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Two Distinct Seasonally Fractionally Differenced Periodic Processes

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Abstract

This article is devoted to study the effects of the S-periodical fractional differencing filter $(1-L^S)^{D_t}$. To put this effect in evidence, we have derived the periodic auto-covariance functions of two distinct univariate seasonally fractionally differenced periodic models. A multivariate representation of periodically correlated process is exploited to provide the exact and approximated expression auto-covariance of each models. The distinction between the models is clearly obvious through the expression of periodic auto-covariance function. Besides producing different auto-covariance functions, the two models differ in their implications. In the first model, the seasons of the multivariate series are separately fractionally integrated. In the second model, however, the seasons for the univariate series are fractionally co-integrated. On the simulated sample, for each models, with the same parameters, the empirical periodic auto-covariance are calculated and graphically represented for illustrating the results and support the comparison between the two models.

1 Introduction

Since their introduction by Gladyshev (1961, 1963) much attention has been given to periodically correlated processes. The interest, for such processes is due to their potential use in modeling of cyclical phenomena appearing in hydrology, climatology and in econometrics. Following pioneer work of Gladyshev (1963), an important part of the literature has been devoted to the periodically correlated discrete time processes. A discrete time process is periodically correlated, if there is a non zero integer S such that

$$E\left(X_{t+S}\right)=E\left(X_t\right) \text{ and } Cov(X_{t_1+S},X_{t_2+S})=Cov(X_{t_1},X_{t_2}).$$

A review of the periodically correlated discrete time processes is proposed in Lund and Basawa (1999), Bentarzi and Hallin (1994) give invertibility conditions for periodic moving average.. A large part of the literature on the subject is devoted to the periodic $ARMA$ ($PARMA$) models, which have the following representation:

$$X_t-\sum_{i=0}^{p_t}\phi_{i,t}X_{t-i}=\sum_{j=0}^{q_t}\theta_{j,t}u_{t-j},\ \ t=0,\pm1,\pm2,....,$$

where u_t is a zero-mean white noise with variance σ_t^2. Among searchers who were interested with the periodic autoregressifs processes not periodically stationary, we cite Boswijk and Franses (1995) which studied the problem of the presence of a unit root in a periodic autoregression model of order p ($PAR(p)$) and Boswijk, Franses and Haldrup (1997) which studied the presence of multiple unit roots in a periodic autoregression model of order p. All work cited above were made under the assumption that the processes are periodically integrated of order zero ($PI(0)$), integrated of order one ($I(1)$) or periodically integrated of order one (($PI(1)$). However currently, it well-known that in the scientific fields mentioned above (hydrology, meteorology, econometrics) much of sets of data that have a certain periodicity; have also a long range dependence (or long memory). Such phenomena can be modeled by stationary processes. The stationary processes with seasonal long memory are well know (see for example Gray, Zhang and Woodward (1989): Garma models; Purter-Hudak (1990):Seasonal $ARFIMA$; Oppenheim, G. and al (2000); ould Haye and al (2003) for references, properties and simulations). Another alternative, to take account of certain periodic phenomena with long memory is to consider nonstationary models (but periodically stationary) such as the periodically correlated processes with long memory. The periodically correlated processes, within the meaning of Gladyshev (1963), with long memory did not receive much attention on behalf of the statisticians and the probabilists. Among works associating periodicity within the meaning of Gladyshev (1963) and the presence of long memory we cite, Hui and Li (1995), Franses and Ooms (1997), Ooms and Franses (2001).

For modelling of the Hong Kong United Christian Hospital attendance series, Hui and Li (1995) propose a 2-periodic correlated process,

$$(1-L)^{d_t} Y_t = u_t, \tag{1.1}$$

where $\{u_t, t \in \mathbb{Z}\}$ is a zero mean white noise with variance σ_t^2, and d_t the 2-periodic fractional parameter. The empirical series y_t concerns seventy five (approximately one and half years) data on the average number of people entering the emergency unit on weekday and weekend.

On the other hand, in order to analyzes the long-memory properties in the conditional mean of the quarterly inflation rate in the United Kingdom Franses and Ooms (1997) propose a 4-periodic correlated process,

$$Y_t = (1-L)^{-d_t} u_t, \tag{1.2}$$

where $\{u_t, t \in \mathbb{Z}\}$ is defined as above and d_t is the 4-periodic fractional parameter.

Finally, for the monthly empirical data, which concern the log transformed data of the monthly mean river flow in cubic feet per second, Ooms and Franses (2001) propose to use the seasonal periodic fractional operator defined, in simple framework as follows,

$$Y_t = (1-L^S)^{-D_t} u_t, \; S = 12 \tag{1.3}$$

where $\{u_t, t \in \mathbb{Z}\}$ is defined as above and $S = 12$.

The main difference between, the one hand, the models (1.1) and (1.2) and the other hand, the model (1.3), is in the unit of lag to which the fractional difference operator is applied. In the models (1.1) and (1.2) the fractional difference operator was applied to weekly and quarterly lags, respectively, corresponding to the basic time interval of the time series analyzed. In the model (1.3) the fractional difference operator was applied to yearly, which is the seasonal lag of the time series analyzed. Indeed, by using a binomial expansion for the difference operator $(1-L)^{d_t}$, $(1-L)^{-d_t}$, $(1-L^S)^{-d_t}$ we can rewrite, respectively, the models (1.1), (1.2) and (1.3) as the following,

$$\sum_{j=0}^{\infty} \frac{\Gamma(j-d_t)}{\Gamma(j+1)\Gamma(-d_t)} Y_{t-j} = u_t, \tag{1.4}$$

$$Y_t = \sum_{j=0}^{\infty} \frac{\Gamma(j+d_t)}{\Gamma(j+1)\Gamma(d_t)} u_{t-j}, \tag{1.5}$$

$$Y_t = \sum_{j=0}^{\infty} \frac{\Gamma(j+D_t)}{\Gamma(j+1)\Gamma(D_t)} u_{t-Sj}. \tag{1.6}$$

where

$$\Gamma(z) = \begin{cases} \int_0^{+\infty} s^{z-1} e^{-z} ds, & \text{if } z > 0 \\ \infty & \text{if } z = 0, \end{cases}$$

if $z < 0$, $\Gamma(z)$ is defined in terms of the above expressions and the recurrence formula $z\Gamma(z) = \Gamma(z+1)$.

While, the invertibility and stationarity conditions of the model (1.3) are known (see Ooms and Franses 2001), apart when $d_t = d$ is a constant, nothing is clear about the models (1.1) and (1.2). More precisely, no thing is clear about the stationarity conditions for the model (1.1), because his infinite moving average representation is unknown and no thing is clear about the invertibility conditions for the model (1.2), because his infinite autoregressive representation is unknown. The model (1.3) is invertible and stationary if $-0.5 < D_t < 0.5$ and it is easy to show in this case that the infinite autoregressive representation of the process y_t is given by

$$\sum_{j=0}^{\infty} \frac{\Gamma(j-D_t)}{\Gamma(j+1)\Gamma(-D_t)} Y_{t-Sj} = u_t.$$

For the model (1.4), at any case, in general, we have,

$$Y_t \neq \sum_{j=0}^{\infty} \frac{\Gamma(j+d_t)}{\Gamma(j+1)\Gamma(d_t)} u_{t-j},$$

and for the model (1.5), at any case, in general, we have,

$$\sum_{j=0}^{\infty} \frac{\Gamma(j-d_t)}{\Gamma(j+1)\Gamma(-d_t)} Y_{t-j} \neq u_t.$$

For the particular periodic $ARFIMA(0, d_t, 0)$, namely $(1-L)^{d_t} y_t = u_t$, $u_t \sim i.i.d(0, \sigma_t^2)$, the infinite moving average representation is unknown. In this paper, we give the closed form of this representation. It is important to known such representation in order to deduce the stationarity condition of this type of model. Unfortunately, the closed form obtained is not easy to handle due to her parametric complexity (see Appendix).

