Envy-free solutions, Non-linear equilibrium and Egalitarian-equivalence for the Package Assignment Problem

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Abstract

The first result in this paper says that given any efficient non-monetary allocation there is a balanced vector of transfers so that the resulting allocation is fair. The second result here says that given any efficient non-monetary allocation there is a pricing function defined on consumption bundles and a balanced vector of transfers so that they together form a non-linear market equilibrium. The first result is used to establish the second. Subsequently we prove the existence of egalitarian equivalent solutions for package assignment problems and shows that they satisfy the "fair share guaranteed" property.

Note: This paper is based on two earlier papers entitled: (a) Envy-free solutions and Non-linear equilibrium for the Package Assignment Problem; (b) The Egalitarian Equivalent Solution for Package Assignment Problems.

Introduction: Fair (or equitable) division of a given amount of resources among a finite number of agents, has concerned humankind since time immemorial. The body of economic literature that is available on the problem is voluminous, though of considerably recent vintage. The compulsions of "being fair" go beyond morality. It has a lot more to do with the prescribed resource allocation being acceptable and self-enforcing for the beneficiaries. Fair division of workload facilitates voluntary compliance by the employees. Equitable division of infrastructure and other facilities reduce tension at work. In society egalitarian distribution of wealth among the people is meant to prevent friction and instability. In recent times fair division solutions have been applied to resource allocation problems in e-commerce.

There is one model of fair division concerned with an infinitely divisible but heterogeneous resource that goes by the name of cake-cutting problems. A good exposition of issues related to cake cutting problems can be found in Robertson and Webb (1998). When the resources are infinitely divisible and homogeneous, the concept of equity that was initially proposed is known as "envy-freeness" by Foley (1967). Rigorous analysis of the concept of envy-freeness begins with the work of Varian (1974). There are many good surveys on this topic including ones that study algorithms/procedures that implement envy-free solutions. A third model of fair division concerns resources that are indivisible and money. The worth of each resource is measured in units of money and the latter is also used for making side-payments or transfers to the agents. Each agent values a bundle of resource differently and this value is measured by the maximum that the agent is willing to pay for the resource. One of the earliest expositions of this model is due to Alkan, Demange and Gale (1991), where there is only one unit of each indivisible resource available for distribution and each agent can consume at most one unit of it. Such a situation is variously referred to as an assignment problem/ game or permutation game in the literature. Alkan et al (1991) showed that in such models there always exists an envy-free allocation and that all envy-free allocations are efficient, i.e. maximizes aggregate willingness to pay. Allocations that are both envy-free and efficient are called fair. While this existence property is a much cited result, what the paper really accomplishes is to show that for assignment problems the theory of efficient and envy-free allocations is no different from the equilibrium theory that was developed for such problems by Koopmans and Beckmann (1957). Since the appearance of the paper by Alkan et al. a large number of proofs have been offered for the existence of market equilibrium and (or what is essentially the same thing) fair allocations for assignment problems, including one by Lahiri (2007).

The next level of generalization involves the fair division of one or more than one type of indivisible good(s) for each of which one or more than one unit is available. Further we do not assume any restriction on the number of units or types of goods that an agent can consume. Once again money is used as a measure of value as well as a unit of payment. A typical agent now has a willingness to pay for a bundle of resources, rather than a single unit of one. Thus complementarities in preferences for goods as well as scale effects may be present. Such a model is what we (following Bikhchandani and Ostroy (2001)) refer to here as a package assignment problem.

One of the major concerns of the package assignment problem is the existence of market equilibrium, where the price per unit of consumption of any commodity is independent of the consumption bundle. There are many examples of package assignment problems, both simple and meaningful, for which market equilibrium does not exist. A very important sub-class of package assignment problems is the so-called "combinatorial allocation problem". Fair allocations for combinatorial allocation problems and the fact that they do not satisfy most properties suugested for such problems have been discussed in Bevia (1998). In a combinatorial allocation problem there is exactly one unit of each indivisible object that is available for distribution. Keslo and Crawford (1982) postulate a *gross substitutability* condition that is sufficient for the existence of market equilibrium in combinatorial allocation problems. Gul and Stacchetti (1997) show that if individual willingness to pay functions exhibit a *no-complementarities* condition, then again a market equilibrium exists for the same class of problems. Bikhchandani and Mamer (1997) show that the aggregate willingness to pay for the combinatorial allocation problem coincides with the optimal value of the corresponding relaxed problem if and only if there is a market equilibrium. This result can be extended to the entire domain of package assignment problems. More recently, Lahiri (2006) shows that a market equilibrium for a package assignment problem exists if and only if the aggregate willingness to pay function is "locally concave" at the initial endowment.

