Age Milestones and Low Interest Rates, an Analytic Approach

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Abstract

Major age milestones like the age of first job, retirement age, or life expectancy, bounding relevant economic periods in a persons' life, have been changing substantially during the last decades. In parallel real interest rates have been significantly declining in relevant world economies, reaching stable negative levels in some cases. We propose an analytic approach to relate those two phenomena by using an overlapping multi-generations model to find expressions for real interest rate elasticities to age parameters. The model formalizes the mechanisms supporting the relation between interest rates and age, sheds light on the relative importance of each age milestone in explaining changes of real interest rates, and how other factors like elasticity of inter-temporal substitution, population and productivity growth, inter-generational altruism, as well as a social security system, may mitigate or amplify those changes.

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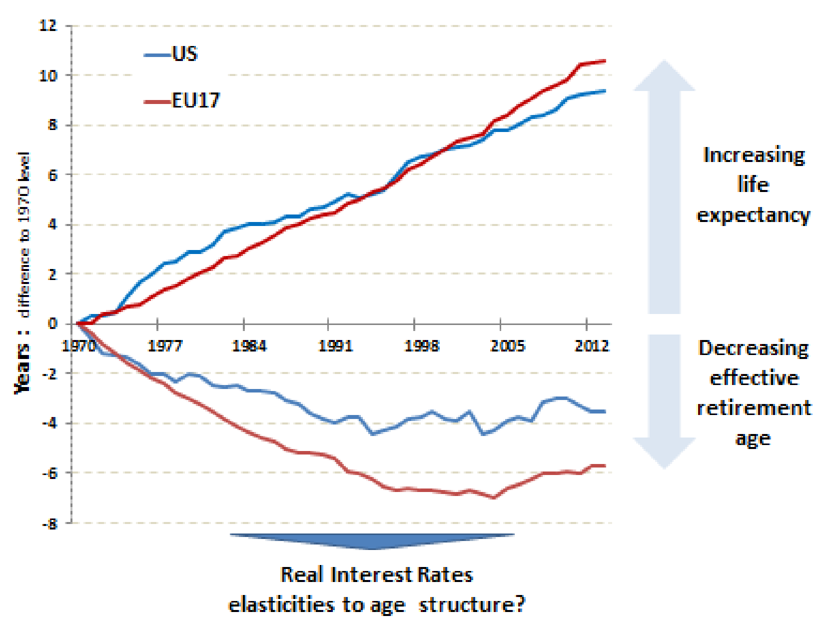
1 Introduction

During the last decades, the age structure of the population in some of World's most relevant economies has changed significantly. For example, although Life expectancy at birth increased by approximately ten years since the 70's both in US and EU, retirement age has declined four and six years respectively, contributing to raise the need to save in those economies (Figure 1). Furthermore, the recent economic crisis tended to affect the average age of first job as firms tend to postpone hiring as a way to adjust down employment level, which could lead to an increase of the borrowing needs of this population segment.

Changes in age milestones determine many aspects of relevant economic periods of a persons' life, which themselves may directly impact real interest rates through changes of borrowing and savings paths. For example, for a higher effective retirement age, people need to save less for their expected retirement period, leading to a contraction of savings and a consequent increase in equilibrium real interest rates. In addition, postponing the age of first job increases the duration of borrowing after adulthood, pushing interest rates upwards too. Increasing both parameters, age of retirement and first job, at the same time and by the same amount, although not changing the duration of the working period, may impact the real interest rate by affecting borrowing and saving paths, and consequently loan market equilibrium and real interest rates.

Figure 1: Life Expectancy and effective Retirement Age: EU and US 1970-2014



Source: World Bank; OECD

Although the impact of age structure in relevant World economies has been a recurrent topic covered in recent literature, in particular to explain the persistent decline of interest rates, economic stagnation and liquidity traps, there has not yet been an attempt, to the best of our knowledge, to formally derive the analytic relations of real interest rates with respect age milestones. The general omission of changing demographic parameters in most current formal economic models ignores a potentially relevant factor influencing equilibrium conditions, and consequently the type and even sign of solutions. For example, an increase of life expectancy can drag the full-employment equilibrium real interest rate from positive to negative. Since a negative level may not be achievable when the nominal interest rate zero lower bound is binding, a first best solution may no more be available in such a model. The same can happen with operative bequest motives, which may become inoperative, for example if the retirement age decreases, or life expectancy increases.

The purpose of this paper is to fill out this gap. By merging an age structure framework with an OLG model, we derive tractable algebraic real interest rate elasticity expressions with respect to each age parameter, to shed light on the demographic formal mechanisms that influence real interest rates, and inspect in particular the examples mentioned above. Moreover we provide a straightforward alternative to heavy computational quantitative models, in order to illustrate the impact of demographic factors on general economic phenomena.

Ikeda and Saito [19] study the effects of demographic changes on the real interest rate in Japan by capturing demographic dynamics by exogenous changes of the ratio of workers to total population. But most of the literature covering the present topic use *perpetual youth* type models inspired by Blanchard and Fischer [5], using transition probabilities between age groups. This approach, that facilitates aggregation of individual agents, thus ensuring analytically more tractable life-cycle models, was adopted, for example, by Carvalho and Ferrero [9] to explain Japan's persistent deflation, using transition probabilities from worker to retired, and from retired to death, and by Carvalho et al. [10] to inspect the mechanisms of how demographics affect real interest rates. Similarly, Aksoy et al. [1] relate macroeconomic trends to demographic structure with a model to which they add an additional transition probability from young to worker, after conducting an empirical study where they found evidence that differences in generation weight across countries explain differences among main macro-economic variables.

Nevertheless, transition probabilities in those models tend to be independent of age, and of time since transition from previous age groups, which makes them less appropriate to derive analytic relations between interest rate and explicit age milestones. This circumstance was recently overcome by Eggertsson and Robbins [14] who used a quantitative overlapping multi-generations model inspired by the work of Auerbach and Kotlikoff [2] to investigate the decline of real interest rates in US.

Similarly, we use an overlapping multi-generations model, where most relevant age milestones are exogenous parameters, allowing to analytically express the real interest rate in terms of age structure changes, surprisingly not affecting algebraic tractability, in order to shed light on relevant demographic mechanisms that are dragging down real interest rates.

In what follows, we begin by outlining an overlapping generations deterministic model in the context of an endowment economy, where agents are economically active after childhood until their age of life expectancy. The number of generations of the model depends already on those two age milestones. At the age of adulthood agents start borrowing to consume. From the age of first job until retirement they receive an income in the form of an endowment, with which they pay back their debt, consume, and save for retirement[1]. During that working period, at a certain moment in time agents have payed back their debts and start saving for retirement. Until that moment agents are borrowers, and after they become savers. The initial savings age is an endogenous variable of the model. During retirement they use their accumulated savings to consume. We derive the equilibrium conditions and aggregate expressions for the main variables of the model, in particular of excess borrowing, in terms of the real interest rate and age milestones, which becomes zero for loan market equilibrium.

In the third section we use the excess borrowing expression at loan market equilibrium in steady state to formalize the analytic relation between the *natural rate of interest*[2] and age structure. We formalize the derivatives of real interest rate with respect to each age parameter, using the partial derivatives of excess borrowing with respect to age milestones, and to the real interest rate. We find that excess borrowing decreases with increasing interest rates if the elasticity of inter-temporal substitution is above a certain acceptable threshold level that depends on the relative duration of retirement. We use this assumption throughout the paper, so that the consistent negative slope of excess borrowing with respect to the real interest rate allows the sign of age milestones elasticities to be determined by the signs of the partial derivative of excess borrowing with respect to each age parameter.

In the fourth section we introduce intergeneration transfers in the form of bequests to children, of gifts to parents, and of a *pay-as-you-go* social security system, as those concepts

[1] In our model the age of *adulthood* and age of *first job* may be different. After adulthood and before the age of first job agents have to borrow in order to consume. In the special case where those two age milestones are set the same, the algebraic expressions are simplified, and the calibrated outputs are not materially different. An alternative not used in our model to date, would be to consider endogenous transfers from parents to children during that specific phase of their lives.

[2] The *natural rate of interest* is defined as the full-employment equilibrium real interest rate.

are closely related to agents age structure, in particular to the age when their children are born. We also analyze how intergeneration transfers parameters affect the elasticities of real interest rates with respect to age milestones. Finally, in section five we calibrate a model with endogenous output and capital to quantify the analytic results of previous sections. We also test the impact of changing capital depreciation on real interest rate elasticities with respect to age milestones, as well as the impact of changes in age structure on the capital-output ratio.

As we have already noted, this paper focuses on the presentation of a framework that allows to derive formal algebraic relations between real interest rates and age milestones, in order to inspect the influence of demographic factors in specific economic mechanisms. In particular, we use our framework to explore how changing age structure can switch an altruistic motive from helping children to supporting parents, or how an increase of life-expectancy, a reduction of retirement age, and postponing of the age of first job explain the decline of real interest rates, further quantifying those phenomena. Although we keep our results focus on the demand side of OLG models, our framework can also be used, for example, to explore secular stagnation mechanisms driven by demographic factors. In particular, in current work in progress, we extend our framework with nominal prices, endogenous output, and nominal wage rigidities, where when the natural rate of interest becomes negative, a second best solution with a sub-optimal stable equilibrium output level is characterized by an endogenous persistent increase of the age of first job.

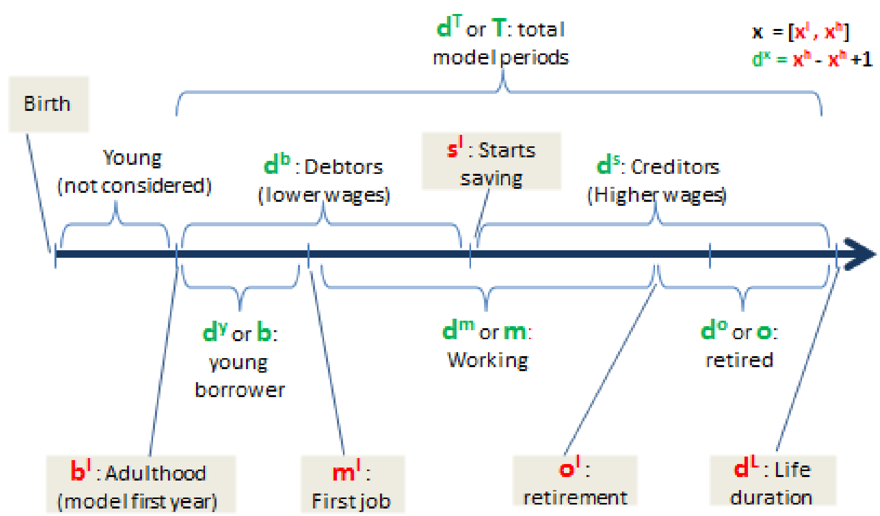
2 An Endowment Economy with age milestones

In this section we describe and solve a multi-generations OLG model where age milestones binding relevant economic periods of households, can exogenously change. We also derive some algebraic tools that simplify the model solution in closed-form expressions, and with which the derivatives of the steady state equilibrium real interest rate with respect to age milestones can be algebraically explicitly derived.

Consider an overlapping generations model in the spirit of Eggertsson and Mehrotra [13] where new generations start every year. Imagine that households live $L \equiv d^L$ years (where d^L stands for *duration of life*), but are considered economically active in the model only after childhood, from the age of adulthood b^l (b^l standing for *lower borrowing age*) until the last year of their lives at age d^L. The number of overlapping generations of the model $T = d^L - b^l + 1$ is then determined by two age milestones, bounding the period that starts at the age of adulthood, and ending at the last year of their lives. We start by considering an endowment economy where agents have no capital to invest in, but where households can lend to one another. After childhood, at age b^l households borrow from other households to consume. During the middle-age period m^l (m^l standing for *lower middle age*) they receive an income in the form of endowment $y_t^{i=age}$ which they use to consume, to pay-back their debts, and to save for retirement by lending to other households. In order to smooth their life-time consumption path, during the first part of their middle age period households are borrowers, becoming savers thereafter until the end of their lives. The *initial saving age* $s^l \in [m^l, o^l[$ is an endogenous parameter of the model. Households are retired from age o^l to d^L, having no endowment and consuming with the proceeds from their savings during that period.