Since the $PARFIMA(p, d_t, q)$ is not easy to handle. The work that we present in this article is concerned only on the Seasonal periodical fractional operator, namely $(1-L^S)^{D_t}$. More Precisely, in this work we are interested in certain theoretical properties of the $SPARFIMA(p, 0, 0)(0, D_t, 0)_S$ (Seasonal periodic $ARFIMA$). The study of the theoretical properties of this class of models remains to be made; because among works which evoke this class, only one exists; that is of Franses and Ooms (2001). The work of Franses and Ooms has to consist in adjusting a $SPARFIMA(p, 0, 0)(0, D_t, 0)_S$ to a set of real data. Precisely the model considered by Ooms and Franses is defined as follows:

$$\phi_t(L)\,(X_t - \mu_t) = \eta_t,\ t \in \mathbb{N}^*, \text{ with } \eta_t = (1 - L^S)^{-D_t} u_t,$$

where μ_t is S-periodical constant such as $\mu_t = \mu_{t+S}$, $\phi_t(L) = 1 - \phi_{t,1}L - \phi_{t,2}L^2 - ... - \phi_{t,p}L^p$. The parameters $\phi_{t,i}$ $i = 1, ..., p$ are periodic functions in t, and η_t a white noise seasonally fractionally integrated of order D_t, where D_t is S-periodical fractional parameter. The model above, if $0 \leq D_t < 0.5$, $\forall t$ can be written as follows:

$$(1 - L^S)^{D_t}\phi_t(L)\,(X_t - \mu_t) = u_t,\ t \in \mathbb{Z}. \qquad (Model(I))$$

There is another class of models $SPARFIMA(p, 0, 0)(0, D_t, 0)_S$ distinct from that used by Franses and Ooms (2001); this class is defined as follows:

$$\phi_t(L)(1 - L^S)^{D_t}\,(X_t - \mu_t) = u_t,\ t \in \mathbb{Z}, \qquad (Model(II))$$

where μ_t, $\phi_t(L)$, D_t are defined like above. These two classes coincide, only if $D_t = D$, $\forall t$, since, generally, the composition of $\phi_t(L)$ and $(1 - L^S)^{D_t}$ is not necessarily commutative. To convince, it is sufficient to notice that the S-variate representation of the model (I) is a $VARFI$ model (vector autoregressive model, driven by fractionally integrated innovation) whereas the multivariate writing of the model (II) is a $FIVAR$ model (fractionally integrated vector autoregression) (see Rebecca Sela and Clifford Mr. Hurvich (2008)). These two distinct classes, generalize the univariate model $ARFIMA$, the first is closely related to the cointegrated processes, whereas the second is closely related to the integrated processes. Consequently, in our case, the model (I) is closely related to the cointegrated season and the model (II) is closely related to the integrated season.

In order to distinguish between the model (I) and (II), we note them, respectively as the following: $PAR(p) - PSFI(D_t)$ and $PSFI(D_t) - PAR(p)$. The rest of this paper is organized as follows: section 2 is devoted to defined two class of processes; the periodic autoregressive of order p process with periodic seasonal fractional integrated of order D_t innovation, namely $PAR(p) - PSFI(D_t)$ and the periodic seasonal fractional integrated process, periodic autoregressive of order p, namely $PSFI(D_t) - PAR(p)$. In section 3, for each model defined in section 2, we provide the exact and approximated expression of the periodic autocovariances function. In the section 4, on the simulated samples for each model, with the same parameters for the model (I) and (II), the empirical periodic autocovariances are calculated and graphically represented for illustrating the theoretical results and comparison between the two models.

Without restricting the generality, we suppose that all processes defined below have zero mean.

2 Representation and notation

2.1 S-periodical seasonally fractionally integrated, periodic autoregressive process ($PSFI(D_t)$–$PAR(p)$)

A periodically correlated process $\{Y_t, t \in \mathbb{Z}\}$ is said S-periodical seasonally fractionally integrated of order D_t, periodic autoregressive of order p; if it has the following representation:

$$\Phi_t(L)(1-L^S)^{D_t}Y_t = u_t, \quad t \in \mathbb{Z}, \tag{2.1}$$

where $\{u_t, t \in \mathbb{Z}\}$ is a zero mean white noise with variance $\sigma_t^2$and $(1-L^S)^{D_t}$ are defined like above. $\phi_t(L) = 1 - \phi_{t,1}L - \phi_{t,2}L^2 - ... - \phi_{t,p}L^p$ where $\phi_{t,1}, ..., \phi_{t,p}$ are S-periodical parameters.
Letting $\underline{\mathbf{Y}}_\tau = (\mathbf{Y}_{1,\tau}, ..., \mathbf{Y}_{s,\tau}, ..., \mathbf{Y}_{S,\tau})'$ and $\underline{\mathbf{u}}_\tau = (\mathbf{u}_{1,\tau}, ..., \mathbf{u}_{s,\tau}, ..., \mathbf{u}_{S,\tau})'$ with $\mathbf{Y}_{s,\tau} = Y_{s+S\tau}$ and $\mathbf{u}_{s,\tau} = u_{s+S\tau}$ then the process (2.1) can be rewritten in the S variate form

$$\mathbf{\Phi}_0 \nabla_S^{\mathbf{D}}(L)\underline{\mathbf{Y}}_\tau - \sum_{i=1}^{\mathbf{P}} \mathbf{\Phi}_i \nabla_S^{\mathbf{D}}(L)\underline{\mathbf{Y}}_{\tau-i} = \underline{\mathbf{u}}_\tau, \tag{2.2}$$

where $\mathbf{P} = \left[\frac{p+1}{S}\right] + 1$, with $[x]$ denotes the smallest integer than or equal to x, $\nabla_S^{\mathbf{D}}(L)$ is defined like above. The autoregressive coefficient matrices are given by

$$(\mathbf{\Phi}_0)_{s,j} = \begin{cases} 1 & s = j, \\ 0 & s < j, \\ -\phi_{s-j,j} & s > j, \end{cases}$$

and

$$(\mathbf{\Phi}_i)_{s,j} = \phi_{iS+s-j,s}, \quad s,j = 1,...,S \text{ and } 1 \le i \le \mathbf{P}.$$

The periodic stationarity condition of the model (2.2) is the same as the stationarity condition of it equivalent fractional integrated vector autoregression, namely $FIVAR$, (Rebecca Sela and Clifford Hurvich (2008)) representation (2.2), which means that the roots of the determinantal equation

$$\det\left(I_S z^{\mathbf{P}} - \sum_{i=1}^{\mathbf{P}} \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_i z^{\mathbf{P}-i}\right) = 0,$$

are less than 1 in absolute value (Hannan (1970), Fuller (1976)) and

$$0 \le D_s < 0.5, \text{for all } s = 1, ..., S,$$

(Hosking (1981)). If the process (2.2) is stationary, then it has an infinite moving average representation given by

$$\begin{aligned}\underline{\mathbf{Y}}_\tau &= \nabla_S^{\mathbf{D}}(L)^{-1}\mathbf{\Phi}(L)^{-1}\underline{\mathbf{u}}_\tau,\\ &= \left(\sum_{j=0}^{\infty}\Psi_j L^j\right)\left(\sum_{j=0}^{\infty}\Pi_j L^j\right)\underline{\mathbf{u}}_\tau,\\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\Psi_k\Pi_{j-k}\right)\underline{\mathbf{u}}_{\tau-j},\\ &= \sum_{j=0}^{\infty}C_j\underline{\mathbf{u}}_{\tau-j}, \qquad (2.3)\end{aligned}$$

where $\nabla_S^{\mathbf{D}}(L) = diag\left((1-L)^{D_1}, \cdots, (1-L)^{D_s}, \cdots, (1-L)^{D_S}\right)$, $\mathbf{\Phi}(L) = \mathbf{\Phi}_0 - \mathbf{\Phi}_1 L - ... - \mathbf{\Phi}_P L^P$ and $[\mathbf{\Phi}(L)]^{-1} = \mathbf{\Pi}(L) = \sum_{j=0}^{\infty}\mathbf{\Pi}_j L^j$, with $(\mathbf{\Pi}_j)_{j\in\mathbb{N}}$ is sequence of absolutely summable matrix i.e. $\sum_{j=0}^{\infty}|\mathbf{\Pi}_j(l,k)| < \infty$, $\forall l \in \{1, ..., S\}$ and $\forall k \in \{1, ..., S\}$. $C_j = \sum_{k=0}^{j}\mathbf{\Psi}_k\mathbf{\Pi}_{j-k}$ with $\mathbf{\Psi}_j$ defined like above. The *ith* element of $\underline{\mathbf{Y}}_\tau$, $\mathbf{Y}_{i,\tau}$ is written as follows