Given the distinct possibility of market equilibrium not existing, the question that was posed by Wurman and Wellman (undated) was whether there is an equilibrium for which instead of a fixed unit price for each commodity there was a price associated with each consumption bundle. The answer to this question was in the affirmative. Bikhchandani and Ostroy (2002) show that given any feasible non-monetary allocation that maximizes the aggregate willingness to pay, i.e. an efficient non-monetary allocation, there are pricing functions one for each agent that is compatible with each agent rationally choosing the consumption bundle assigned to it by the non-monetary allocation.

Issues concerning equitable allocations in a package assignment problem have rarely been raised, since it was realized that the theory of fair allocations is substantially similar to the equilibrium theory. In Moulin (1995) can be found a remarkable result that obtains the existence of fair allocations for package assignment problems as a corollary of the corresponding existence result for assignment problems. However the proof that is available in Moulin (1995) states something much stronger: given any efficient non-monetary allocation there is a vector of transfers, so that the resulting allocation is fair. Since in a package assignment problem an efficient non-monetary allocation always exists the stated result in Moulin (1995) follows from this other stronger result (also due to Moulin (1995)) that we present here as Proposition 1. Using Proposition 1 it is easy to show that given any efficient non-monetary allocation there is pricing function defined on consumption bundles (and common to all agents) that is compatible with each agent rationally choosing the consumption bundle assigned to it by the non-monetary allocation. This is the second result of this note.

Proposition 1 was definitely known and Proposition 2 can be easily recognized. Hence the only token contribution that the presentation of the first two results make is to put in writing a proof of Proposition 2 (which as stated is stronger than the corresponding one in Wurman and Wellman (undated)) using Proposition 1.

In the context of division of infinitely divisible and homogeneous resources, it often happens that envy free allocations may often lead to inequitable distribution of surplus. For instance as shown in Moulin (1995), in any two agent fair division problem with one agent having linear and strictly increasing indifference curves and the other having Cobb-Douglas preferences, at the market equilibrium from equal division the agent with linear indifference curves is exactly as well off as she would be under equal split, where as the other agent may ascend to a higher level of satisfaction. Since market equilibrium from equal division is fair, this in-egalitarian distribution of surplus above equal division does little credit to the concept of envy freeness. An alternative solution concept by Pazner and Schmeidler (1978), often remedies such inequities. The solution due to Pazner and Schmeidler (1978) is known as the egalitarian equivalent solution and it captures the idea of giving "equal" shares of the surplus above equal splits. In this solution, an agent's surplus at an allocation is measured by the fraction of the total resource that she views as just equivalent to the given allocation. An egalitarian equivalent allocation is one where this fraction is the same for all agents. An egalitarian equivalent solution due to Pazner and Schmeidler (1978) is a feasible egalitarian equivalent allocation whose corresponding fraction is the largest among all feasible egalitarian equivalent allocations.

This paper has similar concerns but (once again) in the context of package assignment problems. At the outset it needs to be realized that talking about fractions of the total resource is meaningless when the resource is indivisible. We do not have the facility of convexity that the original model enjoyed. Hence we measure the surplus of an agent at an allocation by the fraction of her willingness to pay for the total resource that is equal to the monetary worth of her consumption at the allocation. An egalitarian equivalent allocation is one for which this fraction is the same for all agents. We call this common fraction the egalitarian equivalence factor (of the allocation). An egalitarian equivalent allocation as defined here is a generalization of an equitable allocation due to Brams and Taylor (1996). An egalitarian equivalent solution is a feasible egalitarian equivalent allocation whose egalitarian equivalence factor maximizes the egalitarian equivalence factor among all such allocations (i.e. among all feasible egalitarian equivalent allocations). Our first result shows that an egalitarian equivalent solution exists and it is efficient.

It may be conjectured that the proof of this result would be identical to the proof of the corresponding result due to Pazner and Schmeidler (1978). That this is not the case follows from the fact that due to indivisibilities we are prevented from making arbitrarily small transfers of non-monetary commodities among the agents. Further, there is no natural bound on the possible monetary transfers we can make among the agents. The two proofs are analogous but materially different.

The final result of this paper shows that the egalitarian solution satisfies what we may refer to as "fair share guaranteed" property. In the divisible good context it is also known as individual rationality fro equal division. Moulin (1995) contains a discussion of this property including its definitions for assignment problems. The first question that arises is: How do we define it for package assignment problems?

We attempt an answer along the following lines. What would a typical agent 'i' consider as an equal division of the total resources? It seems reasonable that equal division of her willingness to pay for the total resource is a good candidate. Accepting this as our benchmark, we propose that any reasonable solution to a fair division problem should allocate each agent at least what she perceives to be an equal division of the total resource. We call this property "fair share guaranteed" and show that the egalitarian equivalent solution satisfies this property. In the fair division literature, "fair share guaranteed" is also known as "proportionality" (see Brams and Taylor (1996) for instance). The egalitarian equivalent solution thus shares with several others a property that is desirable for equitable distribution of resources.

