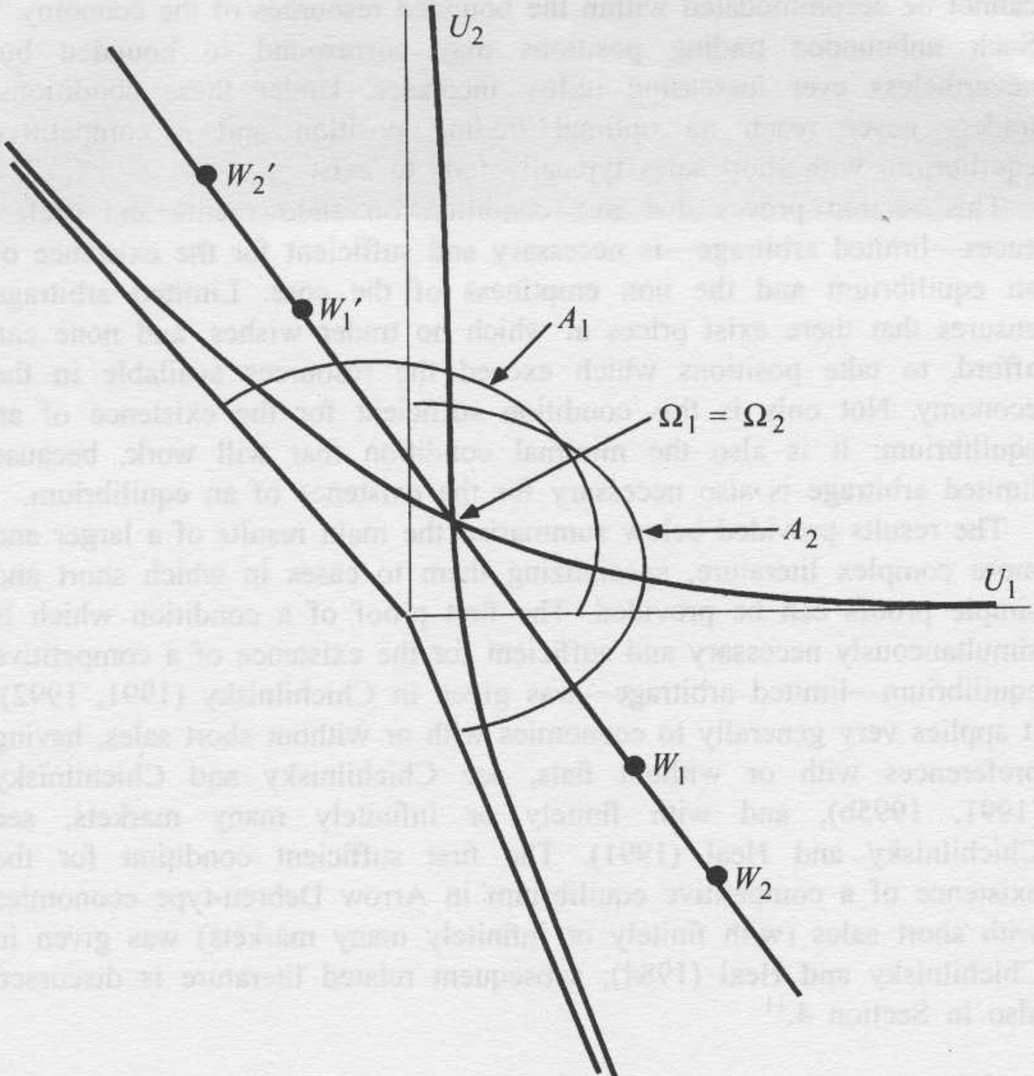

Figure 2: Limited arbitrage does not hold. The global cones are not contained in a half space, and there are sequences of feasible allocations such as (W_1, W_1'), (W_2, W_2') which produce unbounded utilities.

preferences were provided in Arrow and Hahn (1970). They trace this problem to the discontinuity of the excess demand when prices tend to zero and when some trader has a zero endowment of some good, a situation which they consider realistic. This problem is idiosyncratic to economies without short sales.