$$\mathbf{Y}_{i,\tau} = (1-L^S)^{D_i}\left(\mathbf{\Phi}(L)^{-1}\right)_i\underline{\mathbf{u}}_\tau \qquad (2.4)$$

where $\left(\mathbf{\Phi}(L)^{-1}\right)_i$ is the *ith* rows of $\mathbf{\Phi}(L)^{-1}$. From (2.4) we see clearly that,

$$\mathbf{Y}_{i,\tau} \text{ is integrated of order } D_i, i = 1, ..., S.$$

2.2 Periodic autoregressive, S-periodical seasonally fractionally integrated process ($PAR(p)$–$PSFI(D_t)$)

A periodically correlated process $\{Z_t, t \in \mathbb{Z}\}$ is said, periodic autoregressive of order p; S-periodical seasonally fractionally integrated of order D_t if it has the following representation:

$$\Phi_t(L)Z_t = (1-L^S)^{-D_t}u_t, \quad t \in \mathbb{Z}, \qquad (2.5)$$

where $\{u_t, t \in \mathbb{Z}\}$, $(1-L^S)^{D_t}$ and $\phi_t(L)$ are defined like above. Letting $\underline{\mathbf{Z}}_\tau = (\mathbf{Z}_{1,\tau}, ..., \mathbf{Z}_{s,\tau}, ..., \mathbf{Z}_{S,\tau})'$ and $\underline{\mathbf{u}}_\tau = (\mathbf{u}_{1,\tau}, ..., \mathbf{u}_{s,\tau}, ..., \mathbf{u}_{S,\tau})'$ with $\mathbf{Z}_{s,\tau} = Z_{s+S\tau}$ and $\mathbf{u}_{s,\tau} = u_{s+S\tau}$ then the process (2.5) can be rewritten in the S variate form

$$\mathbf{\Phi}_0\underline{\mathbf{Z}}_\tau - \sum_{i=1}^{\mathbf{P}}\mathbf{\Phi}_i\underline{\mathbf{Z}}_{\tau-i} = \nabla_S^{-\mathbf{D}}(L)\underline{\mathbf{u}}_\tau, \qquad (2.6)$$

where $\mathbf{P} = \left[\frac{p+1}{S}\right] + 1$, with $[x]$ denotes the smallest integer than or equal to x, $\nabla_S^{-\mathbf{D}}(L)$, $\mathbf{\Phi}_0$ and $\mathbf{\Phi}_i$, $i = 1, ..., \mathbf{P}$ are defined like above. The model (2.6) is vector autoregression with fractional integrated innovation, namely $VARFI$ (Rebecca Sela and Clifford Hurvich (2008)). The periodic stationarity condition of the model (2.6) is the same than the model (2.2). The *ith* relation of (2.6) is written

$$(\mathbf{\Phi}(L))_i\underline{\mathbf{Z}}_\tau = (1-L^S)^{-D_i}\mathbf{u}_{i,\tau}$$

where $\mathbf{\Phi}(L) = \mathbf{\Phi}_0 - \mathbf{\Phi}_1 L - ... - \mathbf{\Phi}_\mathbf{P} L^\mathbf{P}$ and $(\mathbf{\Phi}(L))_i$ is the ith rows of $\mathbf{\Phi}(L)$, this means that the ith relation of (2.6) is integrated of order D_i. Among the S relations of (2.6), those which are integrated of order lower than $\underset{1 \leq i \leq S}{max} D_i$ are relations of cointegration. If all the values D_i are different, then we can say that there are $(S-1)$ relations of cointegrations. If $D_1 = ... = D_S$ it does not exist any relation of cointegration. Generally, when we have $D_1 < D_2 < ...D_{S-R-1} < (D_{S-R} = D_{S-R+1} = ... = D_S)$ that means that there are $(S-R-1)$ relations of cointegrations between the S seasons. If the model (2.6) is stationary, it has an infinite moving average representation given by

$$\begin{aligned}
\underline{\mathbf{Z}}_\tau &= \Phi(L)^{-1} \nabla_S^{-\mathbf{D}}(L) \underline{\mathbf{u}}_\tau \\
&= \left(\sum_{j=0}^{\infty} \mathbf{\Pi}_j L^j \right) \left(\sum_{j=0}^{\infty} \mathbf{\Psi}_j L^j \right) \underline{\mathbf{u}}_\tau \\
&= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \mathbf{\Pi}_k \mathbf{\Psi}_{j-k} \right) \underline{\mathbf{u}}_\tau \\
&= \sum_{j=0}^{\infty} \mathbf{H}_j \underline{\mathbf{u}}_{\tau-j},
\end{aligned} \tag{2.7}$$

where $\mathbf{H}_j = \sum_{k=0}^{j} \mathbf{\Pi}_k \mathbf{\Psi}_{j-k}$. The ith element of $\underline{\mathbf{Z}}_\tau$, $\mathbf{Z}_{i,\tau}$ is written as follows

$$\mathbf{Z}_{i,\tau} = \sum_{s=1}^{S} \left(\mathbf{\Phi}(L)^{-1} \right)_{i,s} (1-L)^{-D_S} \mathbf{u}_{s,\tau},$$

where $\left(\mathbf{\Phi}(L)^{-1} \right)_{i,s}$ is (i,s) th element of the matrix $\mathbf{\Phi}(L)^{-1}$. $\mathbf{Z}_{i,\tau}$ is written like linear combination of S independent processes, respectively, integrated of order $D_1, ..., D_s, ..., D_S$; consequently $\mathbf{Z}_{i,\tau}$ is integrated of order $\underset{1 \leq i \leq S}{max} D_i$ (Granger 1986).

3 Periodic autocovariances

This section deals with the determination of theoretical periodic autocovariances of periodically correlated processes defined in precedent section.

3.1 $PSFI(D_t) - PAR(p)$ periodic autocovariances

Theorem 1 *Given the stationary S-variate process $\underline{\mathbf{Y}}_\tau$ defined by* (2.2), *we have*

$$\Gamma_{\underline{\mathbf{Y}}_\tau}(h) \sim \Delta \left[h^{D-0.5} \right] \mathbf{A} \Delta \left[h^{D-0.5} \right], \text{ as } h \to \infty \tag{3.1}$$

where the (i,k) *th element of* $S \times S$ *matrix* $\mathbf{A}$ *is:*

$$\frac{\Gamma(1-D_i-D_k)}{\Gamma(D_k)\Gamma(1-D_k)} \mathbf{\Pi}_i' \mathbf{\Omega} \mathbf{\Pi}_k$$

with $\mathbf{\Pi}_i'$ *is the ith rows of the matrix* $\mathbf{\Pi}$ *and* $\mathbf{\Omega} = diag(\sigma_1^2, ..., \sigma_s^2, ..., \sigma_S^2)$.
Proof. *See Ching-Fan Chung (2002).* ■

Corollary 2 *Given the process Y_t defined in* (2.1), *we have:*

$$\gamma^{(s)}(j) \sim (h+\delta)^{D_s+D_{s+\nu-S*\delta}-1}\left[\frac{\Gamma(1-D_s-D_{s+\nu-S*\delta})}{\Gamma(D_{s+\nu-S*\delta})\Gamma(1-D_{s+\nu-S*\delta})}\right]\mathbf{\Pi}'_s\mathbf{\Omega}\mathbf{\Pi}_{s+\nu-S*\delta}, \tag{3.2}$$

where h and ν are integers such as $j = h\times S+\nu$, and $j>0$, i.e. $j \equiv \nu\,[h]$ with $0 \le \nu < S-1$ and δ is defined as follows:

$$\begin{cases} \delta = 0, & \text{if } 1 \le s+\nu < S, \\ \delta = 1, & \text{if } S+1 \le s+\nu \le 2S-1, \end{cases}$$

$\mathbf{\Pi}'_s$ *is the sth rows of the matrix* $\mathbf{\Pi}$ *and* $\mathbf{\Omega} = diag(\sigma_1^2, ..., \sigma_s^2, ..., \sigma_S^2)$.