This paper is an attempt to extend received theory of distributive justice to the domain of package assignment problems. There is far too much about this domain that is already known to suggest that its structure is remarkably different from distribution problems with homogeneous and divisible commodities. At the same time there is not enough known about several conventional solution concepts that have been defined in the context of fair allocation of divisible goods. This is precisely what makes this paper possible.

The Model: Let $Z_+ = \aleph \cup \{0\}$, where $\aleph$ denotes the set of natural numbers. Let there be H > 0 agents and L+1 > 1 commodities, the last one of which is money. The first L commodities are non-monetary consumption goods. Money is both consumed as well as used as a measure of worth/value. Let $w \in Z_+^L$ denote the aggregate initial endowment of the non-monetary goods which is available for distribution among the agents.

For j = 1,…,L, let w_j denote the aggregate amount of commodity j that is initially available in the economy.

We shall refer to elements of Z_+^L as non-monetary consumption vectors/bundles.

For i = 1,…,H, let f^i: $Z_+^L \to \Re_+$ with $f^i(0) = 0$ be the willingness to pay function of agent i. Thus for a non-monetary consumption vector x, $f^i(x)$ denotes the maximum amount of money that agent i is willing to pay for x. We assume that f^i is non-decreasing, i.e. for $x,y \in Z_+^L$: $[x \geq y]$ implies $[\, f^i(x) \geq f^i(y)]$.

The above framework has been known in the literature variously as multi-unit assignment (or auction) and package assignment problems.

It is often represented as the pair $<\{f^i/i= 1,\ldots,H\}, w>$.

Let e denote the vector in $\Re^L$ all whose coordinates are equal to one and for j = 1,…,L, let e^j denote the vector in $\Re^L$ whose j^{th} coordinate is equal to one and all other coordinates are equal to zero.

For $x \in Z_+^L$, let $C(x) = \{y \in Z^L / \ y \leq x\}$.

A non-monetary consumption vector of agent i is denoted by a vector $z^i \in Z_+^L$.

A non-monetary allocation is an array $z = <z^i/\ i = 1,\ldots,H>$ such that for all i = 1,…,H: z^i is a non-monetary consumption vector for agent i.

Given $x \in Z_+^L$, let $F(x) = \{\ z = <z^i/\ i = 1,\ldots,H>/$ z is a non-monetary allocation satisfying $\sum_{i=1}^{H} z^i \leq x\}$. A non-monetary allocation z is said to be feasible if $z \in F(w)$.

An allocation is a pair (z,t) such that z is a non-monetary allocation and $t \in \Re^H$ denotes a vector of transfers to the agents.

An allocation (z,t) is said to be feasible if z is a non-monetary feasible allocation and t is a balanced vector of transfers, i.e. $\sum_{i=1}^{H} t^i = 0$.

The function V: $Z_+^L \to \Re_+$ such that for all x∈ Z_+^L: V(x) = Max{ $\sum_{i=1}^{H} f^i(z^i)$ / z = <z^i/ i = 1,…,H>∈F(x)}, is called the maximum value function.

Since we have assumed that for all i = 1,…,H, f^i is non-decreasing, it must be the case that V is non-decreasing as well (i.e. for all x,y∈Z^L:[x ≥ y] implies [V(x) ≥ V(y)]).

A feasible non-monetary allocation z = <z^i/i = 1,…,H> is said to be efficient if $\sum_{i=1}^{H} f^i(z^i) =$ V(w).

A feasible allocation (z,t) is said to be efficient if z is an efficient non-monetary allocation.

A feasible allocation (z,t) is said to be envy free if for all i, k = 1,…,H: $f^i(z^i) + t^i \geq f^i(z^k) + t^k$.

Thus if (z,t) is envy free then for all i,k = 1,…,H: (a) $f^i(z^i) - f^i(z^k) \geq t^k - t^i$; (b) $f^k(z^k) - f^k(z^i) \geq t^i - t^k$. Thus, (z,t) is envy free implies that there is a real number τ(z) (i.e. possibly depending on z) such that for all i = 1,…,H: $|t^i| \leq \tau(z)$.

Since for a package assignment problem the number of non-monetary allocations is finite, we get the following observation.

There exists a real number τ such that if (z,t) is envy free then: $|t^i| \leq \tau$.

A price vector p is an element of $\Re_+^L \backslash\{0\}$, where for j = 1,…,L, p_j denotes the price of input j.