Here we specialize, instead, in economies with short sales. The problem of the lack of a competitive equilibrium is somewhat different. It is traced to the desire of traders to take arbitrarily long and short positions which cannot be accommodated within the bounded resources of the economy.[10] Such unbounded trading positions may correspond to bounded but nevertheless ever increasing utility increases. Under these conditions, traders never reach an optimal trading position and a competitive equilibrium with short sales typically fails to exist.

This section proves that one condition on endowments and preferences—limited arbitrage—is necessary and sufficient for the existence of an equilibrium and the non emptiness of the core. Limited arbitrage ensures that there exist prices at which no trader wishes, and none can afford, to take positions which exceed the resources available in the economy. Not only is this condition sufficient for the existence of an equilibrium: it is also the minimal condition that will work, because limited arbitrage is also necessary for the existence of an equilibrium.

The results provided below summarize the main results of a larger and more complex literature, specializing them to cases in which short and simple proofs can be provided. The first proof of a condition which is simultaneously necessary and sufficient for the existence of a competitive equilibrium—limited arbitrage—was given in Chichilnisky (1991, 1992); it applies very generally to economies with or without short sales, having preferences with or without flats, see Chichilnisky and Chichilnisky (1991, 1995b), and with finitely or infinitely many markets, see Chichilnisky and Heal (1991). The first sufficient condition for the existence of a competitive equilibrium in Arrow Debreu-type economies with short sales (with finitely or infinitely many markets) was given in Chichilnisky and Heal (1984); subsequent related literature is discussed also in Section 4.[11]

Theorem 3.1. *Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium in the economy E, for any set of initial endowments* $\{\Omega_i\}_{i=1,\dots,H} \in \mathbb{R}^{N\times H}$.

Proof. Necessity first. The proof is by contradiction. Without loss of generality assume that $\forall i,\ \Omega_i = 0$. If p^* is an equilibrium price, then $\forall i,\ x_i \in A_i \Rightarrow (p^*, x_i) > 0$, for otherwise there would exist affordable

[10] When the indifference surfaces of the preferences have a closed set of gradient directions (for example when preferences are linear) then an economy fails to have a competitive equilibrium when it has the resources to accomplish indefinitely large increases in utility, since there are at least two traders who are willing and able to make indefinitely large utility increasing exchanges.

[11] In temporary equilibrium models, which differ from Arrow-Debreu economies in that forward markets are missing, Green (1973) stated conditions on price expectations as necessary and sufficient for the existence of a temporary equilibrium.

consumption bundles whose utility exceeds any other utility level, and therefore an equilibrium cannot exist. Next I will prove that $\forall i$, $x_i \in \partial A_i \Rightarrow (p^*, x_i) > 0$. By contradiction. Assume that $x_i \in \partial A_i$ satisfies $(p^*, x_i) \leqslant 0$, and let y_i be the equilibrium consumption vector for trader i. Now consider the vector $(y_i + \lambda x_i)$, where $\lambda > 1$. Since $(p^*, y_i) \leqslant 0$ and $(\lambda p^*, x_i) = (p^*, \lambda x_i) \leqslant 0$, then $(p^*, y_i + \lambda x_i) = (p^*, y_i) + (\lambda p^*, x_i) \leqslant 0$, so that $y_i + \lambda x_i$ is affordable. Moreover, by assumption (2) on uniformity of the cones, since the vector $x_i \in \partial A_i$, $u_i(y_i + \lambda x_i) > u_i(y_i)$, contradicting the fact that y_i is an equilibrium allocation. Since the contradiction arises from assuming that $\exists\, i$, x_i s.t. $x_i \in \partial A_i$ and $(p^*, x_i) \leqslant 0$, it follows that $\forall i$, $x_i \in \partial A_i = (p^*, x_i) > 0$. Therefore I have shown that $\forall i$, $x_i \in \bar{A}_i = (p^*, x_i) > 0$, so that limited arbitrage is indeed necessary for existence. Sufficiency: in the special case considered here, namely when $X = \mathbb{R}^N$ and when the preferences satisfy conditions (1) to (3), known proofs of existence apply: for example, Theorem 1 of Chichilnisky and Heal (1984, 1993), and the subsequent results[12] of Werner (1987), Nielsen (1989), and Page (1987).