The model age structure is illustrated in Figure 2. Age milestones in red are the boundaries

Figure 2: Relevant Life-cycle Periods and Age Milestones



of life economic periods with durations in green. We can look to an household from an income perspective, starting his journey as a *young borrower* without income who needs to borrow from other agents to be able to consume. The *young borrower's* period has a duration in years of $d^b = m^l - b^l$. He then enters into *middle age*, with a duration in years of $d^m = o^l - ml$, after finding his first job at age m^l, and gets an income in the form of endowment until retirement at age o^l. Thereafter he will be retired for $d^o = T - d^m - d^b = L - m^h$ years. Alternatively we can look to an household from a borrowing/saving perspective, which may facilitate the economic intuition: in the beginning of their journey they are net *borrowers* during d^b years until they pay back their loans, and become savers at the age s^l for $d^s = d^L - s^l + 1$ years. It is the relative weight of borrowers and savers, or the interaction between loan demand and supply, that will determine loan market equilibrium interest rate level. Note that the initial saving age s^l, which determines the relative weight of borrowers and savers, is an endogenous parameter of the model. s^l itself depends on the relative duration of *young borrowers*, middle age and *retirement* periods, with durations respectively given by b, m, and o respectively. In what follows we will use durations notation b, m and model life span $T = b+m+o$ to express most of our findings, where $b = m^l - b^l = b^h - b^l + 1$, $m = m^h - m^l + 1$, and o is retirement duration, here a dependent variable.

Consider then a representative household reaching adulthood at time t, with the following utility function:

$$\max_{c_{t+i}^{b^l+i}} E_t \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i}) \tag{1}$$

Where the $U(c)$ is assumed to be a constant elasticity of inter-temporal substitution utility function expressed by $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. $c_{t+i}^{b^l+i}$ is the consumption of households with age $b^l + i$ at time $t + i$. Furthermore agents borrow and lend to one-another using one year risk-free bonds at an interest rate r_t. We can then write the annual budget constraints faced by an agent reaching adulthood at time t, for the rest of his life.

$$c_t^{b^l} = y_t^{b^l} + b_t^{b^l} \qquad \text{for age} = b^l \tag{2}$$

$$c_{t+i}^{b^l+i} = y_{t+i}^{b^l+i} + b_{t+i}^{b^l+i} - (1 + r_{t+i-1}) b_{t+i-1}^{b^l+i-1} \qquad \text{for age } b^l + i \in]b^l, d^L[\tag{3}$$

$$c_{t+T-1}^{d^L} = y_{t+T-1}^{d^L} - (1 + r_{t+T-2}) b_{t+T-2}^{d^L-1} \qquad \text{for age} = d^L \tag{4}$$

The three budget constraints are similar and can be analyzed from two perspectives. Considering equation (3), an agent with age b^l+i at time $t+i$ takes his endowment $y_{t+i}^{b^l+i}$ together with a new loan $b_{t+i}^{i+1} > 0$ to consume $c_{t+i}^{b^l+i}$ and pay his loan and interest corresponding to the previous period $(1+r_{t+i-1})b_{t+i-1}^{b^l+i-1}$. Alternatively an agent takes his endowment $y_{t+i}^{b^l+i}$ together with his savings $-(1 + r_{t+i-1})b_{t+i-1}^{b^l+i-1}$ from previous year to consume $c_{t+i}^{b^l+i}$ and to save $b_{t+i}^{b^l+i}$ for the next year. In this case $b_{t+i}^{b^l+i}$ is negative. At the age of adulthood (2) there are no loans to pay back from previous year. During the last year of their lives, (4), households do not need to save for the future any longer. Let:

$$y_t^i > 0 \text{ for } i \in [m^l, m^h] \forall t \tag{5}$$

$$y_t^i = 0 \text{ for } i \in [b^l, m^l[\cup]m^h, d^L] \forall t \tag{6}$$

Although endowments are assumed to be strictly positive for the middle age and zero otherwise - equations(5) and (6) - we do not need to impose any special restriction on the duration of the middle age period in order to have distinct borrowing and savings periods. In particular the duration of middle age period $m = d^m$ could coincide with model's time span T. It is the endowment and real interest rate paths that will determine the annual path of agent loans $b_{t+i}^{b^l+i}$ that can be positive or negative in order to smooth households' consumption path, and consequently of aggregate loans throughout all living households, that from now on we define as *excess borrowing*:

$$B_t^{b^l} = \frac{1}{N_t^{b^l}} \sum_{i=0}^{T-1} N_t^{b^l+i} b_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{N_t^{b^l+i}}{N_t^{b^l}} b_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{b_t^{b^l+i}}{\prod_{k=0}^{i-1}(1+g_{t-k})} \tag{7}$$

where $B_t^{b^l}$ is the value of excess borrowing at time t normalized to the size of the younger generation with age b^l at time t, and $1 + g_t = \frac{N_t^{b^l}}{N_{t-1}^{b^l}} = \frac{N_t^{b^l}}{N_t^{b^l+1}}$ is population growth at time t. When excess borrowing is equal to zero then the loan market is in equilibrium. An equilibrium real interest rate solution path ensures that loan market is in equilibrium at any time t. If we can find a closed form expression (at least for steady state equilibria) for excess borrowing as a function of the real interest rate r and other exogenous parameters of the model x , then we could use loan market equilibrium equation $B_t^{b^l}(r, x) = 0$ and the implicit function theorem to find closed form expressions for changes of the equilibrium real interest rate with respect to changes of any parameter of the model x, in particular age milestones:

$$B^{b^l}(r, x) = 0 \Rightarrow r_x \equiv \frac{\partial r}{\partial x}(r, x) = -\frac{\frac{\partial B^{b^l}}{\partial x}(r, x)}{\frac{\partial B^{b^l}}{\partial r}(r, x)} \Leftrightarrow r_x = -\frac{B_x}{B_r}(r, x) \tag{8}$$

Excess borrowing at time t can be expressed[3] as a function of aggregate endowment and consumption at time t, and excess borrowing at time $t - 1$:

$$B_t^{b^l} = C_t^{b^l} - Y_t^{b^l} + \left(\frac{1 + r_{t-1}}{1 + g_t}\right) B_{t-1}^{b^l} \tag{9}$$

Although aggregate endowment at time t, $Y_t^{b^l}$, is determined by the endowment path, population growth and time productivity paths, we need a closed form expression for aggregate consumption which can be obtained by solving the previous optimization problem (1), with a *re-casted* overlapping multi-generations model in present value terms using the next proposition[4]:

[3] see appendix A.

[4] from now on, and by default, when the subscripts and superscript are omitted in aggregates then we are

Proposition 1 *The optimization problem given by equation*(1)*, subject to* (2),(3)*, and* (4) *is equivalent to maximize the same utility function with respect to consumption, subject to the equality of the present values of consumption* $\mathbb{C}_t$ *and endowments* $\mathbb{Y}_t$ *for* T *periods:*

$$\max_{c_{t+i}^{b^l+i}} E_t \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i})$$
$$s.t.\ \mathbb{C}_t = \mathbb{Y}_t$$

The solution expressions for consumption at every age are given by[5]*:*

$$\mathbb{E}_t c_{t+i+1}^{b^{l+i+1}} = \beta_{r_{t+i}}(1+r_{t+i})c_{t+i}^{b^l+i} \tag{10}$$
$$\text{with } c_t^{b^l} = \frac{\mathbb{Y}_t}{f(\beta_{r_t}, T)} \tag{11}$$

Expressions for the present value of consumption $\mathbb{C}_t$*, and aggregate steady state consumption* C*, normalized to the size of the youngest generation* b^l*, are respectively given by:*

$$\mathbb{C}_t = c_t^{b^l} f(\beta_{r_t}, T)$$
$$C = c^{b^l} f(\gamma_r, T)$$

where $\beta_{r_t} = \beta^{\frac{1}{\sigma}}(1+r_t)^{\frac{1-\sigma}{\sigma}} \overset{(\sigma=1)}{=} \beta$, $\gamma_r = \frac{\beta_r(1+r)}{(1+g)(1+z)}$*, and* $f(\beta_{r_t}, T) = \mathbb{E}_t \sum_{i=0}^{T-1} \prod_{k=0}^{i-1} \beta_{r_{t+k}}$*. Note that when the first argument is constant* $f(\beta_r, T) = \sum_{i=0}^{T-1} \beta_r^i = \frac{1-\beta_r^T}{1-\beta_r}$.

Proof: The Euler equations of both problems are equivalent. The model is fully derived in appendix A. ■.

Now that the model is solved for consumption, and that we derived a simple closed form expression for aggregate consumption in steady state, and by assuming aggregate endowment Y can also be represented by a closed form expression (as the endowment path is an exogenous set of parameters), then based on expression(9) we present in the next proposition a steady state closed form expression for excess borrowing, the *corner stone* to explicit the derivatives of equilibrium real interest rate with respect to age milestones:

Proposition 2 *(i) Excess borrowing in steady state is a continuous and differentiable function of the equilibrium real interest rate* $r \in]-1, +\infty[$*, and can be represented by the following*

referring to expressions normalized to the size of generation b^l at time t: $Y \equiv Y_t^{b^l}$.

[5] The model is deterministic.

expression:

$$B(r,x)\begin{cases} \frac{1+r_{gz}}{r-r_{gz}}(Y-C) & for \quad r\neq r_{gz} \\ -(1+r_{gz})\frac{\partial C}{\partial r} & for \quad r=r_{gz} \end{cases} \tag{12}$$

where $1+r_{gz}=(1+g)(1+z)$*, and* x *represents the exogenous parameters of the model.*

(ii) $B(r,x)=0$ *has at least one solution if the no endowment retirement period duration*$\equiv$ o *is lower than the elasticity of inter-temporal substitution times the model duration, or* $\frac{1}{\sigma}>\frac{o}{T-1}$*, , and the duration of the initial no-endowment borrowing period*$\equiv$ b *is strictly lower than* $T-1$*. Moreover,* $\frac{\partial B}{\partial r}(r)<0$ *at least for one solution* r *solving* $B(r,x)=0$*.*

(iii) If $B(r,x)=0$*, the derivative of the real interest rate* r *with respect to any parameter of the model* x *can be expressed by:*

$$\frac{\partial r}{\partial x}\equiv r_x=-\frac{B_x}{B_r}=-\frac{Y_x-C_x}{Y_r-C_r}=-\frac{\log_x Y-\log_x C}{\log_r Y-\log_r C} \tag{13}$$

Proof: (in appendix B) $\blacksquare$.

We have now the tools we need to algebraically formalize and interpret the expressions for the derivatives of steady state real interest rates with respect to age milestones at loan market equilibrium. This will be the purpose of the next section, for which we still need to formally express an endowment path, and respective aggregate and present value endowment expressions.

Note that for a constant endowment path for the total duration of the model, with no population, or productivity growth, and an elasticity of inter-temporal substitution equal to unity, then $1+r=\frac{1}{\beta}$ solves $B(r,x)=0$. This solution is equivalent to the equilibrium real interest rate in steady state of an infinitely lived single agent model, and assumes a uniform distribution of endowment through the duration of the model, with no retirement and no *"no endowment"* borrowing period b for the young households. From this *starting point* the introduction of a retirement period would correspond to a change of the income path that would reduce the steady state equilibrium real interest rate through an expansion of excess savings (equivalent to a contraction of excess borrowing). Moreover, the introduction of a no endowment period in the beginning of an agent's *economic life* would increase r, through an expansion of excess borrowing. Then changing the duration of relevant economic periods, through age-milestones changes, may affect households' income paths, and consequently excess borrowing through households' adjusted borrowing and saving needs in order to smooth households' life-time consumption paths. The resulting contraction or expansion of excess borrowing affects loan market equilibrium real interest rate.