A pair (p^*, (z^*, t^*)) where p^* is a price vector and (z^*, t^*) is a feasible allocation is said to be a Market Equilibrium from Equal Incomes (MEEI) if for all i = 1,…,H:

(i) z^{*i} solves:

Maximize $[f^i(x) - p^{*T}x]$

subject to x ∈C(w);

(ii) $t^i = \frac{1}{H} p^{*T}w - p^{*T}z^i$.

A feasible allocation (z^*, t^*) is said to be an MEEI allocation if there exists a price vector p^* such that (p^*, (z^*, t^*)) is a MEEI.

It is well-known that if L > 1 then no market equilibrium, let alone one from equal incomes need exist for a package assignment problem. If L = 1, then a MEEI exists under very reasonable assumptions on the willingness to pay functions of the agents. However if a package assignment problem admits an MEEI allocation then (as may be easily verified) such an allocation would be both envy-free and efficient.

An allocation (z,t) that is both envy-free and efficient is said to be fair.

The non-existence of MEEI allocations for package assignment problems means that a certain desirable type of fair allocation(s) cannot be guaranteed in our framework.

Fair Allocations: It is well known that if there is just one unit of every commodity that is initially available and each agent can consume at most one commodity at a time, then a fair allocation always exists. Moulin (1995) (Corollary to Theorem 4.1) uses this result to elegantly establish that for package assignment problems a fair allocation always exists. In fact the result that Moulin uses to prove the corollary permits the following stronger version of the existence result a proof of which (due to Aragones (1992)) is being provided for completeness.

Proposition 1: Let z^* be an efficient non-monetary allocation for a package assignment problem. Then there exists a balanced vector of transfers t^* such that (z^*, t^*) is a fair allocation. Since in a package assignment problem an efficient non-monetary allocation always exists, so does a fair allocation.

Proof: Let z^* be an efficient non-monetary allocation for the given package assignment problem. Let $\sigma: \{1,\ldots,H\} \to C(w)$ be defined thus: $\sigma(i) = z^{*i}$ for $i = 1,\ldots,H$.
For $i, j \in \{1,\ldots,H\}$, let $q_{ij} = f^i(\sigma(i)) - f^i(\sigma(j))$.
For $i \in \{1,\ldots,H\}$, let $r_i =$
$\min\{ \sum_{t=1}^{k} q_{i_{t-1}i_t} \,/\, k \text{ is a natural number}, i_t \in \{1,\ldots,H\} \text{ for } t=0,\ldots,k \text{ and } i_0 = i \}$.
In the above for $i_t \in \{1,\ldots,H\}$, repetitions are allowed, although as we shall see, that the minimum is attained when there are no repetitions.
Suppose $< q_{i_{t-1}i_t} \,/\, i_t \in \{1,\ldots,H\} \text{ for } t=0,\ldots,k \text{ (where } k \text{ is a natural number) and } i_0 = i >$ is a finite sequence.
Suppose $<i_j, i_{j+1},\ldots, i_{s-1}, i_s>$ is a subsequence of consecutive elements from $<i_o,\ldots,i_k>$ such that $i_j = i_s$, other terms being distinct, i.e. is a minimal cycle.
Let $\sigma^0: \{1,\ldots,H\} \to C(w)$ be such that $\sigma^0|\{1,\ldots,H\}\backslash\{i_j,\ldots,i_s\} = \sigma|\{1,\ldots,H\}\backslash\{i_j,\ldots,i_s\}$ and $\sigma^0(i_t) = \sigma(i_{t+1})$ for $t \in \{j,\ldots, s-1\}$.
Since σ is efficient, $\sum_{i=1}^{H} f^i(\sigma(i)) - \sum_{i=1}^{H} f^i(\sigma^0(i)) \geq 0$.
Thus $\sum_{t=j}^{s-1} q_{i_t i_{t+1}} \geq 0$.
Thus $\sum_{t=0}^{k-1} q_{i_t i_{t+1}} \geq \sum_{t=0}^{k-1} q_{i_t i_{t+1}} - \sum_{t=j}^{s-1} q_{i_t i_{t+1}} \geq r_i$.
Hence for the purpose of calculating r_i, there will be no difference if we considered the sequence $< q_{i_{t-1}i_t} \,/\, i_t \in \{1,\ldots,H\} \text{ for } t=0,\ldots,j,s+1,\ldots,k >$ instead of the original sequence.
Hence for the purpose of calculating r_i we can omit all cycles from a sequence and consider sequences of distinct indices and the one element sequence $\{q_{ii}\}$.
Since the number of such sequences is finite it must be the case that $r_i > -\infty$. Since $q_{ii} = 0$, $r_i \leq 0$.

Consider the vector of transfers t^*, where $t_i^* = \frac{1}{H}\sum_{k=1}^{H} r_k - r_i$.

It is balanced, i.e. $\sum_{i=1}^{H} t^{*i} = 0$.