Intuitively Theorem 3.1 is a reasonable result: in an economy without limited arbitrage, such as that in Figure 2, traders wish to take unboundedly large and opposed trading positions, and cannot reach an equilibrium. Desired trades are just too diverse to be accommodated within the same economy. As a corollary of Theorem 3.1 one obtains:

Theorem 3.2. *Limited arbitrage is necessary and sufficient for the non-emptiness of the core in the economy E, for any set of initial endowments* $\{\Omega_i\}_{i=1,\ldots,H}\mathbb{R}^{N\times H}$.

Proof. Since a competitive equilibrium is in the core, the sufficiency of limited arbitrage follows directly from Theorem 3.1. Necessity: a core allocation is Pareto efficient, and it is therefore a competitive equilibrium for some initial endowments: this is the second welfare theorem. For these initial endowments, therefore, Theorem 3.1 implies that limited arbitrage must be satisfied. But limited arbitrage is satisfied simultaneously at all initial endowments, because of the uniformity assumption (2). Therefore limited arbitrage is satisfied at the original initial endowments as well. Therefore limited arbitrage is necessary for the existence of a core allocation.

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12 The proofs of existence of an equilibrium in Chichilnisky (1992, 1995a) include also more general economies: markets with as well as without short sales, preferences which may or may not be strictly convex, and finitely or infinitely many markets.

4. SUFFICIENT CONDITIONS FOR EXISTENCE OF AN EQUILIBRIUM WITH SHORT SALES

In Theorem 1 we invoked prior results which provided sufficient conditions for the existence of an equilibrium in Arrow Debreu markets with short sales. This section shows the connection between different results.

4.1 The literature

In 1984 Chichilnisky and Heal (1984, 1993) defined a **no-arbitrage condition C**, and proved that it is sufficient for the existence of an Arrow Debreu equilibrium in economies with or without short sales, with finitely or infinitely many markets, and with preferences which are or are not strictly convex. Subsequently, in Arrow Debreu economies with strictly convex preferences and with finitely many markets, sufficient conditions for the existence of a competitive equilibrium with short sales[13] were given by Werner (1987) and then by Nielsen (1989). In the context of Hart's model which does not have the generality of the Arrow Debreu market considered here, in Chichilnisky and Heal (1984, 1993) and in Werner (1987), Page (1987) uses the same cones and the same no-arbitrage condition defined earlier by Werner (1987) to state existence results.

The results of Chichilnisky and Heal (1984, 1993) are prior and more general than the rest: their paper was submitted for publication on February 28, 1984 at least a year and a half earlier than the rest, as recorded in the printed versions. Proposition 4.2 shows that the results of Chichilnisky and Heal (1984, 1993) are more general than those of Werner (1987), which were submitted for publication in July 1985, as well as those published later in Nielsen (1989). The subsequent results of Page (1987) are less general than those of Chichilnisky and Heal (1984,

[13] In the particular case covered in this paper, which excludes the Arrow-Debreu model in its classic form (because it admits short sales) and which excludes also the case of preferences which may have flats and may have non-uniform recession cones, Werner (1987) provided a proof of sufficiency and stated, without proof, that the condition which he calls no-arbitrage is necessary for existence. The argument which Werner (1987) provides for necessity is not complete. The point is that the no-arbitrage condition in his Theorem 1 is defined in terms of the sets S_i: it is $\bigcap_i S_i \neq \phi$, while the hint he gives for necessity after the proof of sufficiency is stated, instead, in terms of sets D_i: it is $\bigcap_i D_i \neq \phi$. A proof that $S_i = D_i$ is needed, but the one provided in the Proposition 2 assumes that all sets W_i are the same (see last three lines of (i) in the Proof of Proposition 2, p. 1410), while the assumption made in the paper is, instead, that all recession cones R_i are the same, see Assumption 3, on page 1408. However, $R_i \neq W_i$, see p. 1408, two lines above Assumption 3. Therefore there is no proof of necessity in Werner (1987).