Loan demand and supply

An equivalent way to understand the dynamics of excess borrowing in steady state is to split its expression $B_t^{b^l}$ into loan demand L_t^d and supply L_t^s. Loan demand for a given interest rate r is defined as excess borrowing of all households until the age they start to be savers, s^l. And loan supply is the negative expression of excess borrowing of all households that are net savers, with age equal or above s^l:

$$L_t^d(r, v^i) = B_t^{d^b(r),b^l}(r, v^i) \text{ , where } d^b(r) = [b^l, s^l[\tag{14}$$

$$L_t^s(r, v^i) = -B_t^{d^s(r),b^l}(r, v^i) \text{ , where } d^s(r) = [s^l, d^L[\tag{15}$$

Note that $s^l \equiv s^l(r, v^i)$ is an endogenous variable of the model, which depends on r and age milestones v^i. Excess borrowing expressed in terms of loan demand and supply, is given by:

$$B_t^{b^l} = B_t^{d^b,b^l} + B_t^{d^s,b^l} = L_t^{d,b^l} - L_t^{s,b^l} \tag{16}$$

and loan market equilibrium can now be expressed by:

$$B_t^{b^l} = 0 \Leftrightarrow L_t^{d,b^l} = L_t^{s,b^l} \tag{17}$$

Where loan demand and supply may be expressed in terms of aggregate income and consumption during the respective periods[6]:

$$L^{d,b^l} = \frac{1 + r_{gz}}{r - r_{gz}}(Y^{d^b,b^l} - C^{d^b,b^l}) \tag{18}$$

$$L^{s,b^l} = -\frac{1 + r_{gz}}{r - r_{gz}}(Y^{d^s,b^l} - C^{d^s,b^l}) \tag{19}$$

This equivalent representation of excess borrowing can be helpful when interpreting how changes in age milestones affect equilibrium real interest rates, by comparing graphically steady state changes of loan demand and supply, as in Eggertsson and Mehrotra [13].

In the next section we analytically express and interpret how age milestone changes affect steady state equilibrium real interest rates.

[6]expressions derived in appendix

3 Real interest rate derivatives with respect to age parameters

In this section we present and interpret the derivatives of real interest rate with respect to age milestones $r_{v_i} \equiv \frac{\partial r}{\partial v_i}$. As age milestones are exogenous parameters of our model, closed form expressions for the derivatives of real interest rate with respect to age milestones are given directly using the tools described above, once we characterize households' endowment path with continuous and differentiable closed form expressions for present value of endowment $\mathbb{Y}$, and aggregate endowment Y.

We can interpret $r_{v_i} \equiv \frac{\partial r}{\partial v_i}$ as by how much r would have to change to compensate for the impact in excess borrowing of a change of a given age milestone v_i, so that loan market remains in equilibrium. By reasonably assuming that excess borrowing is a decreasing function of the real interest rate [7], or $B_r \equiv \frac{\partial B}{\partial r} < 0$, then an increase $dr > 0$ of the real interest rate would have a contraction impact on excess borrowing B, by $\frac{\partial B}{\partial r} dr < 0$. In order for loan market to remain in equilibrium B would have to increase back to 0, through a change ∂v_i of any given age milestone v_i, which should have an expansion effect on excess borrowing $\frac{\partial B}{\partial v_i} dv_i > 0$, such that:

$$dB(r, v_i) = \frac{\partial B}{\partial v^i} dv^i + \frac{\partial B}{\partial r} dr = 0 \tag{20}$$

Note that the expansion effect $\frac{\partial B}{\partial v_i} dv_i > 0$ implies that the change of age milestone ∂v_i has the same sign of the derivative of excess borrowing with respect to the age milestone, which means that if excess borrowing is a negative function of a given age milestone, then a decrease of this age milestone is required to compensate for an increase of the real interest rate, which is the same to say that r_{v_i} and B_{v_i} have the same sign, when $B_r < 0$. This is directly observed from the general expression for r_{v_i} given by:

$$r_{v^i}(r, v) = \frac{dr}{dv^i} = -\frac{\frac{\partial B}{\partial v_i}}{\frac{\partial B}{\partial r}} = -\frac{B_{v_i}}{B_r}(r, v) \tag{21}$$

We also inspect how this relation is affected by other relevant parameters of the model like changes in time related productivity z_t, age related productivity ρ_t^i, and elasticity of intertemporal substitution $\frac{1}{\sigma}$. With the same denominator B_r we expect that the derivatives of the real interest rate with respect to the exogenous parameters of the model keep a similar proportional relation. While B_v sets the sign of the real interest rate derivatives with respect to an age milestones, and the relative magnitude of the derivative with respect to others,

[7] From the proof of proposition 2 there are always more solutions for $B(r) = 0$ where $B_r < 0$ than otherwise. Although $B_r < 0$ for a wide calibration range, we could not find (yet) a sufficient condition for a unique solution where $B_r < 0$.

B_r is the same denominator of those expressions setting a common amplitude factor. For example, for lower elasticities of inter-temporal substitution $\frac{1}{\sigma}$, aggregate consumption, and consequently excess borrowing, are expected to change less with real interest rate changes (B_r is flatter). Then, a stronger real interest rate reaction is required to compensate for the impact on excess borrowing of changing an age milestone, relative to a higher EIS. This phenomena can be observed in the calibration section.

B_{v_i} is the term that determines the relative signs and magnitudes of the derivatives with respect to each other, besides determining the sign of the derivative itself (given the sign of their common denominator B_r). Inspecting B_{v_i} is the purpose of the next proposition.

The missing pieces to derive a tractable closed-form expression for steady state excess borrowing are the closed-form expressions for present value and aggregate endowment. With a sufficiently generic endowment path, with no-endowment periods at the beginning and at the end of the model time span, with durations respectively given by $b \equiv d^b \geq 0$ and $o \equiv d^o \geq 0$, time and age type productivity growth rates such that $y^i_{t+1} = (1+z)y^i_t$, and $y^{i+1}_t = (1+\rho)y^i_t$, the endowment present value and aggregate expressions in steady state are respectively given by:

$$\mathbb{Y}_t = \frac{y_t^{m^l}}{(1+r_z)^b} f\left(\frac{1+\rho}{1+r_z}, m\right) \tag{22}$$

$$Y_t = \frac{y_t^{m^l}}{(1+g)^b} f\left(\frac{1+\rho}{1+g}, m\right) \tag{23}$$

where $1 + r_z = \frac{1+r}{1+z}$.

Proposition 3 *for $r > -1$ solving $B(r, v) = 0$, the partial derivatives of excess borrowing and equilibrium real interest rate with respect to age milestones and durations, can be expressed by,*

$$B_{v_i} = \frac{1+r_{gz}}{r - r_{gz}} Y \left(Log_{v_i} Y - Log_{v_i} C\right) \tag{24}$$

$$r_{v_i} = -\frac{B_{v_i}}{B_r} = \frac{Y_{v_i} - C_{v_i}}{C_r} = \frac{Log_{v_i} Y - Log_{v_i} C}{Log_r C} \tag{25}$$

(i) The derivatives of the natural rate of interest with respect to age milestones can be expressed in terms of the derivatives of the natural rate of interest with respect to the durations of the young borrowing period b, the duration of middle age m, and the duration of the model T (here the duration of retirement o is a dependent variable):

- *Adulthood:* $r_{b^l} = -r_b - r_T$

- *First job:* $r_{m^l} = r_b - r_m$
- *Retirement:* $r_{o^l} = r_m$
- *Life expectancy:* $r_L = r_T$

(ii) for a sufficiently generic households' endowment path expressed by $y_t^i > 0$ *for* $i \in [m^l, m^h]$*, and* $y_t^i = 0$ *for* $i \in [b^l, m^l[\cup]m^h, d^L]$*, where time and age related productivity growth rates, for* $i \in [m^l, m^h]$*, respectively given by* $1 + z_t^i = \frac{y_{t+1}^i}{y_t^i}$*, and* $1 + \rho_t^i = \frac{y_t^{i+1}}{y_t^i}$*, the partial derivatives* B_{v_i} *and* r_{v_i} *have the following signs:*

♦ *Partial derivatives with respect to durations:*

- *Young borrower* $B_b > 0 \overset{B_r<0}{\Rightarrow} r_b > 0$
- *Middle age* $B_m > 0 \overset{B_r<0}{\Rightarrow} r_m > 0$
- *Model duration* $B_T < 0 \overset{B_r<0}{\Rightarrow} r_T < 0$

♦ *Partial derivatives with respect to age milestones:*

- *Adulthood:* $B_{b^l} < 0 \overset{B_r<0}{\Rightarrow} r_{b^l} < 0$
- *First job:* $B_{m^l} > 0 \overset{B_r<0}{\Rightarrow} r_{m^l} > 0$
- *Retirement:* $B_{o^l} > 0 \overset{B_r<0}{\Rightarrow} r_{o^l} > 0$
- *Life expectancy:* $B_{d^L} < 0 \overset{B_r<0}{\Rightarrow} r_{d^L} < 0$

Proof: in appendix C.■

As the relative signs of the partial derivatives of real interest rate are determined by the partial derivatives of excess borrowing with respect to age milestones and durations, given by (24), we next present those expressions for interpretation. The partial derivatives of excess borrowing w.r.t. periods duration d are given by:

$$B_b = Y\beta_r \triangle \log(\gamma_r, \beta_r) \qquad > 0 \qquad (26)$$

$$B_m = Y\frac{1+\rho}{1+r_z}\triangle H^m\left(\frac{1+\rho}{1+g}, \frac{1+\rho}{1+r_z}\right) \qquad > 0 \qquad (27)$$

$$B_T = -Y\beta_r \triangle H^T(\gamma_r, \beta_r) \qquad < 0 \qquad (28)$$

and the partial derivatives of Excess Borrowing w.r.t. age milestones:

$$B_{b^l} = -Y\beta_r \triangle H^{-T}(\gamma_r, \beta_r) \qquad = -B_b - B_T < 0 \tag{29}$$

$$B_L = -Y\beta_r \triangle H^{T}(\gamma_r, \beta_r) \qquad = B_T < 0 \tag{30}$$

$$B_{m^l} = Y\frac{1+\rho}{1+r_z}\triangle H^{-m}\left(\frac{1+\rho}{1+g}, \frac{1+\rho}{1+r_z}\right) \qquad = B_b - B_m > 0 \tag{31}$$

$$B_{o^l} = Y\frac{1+\rho}{1+r_z}\triangle H^{m}\left(\frac{1+\rho}{1+g}, \frac{1+\rho}{1+r_z}\right) \qquad = B_m > 0 \tag{32}$$

where $\triangle f(x,y) = \frac{f(x)-f(y)}{x-y} > 0$ if $f' > 0$; $H^a(x) \equiv \frac{1}{a}\frac{\log x^{-a} - log 1}{x^{-a}-1}$, and $H^{a'} > 0$[8].

All age-milestones expressions have a similar look. (i) The superscript parameter of function H is the duration the period affected by the change of the age milestone. The sign of the superscript corresponds to the change sign of the period durtation with an increase of the age milestone. For example, if b^l increases then the duration of the model T will decrease, and the superscript $-T$ is used for the function H. (ii) We can note also that the ratio of any two arguments is the same, or $\frac{\gamma_r}{\beta_r} = \frac{\frac{1+\rho}{1+g}}{\frac{1+\rho}{1+r_z}} = \frac{1+r}{1+r_{gz}}$.

3.1 Durations of relevant lifetime economic periods

Model duration T: An increase of model duration T expands the expected retirement duration for the same amount, $o = T - m - b \Rightarrow \partial o = \partial T$, assuming retirement duration o is the dependent variable. This increases households savings needs during middle age for the same steady state real interest rate level, corresponding to a reduction of excess borrowing. Consequently the partial derivative of excess borrowing with respect to T is negative. For the loan market to remain in equilibrium a positive compensation of excess borrowing is required by an adjustment of the real interest rate, which must be negative if the partial derivative of excess borrowing with respect to the natural rate of interest is negative too, our base case by assumption. A positive change in life expectancy L, calling for a negative change in r leads to a negative $r_L = \frac{\partial r}{\partial L}$.

Endowment duration m: An increase of endowment duration m contracts the expected retirement duration for the same amount, $o = T - m - b \Rightarrow \partial o = -\partial m$. This reduces households savings needs during middle age, corresponding to an increase of excess borrowing. The mechanism is the opposite as the one described above for T.

Initial no-endowment duration b: An increase of b has a double positive effect on excess

[8] see proof of proposition 3.

borrowing. The first by expanding initial borrowing period by ∂b with a positive effect on excess borrowing, and a second by contracting the retirement period by $\partial o = -\partial b$ which reduces household saving needs with a further positive impact on excess borrowing.