Let us show now that for all $i,k \in \{1,\ldots,H\}$: $f^i(\sigma(i))+t^{*i} \geq f^i(\sigma(k))+t^{*k}$.

Let $r_k = \sum_{t=0}^{K-1} q_{i_t i_{t+1}}$, where $i_o = k$.

Then $[f^i(\sigma(i))+t^{*i}] - [f^i(\sigma(k))+t^{*k}] = [f^i(\sigma(i))-f^i(\sigma(k))] + [t^{*i}- t^{*k}] = q_{ik} + r_k - r_i = q_{ik} + \sum_{t=0}^{K-1} q_{i_t i_{t+1}} - r_i \geq 0$.

Thus (z^*, t^*) is envy-free. Since z^* is efficient it follows that (z^*, t^*) is fair. Q.E.D.

The following observation concerning envy-free allocations for a package assignment problem is noteworthy.

Claim 1: An allocation (z^*, t^*) is envy free if and only if for all (coalitions) $S,T \subset \{1,\ldots,H\}$ with $S, T \neq \phi$: $\sum_{i\in S}[f^i(z^{*i})+t^{*i}] \geq \sum_{i\in S}\{\frac{1}{|T|}\sum_{k\in T}[f^i(z^{*k})+t^{*k}\}]$.

Proof: Suppose that for all (coalitions) $S,T \subset \{1,\ldots,H\}$ with $S, T \neq \phi$:

$$\sum_{i\in S}[f^i(z^{*i})+t^{*i}] \geq \sum_{i\in S}\{\frac{1}{|T|}\sum_{k\in T}[f^i(z^{*k})+t^{*k}\}].$$

Then taking $|S| = |T| = 1$ it follows that (z^*,t^*) is envy free.

Now suppose that (z^*, t^*) is envy-free. Let $S,T \subset \{1,\ldots,H\}$ with $S, T \neq \phi$.

Thus for all $i\in S$ and $k\in T$: $f^i(z^{*i}) + t^{*i} \geq f^i(z^{*k}) + t^{*k}$. Taking the sum over all $k\in T$ and dividing by $|T|$ we get $f^i(z^{*i}) + t^{*i} \geq \frac{1}{|T|}\sum_{k\in T}[f^i(z^{*k})+t^{*k}]$ for all $i\in S$.

The desired inequality in Claim 1 follows by taking the sum of the last inequalities over $i\in S$. Q.E.D.

Non-Linear Market Equilibrium: A non-linear price function, or simply a pricing function(to put it briefly) is a function $P:C(w)\rightarrow\Re_+$ such for $x\in C(w)$, $P(x)$ is the price of the non-monetary consumption bundle x.

Faced with a pricing function P, an agent chooses a consumption bundle in C(w) that maximizes his willingness to pay minus the price of the bundle.

A pair $(P, (z^*, t^*))$ where P is a pricing function and (z^*, t^*) an efficient allocation is said to be a Non-linear Market Equilibrium from Equal Incomes (NMEEI) if for all i = 1,…,H:

(i) $f^i(z^{*i}) - P(z^{*i}) \geq f^i(x) - P(x)$ for all $x\in C(w)$;

(ii) $t^{*i} = \frac{\sum_{k=1}^{H} P(z^{*k})}{H} - P(z^{*i})$.

In a recent paper by Wellman and Wurman (undated), it has been claimed that in every package assignment problem, where there is exactly one unit of each good available, a non-linear market equilibrium exists. In view of Proposition 1 we are able to establish a stronger result that says, given any efficient non-monetary allocation, we can find a pricing function and a corresponding balanced vector of transfers, in conformity with the requirements of a NMEEI.

Proposition 2: Let z^* be an efficient non-monetary allocation for a given package assignment problem. (i)Then there exists a pricing function P and a balanced vector of transfers t^* such that $((z^*, t^*), P)$ is a NMEEI. (ii) If there exists a pricing function P and a balanced vector of transfers t^* such that $((z^*, t^*), P)$ is a NMEEI, then (z^*, t^*) is an envy-free allocation.