1993), since they apply only to Hart's model. In addition, Chichilnisky and Heal (1984, 1993) has a much larger domain: it applies to finite or infinite dimensional economies, and to preferences which may or may not be strictly convex, while the other three papers are finite dimensional, and apply only to strictly convex preferences.

In the following I restrict the discussion to the finite dimensional case with strictly convex preferences to facilitate the comparison.

4.2 Different no-arbitrage conditions

The main problem for the existence of an equilibrium with short sales is that the set of feasible allocations is unbounded. This can lead traders to take arbitrarily long and short positions, and to the possible loss of equilibrium. To control this problem, Chichilnisky and Heal (1984, 1993) observed that, although feasible allocations are unbounded, **individually rational** feasible allocations may define a bounded set. Intuitively this happens when certain traders are relatively more risk averse than others, and wish to avoid very short or long positions; under these conditions Chichilnisky and Heal (1984, 1993) proves the existence of an equilibrium. They formalized this by requiring:

> **No-arbitrage Condition C** (Chichilnisky and Heal (1984, 1993)): If along a sequence of feasible allocations some trader's allocation grows without bound, then the utility of some other trader's eventually drops below the level of his/her initial endowment.

Subsequently, Werner (1987) and Nielsen (1989) required a no-arbitrage condition, the same in both papers, which they proved to be sufficient for the existence of an equilibrium. Within finite dimensional economies with strictly convex preferences they required:

> **No-arbitrage condition (*)**: There exists a price p such that $(p^*, x_i) > 0$ $\forall x \in R_i$ for $i = 1, 2, \ldots, H$.

In this definition R_i is trader ith s recession cone, $R_i = \{x \in \mathbb{R}^N: u_i(\Omega_i + \lambda x) \geqslant u_i(\Omega_i)$ for all $\lambda > 0\}$; this cone was used by Rockafellar (1970) and subsequently by Werner (1987), Page (1987) and Nielsen (1989). In order to compare the results it is useful to compare the families of cones used in the different concepts.

4.3 Global cones and recession cones

Example 1. *The recession cone R_i is generally different from the global cone A_i, see also Chichilnisky (1995a). For example with Leontief preferences the global cone A_i is the open positive orthant, while the recession cone is the closed orthant, namely the closed upper contour of the preference.*

Example 2. *The recession cone R_i is also generally different from the closure of the global cone $\bar{A}_i$. For example, in Figure 3 below there is a fan of different directions each of which becomes eventually a subset of a different indifference surface. Each of these directions belongs to the recession cone R_i. On the other hand, the closure of the global cone includes only one such direction.*

Example 3. *Under the special assumptions of this paper (1), (2) and (3), $R_i = \bar{A}_i$. This is because, under these assumptions, there is only one utility*