Proof. The proof of the corollary, rises directly from theorem 1. From theorem 1, we have :

$$\Gamma^{(i,k)}_{\underline{\mathbf{Y}}_\tau}(h) \sim h^{(D_i+D_k-1)}\frac{\Gamma(1-D_i-D_k)}{\Gamma(D_k)\Gamma(1-D_k)}\mathbf{\Pi}'_i\mathbf{\Omega}\mathbf{\Pi}_k, \tag{3.3}$$

where $\Gamma^{(i,k)}_{\underline{Y}_\tau}(h) = Cov(\mathbf{Y}_{i,\tau}, \mathbf{Y}_{k,\tau+h})$ are the (i,k) *th* element of the covariance matrix of $\Gamma_{\underline{\mathbf{Y}}_\tau}(h)$. Moreover, it is known that

$$\begin{aligned} \gamma^{(s)}(j) &= Cov(Y_{S\tau+s},\ Y_{S\tau+s+j}) \\ &= Cov(\mathbf{Y}_{s,\tau}, \mathbf{Y}_{s+j,\tau}). \end{aligned} \tag{3.4}$$

Putting $j = Sh+\nu$ with $0 \le \nu < S-1$, by replacing j by $Sh+\nu$ in (3.4), we have

$$\gamma^{(s)}(j) = Cov(\mathbf{Y}_{s,\tau}, \mathbf{Y}_{s+\nu,\tau+h}). \tag{3.5}$$

According to the value of $(s+\nu)$, the equality (3.5), becomes

$$\gamma^{(s)}(j) = \begin{cases} \Gamma^{(s,s+\nu)}_{\underline{\mathbf{Y}}_\tau}(h), & \text{if } 1 \le s+\nu < S, \\ \Gamma^{(s,s+\nu-S)}_{\underline{\mathbf{Y}}_\tau}(h+1), & \text{if } (S+1) \le s+\nu \le 2S-1. \end{cases}$$

By using the approximation (3.3), we have:

$$\gamma^{(s)}(j) \sim \begin{cases} h^{(D_s+D_{s+\nu}-1)}\frac{\Gamma(1-D_s-D_{s+\nu})}{\Gamma(D_{s+\nu})\Gamma(1-D_{s+\nu})}\mathbf{\Pi}'_s\mathbf{\Omega}\mathbf{\Pi}_{s+\nu}, \\ \text{if } 1 \le s+\nu < S, \\ \\ (h+1)^{(D_s+D_{s+\nu-S*\delta}-1)}\frac{\Gamma(1-D_s-D_{s+\nu-S*\nu})}{\Gamma(D_{s+\nu})\Gamma(1-D_{s+\nu-S*\delta})}\mathbf{\Pi}'_s\mathbf{\Omega}\mathbf{\Pi}_{s+\nu-S*\delta}, \\ \text{if } (S+1) \le s+\nu \le 2S-1, \end{cases}$$

where $\mathbf{\Pi}'_s$ is the *sth* rows of the matrix $\mathbf{\Pi}$ and $\mathbf{\Omega} = diag(\sigma_1^2, ..., \sigma_s^2, ..., \sigma_S^2)$. From corollary 1; emerges several remarks, the most important are ■

Remark 3 *: The periodic autocovariances $\gamma^{(s)}(j)$ $s = 1, ..., S$ taper off at different hyperbolic rates. If we suppose that $\min_{1\le i\le S} D_i = D_1$ and $\max_{1\le i\le S} D_i = D_S$ (this does not restrict the generality) than $\gamma^{(1)}(j)$, with $j \equiv 0[S]$ has the more speedy taper off hyperbolic rate ($\propto h^{2D_1-1}$) and $\gamma^{(S)}(j)$, with $j \equiv 0[S]$ has the lowest taper off hyperbolic rate ($\propto h^{2D_S-1}$).*

This remark will be largely clarified graphically (see section 4, couples of figures $(1a, 1b)$ to $(5a, 5b)$. The advantage which offer by the periodic process is the possibility of representing the graph of the autocovariances in various manners. The autocovariances functions $\gamma^{(s)}(j)$, $s = 1, ..., S$ can be represented in the same plot (hui ad Li 1995), or separately. For $j = Sh + \nu$, with $0 \leq \nu < S - 1$ we can also represented $\gamma^{(s)}(Sh + \nu)$, $\nu = 0, ..., S - 1$ in the same plot. These are the three kinds of graphs which we will use in the next section.

3.2 $PAR(p) - PSFI(D_t)$ periodic autocovariances

Before stating the main result of this section, we need some further notation. Let $D_{\max} = \max\limits_{1 \leq s \leq S} D_s$ and define Before stating the main result of this section, we need some further notation. Let $D_{\max} = \max\limits_{1 \leq s \leq S} D_s$ and define $F = \{1, ..., s, ..., S\}$, $F_1 = \{i,\ i \in F \ / \ D_i = D_{\max}\}$, with $|F_1| = R$ and $F_2 = \{i,\ i \in F \ / \ D_i < D_{\max}\}$, with $|F_2| = S - R$. We have $F_1 \cap F_2 = \varnothing$ and $F_1 \cup F_2 = F$.

Theorem 4 *Given the stationary S-variate process* $\underline{\mathbf{Z}}_\tau$ *defined by* (2.6), *we have*

$$\Gamma_{\underline{\mathbf{Z}}_\tau}(j) \sim j^{2D_{\max}-1}\mathbf{A},\ as\ j \to \infty \tag{3.6}$$

where the (i, k) *th element of* $S \times S$ *matrix* $\mathbf{A}$, *is:*

$$\mathbf{A}(l, m) = \frac{\Gamma(1 - 2D_{\max})}{\Gamma(D_{\max})\Gamma(1 - D_{\max})} \sum_{i \in F_1} \mathbf{\Pi}(l, i)\mathbf{\Pi}(m, i)\sigma_i^2$$

where $\mathbf{\Pi} = [\mathbf{\Phi}(1)]^{-1} = \sum\limits_{j=0}^{\infty} \mathbf{\Pi}_j$ *and* $\mathbf{\Pi}(l, i)$ *is* (l, i) *th element of* $\mathbf{\Pi}$.

Proof. See Ching-Fan Chung (2002). ■

The corollary below, gives the approximated expression, as $j \to \infty$, of the periodic autocovariances function, $\gamma^{(s)}(j) = cov(Z_{S\tau+s},\ Z_{S\tau+s+j})$ of the process Z_t, defined in (2.5).

Corollary 5 *Given the process* Z_t *defined in* (2.5), *we have,*

$$\gamma^{(s)}(j) \sim (h+\delta)^{2D_{\max}-1} \left[\frac{\Gamma(1 - 2D_{\max})}{\Gamma(D_{\max})\Gamma(1 - D_{\max})}\right] \sum_{i \in F_1} \mathbf{\Pi}(s, i)\mathbf{\Pi}(s+\nu-S\delta, i)\sigma_i^2 \tag{3.7}$$

where h *and* ν *are integers such as* $j = hS + \nu$, *and* $j > 0$, *i.e.* $j \equiv \nu\,[h]$ *with* $0 \leq \nu < S - 1$, *and* δ *is defined as follows:*

$$\begin{cases} \delta = 0, & if\ 1 \leq s + \nu < S \\ \delta = 1, & if\ S + 1 \leq s + \nu \leq 2S - 1 \end{cases}$$

and $\mathbf{\Pi}(i, s)$ *is the* (i, s) *th element of the matrix* $\mathbf{\Pi} = [\mathbf{\Phi}(1)]^{-1} = \sum\limits_{j=0}^{\infty} \mathbf{\Pi}_j$.