The partial derivatives of real interest rates and excess borrowing share the same sign, if $B_r < 0$, and are analytically expressed by:

$$r_b = \frac{\log \gamma_r - \log \beta_r}{\log_r C} > 0 \tag{33}$$

$$r_m = \frac{H^T\left(\frac{1+\rho}{1+g}\right) - H^T\left(\frac{1+\rho}{1+r_z}\right)}{\log_r C} > 0 \tag{34}$$

$$r_T = -\frac{H^T\gamma_r - H^T\beta_r}{\log_r C} < 0 \tag{35}$$

3.2 Age milestones bounding the duration of the model

Age milestones b^l and $b^L \equiv L$ limit the model duration $d^T \equiv T = L - b^l + 1$.

Age milestone L: When only life expectancy [9] increases among all age milestones, then the model duration T increases by the same amount, $T = L - b^l + 1 \Rightarrow \partial T = \partial L$, increasing the expected retirement duration too[10], $o \equiv d^o = L - m^h + 1 \Rightarrow \partial o = \partial L = \partial T$, impacting negatively excess borrowing by the same mechanism described above for the model duration T.

Age milestone b^l: Furthermore, increasing the age of adulthood b^l shortens the model duration T by the same amount, as well as the duration of borrowing period $d^b = m^l - b^l$. This contraction of the initial borrowing period leads to a reduction of excess borrowing for the same real interest rate level.

$B_{b^l} = -B_b - B_T$: Note that an increase of adulthood age can be interpreted as a combination of changes in two periods: a contraction of the young borrowing period b, combined with a contraction of the total duration of the model T when the retirement duration is a dependent variable. As we have seen above, a contraction of b alone as a double negative effect on excess borrowing. The first one is through a reduction of loans demand for the same interest rate level, and a second one is an increase of loans supply through an expansion of the retirement period, in order to keep the durations of middle age and model period, m and T, unchanged.The second effect, the retirement duration increase, is directly offset by the

[9] although we use the term *expectancy* the model is deterministic

[10] We are inspecting partial derivatives, which means that only one parameter changes.

reduction of T which only impacts retirement duration. The combined effect is a contraction of the borrowing period b, leaving m and o unchanged. So r_{b^l} would be equal to $-r_b$ if the dependent duration parameter was T.

The partial derivatives of excess borrowing with respect to age milestones bounding the duration of the model T are both negative. The consequence is the same, although triggered by different mechanisms: (i) Increasing life expectancy that expands loan supply, and (ii)increasing the age of adulthood that contracts loan demand. The partial derivatives of real interest rates have the same negative sign, if $B_r < 0$, and are analytically expressed by:

$$r_{b^l} = -\frac{H^{-T}(\gamma_r) - H^{-T}(\beta_r)}{\log_r C} = -r_b - r_T \tag{36}$$

$$r_L = -\frac{H^{T}(\gamma_r) - H^{T}(\beta_r)}{\log_r C} = r_T \tag{37}$$

3.3 Age milestones bounding labor income duration

Age milestones m^l and $o^L \equiv L$ limit the endowment duration $d^m \equiv m = o^l - m^l$.

Age milestone o^l: An increase of the retirement age o^l contracts expected retirement duration, reducing households' saving needs, equivalent to increasing excess borrowing for the same real interest rate level. The mechanism is the same as a reduction of life expectancy described above.

Age milestone m^l: Furthermore, increasing the age of first job m^l expands the duration of the borrowing period $d^b = m^l - b^l$, which causes an increase of excess borrowing for the same real interest rate level.

$B^l_m = B_b - B_m$: Note that an increase of the age of first job can be interpreted as a combination of changes in two periods: an expansion of the young borrowing period b, combined with a contraction of the middle age duration m, when the retirement duration is a dependent variable. As we have seen above, an expansion of b alone as a double positive effect on excess borrowing. The first one is via an increase of loans demand for the same interest rate level, and a second one is via a reduction of loans supply through an contraction of the retirement period, in order to keep the durations of middle age and the model, m and T, unchanged. The second effect, the reduction of the retirement duration, is directly canceled by the reduction of the middle age period m which increases back the retirement duration by the same amount. The combined effect corresponds to an expansion of the borrowing period b that leaving retirement duration o unchanged. r_{m^l} would be equal to r_b if T was the dependent variable.

The partial derivatives of excess borrowing with respect to age milestones bounding endowment duration are both positive. Same consequence triggered by different mechanisms: Increasing retirement age that contracts loan supply, and increasing age of first job that expands loan demand. The partial derivatives of real interest rates have the same negative sign, if $B_r < 0$, and are analytically expressed by:

$$r_{m^l} = \frac{H^{-m}\left(\frac{1+\rho}{1+g}\right) - H^{-m}\left(\frac{1+\rho}{1+r_z}\right)}{\log_r C} = r_b - r_m \tag{38}$$

$$r_{o^l} = \frac{H^{m}\left(\frac{1+\rho}{1+g}\right) - H^{m}\left(\frac{1+\rho}{1+r_z}\right)}{\log_r C} = r_m \tag{39}$$

Equation (25) can also be used to formalize algebraically the partial derivatives of the natural rate of interest with respect to other exogenous parameters of the model, namely the population growth rate g, total factor productivity growth z, and age dependent endowment growth rate ρ, whose relative signs are given by expression (24).

4 Intergeneration transfers

Until now we assumed that agents only interact with each-other by borrowing and lending. We now introduce transfers between generations in the form of bequests to children, gifts to parents, and a *pay-as-you-go* social security system, as those concepts are closely related to agents age structure. We will see how endogenous bequest and gifts are affected by changes in age structure, and how a social security tax may mitigate or amplify those changes.

4.1 Intergenerational Altruism

Imagine there are intergenerational altruistic linkages between parents and children, taking the form of transfers between agents during the last year of their lives and their direct descendants. We start by assuming that transfers are positive corresponding to positive bequests left by parents to their children, but we also analyze the case of children caring about their parents wealth. We start by recasting the budget constraints and derive a re-casted expression for excess borrowing as a function of the excess borrowing expression without intergenerational transfers, which will be valid for the two alternative preference functions used later in the sub-section. Then we endogenize the bequest motive by adjusting agents' preferences, to inspect how bequests and gifts are affected by changes of age milestones, and vice-versa. We use two modeling methods: first a *Warm glow* bequest motive type, where parents value the bequest itself. This bequest motive is the one we use to calibrate the model in the last section. And second, an *altruism* bequest motive type, where parents value their children's utility.

(i) Recasting budget constraints with bequests

Let inter-generational altruism be represented by the bequest level Q_t^T left or received by an agent to or from his direct descendants during the last year of his live T at time t. A positive Q refers to *forward* altruism from parents to children, where $Q^f \equiv Q > 0$. And a negative Q refers to *backward* altruism from children to parents, where $Q^b \equiv -Q > 0$. Without loss of generality, and in order to keep the model tractable the same mechanism is used in both cases. Let $Q_t \equiv Q_t^T$, μ be the age difference between parents and children, and n the number of children per agent. Note that $n = (1+g)^\mu$. The budget constraints have adjusted expressions when bequests are received at the age $b^l + T - 1 - \mu$, and at life expectancy $b^l + T - 1$:

$$c_{t+T-1-\mu}^{T-\mu} = y_{t+T-1-\mu}^{T-\mu} + b_{t+T-1-\mu}^{T-\mu} - (1 + r_{t+T-2-\mu})b_{t+T-2-\mu}^{T-1-\mu} + \frac{Q_{t+T-1-\mu}}{n} \tag{40}$$

$$c_{t+T-1}^{T} = y_{t+T-1}^{T} - (1 + r_{t+T-2})b_{t+T-2}^{T-1} - Q_{t+T-1} \tag{41}$$

Adjusted present value budget constraint and excess borrowing expressions in steady state are given by:

$$\mathbb{C}_t^{b^l} = \mathbb{Y}_t^{b^l} + \frac{r - r_{gz}}{1 + r_{gz}} M^q Q_t \tag{42}$$

$$B_t^q = B_t - Q_t G^q \text{ , where } G^q = M^q \frac{f(\gamma_r, T)}{f(\beta_r, T)} \tag{43}$$

$$\text{where } M_q = \frac{(1 + r_{gz})^{T-\mu}}{(1 + r)^{T-1}} \triangle x^\mu (1 + r, 1 + r_{gz}) \tag{44}$$

$$\text{and } \triangle x^\mu(y, z) = \frac{y^\mu - z^\mu}{y - z} > 0 \tag{45}$$

Because M_q is positive , G_q is also positive, from where it is straightforward to derive the impact of bequests on the natural rate of interest from a *no-bequest* initial state: Let r^q solve $B^q(r^q) = 0$ with an operative bequest motive, and Let r solve $B(r) = 0$ otherwise, with an inoperative bequest motive. Note that $B(r^q)$ has the same sign of Q because $B^q(r^q) = 0 \Leftrightarrow B(r^q) = QG^q(r^q)$ and $G^q(r^q) > 0$. Because we are assuming that $B(r)$ is a negative function of r we have:

$$Q > 0 \Rightarrow B(r^q) > 0 \Rightarrow r^q < r \tag{46}$$

$$Q < 0 \Rightarrow B(r^q) < 0 \Rightarrow r^q > r \tag{47}$$

From where bequest from parents to children reduce the equilibrium real interest rate from a *no-bequest* motive, and gifts from children to parents would have an opposite effect. By endogenizing the bequest motive we can derive Q, as well as all adjusted expressions of previous section.

(ii) Warm glow bequest motive:

To derive an expression for bequest Q using a *warm glow* motive, the previous utility function is adjusted according to the literature, by adding a bequest term to agent's utility function:

$$U_t = \max_{c_{t+i}^{1+i}} E_t \sum_{i=0}^{T-1} \beta^i u(c_{t+i}^{1+i}) + \beta^{T-1} \frac{u(Q_{t+T-1}^T)}{1+\phi} \tag{48}$$

From FOC Q_{t+T-1} and c_{t+T-1} we get a steady state expression for bequests as a function of the present value of consumption $\mathbb{C}_t^{b^l}$:

$$Q_t = \frac{\mathbb{C}_t^{b^l}}{(1+\phi)^{\frac{1}{\sigma}}} \frac{\gamma_r^{T-1}}{f(\beta_r, T)} > 0 \tag{49}$$

By combining the previous equation with the present value budget constraint (42) we obtain an expression for bequest Q_t given by:

$$Q_t = \frac{\mathbb{Y}_t}{(1+\phi)^{\frac{1}{\sigma}} \frac{f(\beta_r,T)}{\gamma_r^{T-1}} - \frac{r - r_{gz}}{1+r_{gz}} M_q} \tag{50}$$

Those new tools would be enough to derive new closed-form expressions for the partial derivatives of the real interest rates with respect to age milestones. We approach quantitatively that topic in the last section of this paper.