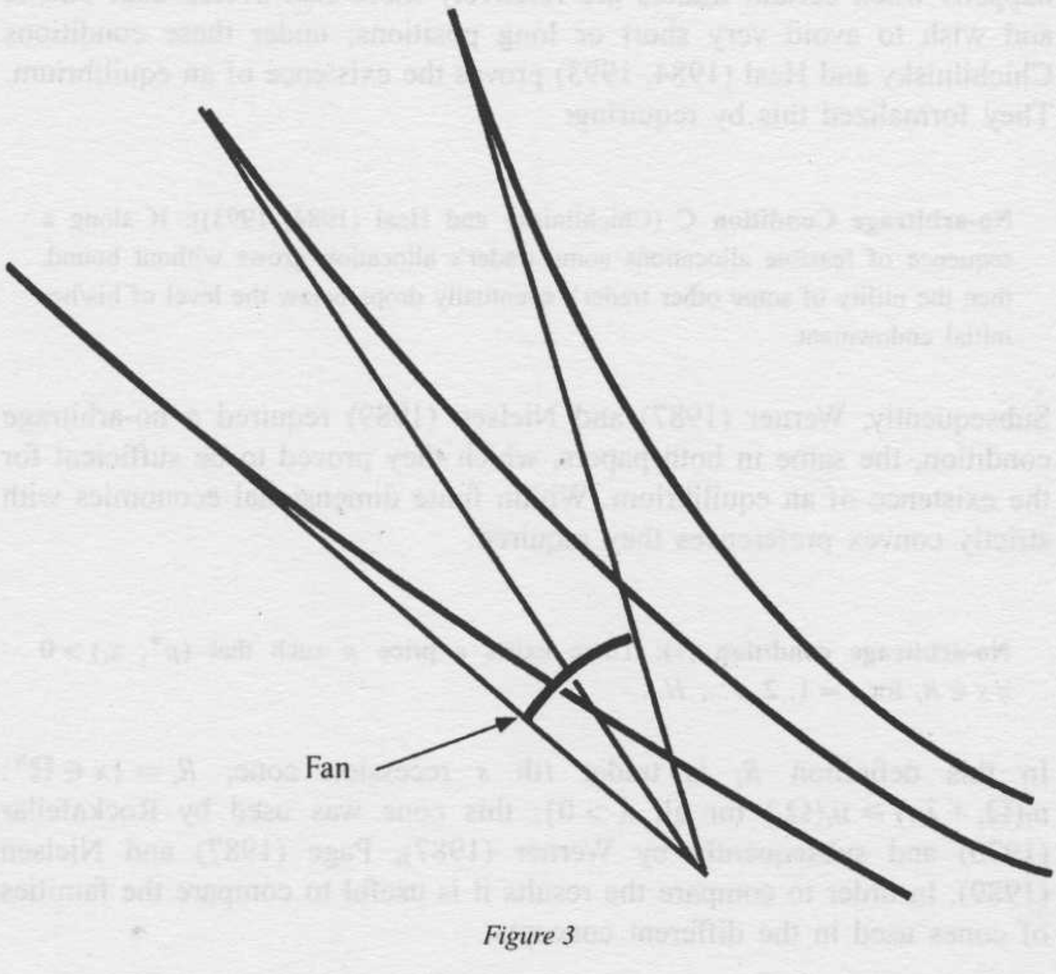

Figure 3

value associated to directions in the boundary of the global cone, ∂A_i, and therefore the type of examples provided above cannot occur. For a proof of this, see Proposition 2 of Chichilnisky (1995a).

Since in our case $R_i = \bar{A}_i$, one has:

Proposition 4.1. *Under assumptions (1), (2) and (3), limited arbitrage coincides with the no-arbitrage condition* (*).

This proposition does not hold when the closure of the global cone $\bar{A}_i$ is different from the recession cone R_i, see e.g. Chichilnisky (1995a).

4.4 Existence of equilibrium with short sales and the no-arbitrage condition C

The following result establishes that the results on the existence of an equilibrium with short sales of Chichilnisky and Heal (1984, 1993) contain the results of Werner (1987) and Nielsen (1989) as special cases. It does so by establishing that the no-arbitrage condition C of Chichilnisky and Heal (1984, 1993) is weaker than $(*)$ and than limited arbitrage when preferences are strictly convex.

Proposition 4.2. *The no-arbitrage condition C is weaker than limited arbitrage and than no-arbitrage condition (*) when preferences satisfy (1), (2) and (3). In particular, the earlier results on the existence of a competitive equilibrium in economies with short sales of Chichilnisky and Heal (1984, 1993) contain as special cases those of Werner (1987) and Nielsen (1989).*

Proof. The proof is by contradiction. Assume that the no-arbitrage condition $(*)$ is satisfied but the set of individually rational feasible allocations is not bounded, then there exists a sequence of allocations denoted $x^j_{j=1,2,\ldots}$ where $x^j = x^j_1, \ldots, x^j_H \in \mathbb{R}^{N\times H}$ such that $\forall j$, $\sum_{i=1}^H (x^j_i - \Omega_i) = 0$, for all i, $u_i(x^j_i) \geqslant u_i(\Omega_i)$ and, for some i, $\lim_j x^j_i = \infty$. Let J be the set of traders for which the norms of the allocations are unbounded: $i \in J \Leftrightarrow \lim_j \|x^j_i\| = \infty$. Since $\sum_{i=1}^H (x^j_i - \Omega_i) = 0$, there must exist at least two traders in J. For each $i \in J$ consider a convergent subsequence of the sequence of normalized vectors $x^j_i/\|x^j_i\|$, and use the same notation for the subsequence. Now observe that when preferences u_i have no flats, assumption (3), a ray defined by a vector v starting from Ω_i, is either in $\bar{A}_i$ (which is not empty by assumption (1)) and therefore u_i is always strictly increasing along this ray, or else $v \in \bar{A}_i^c$, the complement of $\bar{A}_i$, and u_i eventually decreases below the level achieved at $\{\Omega_i\}$. This implies that $\forall\, i \in J$, $x_i = \lim_j x_i^j/\|x_i^j\| \in \bar{A}_i$, because by assumption $\lim_j x_i^j = \infty$ and $u_i(x_i^j) \geqslant u_i(\Omega_i)$: since $\lim_j \|x_i^j\| = \infty$, if $x_i \notin \bar{A}_i$, then $\lim_j u_i(x_i^j) < u_i(\Omega_i)$, which is a contradiction.