Proof: Let z^* be as in the statement of the theorem.
(i) By Proposition 1, there exists a balanced vector of transfers t^* such that (z^*, t^*) is a fair allocation.
Let $t^0 = \max\{t^{*i}/ i\in\{1,\ldots,H\}\}$. Clearly $t^0 \geq 0$. Define P: $C(w)\rightarrow\Re_+$ as follows: (a) $P(z^{*i}) = t^0 - t^{*i}$; (b) $P(x) = \max\{\max \{f^i(x) - f^i(z^{*i})+ P^*(z^{*i})/ i = 1,\ldots,H\}, 0\}$ if $x\in C(w)\backslash\{z^{*i}/ i = 1,\ldots,H\}$.
First let us verify that P is well-defined, i.e. if $z^{*i} = z^{*k}$, then $P(z^{*i}) = P(z^{*k})$. This follows from the fact that (z^*, t^*) is envy free, so that $f^i(z^{*i}) + t^{*i} \geq f^i(z^{*k}) + t^{*k}$, $f^k(z^{*k}) + t^{*k} \geq f^k(z^{*i}) + t^{*i}$ and $z^{*i} = z^{*k}$ together imply $t^{*i} = t^{*k}$ and thus $P(z^{*i}) = P(z^{*k})$.
Now (z^*, t^*) is envy free, for all i, k = 1,…,H: $f^i(z^{*i}) - P(z^{*i}) = f^i(z^{*i}) + t^{*i} - t^0 \geq f^i(z^{*k}) + t^{*k} - t^0 = f^i(z^{*k}) - P(z^{*k})$.
Suppose $x\in C(w)\backslash\{z^{*i}/ i = 1,\ldots,H\}$. Thus for i = 1,…,H: $P(x) \geq f^i(x) - f^i(z^{*i}) + P(z^{*i})$ implies $f^i(z^{*i}) - P(z^{*i}) \geq f^i(x) - P(x)$ for all $x\in C(w)\backslash\{z^{*k}/ k = 1,\ldots,H\}$.
Thus, $f^i(z^{*i}) - P(z^{*i}) \geq f^i(x) - P(x)$ for all $x\in C(w)$.
Finally, $P(z^{*i}) = t^0 - t^{*i}$ for all i = 1,…,H and t^* is a balanced vector of transfers implies that $t^0 = \dfrac{\sum_{k=1}^{H} P(z^{*k})}{H}$.

Hence $t^{*i} = \dfrac{\sum_{k=1}^{H} P(z^{*k})}{H} - P(z^{*i})$ for all i = 1,…,H.
This proves(i).
The proof of (ii) is immediate. Q.E.D.

Note: (a) The above proof is along the lines of the proof of Theorem 4.1(iii) in Moulin (1995). Our result extends the result in Moulin to a larger domain of assignment problems. The pricing function defined in the proof of Proposition 2, resembles the one defined in the proof of the corresponding result in Wurman and Wellman (undated).
(b)Unlike Bikhchandani and Ostroy (2002) our pricing function in a non-linear market equilibrium does not discriminate between agents.

Egalitarian Equivalent Solutions: An allocation (z,t) is said to be egalitarian equivalent if there exists $\lambda \geq 0$ such that for all i = 1,…,n: $t^i + f^i(z^i) = \lambda f^i(w)$.
The real number λ is said to be the egalitarian equivalence factor for (z,t).

A feasible allocation (z^*,t^*) is said to be an egalitarian equivalent solution if whenever (z,t) is a feasible egalitarian equivalent allocation whose egalitarian equivalence factor is λ, then $\lambda^* \geq \lambda$, where λ^* is the egalitarian equivalence factor for (z^*, t^*).

Thus (z^*, t^*) is an egalitarian equivalent solution if it is feasible, egalitarian equivalent and its egalitarian equivalence factor exceeds that of any other egalitarian equivalent allocation.

Proposition 3: Given any package assignment problem, we can always find an egalitarian equivalent solution for it. Further, such a solution is always efficient.

Proof: Suppose (z,t) is an egalitarian equivalent allocation and suppose its egalitarian equivalence factor exceeds one. If (z,t) were feasible, then we would get $\sum_{i=1}^{H} f^i(z^i) >$ $\sum_{i=1}^{H} f^i(w)$ implying $f^i(z^i) > f^i(w)$ for at least one $i\in\{1,\ldots,H\}$. Since all the willingness to pay functions have been assumed to be weakly increasing, there exists at least one $i\in\{1,\ldots,H\}$ such that *it is not the case that* $z^i \leq w$. This contradicts feasibility of (z,t). Hence if the egalitarian equivalence factor exceeds one, then the corresponding allocation cannot be feasible.

Let $I = \{\lambda\in[0,1]/$ there exists a feasible allocation (z,t) with $f^i(w) \geq t^i + f^i(z^i) \geq \lambda f^i(w)$ for $i = 1,\ldots,H\}$.

Since $0\in I$(the allocation where no transfer takes place and no one gets anything has egalitarian equivalence factor zero), I is non-empty.