We next inspect an alternative bequest motive where agents consider the utility of their descendants in their preference function. In that case, equilibrium real interest rates are constant while the bequest motive is active. We deriving the impact of changing age milestones on the bequest level similarly to previous sections:

$$\frac{\partial Q}{\partial v_i} \equiv Q_{v_i} = -\frac{B_{vi}^q}{B_Q^q} = \frac{B_{vi}^q}{G^q} \tag{51}$$

Q_{v_i} would have the sign of B^q_{vi}, since $G^q > 0$. And if B^q_{vi} and B_{vi} have the same sign then the results for Q_v would be the same as the ones for r_v. For example, an increase of life expectancy would reduce bequests left to children, and an increase of the retirement age would have the opposite effect. Let's then briefly study the model:

(iii) Altruism bequest motive:

We now assume that bequests reflect agent's concern for the welfare of their descendants, by weighting children utility in agent's utility function, in the spirit of Barro [3]. By using the same mechanism we also examine agents concern with their parents, following the approach of Blanchard and Fischer [5]. The utility function takes the expression given below for both cases, where the utility of descendants is discounted in agent's utility with a lag of μ years (age difference between parents and children). The discount factor is β weighted by ϕ^f a selfish parameter (Barro and Sala-i Martin [4]) greater than one when parents prefer an additional unit of self consumption to a unit of children consumption in the same year. We assume that the utility of descendants in agents' preference function is not affected by the number of children (Blanchard and Fischer [5]):

$$U_t^0 = \max_{c_{t+i}^{b^l+i}} E_t \sum_{i=0}^{T-1} \beta^i u(c_{t+i}^{b^l+i}) + \left(\frac{\beta}{1+\phi^f}\right)^{\mu} U_{t+\mu}^1 = V_t^0 + \left(\frac{\beta}{1+\phi^f}\right)^{\mu} U_{t+\mu}^1 \Leftrightarrow \tag{52}$$

Where $V_t^0 = \max_{c_{t+i}^{1+i}} E_t \sum\limits_{i=0}^{T-1} \beta^i u(c_{t+i}^{1+i})$. We can solve (52) recursively forward, and get:

$$U_t^0 = \sum_{j=0}^{\infty} \left[\left(\frac{\beta}{1+\phi^f}\right)^{\mu}\right]^j V_{t+j\mu}^j \tag{53}$$

The budgets constraints of this maximization problem are the same considered at the beginning of the current section, given by expressions (40)(41). With no restrictions on the sign and level of the intergenerational transfer from an agent to his descendants during the last year of his life, the equilibrium real interest rate would have the following expression in steady state:

$$1 + r^{\phi^f} = \frac{1+\phi^f}{\beta}(1 + r_{gz}) \tag{54}$$

The bequest parameter Q^f is directly derived from loan market equilibrium expression given by equation (43):

$$B^q(r^{\phi^f}) = 0 \Leftrightarrow B(r^{\phi^f}) - Q^f G^q(r^{\phi^f}) = 0 \Rightarrow Q^f = \frac{B(r^{\phi^f})}{G^q(r^{\phi^f})} \tag{55}$$

We have seen that $G^q > 0$. Then a positive forward bequest Q^f implies that $B(r^{\phi f}) > 0$, and consequently $r^{\phi f} < r$, where r is the natural rate of interest of the maximization problem without bequest, since we are assuming that excess borrowing without bequest $B(r)$ is decreasing with r and $B(r) = 0$. Then, if $r^{\phi f} < r$, the natural rate of interest with forward altruism r^f, and Q^f are given by:

$$r^{\phi f} < r \Rightarrow r^f = r^{\phi f}, \text{ and } Q^f = \frac{B(r^{\phi f})}{G(r^{\phi f})} > 0 \tag{56}$$

Otherwise, if $B(r^{\phi f}) \leq 0 \Rightarrow r^{\phi f} \geq r$, then there is no loan market equilibrium with a positive bequest. Consequently $Q^f = 0$, and the excess borrowing expression with a forward altruism bequest motive B^q is the same as in the problem with an inoperative bequest motive, and the same for the natural rate of interest:

$$r^{\phi f} \geq r \Rightarrow r^f = r, \text{ and } Q^f = 0 \tag{57}$$

To summarize, parents leave bequests to their children only if their natural rate of interest without the possibility of bequests is greater than $r^{\phi f}$. Otherwise they will leave no bequests to future generations independently of their degree of altruism.

$$r^f = \min(r, r^{\phi f}) \tag{58}$$

$$Q^f = \frac{B}{G^q}(r^f) \geq 0 \tag{59}$$

Note that the derivatives of bequest and excess borrowing with an inoperative bequest motive with respect to age milestones bounding the endowment period, Q_v and B_v, have the same sign[11]. Then the changes in age milestones that affect the natural rate of interest without bequest will affect bequest levels in the same direction, when the motive is active. For example increasing retirement age will motivate parents to increase the bequest to their children. In the next section we confirm that the same result is robust also for age milestones bounding the duration of the model, L and b^l. Then an increase of life expectancy would reduce bequests left to children.

Backward Altruism: Agents concerned with their parents' wealth

We now assume instead that agents are concerned with their parents wealth, by helping them during the last year of their parents' lives. The budget constraints are the same but Q is now negative. The backward transfer parameter from children to parents is given by

[11] G^q is constant with respect to m^l and o^l.

$Q^b = -Q$, and the backwards altruistic parameter by ϕ^b. The mechanism is the same as before, with utility function represented by:

$$U_t^0 = V_t^0 + \left(\frac{\beta}{1+\phi^b}\right)^{-\mu} U_{t-\mu}^{-1} \Leftrightarrow \tag{60}$$

$$U_t^0 = \sum_{j=0}^{-\infty} \left[\left(\frac{\beta}{1+\phi^b}\right)^{\mu}\right]^j V_{t+j\mu}^j \tag{61}$$

The resulting expressions for the natural rate of interest and transfers from agents to their parents are a mirror of the above:

$$r^b = \max(r, r^{\phi^b}) \tag{62}$$

$$Q^b = -\frac{B}{G^q}(r^b) = -Q \geq 0 \tag{63}$$

where,

$$1 + r^{\phi^b} = \frac{1+\phi^b}{\beta}(1 + r_{gz}) \tag{64}$$

Agents help their parents only if their natural rate of interest with an inoperative bequests motive is lower than r^{ϕ^b}. Furthermore, and because the derivatives of Q^b and excess borrowing without bequest $B(r)$ with respect to age milestones have opposite signs, the changes in age milestones that affect the natural rate of interest with an inoperative bequest motive will affect bequest levels in the opposite direction too. For example increasing retirement age will motivate agents to be less generous with their parents, and a longer life will have the opposite effect.

Two Sided Altruism: Agents concerned with children and parents wealth

We now combine the motivations above by assuming that agents have the choice of helping their children or their parents. This assumption simplifies the maximization problem as the same budget constraints can be used, where Q can be positive or negative, but not both at the same time. We further assume that agents will prioritize supporting their parents at the end of their lives, to future bequests for their children:

$$\phi^b \leq \phi^f \Leftrightarrow r^{\phi^b} \leq r^{\phi^f} \tag{65}$$

Combining the previous results for one sided forward and backward altruism we find the following expression for the natural rate of interest r^ϕ:

$$r^\phi = \begin{cases} r^{\phi^b} & \text{for} \quad r < r^{\phi^b} & \Rightarrow Q^b > 0, Q^f = 0 \\ r & \text{for} \quad r \in [r^{\phi^b}; r^{\phi^f}] & \Rightarrow Q^b = 0, Q^f = 0 \\ r^{\phi^f} & \text{for} \quad r > r^{\phi^f} & \Rightarrow Q^b = 0, Q^f > 0 \end{cases} \tag{66}$$

When the natural rate of interest with an inoperative bequest motive changes significantly agents may change their altruistic behavior between their children and their parents. For example, a significant increase in life expectancy could change the motivation of agents from leaving a bequest to their children to helping their parents. Or, the increase of retirement age could have the opposite effect: parents would need less help, and more wealth would be available to help children. Furthermore, factors that contribute to lower the natural rate of interest without intergenerational altruism, like the increase of life expectancy, will decrease the propensity to leave a bequest to the next generation, and increase the willingness to help the previous one. Moreover in this model the natural rate of interest with and inoperative bequest motive increases with productivity. Assuming wealthier societies and agents are more productive, then the poorer would be more inclined to help their parents, and the richer their children.

We next introduce in our model a *social security tax* , to analyze how the equilibrium real interest rate, as well as bequests and gifts, are affected in the presence of social security transfers.

4.2 Social Security

We now introduce a *pay-as-you-go* social security system, where agents pay a tax τ on their income/endowment while they are working, and receive a pension after retirement equal to total collected social security contributions divided by the number of retired agents in each year. The budget constraints are now given by:

$$i \in [m^l, m^h] : c^i_{t+i-1} = (1-\tau) y^i_{t+i-1} + b^i_{t+i-1} - (1 + r_{i-1}) b^{i-1}_{t+i-2} \tag{67}$$

$$i \in [o^l, T-1] : c^i_{t+i-1} = b^i_{t+i-1} - (1 + r_{i-1}) b^{i-1}_{t+i-2} + \tau \frac{Y_{t+i-1}}{N^o_{t+i-1}} \tag{68}$$

$$i = T : c^T_{t+T-1} = -(1 + r_{T-1}) b^{T-1}_{t+T-1} + \tau \frac{Y_{t+T-1}}{N^o_{t+T-1}} \tag{69}$$

The present value budget constraint in steady state is now expressed by:

$$\mathbb{C}_t^{b^l,L} = \mathbb{Y}_t^{b^l,L}\left(1 - \tau M_\tau\right) \tag{70}$$

$$\text{where } M_\tau = 1 - \frac{Y_t^{b^l}}{\mathbb{Y}_t^{b^l,L}}\left(\frac{1+g}{1+r_z}\right)^{m+b}\frac{f\left(\frac{1}{1+r_z}, o\right)}{f\left(\frac{1}{1+g}, o\right)} \tag{71}$$

$$\text{and } \frac{Y_t^{b^l}}{\mathbb{Y}_t^{b^l,L}} = \left(\frac{1+g}{1+r_z}\right)^{-b}\frac{f\left(\frac{1+\rho}{1+g}, m\right)}{f\left(\frac{1+\rho}{1+r_z}, m\right)} \tag{72}$$

Using expression (70) in (11), we derive excess borrowing with social security based on (12), as the sum of the corresponding expression without social security with a positive term:

$$B^\tau(r, v, \tau) = B(r, v) + \tau G^\tau(r, v) \tag{73}$$

where $G^\tau(r, v)$ is positive [12]:

$$G^\tau(r, v) = \left(\frac{1+g}{r_z - g}\right)\mathbb{Y}_t^{b^l,L} M_\tau \frac{f(\gamma, T)}{f(\beta_r, T)} \tag{74}$$

The introduction of a social security system of this type corresponds to a forced redistribution of wealth from workers to the old. The need to save for retirement is expected to decrease. This fact is reflected by the positive term $\tau G^\tau(r, x)$ that expands excess borrowing, causing the steady state natural rate of interest r_n^τ to be greater than the one with no social security, r_n:

$$B^\tau(r_n^\tau, v, \tau) = 0 \Leftrightarrow B(r_n^\tau, v) = -\tau G^\tau(r, v) < 0 \Rightarrow r_n < r_n^\tau \tag{75}$$

or directly from (73), the derivative of excess borrowing B^τ with respect to τ is positive:

$$B^\tau(r, v, \tau) = \frac{\partial B^\tau}{\partial \tau}(r, v, \tau) = G^\tau(r, v) > 0 \tag{76}$$

The derivative of the natural rate of interest with respect to the social security tax given is positive[13] and is expressed by:

$$r_\tau(r_n^\tau, v, \tau) \equiv \frac{dr}{d\tau}(r_n^\tau, v) = -\frac{G^\tau(r_n^\tau, v)}{B_r(r_n^\tau, v) + \tau G_r^\tau(r_n^\tau, v)} > 0 \tag{77}$$

[12] Note that $\frac{M_\tau}{r_z - g} > 0 \Leftrightarrow \frac{h_3(1+r_z) - h_3(1+g)}{r_z - g} > 0$ is true, because $h_3(x) = x\frac{f\left(\frac{x}{1+\rho}, m\right)}{f\left(\frac{1}{x}, o\right)}$ increases with x.

[13] We continue to assume that excess borrowing in the presence o this social security system has a negative slope with respect to the real interest rate.

Social Security and inter-generations Altruism

We now inspect how intergenerational altruism is affected by a social security tax, an exogenous compulsory transfer from younger to older generations. If agents care for their parents wealth without any social security system in place it is expectable that any social security income would reduce the perceived level of help parents would need when old. Furthermore, if a social security system is sufficiently generous it is also expectable that elder agents become more motivated to help their children.

Let's first look to how a *pay-as-you-go* social security may affect backward altruism from agents to parents. The optimization problem is now given by agent's maximizing inter-generations altruism utility expression (52) subject to the budget constraints resulting from the direct combination of expressions (2),(3),(40),(41), with (67),(68),(69). The new excess borrowing expression that takes into account bequests and social security is an intuitive combination of the previous expressions, such that:

$$B^{\tau,b}(r^{\phi^b}, v, Q^b, \tau) = B(r^{\phi^b}, v) + \tau G^{\tau}(r^{\phi^b}, v) + Q^b G(r^{\phi^b}, v) = 0 \tag{78}$$

$$\text{and consequently, } Q^b = -\frac{B(r^{\phi^b}, v) + \tau G^{\tau}(r^{\phi^b}, v)}{G(r^{\phi^b}, v)} = \frac{B^{\tau}(r^{\phi^b}, v, \tau)}{G(r^{\phi^b}, v)} \geq 0. \tag{79}$$

Where Q^b is positive by assumption. Consequently,

$$Q^b > 0 \Leftrightarrow B^{\tau}(r^{\phi^b}, v, \tau) < 0 \Leftrightarrow \begin{cases} r_n^{\tau} < r^{\phi^b} \\ \tau < \tau^b = -\frac{B(r^{\phi^b}, v)}{G^{\tau}(r^{\phi^b}, v)} \Rightarrow B(r^{\phi^b}, v) < 0 \Rightarrow r_n < r^{\phi^b} \end{cases} \tag{80}$$

Children will care for their parents wealth when the social security tax is lower than the threshold τ^b. In that case the natural rate of interest is equal to r^{ϕ^b}, and the gift parameter Q^b from children to parents will change with the social security tax according to:

$$\frac{dQ^b}{d\tau} \equiv q_{\tau}^b = -\frac{G^{\tau}(r^{\phi^b}, v)}{G(r^{\phi^b}, v)} < 0 \tag{81}$$

A higher social security tax discourages transfers from children to parents, until a point when it starts encouraging bequests from parents to children. This is the mechanism we analyze next.