Proof. The proof of the corollary, rises directly from theorem 3. From theorem 3, we have :

$$\Gamma_{\underline{\mathbf{Z}}_\tau}^{(l,k)}(j) \sim j^{2D_{\max}-1} \frac{\Gamma(1 - 2D_{\max})}{\Gamma(D_{\max})\Gamma(1 - D_{\max})} \sum_{i \in F_1} \mathbf{\Pi}(l, i)\mathbf{\Pi}(k, i)\sigma_i^2 \tag{3.8}$$

where $\Gamma^{(l,k)}_{\underline{\mathbf{Z}}_\tau}(j) = Cov(\mathbf{Z}_{l,\tau}, \mathbf{Z}_{k,\tau+j})$ is the (i,k) th element of the covariance matrix $\Gamma_{\underline{\mathbf{Z}}_\tau}(j)$. Moreover, it is known that

$$\begin{aligned} \gamma^{(s)}(j) &= Cov(Z_{S\tau+s}, Z_{S\tau+s+j}) \\ &= Cov(\mathbf{Z}_{s,\tau}, \mathbf{Z}_{s+j,\tau}) \end{aligned} \tag{3.9}$$

Putting $j = Sh + \nu$ with $0 \leq \nu < S-1$, by replacing j by $Sh+\nu$ in (3.9), we have

$$\gamma^{(s)}(j) = Cov(\mathbf{Z}_{s,\tau}, \mathbf{Z}_{s+\nu,\tau+h}) \tag{3.10}$$

According to the value of $(s+\nu)$, the equality (3.10), becomes

$$\gamma^{(s)}(j) = \begin{cases} \Gamma^{(s,s+\nu)}_{\underline{\mathbf{Z}}_\tau}(h), & \text{if } 1 \leq s+\nu < S \\ \Gamma^{(s,s+\nu-S)}_{\underline{\mathbf{Z}}_\tau}(h+1), & \text{if } (S+1) \leq s+\nu \leq 2S-1 \end{cases}$$

By using the approximation (3.8), we have,

$$\gamma^{(s)}(j) \sim \begin{cases} h^{(2D_{\max}-1)} \frac{\Gamma(1-2D_{\max})}{\Gamma(D_{\max})\Gamma(1-D_{\max})} \sum_{i\in F_1} \mathbf{\Pi}(s,i)\mathbf{\Pi}(s+\nu,i)\sigma_i^2, \\ \qquad \text{if } 1 \leq s+\nu < S \\ \\ (h+1)^{(2D_{\max}-1)} \frac{\Gamma(1-2D_{\max})}{\Gamma(D_{\max})\Gamma(1-D_{\max})} \sum_{i\in F_1} \mathbf{\Pi}(s,i)\mathbf{\Pi}(s+\nu-S,i)\sigma_i^2, \\ \qquad \text{if } (S+1) \leq s+\nu \leq 2S-1 \end{cases}$$

■

Remark 6 *If $D_1 = D_2 = ... = D_S$ the periodic autocovariances $\gamma^{(s)}(j)$ of the model* (2.2) *coincide with those of model* (2.8)

Remark 7 *From corollary 4; we see that the periodic autocovariances $\gamma^{(s)}(j)$ $s = 1, ..., S$ taper off at the same hyperbolic rates.*

4 Simulation

In this section we compare the finite sample of the periodic autocovariances $\gamma^{(s)}(j)$ $s = 1, ..., 4$ of the models (1.3), (2.1) and (2.5) for different value of $\underline{D} = (D_1, D_2, D_3, D_4)$. The sample size for each model is $T = 1000$.

The model we consider for the simulation study are

- **Model A**

$$(1-L^4)^{D_t} X_t = \varepsilon_t$$

which has the following S-variate representation

$$\begin{pmatrix} (1-L)^{D_1} & 0 & 0 & 0 \\ 0 & (1-L)^{D_2} & 0 & 0 \\ 0 & 0 & (1-L)^{D_3} & 0 \\ 0 & 0 & 0 & (1-L)^{D_4} \end{pmatrix} \underline{\mathbf{X}}_\tau = \underline{\mathbf{u}}_\tau$$

- **Model B**

$$\Phi_t(L)(1-L^4)^{D_t}Y_t = u_t$$

which has the following S-variate representation

$$\begin{pmatrix} 1 & 0 & 0 & -0.7 \\ -0.8 & 1 & 0 & 0 \\ 0 & -0.6 & 1 & 0 \\ 0 & 0 & -0.4 & 1 \end{pmatrix} \begin{pmatrix} (1-L)^{D_1} & 0 & 0 & 0 \\ 0 & (1-L)^{D_2} & 0 & 0 \\ 0 & 0 & (1-L)^{D_3} & 0 \\ 0 & 0 & 0 & (1-L)^{D_4} \end{pmatrix} \underline{\mathbf{Y}}_\tau = \underline{\mathbf{u}}_\tau$$

- **Model C**

$$(1-L^4)^{D_t}\Phi_t(L)Z_t = u_t$$

which has the following S-variate representation

$$\begin{pmatrix} (1-L)^{D_1} & 0 & 0 & 0 \\ 0 & (1-L)^{D_2} & 0 & 0 \\ 0 & 0 & (1-L)^{D_3} & 0 \\ 0 & 0 & 0 & (1-L)^{D_4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -0.7 \\ -0.8 & 1 & 0 & 0 \\ 0 & -0.6 & 1 & 0 \\ 0 & 0 & -0.4 & 1 \end{pmatrix} \underline{\mathbf{Z}}_\tau = \underline{\mathbf{u}}_\tau$$

where $\underline{\mathbf{u}}_\tau$ are $i.i.d$ $N(\mathbf{0}, \mathbf{\Omega})$ with $\mathbf{\Omega} = diag(1,1,1,1)$.

4.1 Simulated Autocovariances of model A

In figures 1 to 4, we represent the empirical autocovariances function $\gamma^{(s)}(j)$ $s = 1, ..., 4$ in the same plot, for the model A for different value of

$\underline{D} = (D_1, D_2, D_3, D_4)$.

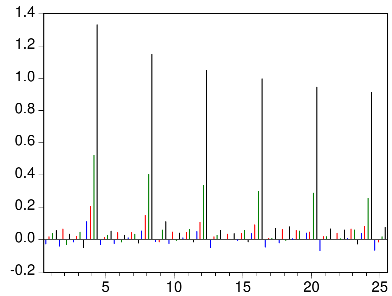


Figure 1: The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 25 with $\underline{D} = (0.1,\ 0.2,\ 0.3,\ 0.4)$ for model A

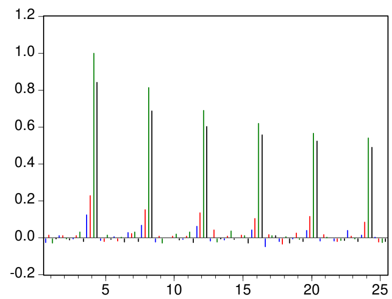


Figure 2: The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 25 with $\underline{D} = (0.1,\ 0.2,\ 0.4,\ 0.4)$ for model A

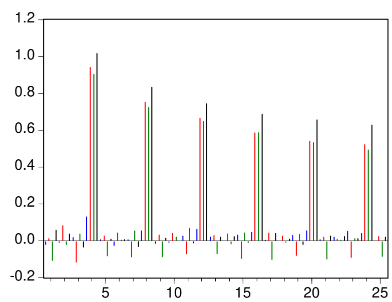


Figure 3: The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 25 with $\underline{D} = (0.1,\ 0.4,\ 0.4,\ 0.4)$ for model A

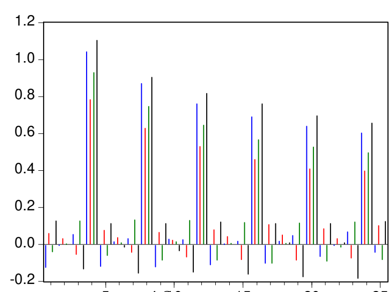
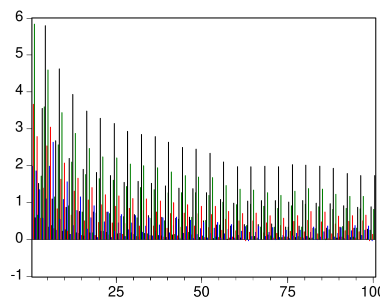


Figure 4: The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 25 with $\underline{D} = (0.4,\ 0.4,\ 0.4,\ 0.4)$ for model A

The figures 1, illustrate well the theoretical result of theorem 1 and also states that the periodicity is caused by the fractional parameters $\underline{D} = (0.1, 0.2, 0.3, 0.4)$ (the auto-covariances $\gamma^{(s)}(j)$ $s = 1, ..., 4$ for lag $j \equiv 0[4]$ taper off, respectively, at hyperbolic rates, according the value of $\underline{D}$.