Let $<\lambda^k/ k\in \aleph>$ be a sequence in I with $\lim_{k\to\infty} \lambda^k = \lambda$. Let $<(z^{(k)}, t^{(k)})/ k\in \aleph>$ be a sequence of feasible allocations such that for all $i = 1,\ldots,H$ and $k\in \aleph$: $t^{(k)i} + f^i(z^{(k)i}) \geq \lambda^{(k)}f^i(w)$. Since $\lim_{k\to\infty} \lambda^k = \lambda$, there exists a real number M such that for all $i = 1,\ldots,H$ and $k\in \aleph$: $\max \{f^i(w)/ i = 1,\ldots,H\}\geq t^{(k)i} + f^i(z^{(k)i}) \geq M$. Thus $<t^{(k)}/ k\in \aleph>$ must be a bounded sequence. Hence it admits a convergent subsequence. Further since the number of feasible non-monetary allocations are finite, there exists a non-monetary allocation z which repeats itself infinitely often in $<z^{(k)}/ k\in \aleph>$. Thus there exists a subsequence of $<(z^{(k)}, t^{(k)})/ k\in \aleph>$ which converges to a feasible allocation (z,t). Clearly, $f^i(w) \geq t^i + f^i(z^i) \geq \lambda f^i(w)$ for $i = 1,\ldots,H$. Thus I is closed.

Let $\lambda^* = \sup\{\lambda\in I\}$. Thus $\lambda^*\in I$.

Hence there exists a feasible allocation (z^*, t^*) such that $f^i(w) \geq t^{*i} + f^i(z^{*i}) \geq \lambda^* f^i(w)$ for $i = 1,\ldots,H$.

Towards a contradiction suppose $t^{*i} + f^i(z^{*i}) > \lambda^* f^i(w)$ for some i. Thus $\lambda^* < 1$. Let $I^+ = \{i/ t^{*i} + f^i(z^{*i}) > \lambda^* f^i(w)\}$. Clearly I^+ is non-empty. If $I^+ = I$, then it is possible to find $\lambda > \lambda^*$ such that $\lambda\in I$, contradicting the maximality of λ^*. Thus $I \setminus I^+$ is non-empty. Let $\varepsilon > 0$ be such that $t^{*i} + f^i(z^{*i}) > t^{*i} + f^i(z^{*i}) - \frac{\varepsilon}{|I^+|} > \lambda^* f^i(w)$ for all $i\in I^+$. Thus, $t^{*i} +$ $f^i(z^{*i}) + \frac{\varepsilon}{|I \setminus I^+|} > \lambda^* f^i(w)$ for all $i\in I\setminus I^+$. Let t be the balanced vector of transfers

where $t^i = t^{*i} - \frac{\varepsilon}{|I^+|}$ for all $i \in I^+$ and $t^i = t^{*i} + \frac{\varepsilon}{|I \setminus I^+|}$ for all $i \in I \setminus I^+$. (z^*, t) is feasible and there exists $\lambda > \lambda^*$ such that $f^i(w) \geq t^i + f^i(z^{*i}) \geq \lambda f^i(w)$ for i = 1,…,H. This again contradicts the maximality of λ^*. Thus, $f^i(w) \geq t^{*i} + f^i(z^{*i}) = \lambda^* f^i(w)$ for i = 1,…,H.
Now suppose (z^*, t^*) is not efficient. Thus there exists a feasible non-monetary allocation z such that $\sum_{i=1}^{H} f^i(z^i) > \sum_{i=1}^{H} f^i(z^{*i})$. Let $I^+ = \{i/ t^{*i} + f^i(z^i) > t^{*i} + f^i(z^{*i})\}$.
Clearly I^+ is non-empty. If $I^+ = I$, then it is possible to find $\lambda > \lambda^*$ such that $\lambda \in I$, contradicting the maximality of λ^*. Thus $I \setminus I^+$ is non-empty.
Since $\sum_{i=1}^{H} f^i(z^i) > \sum_{i=1}^{H} f^i(z^{*i})$ it must be the case that $\sum_{i \in I^+} [f^i(z^i) - f^i(z^{*i})] >$ $\sum_{i \in I \setminus I^+} [f^i(z^{*i}) - f^i(z^i)]$.
Thus for $i \in I^+$ there exists $\varepsilon^i > 0$ and for $i \in I \setminus I^+$ there exists $\delta^i \geq 0$ such that $\sum_{i \in I^+} \varepsilon^i =$ $\sum_{i \in I \setminus I^+} \delta^i$, $f^i(z^i) - \varepsilon^i > f^i(z^{*i})$ for $i \in I^+$ and $f^i(z^i) + \delta^i > f^i(z^{*i})$ for $i \in I \setminus I^+$.
Indeed, let $\varepsilon > 0$ be such that $\sum_{i \in I \setminus I^+} [f^i(z^{*i}) - f^i(z^i)] + \varepsilon < \sum_{i \in I^+} [f^i(z^i) - f^i(z^{*i})]$.
Let $\varepsilon^i = \frac{f^i(z^i) - f^i(z^{*i})}{\sum_{i \in I^+} [f^i(z^i) - f^i(z^{*i})]} \left(\sum_{i \in I \setminus I^+} [f^i(z^{*i}) - f^i(z^i)] + \varepsilon \right)$ for $i \in I^+$ and $\delta^i =$ $\frac{f^i(z^{*i}) - f^i(z^i)}{\sum_{i \in I \setminus I^+} [f^i(z^{*i}) - f^i(z^i)]} \sum_{i \in I^+} \varepsilon^i$ for $i \in I \setminus I^+$. It is easy to see that with these values of ε's and δ's the desired requirements are satisfied.
Let t be a balanced vector of transfers such that $t^i = t^{*i} - \varepsilon^i$ for $i \in I^+$ and $t^i = t^{*i} + \delta^i$ for $i \in I \setminus I^+$.
(z^*, t) is feasible and there exists $\lambda > \lambda^*$ such that $f^i(w) \geq t^i + f^i(z^{*i}) \geq \lambda f^i(w)$ for i = 1,…,H. Once again maximality of λ^* is contradicted. Thus (z^*, t^*) is efficient. Q.E.D.