If agents care for their children wealth without any social security system in place it is expectable that any social security income further increases the propensity to help the next generation. The optimization problem is the same as before, and forward altruistic transfers from children to parents require that:

$$B^{\tau,f}(r^{\phi f}, v, Q^f, \tau) = B(r^{\phi f}, v) + \tau G^\tau(r^{\phi f}, v) - Q^f G(r^{\phi f}, v) = 0 \tag{82}$$

$$\text{and } Q^f = \frac{B(r^{\phi f}, v) + \tau G^\tau(r^{\phi f}, x)}{G(r^{\phi f}, v)} = \frac{B^\tau(r^{\phi f}, v, \tau)}{G(r^{\phi f}, v)} \geq 0 \tag{83}$$

Consequently,

$$Q^f > 0 \Leftrightarrow B^\tau(r^{\phi f}, v, \tau) > 0 \Leftrightarrow \begin{cases} r^{\phi f} < r_n^\tau \\ \tau > \tau^f = -\frac{B(r^{\phi f}, v)}{G^\tau(r^{\phi f}, v)} \Leftarrow B(r^{\phi f}, x) > 0 \Leftarrow r^{\phi f} < r_n \end{cases} \tag{84}$$

Parents will care for their children wealth when the social security tax is above a threshold τ^f. In that case bequest Q^f from agents to their children will change with the social security tax according to:

$$\frac{dQ^f}{d\tau} = q_\tau^f = \frac{G^\tau(r^{\phi f}, x)}{G(r^{\phi f}, x)} > 0 \tag{85}$$

An increase of the social security tax encourages forward altruism, which is also an intuitive result.

5 Quantitative calibration

In the previous sections we formalized algebraically the relation of changes of equilibrium real interest rates and evolving age milestones. Now we explain how those milestone changes may account for around one third of the real interest rates reduction in US in recent years, by calibrating our model with capital during the period between 1985 and 2005.

We start by formally deriving the model with capital. Then we parametrized it to match some initial conditions in the initial steady state, namely the real interest rate, the capital output ratio $\frac{K}{Y}$, and the bequests to output ratio, using explicit values for age milestones in 1985. We use this calibrated version to calculate the derivatives of the real interest rate with respect to each age milestone $\frac{\partial r}{\partial v_i}$, and check how those values vary with changes of other parameters and variables of the model. Finally we estimate how much of the real interest rate reduction between 1985 and 2005 is explained by the age milestone changes observed during the period between 1985 and 2005 using our model.

5.1 OLG model with endogenous output and capital

Formally the model with endogenous output and capital is derived in a similar way[14]. Although the derivatives of the real interest rate with respect to age milestones can be easily derived, we do not give in this section their algebraic representation.

The household maximization problem, without bequests an social security for now, has the same appearance has before, and is given by:

$$\max_{c_{t+i}^{b^l+i}} E_t \left\{ \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i}) \right\} \tag{86}$$

$$\text{s.t. } c_t^{b^l} = w_t^{b^l} l_t^{b^l} - a_t^{b^l} \tag{87}$$

$$c_{t+i}^{b^l+i} = w_{t+i}^{b^l+i} l_{t+i}^{b^l+i} + (1 + r_{t+i-1}) a_{t+i-1}^{b^l+i-1} - a_{t+i}^{b^l+i} \tag{88}$$

$$c_{t+T-1}^{o^h} = w_{t+T-1}^{o^h} l_{t+T-1}^{o^h} + (1 + r_{t+T-2}) a_{t+T-2}^{o^h-1} \tag{89}$$

where the same utility function is used, $U(c) = \frac{C^\sigma - 1}{1-\sigma}$. Assets $a_t = k_t - b_t$ are composed by capital k_t that households rent to firms, and loans to other households $-b_t$. The capital portion of assets is always positive but the loans to other households $-b_t$ can be positive or negative. In any case, agents are called borrowers when $a_t < 0$ and savers otherwise. While employed each household is given an exogenous annual labor endowment that is assumed to increase with work experience at a constant rate[15]. The expression for labor endowment at age i is given by:

$$l_t^{m^l+i} = l_t^{m^l} (1 + \rho)^i \tag{90}$$

Without loss of generality we assume that the labor endowment in the beginning of the working period $l^{m^l} = 1$. Regarding households' asset composition $a_t = k_t - b_t$, we can rewrite the budget constraints in terms of loans and capital:

$$c_{t+i}^{b^l+i} = w_{t+i}^{b^l+i} l_{t+i}^{b^l+i} + \left[(1-\delta) k_{t+i-1}^{b^l+i-1} + (1 + r_{t+i-1}^k) k_{t+i-1}^{b^l+i-1} - (1 + r_{t+i-1}) b_{t+i-1}^{b^l+i-1} \right] - \left[k_{t+i}^{b^l+i} - b_{t+i}^{b^l+i} \right] \tag{91}$$

where from the first order conditions of k_t and b_t, the *No-Arbitrage Condition (NAC)* is given by:

$$r_t^k = r_t + \delta \tag{92}$$

Similarly to the previous sections, solving the previous household optimization problem for

[14] Appendix D.

[15] We use this assumption to ensure algebraic tractability.

consumption, without the bequests and social security modules, is equivalent to solve:

$$\max_{c_{t+i}^{b^l+i}} E_t \left\{ \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i}) \right\} \tag{93}$$

$$\text{s.t. } \mathbb{C}_t^{b^l} = \mathbb{W}_t^{b^l} \tag{94}$$

where $\mathbb{W}_t^{b^l}$ is the present value of wages instead of endowments. By introducing bequests and/or social security in the model, the present value budget constraint becomes:

$$\mathbb{C}_t^{b^l} = \mathbb{W}_t^{b^l}(1 + M_Q - \tau M_\tau) \tag{95}$$

where M_Q and M_τ are the same as derived in the previous section, with expressions respectively given by equations (44) and (72). The loan market equilibrium condition is still given by excess borrowing $B_t^{b^l} = 0$, which in steady state has the following representation:

$$B_t^{b^l} = \begin{cases} \frac{1+r_{gz}}{r-r_{gz}}(W_t^{b^l} - C_t^{b^l}) + K_t^{b^l} & \text{for} \quad r \neq r_{gz} \\ \\ -(1+r_{gz})w_t \frac{\partial \frac{C^{b^l}}{w_t}}{\partial r} + K_t^{b^l} & \text{for} \quad r = r_{gz} \end{cases} \tag{96}$$

Where $1 + r_{gz} = (1+g)(1+z_\alpha)$, and $1 + z_\alpha = (1+z)^{\frac{1}{\alpha}}$. B^{b^l} is continuous and differential function for $r \in]-\delta, +\infty[$, from where the derivatives of the equilibrium real interest rate with respect to age milestones are also given by:

$$r_{v_i} \equiv \frac{\partial r}{\partial v_i} = -\frac{\frac{\partial B_t^l}{\partial v_i}}{\frac{\partial B_t^l}{\partial r}} \tag{97}$$

The model with capital with and without bequests and social security, is fully derived in appendix.

5.2 Quantifying derivatives of r with respect to age milestones

We start by parameterizing the initial steady state of the model with capital and bequests, as in table(1) for US in 1985. We use values for average life expectancy and retirement age from Knoema[16]. We ensure that the capital to output ratio $\frac{K}{Y} = \frac{1-\alpha}{r+\delta}$ is approximately[17] 3.

[16] We assume that the ages of adulthood and first job are respectively equal to 21 and 23. We could alternatively have assumed, with similar results, that the age of adulthood was 18 instead, and that those two initial age milestones coincide, which might change the calibration of β.

Table 1: Initial steady state parameters for 1985

Description	Parameter	Value
Emancipation	b^l	21.0
First job	m^l	23.0
Retirement	o^l	65.8
Life expectancy	d^L	74.6
Discount rate	β	0.978
Population growth	g	0.9%
TFP growth rate	z	0.0%
Age related productivity growth rate	ρ	0.0%
Intertemporal substitution	$\frac{1}{\sigma}$	0.5
Labor share	α	0.6
Depreciation rate	δ	0.1
Bequest parameter (warm glow)	ϕ	−0.6
Age children born	μ	25
Social security tax	τ	12.4%
Initial real interest rate	r	4.4%
Estate size/Output	Q/Y	2.2%
Capital to output ratio	K/Y	2.78

We use the bequest parameter ϕ to match the ratio $\frac{Q}{Y}$ around[18] 0.02. We do not consider a social security tax in the base case scenario of the model. We derive the value of β by solving $B_t^{b^l}(r = 4.4\%, v_i) = 0$, where 4.4% is the real interest rate corresponding to the initial steady state in 1985.

We then use the above parameters to calculate the derivatives of the real interest rate with respect to each age milestone using the expressions derived in previous sections, in several scenarios starting with an endowment economy and successively adding bequests, capital, and social security[19]. In table(2) we use the same parameters of our base case scenario, with endogenous output, capital, and a bequest motive, to calculate the derivatives of real interest rates with respect to age milestones, with and without capital, bequest and social security. Since all parameters besides ϕ, α, and τ are unchanged for all scenarios, the *natural rates of interest* r changes accordingly: r is higher when capital is introduced, lower when bequest are operative, and higher when social security is considered. Furthermore, we observe the following:

[17]Brinca et al. [6].

[18]Hendricks [18]

[19]Derivatives are calculated using each corresponding expressions for endowment economy scenarios, and compared with numerical estimations for all scenarios.

Table 2: Derivatives with same parametrization, $\beta = 0.978$: changing r

Scenario	Derivatives of r with respect to age b^l	m^l	o^l	L	r
E≡ Endowment economy	-0.85%	0.70%	0.64%	-0.48%	-0.93%
E+Q≡ Bequests	-0.89%	0.68%	0.68%	-0.48%	-1.62%
E+Q+τ ≡ Social security tax	-0.64%	0.58%	0.38%	-0.29%	2.34%
K	-0.62%	0.72%	0.30%	-0.35%	4.83%
Base case: K+Q	-0.62%	0.69%	0.31%	-0.33%	4.40%
K+Q+τ	-0.59%	0.69%	0.26%	-0.31%	6.42%

- The model can generate negative steady state equilibrium real interest rates, as can be observed in the Endowment Economy scenario of table(2).
- The signs of derivatives of real interest rates with respect to age milestones, r_{v_i}, are the ones expected: increasing life expectancy, and increasing age of adulthood have a negative impact on the real interest rate, respectively because savings expand, and borrowing contracts. Increasing the retirement age, as well as the age of first job, both impact positively the real interest rate, respectively because savings contract, and borrowing expands.
- When the capital weight in the model increases, an increase of the real interest rate ∂r has a greater expansion impact on loan supply, $S = -B$, to compensate for the contraction of capital, requiring bigger changes in age milestones to sustain loan market in equilibrium. This reduces, in absolute terms, the derivatives of real interest rates with respect to age milestones, when capital is introduced in the model.

In table(3) we calibrate all the scenarios to match the steady state natural rate of interest of the base case, by adjusting β. We observe that the derivatives corresponding to higher age milestones (retirement age, and life expectancy) become more similar across scenarios, and that in general the orders of magnitude do not change significantly.

We continue to test the robustness of the model in table (4), where we can also observe that the derivatives of the real interest rates with respect to age milestones, are generally less sensible to changes of the main parameters of the model, with the exception of the elasticity of inter-temporal substitution $\frac{1}{\sigma}$.