4.2 Simulated autocovariances of model B

For $\underline{D} = (0.1, 0.2, 0.3, 0.4)$, the figures $(1a)$ and $(1b)$ represents the empirical autocovariances $\widehat{\gamma}^{(s)}(j)$ $s = 1, ..., 4$, respectively, in spike graph and in line graph of the model B. The couples of figures $(2a, 2b)$ to $(5a, 5b)$ represents the empirical autocovariances $\widehat{\gamma}^{(1)}(4h+\nu)$, $\nu = 0, ..., 3$ to $\widehat{\gamma}^{(4)}(4h+\nu)$, $\nu = 0, ..., 3$, for $h = 1$ to 25, respectively, in spike graph and line graph, for the model B.



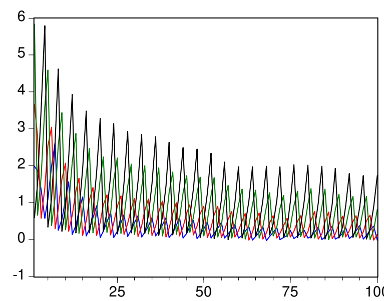
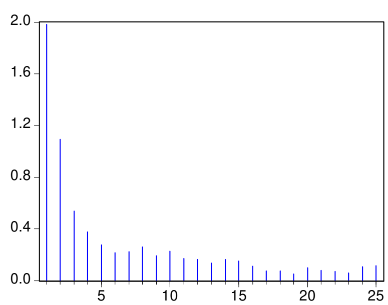


Figure $1a$ Figure $1b$

The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 100, of model B, with $\underline{D} = (0.1,\ 0.2,\ 0.3,\ 0.4)$, taper off at different hyperbolic rates



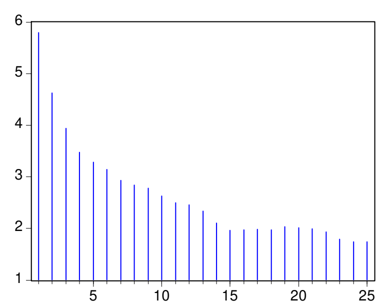


figure $2a$ figure $2b$

The figure $2a$ and $2b$ represents, respectively, the speedy and the lowest taper off hyperbolic rate of autocovariances of model (B)

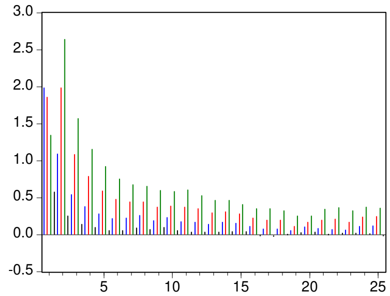


Figure 2a

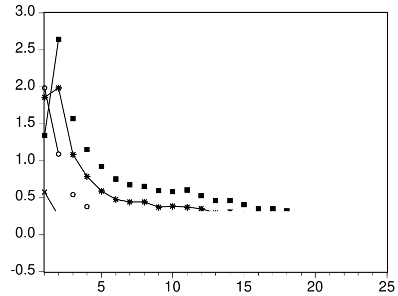


Figure 2b

The periodic autocovariances $\gamma^{(1)}(4h+\nu),\ \nu=0,...,3,$ for fixed h ($h=1$ to 25) have tendency to increase according with the value of $D_1+D_{1+\nu-4\delta}$

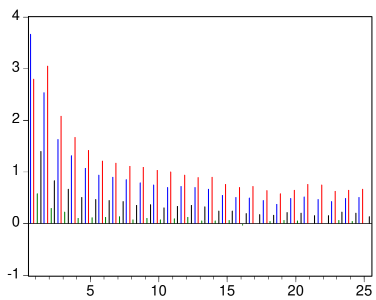


Figure 3a

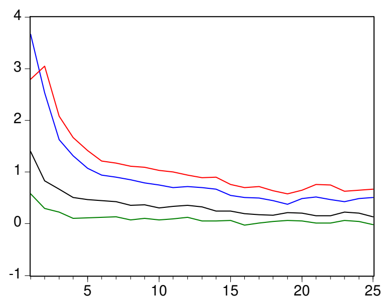


Figure 3b

The periodic autocovariances $\widehat{\gamma}^{(2)}(4h+\nu),\ \nu=0,...,3,$ for fixed h ($h=1$ to 25) have tendency to increase according with the value of $D_2+D_{2+\nu-4\delta}$

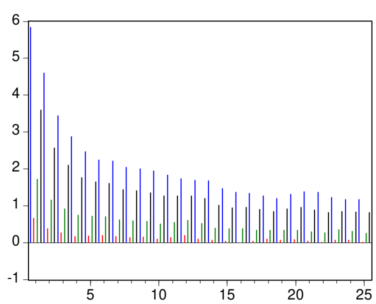


Figure 4a

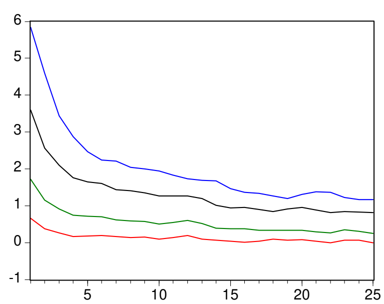


Figure 4b

The periodic autocovariances $\gamma^{(3)}(4h+\nu),\ \nu=0,...,3,$ for fixed h ($h=1$ to 24) have tendency to increase according with the value of $D_3+D_{3+\nu-4\delta}$

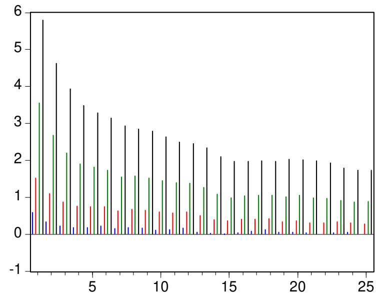


figure 5a

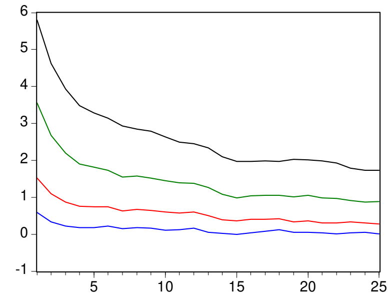


figure 5b

The periodic autocovariances $\widehat{\gamma}^{(4)}(4h+\nu),\ \nu = 0,...,3,$ for fixed h ($h = 1$ to 25) have tendency to increase according with the value of $D_4 + D_{4+\nu-4\delta}$

4.3 Simulated autocovariances of model C

The figures $(1c)$ and $(2c)$ represents the empirical autocovariances $\widehat{\gamma}^{(s)}(j)$ $s = 1,...,4$, respectively, in spike graph and in line graph of the model C. The difference between the periodic autocovariances $\gamma^{(s)}(j),\ s = 1,...,4,$ for lag $j = 1$ to 100, decreases at the same manner, mainly because they taper off at the same hyperbolic rates

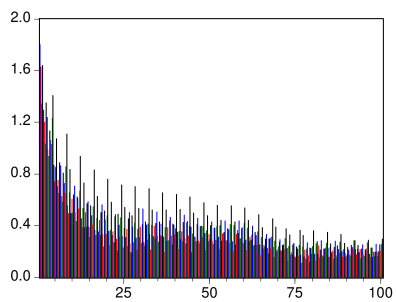


Figure $1c$

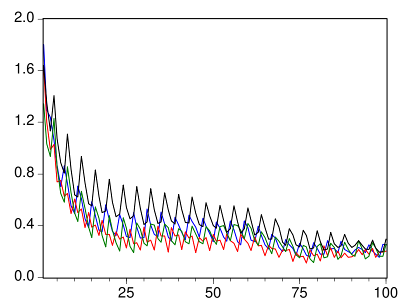


Figure $2c$

The periodic autocovariances $\widehat{\gamma}^{(s)}(j),\ s = 1,...,4,$ for lag $j = 1$ to 100 and $\underline{D} = (0.1,\ 0.2,\ 0.4,\ 0.4),$ taper off at the same hyperbolic rates

4.4 Simulated comparison between autocovariances of model B and C

In order to compare, both autocovariances $\widehat{\gamma}^{(s)}(j),\ s = 1,...,4$ for model (B) and model (C) we represent them graphically in the same scale for different value of $\underline{D} = (D_1, D_2, D_3, D_4)$ (see below).