The following result shows that the egalitarian equivalent solution satisfies a much desirable equity concept known variously "fair-share guaranteed" or "indidual rationality from equal-share".

Proposition 4: Let (z^*, t^*) be an egalitarian equivalent solution. Then for all i = 1,…,H: $t^{*i} + f^i(z^{*i}) \geq \frac{1}{H} f^i(w)$.

Proof: Since for all i = 1,…,H the allocation where 'i' gets the entire initial endowment and no one gets anything else at all is feasible it is clear that $V(w) \geq f^i(w)$ for all i = 1,…,H.

Let (z^*,t^*) be an egalitarian equivalent solution and λ^* the egalitarian equivalence factor. We need to show that $\lambda^* \geq \frac{1}{H}$.

Towards a contradiction suppose that $\lambda^* < \frac{1}{H}$. Thus, for all i= 1,…,H: $t^{*i} + f^i(z^{*i}) < \frac{1}{H} f^i(w)$. Summing over i and appealing to Proposition 3, we get by the efficiency of (z^*, t^*) that $V(w) < \frac{1}{H}\sum_{i=1}^{H} f^i(w) \leq V(w)$. The last inequality follows from the fact that $V(w) \geq f^i(w)$ for all i = 1,…,H. However this leads to V(w) < V(w) which is not possible. Hence $\lambda^* \geq \frac{1}{H}$. Q.E.D.

A feasible allocation (z,t) is said to be fair-share guaranteed if for all i = 1,…,H: $t^i + f^i(z^i) \geq \frac{1}{H} f^i(w)$.

If a feasible allocation is fair-share guaranteed then each agent gets at least what she would be getting if her willingness to pay for the entire initial endowment was divided equally among all the agents. In a sense each agent is assured an equal share of the entire endowment viewed from her perspective.

Note: If $(P, (z^*, t^*))$ is a Non-linear Market equilibrium from Equal Incomes and z^* is an efficient non-monetary allocation then (z^*, t^*) is fair-share guaranteed. The reasoning behind this observation is as follows. Since $(P, (z^*, t^*))$ is a NMEEI, z^* is an efficient non-monetary allocation (i.e. $V(w) = \sum_{i=1}^{H} f^i(z^{*i})$) and for all i = 1,…,H:

(i) $f^i(z^{*i}) - P(z^{*i}) \geq f^i(x) - P(x)$ for all $x \in C(w)$;

(ii) $t^{*i} = \frac{\sum_{k=1}^{H} P(z^{*k})}{H} - P(z^{*i})$.

Thus for all i, k = 1,…,H: $f^i(z^{*i}) - P(z^{*i}) \geq f^i(z^{*k}) - P(z^{*k})$. Keeping i fixed and summing over k, we get $H[f^i(z^{*i}) - P(z^{*i})] \geq V(w) - \sum_{k=1}^{H} P(z^{*k})$. However, as observed in the proof of Proposition 4, $V(w) \geq f^i(w)$. Thus, $H[f^i(z^{*i}) - P(z^{*i})] \geq f^i(w) - \sum_{k=1}^{H} P(z^{*k})$. Dividing this inequality though out by H and using the fact that $t^{*i} = \frac{\sum_{k=1}^{H} P(z^{*k})}{H} - P(z^{*i})$, we get $f^i(z^{*i}) + t^{*i} \geq \frac{1}{H} f^i(w)$, as was desired.

This observation should be contrasted with the one available in Bevia (1998) which says that envy-free solutions may not be included in the set of “Identical Preferences Lower Bound solution” where the latter property is meant to convey a similar idea to what the fair-share guaranteed property does in our paper.

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