As we can observe in table(5), the constant relative risk aversion coefficient σ is a determinant factor driving the magnitude of the derivatives. Excess borrowing is more rigid for higher levels of σ, calling for a lower change of an age milestone in absolute terms, to

Table 3: Derivatives setting $r = 4.40\%$, adjusting β for equilibrium

Scenario	Derivatives of r with respect to age b^l	m^l	o^l	L	β
E≡ Endowment economy	−1.06%	0.99%	0.44%	−0.32%	0.944
E+Q≡ Bequests	−1.10%	1.00%	0.44%	−0.30%	0.942
E+Q+τ ≡ Social security tax	−0.75%	0.73%	0.35%	−0.27%	0.962
K	−0.58%	0.68%	0.31%	−0.36%	0.981
Base case: K+Q	−0.62%	0.69%	0.31%	−0.33%	0.978
K+Q+τ	−0.45%	0.54%	0.26%	−0.31%	0.996

Table 4: Robustness Analysis

parameter≡ x	∂x	$y = r_{v_i}(x+\delta x)/r_{v_i}(x)$ y_{b^l}	y_{m^l}	y_{o^l}	y_L
$\frac{1}{\sigma}$	0.5 → 1.0	0.4	0.4	0.5	0.5
g	0.9% → 0.0%	0.9	0.9	1.1	1.1
z	0.0% → 1.0%	1.3	1.3	1.0	1.0
ρ	0.0% → 1.0%	1.1	1.0	1.1	1.1
δ	0.1 → 0.2	1.2	1.1	1.4	1.2
α	0.6% → 0.7%	1.0	1.0	1.2	1.0
ϕ	−0.6 → 0.0	1.0	1.0	1.0	1.0
τ	0.0% → 12.4%	1.0	1.0	0.8	0.9

Table 5: Relative Risk Aversion

σ	Derivatives of r with respect to age b^l	m^l	o^l	L
0.5	-0.14%	0.15%	0.07%	-0.07%
1	-0.30%	0.31%	0.14%	-0.15%
2	-0.62%	0.69%	0.31%	-0.33%
3	-0.96%	1.19%	0.50%	-0.54%
4	-1.32%	1.91%	0.74%	-0.79%

Table 6: Simulation Results

Scenario	Age milestones				natural rate of interest: r			
	b^l	m^l	o^l	L	$\sigma = 1$	$\sigma^* = 2$	$\sigma = 3$	$\sigma = 4$
Base case: 1985	21.0	23.0	65.8	74.6	4.4%	4.4%	4.4%	4.4%
2005	21.0	23.0	64.6	77.5	3.7%	2.5%	1.2%	-0.3%
2015	23.0	25.0	64.6	79.0	3.5%	2.2%	0.9%	-0.6%
$\triangle(1985, 2005)$	0.0	0.0	-1.2	$+2.9$	-0.7%	-1.9%	-3.2%	-4.7%
$\triangle(1985, 2015)$	$+2.0$	$+2.0$	-1.2	$+4.4$	-0.9%	-2.2%	-3.5%	-5.0%

$\sigma^* = 2$ is the base case scenario.

compensate the same real interest rate change, in order to keep loan market in equilibrium. Consequently the derivative corresponding to a greater σ will be greater. In table(5) we observe a relation of quasi proportionality between real interest rate derivatives and σ: The elasticity of intertemporal substitution then seems a decisive factor when age structure is considered, as it can greatly influence the conclusions and quantitative results from a calibrated model. This becomes clear in table (6) where, in our base case scenario with $\sigma = 2$, changes in age milestones could explain around half of the real interest rate decline from 1985 to 2005/2015 in US, considering a real interest rate equal to 4.4% in 1985 and around zero in 2015[20]. We use observed values for average retirement age and life expectancy, as well population annual growth rate [21]. But if we calibrate the model using $\sigma = 1$, the same age milestone changes would account for around one fourth of the total real interest rate decline during the period in study. And if σ takes the value of 4 then the demographic changes during the same period would be able to explain the full real interest rate decline.

We can conclude with the observation that the constant relative risk aversion coefficient

[20]We assume that young agents postpone by two years their adulthood, first job, and first child in 2015, relativity to 2005.

[21]The reduction of population annual growth rate from 0.9% in 1985 to 0.7% in 2005 explains a reduction of the real interest rate of around -0.2%.

may work as an amplification factor of the impact of demographic changes on the *natural rate of interest*.

6 Final remarks

In this paper we formalized the relation between real interest rates and relevant age milestones of a person's life, using an overlapping multi-generations model where one generation correspond to one year.

Although the impact of age structure in relevant World economies has been a recurrent topic covered in recent literature, in particular to explain the persistent decline of interest rates, economic stagnation and liquidity traps, there has not yet been an attempt, to the best of our knowledge, to formally derive the analytic relations of real interest rates with respect age milestones. The general omission of changing demographic parameters in most current formal economic models ignores a potentially relevant factor influencing equilibrium conditions, and consequently the type and even sign of solutions. For example, an increase of life expectancy can drag the full-employment equilibrium real interest rate from positive to negative. Since a negative level may not be achievable when the nominal interest rate zero lower bound is binding, a first best solution may no more be available in such a model. The same can happen with operative bequest motives, which may become inoperative, for example if the retirement age decreases, or life expectancy increases. The purpose of this paper is to fill out this gap. By merging an age structure framework with an OLG model, we derive tractable algebraic real interest rate elasticity expressions with respect to each age parameter, to shed light on the demographic formal mechanisms that influence real interest rates, and inspect in particular the examples mentioned above. Moreover we provide a straightforward alternative to heavy computational quantitative models, in order to illustrate the impact of demographic factors on general economic phenomena.

The main underlying mechanism relating age milestones and real interest rates in our model relies on the relative place and duration of labor income with respect to life expectancy when agents smooth consumption. For example, a longer retirement period, resulting from a reduction of the retirement age or an increase of life expectancy, makes households save more, thus expanding the supply of loans which drags down the real interest rate that ensures equilibrium in the loan market. We also inspect how the exogenous parameters of the model, namely the elasticity of inter-temporal substitution, productivity growth, income growth path of households, and population growth, may amplify or mitigate the impact of age milestones changes on the natural rate of interest. In addition we inspected how inter-generation transfers are affected by age milestone changes. For example, why and how increasing life expectancy may reduce endogenous bequest levels, decrease the propensity to help children, or increase the willingness to help parents.

Laterally to our main contribution in this paper, the analytic formulation of interest rate changes with respect to age milestones, we also developed a tractable algebraic framework to solve overlapping multi-generations optimization problems with a demographic structure.

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A Proposition 1: Present Value and Aggregate Consumption

By expressing b_{t+T-2}^{T-1} in the third budget constraint (4) in terms of endowment and consumption, and recursively substituting the expressions for b_{t+i}^{i} we can derive the following equality between present values of expected households' consumption and income paths, beginning adulthood at time t:

$$\mathbb{C}_t^{b^l,d^L} = \mathbb{Y}_t^{b^l,d^L} \tag{A.1}$$

where $\mathbb{C}_t^{l,h}$ is the present value of expected future and present consumption of an agent with age l at time t, until age h at time $t+(l-h)$ - and the same for endowment - given by expressions:

$$X_t^{l,h} = \mathbb{E}_t \sum_{i=0}^{h-l} \frac{x_{t+i}^{l+i}}{(1+r)_t^i}, \text{ where } (1+r)_t^i = \prod_{j=0}^{i-1}(1+r_{t+j}) \tag{A.2}$$

We omit the present value upper bound when it is equal to d^L, or $\mathbb{X}_t^{l,h} \equiv \mathbb{X}_t^l$. Equation (A.1) is intuitive, and corresponds to a *present value* budget constraint derived from the previous ones, (2) to (4). It means that the present value of lifetime consumption of an household is equal to the present value of its endowment path. The borrowing/saving path of an household, or its loan path, is an enabler of its optimal consumption path, given its income path.

A more general version of the present value budget constraint, at any given age, is given by the following expression:

$$\mathbb{C}_t^{b^l+i} = \mathbb{Y}_t^{b^l+i} - (1+r_{t-1})b_{t-1}^{b^l+i-1} \tag{A.3}$$

This expression is also self-explanatory: The forward present value of consumption at time t is equal to the present value of the endowment path minus debt and interest costs from previous period (or plus savings and interest income from previous period). Equations (A.1) and (A.3) will be frequently used, in particular to derive expressions for equilibrium consumption, for which we previously derive from consumption *First Order Conditions* the consumption general Euler equation for this model, given by:

$$\mathbb{E}_t \frac{c_{t+1}^{b^l+i+1}}{1+r_t} = \beta_{r_t} c_t^{b^l+i} \tag{A.4}$$

where,

$$\beta_{r_t} = \beta^{\frac{1}{\sigma}}(1+r_t)^{\frac{1-\sigma}{\sigma}} \overset{(\sigma=1)}{=} \beta \tag{A.5}$$

By using the Euler equation (A.4) in the present value general expression given by (A.2), we get an expression for the present value of consumption dependent on household's effective consumption at time t and expected real interest path, given by:

$$\mathbb{C}_t^{b^l+i} = c_t^{b^l+i} \sum_{j=0}^{T-i-1} \prod_{k=0}^{j-1} \beta_{r_{t+k}} = c_t^{b^l+i} f(\beta_{r_t}, T-i) \tag{A.6}$$

where $f(x_t, n) = \sum_{j=0}^{n-1} \prod_{k=0}^{j-1} x_{t+k}$. Note that when x is constant $f(x, n) = \sum_{j=0}^{n-1} x^j = \frac{1-x^n}{1-x}$. This simplifies the algebra in the special case of a log utility function for $\sigma = 1$ and $\beta_r = \beta$, as well as when formulating expressions for steady state. Combining the previous equation (A.6) at the age of adulthood with the present value budget constraint given by (A.1), we can express consumption at the age of adulthood in terms of expected real interest rate and income paths:

$$c_t^{b^l} = \frac{\mathbb{Y}_t^{b^l}}{f(\beta_{r_t}, T)} \tag{A.7}$$

A useful expression when deriving aggregate consumption in steady state, later on.

Deriving $\mathbb{Y}$: Let households time related productivity be given by:

$$1 + z_t^{b^l+i} = \frac{y_{t+1}^{b^l+i}}{y_t^{b^l+i}} \tag{A.8}$$

Then a general expression for endowment present value at time t and its corresponding steady state version are given by:

$$\mathbb{Y}_t^{b^l} = \sum_{i=0}^{T-1} \frac{y_{t+i}^{b^l+i}}{\prod_{j=0}^{i-1}(1 + r_{t+b+j})} = \sum_{i=0}^{T-1} y_t^{b^l+i} \prod_{j=0}^{i-1} \frac{1 + z_{t+j}}{1 + r_{t+j}} \tag{A.9}$$

$$\text{In steady state: } \mathbb{Y}_t^{b^l} = \sum_{i=0}^{T-1} y_t^{b^l+i} \left(\frac{1+z}{1+r}\right)^i \tag{A.10}$$

For now we solve the model with a generic endowment path. Later on we will use a more specific expression for it, in order to find explicit expressions for the derivatives of the real interest rate w.r.t. age milestones.

We express household consumption at age $b^l + i$ in a similar way, by using expressions (A.3) and (A.6). It depends on the present value of income from age $b^l + i$ until the end of life, on expected real interest rate path, but now takes into account the real interest rate and net assets from previous year.

$$c_t^{b^l+i} = \frac{\mathbb{Y}_t^{b^l+i} - (1+r_{t-1})b_{t-1}^{b^l+i-1}}{f(\beta_{r_t}, T-i)} \tag{A.11}$$

The borrowing level of an agent with age $b^l + i$ at time t, $b_t^{b^l+i}$ can be determined recursively using budget constraints (2) and (3), and using consumption expression (A.11). A simplified expression using determined past and present consumption and endowment, with recursion on the second budget constraint (3) is given by:

$$b_t^{b^l+i} = \mathbb{C}_{-t}^{b^l, b^l+i} - \mathbb{Y}_{-t}^{b^l, b^l+i} \tag{A.12}$$

where $\mathbb{C}_{-t}^{l,h}$ is the present value of past and present consumption of an agent with age h at time t, from age l, and the same for endowment, given by expressions:

$$\mathbb{C}_{-t}^{l,h} = \mathbb{E}_t \sum_{j=0}^{h-l} \left(c_{t-j}^{h-j} \prod_{k=1}^{j} (1+r_{t-k}) \right) \text{ and } \mathbb{Y}_{-t}^{l,h} = \mathbb{E}_t \sum_{j=0}^{h-l} \left(y_{t-j}^{h-j} \prod_{k=1}^{j} (1+r_{t-k}) \right) \tag{A.13}$$

Equation (A.12) means that the borrowing level of an agent with age $b^l + i$ at time t is equal to the present values of past and present consumption and endowment levels since the beginning of his economic life, ate age b^l. The term $(1 + r_{t-1})b_{t-1}^{b^l+i-1}$ is then determined by past endowment and consumption paths of an household.