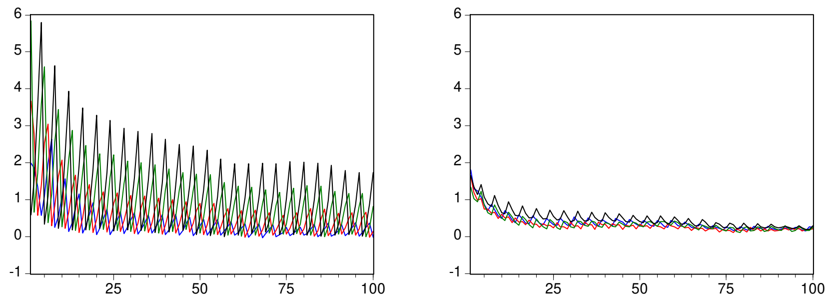


Figure 5: The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 100, $\underline{D} = (0.1,\ 0.2,\ 0.3,\ 0.4)$ for respectively, model (B) and model (C)

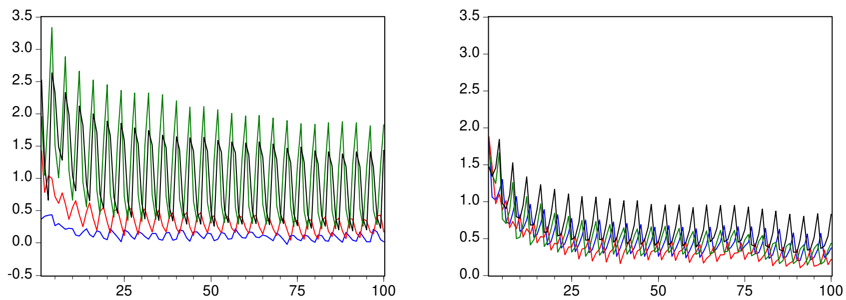


Figure 6: The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 100, $\underline{D} = (0.1,\ 0.2,\ 0.4,\ 0.4)$ for respectively, model (B) and model (C)

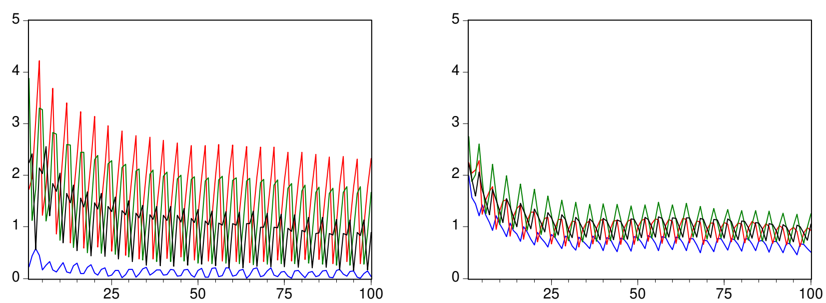


Figure 7: The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 100, $\underline{D} = (0.1,\ 0.4,\ 0.4,\ 0.4)$ for, respectively, model (B) and model (C)

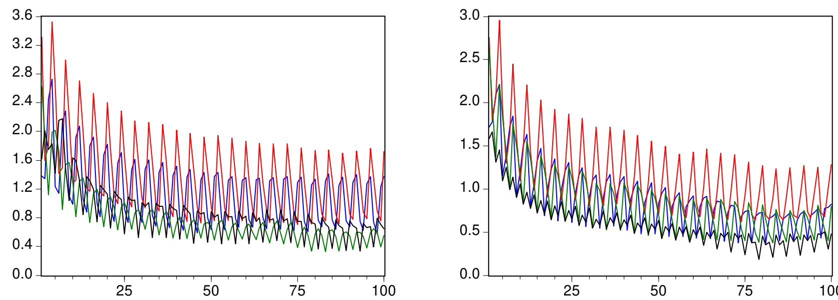


Figure 8: The periodic autocovariances $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$, for lag $j = 1$ to 100, $\underline{D} = (0.4,\ 0.4,\ 0.4,\ 0.4)$ for, respectively, model (B) and model (C)

In figures 5 to 8 we plot the autocovariance sequences $\widehat{\gamma}^{(s)}(j)$, $s = 1, ..., 4$ of model B and model C in the same scale and with identical parameters ($\Phi(L)$, Ω, and $\underline{D}$). The autocovariances sequences differ dramatically. Rebecca Sela and Clifford Hurvich (2008) presents a similar conclusion for cross-covariance sequences of bivariate $FIVAR(1, \underline{D})$ and $VARFI(1, \underline{D})$ processes with the same parameters. They point out that the first model have the series integrated separately (in our case the seasons are integrated separately) and in the second there is cointegration relation between the two series (in our case there are 3 cointegrations relations between the four seasons). This fact, does not explain clearly why there is such difference between the autocovariances of model B and model C. Further more, the taper off hyperbolic rates of the autocovariances of model (C) is equal than the lowest tapper off hyperbolic rate of the autocovariances of model (B), so why the autocovariance sequences differ dramatically? The explanation is in explicit results of corollary 3.1 and corollary 3.2. Generally, in the literature of long memory models, attention is focused on the fractional parameters (which associate with hyperbolic tapper off of autocovariance) rather than on autoregressive or moving average parameters and $V(\varepsilon_t)$ included in expression of autocovariance. In the expression (3.4), the autoregressive parameters and $V(\varepsilon_t)$ appears in the following form: $\pi'_s \Omega \pi_{s+\nu-S*\delta}$ and in expression (3.6) it appears in the following form: $\Pi(s, S)\Pi(s + \nu - S * \delta, S)\sigma^2_S$. From model (B) and model (C), the set, of possible values, of these two quantities are respectively:

$$\begin{pmatrix} 2.131 & 1.8989 & 1.4628 & 1.3938 \\ 1.8989 & 2.6744 & 1.8634 & 1.3923 \\ 1.4628 & 1.8634 & 2.2734 & 1.2975 \\ 1.3938 & 1.3923 & 1.2975 & 1.6743 \end{pmatrix} \tag{4.1}$$

and

$$\begin{pmatrix} 0.65398 & 0.52318 & 0.31391 & 0.93428 \\ 0.52318 & 0.41854 & 0.25113 & 0.74742 \\ 0.31391 & 0.25113 & 0.15068 & 0.44845 \\ 0.93428 & 0.74742 & 0.4445 & 1.3347 \end{pmatrix} \tag{4.2}$$

It seen that all, possible values, of $\pi_s' \Omega \pi_{s+\nu-S*\delta}$ are greater than 1 (some are greater than 2, see the diagonal of matrix (4.1)). On the other hand, all values of $\Pi(s,S)\Pi(s+\nu-S*\delta,S)\sigma_S^2$ are lower than 1 (except the last value in the diagonal of matrix (4.2).

5 Conclusion

For Seasonal-Periodic-$ARFIMA(p,0,0)(0,D,0)$ model, allowing the seasonal fractional parameter D to be S-periodic rather than constant we have highlighted the existence of two distinct models (see section 1, model(I) and model (II)). For these two distinct models we have established the exact and approximated expression of the periodic autocovariance. On the simulated sample, for each model, the empirical periodic autocovariance are calculated and graphically represented.

It is clear, through, theoretical and simulated results that it is not easy to distinguish between these two models (the shape of the autocovariance for each model is not sufficient). If we consider the general model, namely, Seasonal-Periodic-$ARFIMA(p,d_t,q)(P,D_t,Q)$ the situation becomes more complex to handle, because the number of different models we can distinguish is more than two models. Furthermore, the non seasonal part of the general model (i.e. $PARFIMA(p,d_t,q)$) did not receive much attention on behalf of the statisticians and the probabilists.