Let $G = 1, 2, \ldots, H - J$ be the complement of the set J. By definition, $\lim \sum_{i \in G}(x_i^j - \Omega_i) = M < 0, \infty$, and by construction $\lim_j \sum_{i \in J}(x_i^j - \Omega_i) + \lim_j \sum_{i \in G}(x_i^j - \Omega_i) = \lim \sum_{i \in J}(x_i^j - \Omega_i) + M = 0$, so that $\lim \sum_{i \in J}(x_i^j - \Omega_i - M/\#J) = 0$, where $\#J$ is the cardinality of the set J. Observe that by assumption (2) of uniformity, since

$$\lim \frac{x_i^j}{\|x_i^j\|} \in \bar{A}_i \quad \text{then} \quad z_i = \lim \frac{x_i^j - \Omega_i - \dfrac{M}{\#J}}{\left\| x_i^j - \Omega_i - \dfrac{M}{\#J} \right\|} \in \bar{A}_i \quad \text{for all } i.$$

Consider now the cone in $\mathbb{R}^N$ defined by all strictly positive linear combinations of the vectors $z_{i,\, i \in J}$. Such a cone must either be contained strictly in a half-space of $\mathbb{R}^N$, or must equal a whole subspace of $\mathbb{R}^N$. Since $\lim_j \sum_{i \in J}(x_i^j - \Omega_i) + M = 0$ this implies that the strictly positive combination of the vectors $z_{i,\, i \in J}$ cannot be contained strictly in a half-space of $\mathbb{R}^N$, therefore they must define a subspace of $\mathbb{R}^N$. In particular, for some $j \in J$ they must contain the vector $-z_j$, i.e.

$$\sum_{i \in J} \lambda_i z_i = -z_j, \qquad \text{for some } \lambda_i > 0, \text{ where } z_i \in \bar{A}_i \tag{1}$$

But the no-arbitrage condition $(*)$ requires that $\exists\, p \in \bigcap_i^H D_i$ and, in particular, that $\forall\, z_i \in \bar{A}_i$, $(p,\ z_i) > 0$, which contradicts (1). Since the contradiction arose from assuming that the set of feasible and individually rational allocations is not bounded, the set of such allocations must be bounded. Therefore the no-arbitrage $(*)$ implies the no-arbitrage condition C, as we wished to prove.

Remark 1. *Under the assumptions of this paper (1), (2) and (3), the 1984 no-arbitrage Condition C of Chichilnisky and Heal (1984, 1993) is necessary and sufficient for the existence of a competitive equilibrium and the nonemptiness of the core in economies with short sales.*

Proof. This follows directly from Theorem 3.1 and Proposition 4.2. Condition C is sufficient for the existence of an equilibrium by Theorem 1 of Chichilnisky and Heal (1984, 1993). Necessity is established as follows. Limited arbitrage is necessary for the existence of an equilibrium by Theorem 3.1. Therefore, if an equilibrium exists, limited arbitrage must be satisfied. By Proposition 4.2 limited arbitrage implies the no-arbitrage Condition C. Therefore Condition C is satisfied when an equilibrium exists: i.e. Condition C is necessary for existence. The result on the core follows directly from Theorem 3.2.