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6 Appendix A

Proposition: *The infinite moving average representation of the process,* $\{y_t, t \in \mathbb{Z}\}$ *, defined by* (1.1)*, is given by*

$$y_t = u_t + \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} (-1)^k \sum_{i_1+i_2+\cdots+i_k=j} \pi_{i_1}(-d_t)\pi_{i_2}(-d_{t-i_1})\pi_{i_3}(-d_{t-i_1-i_2}) \cdots \pi_{i_k}\left(-d_{t-\sum_{l=1}^{k-1} i_l}\right)\right) u_{t-j},$$

where $\pi_0(-d_t) = 1$ and $\pi_{i_1}(-d_t) = \frac{-d_t(-d_t+1)\cdots(-d_t+i_1-1)}{(i_1)!}$. The number terms in the sum " $\sum_{i_1+i_2+\cdots+i_k=j}$ " is equal 2^{j-1}. The number 2^{j-1} represent the cardinal sets of k positive integers, namely, $(i_1, i_2, \cdots, i_k)$, which when summed together give j.

Proof. Putting

$$\Pi_0(-d_t) = 1 \text{ and } \frac{\Gamma(j-d_t)}{\Gamma(j+1)\Gamma(-d_t)} = \Pi_j(-d_t),$$

we can rewrite (1.4) as

$$y_t + \sum_{j=1}^{\infty} \Pi_j(-d_t) y_{t-j} = u_t. \qquad (A1)$$

■

More generally, we have

$$y_{t-j} + \sum_{k=1}^{\infty} \Pi_j(-d_{t-j}) y_{t-j-k} = u_{t-j}.$$

Suppose that the infinite moving average representation of $(A1)$ is given by

$$y_t = \Psi_0(t) u_t + \sum_{j=1}^{\infty} \Psi_j(t) u_{t-j}, \text{ with } \Psi_0 = 1, \tag{A2}$$

we have for the lagged variable y_{t-j},

$$y_{t-j} = u_{t-j} + \sum_{k=1}^{\infty} \Psi_k(t-j) u_{t-j-k}. \tag{A3}$$

By replacing y_{t-j} by $u_{t-j} + \sum_{k=1}^{\infty} \Psi_k(t-j) u_{t-j-k}$ in $(A1)$, we obtain

$$Y_t + \sum_{j=1}^{\infty} \Pi_j(-d_t) u_{t-j} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Pi_j(-d_t) \Psi_k(t-j) u_{t-j-k} = u_t. \tag{A4}$$

Putting $k' = k + j$, $(A4)$ becomes

$$Y_t + \sum_{j=1}^{\infty} \Pi_j(-d_t) u_{t-j} + \sum_{j=1}^{\infty} \sum_{k'=j+1}^{\infty} \Pi_j(-d_t) \Psi_{k'-j}(t-j) u_{t-k'} = u_t. \tag{A5}$$

Let $\Phi_{j,k'}(t-j) = \Pi_j(-d_t) \Psi_{k'-j}(t-j)$, then we can rewrite $(A5)$ as,

$$Y_t + \Pi_1(-d_t)u_{t-1} \sum_{j=2}^{\infty} \Pi_j(-d_t)u_{t-j} + \sum_{k'=2}^{\infty} \Phi_{1,k'}(t-1)u_{t-k'} + \sum_{k'=3}^{\infty} \Phi_{2,k'}(t-2)u_{t-k'} + \cdots$$
$$+ \sum_{k'=j+1}^{\infty} \Phi_{j,k'}(t-j)u_{t-k'} + \cdots = u_t, \quad (A6)$$

We can rewrite ($A6$) as,

$$\begin{aligned} Y_t &+ \Pi_1(-d_t)u_{t-1} + (\Pi_2(-d_t) + \Phi_{1,2}(t-1))\, u_{t-2} \\ &+ (\Pi_3(-d_t) + \Phi_{1,3}(t-1) + \Phi_{2,3}(t-2))\, u_{t-3} \\ &+ (\Pi_4(-d_t) + \Phi_{1,4}(t-1) + \Phi_{2,4}(t-2) + \Phi_{3,4}(t-3))\, u_{t-4} \\ &\vdots \\ &+ (\Pi_h(-d_t) + \Phi_{1,h}(t-1) + \Phi_{2,h}(t-2) + \cdots + \Phi_{h-1,h}(t-h+1))\, u_{t-h} \\ &+ \vdots \end{aligned} \quad (A7)$$

Let $\Pi_h(-d_t) + \Phi_{1,h}(t-1) + \Phi_{2,h}(t-2) + \cdots + \Phi_{h-1,h}(t-h+1) = \beta_h(t-h+1)$, for $h \geq 1$, then we can rewrite ($A7$) as

$$Y_t + \beta_1(t)u_{t-1} + \beta_2(t-1)u_{t-2} + \cdots + \beta_h(t-h+1)u_{t-h} + \cdots = u_t. \quad (A8)$$

From ($A8$), the infinite moving average representation of the process y_t is

$$Y_t = u_t + \sum_{j=1}^{\infty} \left(-\beta_j(t)\right) u_{t-j}. \quad (A9)$$

By identification between ($A2$) and ($A9$), we obtain

$$\Psi_j(t) = -\beta_j(t-j+1). \quad (A10)$$

From ($A10$), the first three coefficients, ($\Psi_1(t)$, $\Psi_2(t)$, $\Psi_3(t)$) are:

- $\Psi_1(t) = -\beta_1(t),$
  $= -\Pi_1(-d_t).$
- $\Psi_2(t) = -\beta_2(t-1)$
  $= -\left(\Pi_2(-d_t) + \Phi_{1,2}(t-1)\right)$
  $= -\Pi_2(-d_t) - \Pi_1(-d_t)\Psi_1(t-1)$
  $= -\Pi_2(-d_t) + \Pi_1(-d_t)\Pi_1(-d_{t-1})$
  $= \sum_{k=1}^{2} (-1)^k \left(\sum_{\substack{i_1=2 \\ i_1 \neq 0}} \pi_{i_1}(-d_t) + \sum_{\substack{i_1+i_2=2 \\ i_1\neq 0;\ i_2 \neq 0}} \pi_{i_1}(-d_t)\pi_{i_2}(-d_{t-i_1}) \right)$
  with $i_l \neq 0$, $l \in \overline{1,2}$
- $\Psi_3(t) = -\beta_3(t-2)$
  $= -\left(\Pi_3(-d_t) + \Phi_{1,3}(t-1) + \Phi_{2,3}(t-2)\right)$
  $= -\left(\Pi_3(-d_t) + \Pi_1(-d_t)\Psi_2(t-1) + \Pi_2(-d_t)\Psi_1(t-2)\right)$
  $= -\Pi_3(-d_t) + \Pi_1(-d_t)\Pi_2(-d_{t-1}) - \Pi_1(-d_t)\Pi_1(-d_{t-1})\Pi_1(-d_{t-2}) + \Pi_2(-d_t)\Pi_1(-d_{t-2})$
  $= \sum_{k=1}^{3} (-1)^k \sum_{i_1=3} \pi_{i_1}(-d_t) + \sum_{i_1+i_2=3} \pi_{i_1}(-d_t)\pi_{i_2}(-d_{t-i_1}) + \sum_{i_1+i_2+i_3=3} \pi_{i_1}(-d_t)\pi_{i_2}(-d_{t-i_1})\pi_{i_3}(-d_{t-i_1-i_2}),$

with $i_l \neq 0$, $l \in \overline{1,3}$.More generally, we have,

$$\begin{aligned}\Psi_j(t) &= -\beta_j(t-j+1)\\ &= \left(\sum_{k=1}^{j}(-1)^k \sum_{i_1+i_2+\cdots+i_k=j} \pi_{i_1}(-d_t)\pi_{i_2}(-d_{t-i_1})\pi_{i_3}(-d_{t-i_1-i_2})\cdots\pi_{i_k}(-d_{t-j+i_k})\right),\end{aligned} \quad (A11)$$

with $i_l \neq 0$, $l \in \overline{1,k}$

When $d_t = d = constant$, we have $(\Psi_1(t),\ \Psi_2(t),\ \Psi_3(t)) = \left(d,\ \frac{d(d+1)}{1},\ \frac{d(d+1)(d+2)}{2}\right)$.