Observe that Proposition 4.1 does not hold when assumption (3) is not satisfied, i.e. when preferences have flats:

Remark 2. *Limited arbitrage does not imply that the individually rational feasible allocations form a bounded set when preferences have "flats". In particular, limited arbitrage does not imply Condition (C) in this context. Consider two identical linear preferences in $\mathbb{R}^2$. Limited arbitrage is satisfied because the two preferences are identical. However, Condition C is not, because the set of individually rational feasible allocations is unbounded.*

REFERENCES

Arrow, K. and Hahn, F. (1970) *Competitive Equilibrium Analysis* (San Francisco and New York: Holden Day).

Chichilnisky, G. (1991) "Markets, Arbitrage and Social Choices", Discussion Paper No. 649, Columbia University, December 1991, presented at the conference "Columbia Celebrates Arrow's Contributions" Columbia University, October 27 1991, *CORE Discussion Paper* No. 9342, 1993, forthcoming also in *Social Choice and Welfare*.

Chichilnisky, G. (1992) "Limited Arbitrage is Necessary and Sufficient for the Existence of a Competitive Equilibrium With or Without Short Sales", *Economic Theory* 5, January 1995, pp. 79–108.

Chichilnisky, G. (1993a) "Limited Arbitrage is Necessary and Sufficient for the Existence of the Core", Working Paper, Departments of Economics and Operations Research, Stanford University, July 1993, presented and circulated at the Winter Meetings of the Econometric Society, Boston, January 3–5, 1994.

Chichilnisky, G. (1993b) "Intersecting Families of Sets and the Topology of Cones in Economics", *Bulletin of the American Mathematical Society* 29, pp. 189–207.

Chichilnisky, G. (1993c) "On Strategic Control" *Quarterly Journal of Economics*, February, pp. 285–90.

Chichilnisky, G. (1994a) "Social Diversity, Limited Arbitrage and Gains from Trade: A Unified Perspective on Resource Allocation", *American Economic Review*, pp. 427-34.

Chichilnisky, G. (1994b) "Limited Arbitrage is Necessary and Sufficient for the Existence of an Equilibrium and the Core and it Limits Voting Cycles", *Economic Letters* 46, pp. 321–331.

Chichilnisky, G. (1995a) "A Unified Perspective on Resource Allocation: Limited Arbitrage is Necessary and Sufficient for the Existence of a Competitive Equilibrium, the Core and Social Choice", CORE Discussion Paper 9527, Universite Catolique de Louvain, May 1995, and forthcoming in *Social Choice Reexamined*, eds. K. Arrow, A. Sen and T. Suzumura, (London: Macmillan).

Chichilnisky, G. (1995b) "Concave Utilities have Uniform Global Cones in Finite or Infinite Dimensions", Working Paper, Columbia University, August 1995.

Chichilnisky, G. and Heal, G. (1984) "Existence of a Competitive Equilibrium in L_p and Sobolev Spaces without Bounds in Short Sales", IMA Preprint Series No. 79, Institute for Mathematics and its Applications, University of Minnesota, 1984.

Chichilnisky, G. and Heal G. (1991) "Limited Arbitrage is Necessary and Sufficient for the Existence of Competitive Equilibrium and the Core in Sobolev Spaces", Working Paper, Columbia Business School, *First Boston Working Paper Series*.

Chichilnisky, G. and Heal, G. (1993) "Existence of a Competitive Equilibrium in Sobolev Spaces without Bounds in Short Sales", *Journal of Economic Theory* 59, pp. 364–84.

Green, J. (1973) "Temporary Equilibrium in a Sequential Trading Model with Spot and Future Transactions", *Econometrica* 41, pp. 1103–1123.

Nielsen, L. (1989) "Asset Market Equilibrium with Short Selling" *Review of Economic Studies* 56, pp. 467–474.

Page, F. (1987) "Notes and Comments to the Editor: On Equilibrium in Hart's Securities Exchange Model", *Journal of Economic Theory* 41, pp. 392–404.

Rockafellar, R. T. (1970) *Convex Analysis*, (Princeton: Princeton University Press).

Werner, J. (1987) "Arbitrage and the Existence of a Competitive Equilibrium", *Econometrica* 55, pp. 1403–1418.

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