Those tools will be useful to derive the transition dynamics of the model, namely when age milestones are changed.

To summarize, we derived expressions for consumption and borrowing levels of households with any age $b^l + i$ at time t. Consumption $c_t^{b^l+i}$ is expressed in terms of future income, expected future real interest rate path, and loans from previous year. And borrowing level at age $b^l + i$ and time t, $b_t^{b^l+i}$, is expressed in terms of present and past consumption, income and interest rate levels. c_t^i and b_t^i are then completely determined given an expected real interest rate path that solves loan market equilibrium, or excess borrowing equal to zero at any time t.

Assuming that the size of each generation at time t with age b^l+i is N_t^i we define adulthood growth rate by $1 + g_t = \frac{N_t^{b^l}}{N_{t-1}^{b^l}} = \frac{N_t^{b^l}}{N_t^{b^l+1}}$. Equilibrium in the bond market is given by:

$$\forall t : B_t^{b^l} = 0 \tag{A.14}$$

where $B_t^{b^l}$ is excess borrowing normalized to the size of generation of age b^l:

$$B_t^{b^l} = \frac{1}{N_t^{b^l}} \sum_{i=0}^{T-1} N_t^{b^l+i} b_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{N_t^{b^l+i}}{N_t^{b^l}} b_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{b_t^{b^l+i}}{\prod_{k=0}^{i-1}(1+g_{t-k})} \tag{A.15}$$

Excess borrowing $B_t^{b^l}$ can be expressed in terms of aggregate endowment and consumption at time t, and excess borrowing at $t-1$, by replacing the second budget constraint (3) in (A.15):

$$B_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i} - y_t^{b^l+i} + (1+r_{t-1}) b_{t-1}^{b^l+i-1}}{\prod_{k=0}^{i-1}(1+g_{t-k})} = C_t^{b^l} - Y_t^{b^l} + \left(\frac{1+r_{t-1}}{1+g_t}\right) B_{t-1}^{b^l} \tag{A.16}$$

where aggregate endowment normalized to the size of generation b^l at time t is expressed by:

$$Y_t^{b^l} = \sum_{i=0}^{T-1} \frac{y_t^{b^l+i}}{\prod_{j=0}^{i-1}(1+g_{t-j})} \tag{A.17}$$

$$\text{steady state, } Y_t^{b^l} = \sum_{i=0}^{T-1} y_t^{b^l+i} \frac{1}{(1+g)^i} \tag{A.18}$$

Note that directly from endowment present value and aggregate expressions (A.10) and (A.18):

$$\frac{1+z}{1+r} = \frac{1}{1+g} \Leftrightarrow 1+r = (1+g)(1+z) = 1+r_{gz} \tag{A.19}$$

$$\Rightarrow \mathbb{Y}(r_{gz}) = Y \tag{A.20}$$

Moreover, aggregate consumption in period t normalized to the size of the youngest generation with age b^l is given by:

$$C_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i}}{\prod_{j=0}^{i-1}(1+g_{t-j})} \tag{A.21}$$

$$\text{in steady sate, } C_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i}}{(1+g)^i} \tag{A.22}$$

Note that in steady state, $c_t^{b^l+i}$ can be expressed in terms of $c_t^{b^l}$:

$$c_t^{b^l+i} = c_{t-i}^{b^l} \left[\beta_r(1+r)\right]^i \tag{A.23}$$

and that $c_{t-i}^{b^l}$ can be expressed in terms of $c_t^{b^l}$:

$$c_{t-i}^{b^l} = \frac{\mathbb{Y}_{t-i}^{b^l}}{f(\beta_r, T)} = \frac{1}{(1+z)^i}\frac{\mathbb{Y}_t^{b^l}}{f(\beta_r, T)} = \frac{c_t^{b^l}}{(1+z)^i} \tag{A.24}$$

$$\Rightarrow c_t^{b^l+i} = c_{t-i}^{b^l}\left[\beta_r(1+r)\right]^i = \tag{A.25}$$

$$= c_t^{b^l}\left(\frac{\beta_r(1+r)}{1+z}\right)^i \tag{A.26}$$

Then, in steady state:

$$C_t^{b^l} = \sum_{i=0}^{T-1}\frac{c_t^{b^l+i}}{(1+g)^i} = c_t^{b^l}\sum_{i=0}^{T-1}\left[\frac{\beta_r(1+r)}{(1+g)(1+z)}\right]^i = c_t^{b^l} f(\gamma_r, T) = \mathbb{Y}_t^{b^l}\frac{f(\gamma_r, T)}{f(\beta_r, T)} \tag{A.27}$$

where $\gamma_r = \frac{\beta_r(1+r)}{(1+g)(1+z)}$. Note that if $r = r_{gz}$ the $\gamma_r = \beta_r$. We have seen that $\mathbb{Y}(r_{gz}) = Y$, then in steady state:

$$C(r_{gz}) = Y \tag{A.28}$$

This is an important point in proof of the continuity of excess borrowing in proposition 2.

We define an equilibrium as a set of processes $\{c_t^i, b_t^i, r_t\}$ $\forall i \in [b^l, d^L]$ that solve (1) subject to (2),(3),(4),(A.4), and (A.14), given an exogenous process for $\{b_t^l, m_t^l, o_t^l, d_t^L, g_t, z_t, \rho_t^i\}$, $\forall t$. Inspired by Auerbach and Kotlikoff [2], the effect of changing age milestones at time t on existing cohorts is the same as if they were born again, behaving like members of a new generation but with a shorter life expectancy, and initial assets resulting from prior accumulation.

B Proposition 2: *Excess Borrowing* Steady State Properties

(i) The steady state expression for excess borrowing normalized to the size of the younger generation in the model, with age b^l, is derived directly from (A.16):

$$B_t^{b^l}(r, x) = \begin{cases} \frac{1+r_{gz}}{r-r_{gz}}(Y_t^{b^l} - C_t^{b^l}) & \text{for} \quad r \neq r_{gz} \\ -(1+r_{gz})\frac{\partial C^{b^l}}{\partial r} & \text{for} \quad r = r_{gz} \end{cases}, \text{ where } 1 + r_{gz} = (1+g)(1+z) \tag{B.1}$$

The lower line of the expression ensures that $B^{b^l}(r, x)$ is continuously differentiable for $r \in]-1, +\infty[$, in particular for $r = r_{gz}$.

Proof (i): in steady state $r_t = r$ and so on for g_t and z_t:

$$B_t = C_t - Y_t - (1+r_{t-1})B_{t-1} = C_t - Y_t - (1+r_{t-1})\frac{B_t}{(1+g)(1+z)} \tag{B.2}$$

$$\Leftrightarrow B\left[1 - \frac{1+r}{(1+g)(1+z)}\right] = C - Y \tag{B.3}$$

$$\Leftrightarrow B = \frac{1+r_{gz}}{r-r_{gz}}(Y-C)\blacksquare \tag{B.4}$$

Continuity and differentiability of excess borrowing when $r = r_{gz}$

Aggregate endowment is a constant function of r, and aggregate consumption is continuous and differentiable for $r \in]-1,+\infty[$. Then we just have to prove the continuity and differentiability of steady state excess borrowing for $r = r_{gz}$. For that we use equation (A.28), or $Y = C(r_{gz})$:

$$\lim_{r\to r_{gz}} B(r) = \lim_{r\to r_{gz}} \frac{1+r_{gz}}{r-r_{gz}}\left[Y - C(r)\right] \tag{B.5}$$

$$= -(1+r_{gz}) \lim_{r\to r_{gz}} \frac{C(r) - C(r_{gz})}{r - r_{gz}} \tag{B.6}$$

$$= -(1+r_{gz})\frac{\partial C}{\partial r}(r_{gz}) \tag{B.7}$$

Then $B(r_{gz}) = -(1+r_{gz})\frac{\partial C}{\partial r}(r_{gz})$ ensures the continuity of excess borrowing for $r \in]-1,+\infty[$ in particular for $r = r_{gz}$. Likewise, $B'(r_{gz}) = -(1+r_{gz})C''(r_{gz})$ ensures the differentiability of excess borrowingfor $r \in]1,+\infty[$ in particular for $r = r_{gz}\blacksquare$.

(ii) Existence and properties of a solution B=0: $B(r,x) = 0$ has at least one solution if the no endowment retirement period duration$\equiv$ o is lower than the elasticity of inter-temporal substitution times the model duration, or $\frac{1}{\sigma} > \frac{o}{T-1}$, and the duration of the initial no-endowment borrowing period$\equiv$ b is strictly lower than $T-1$. Moreover, $\frac{\partial B}{\partial r}(r) < 0$, at least for one solution r solving $B(r) = 0$.

Proof: We prove that if this condition is verified then excess borrowing tends to $+\infty$ when r tends to -1^+, and to $-\infty$ when r tends to $+\infty$. Then, because of the continuity and differentiability of excess borrowing there exists at least one solution r such that $B(r) = 0$, where $B'(r) \geq 0$:

With endowment levels equal to zero at the beginning and end of life, aggregate and present value expressions can be given by:

$$Y_t^{b^l} = \frac{1}{(1+g)^i} \sum_{i=0}^{m-1} \frac{y_t^{m^l+i}}{(1+g)^i} \tag{B.8}$$

$$\mathbb{Y}_t^{b^l} = \left(\frac{1+z}{1+r}\right)^b \sum_{i=0}^{m-1} y_t^{m^l+i} \left(\frac{1+z}{1+r}\right)^i \tag{B.9}$$

where b is the duration of the initial no-endowment period, m the duration of the period while $y^i > 0$, and m^l the age where the endowment period begins.

(i) $B(r = -1^+) > 0$? Because the term $\frac{1+r_{gz}}{r-rgz} < 0$ if $r = -1^+$, and Y is constant in terms of r, then it is sufficient that $C(-1^+) = \mathbb{Y}(-1^+)\frac{f(\gamma_{-1^+},T)}{f(\beta_{-1^+},T)} = +\infty$.If $\frac{1}{\sigma} > 2$ then $\beta_{r=-1^+} = \beta^{\frac{1}{\sigma}}(1+(-1^+))^{\frac{1}{\sigma}-1} = 0^+ \Rightarrow f(\beta_{-1^+},T) = 1 \Rightarrow C(-1^+) = +\infty \rightarrow B(-1^+) = +\infty$. If $\frac{1}{\sigma} < 2$ then $f(\beta_{-1^+},T) = +\infty$. $f(\beta_r,T) = \beta_r^{T-1} f(\beta_r^{-1},T)$. $f(\beta_{-1^+}^{-1},T) = 1 \Rightarrow f(\beta_{-1^+},T) \rightarrow \beta^{\frac{T-1}{\sigma}}(1+r)^{(T-1)(\frac{1}{\sigma}-1)} \Rightarrow C(-1^+) \rightarrow \frac{y_t^{b^l}(1+r)^{-b}(1+r)^{-(m-1)}}{\beta^{\frac{T-1}{\sigma}}(1+r)^{(T-1)(\frac{1}{\sigma}-1)}} = y_t^{b^l}\beta^{\frac{1-T}{\sigma}}(1+r)^{T-b-m-\frac{T-1}{\sigma}} = +\infty$ if $T-b-m-\frac{T-1}{\sigma} = o - \frac{T-1}{\sigma} < 0 \Leftrightarrow \frac{o}{T-1} < \frac{1}{\sigma}$.

(ii) $B(r = +\infty) < 0$? Because the term $\frac{1+r_{gz}}{r-rgz} > 0$ if $r = +\infty$, and Y is constant in terms of r, then it is sufficient that $C(+\infty) = \mathbb{Y}(+\infty)\frac{f(\gamma_{+\infty},T)}{f(\beta_{+\infty+},T)} = +\infty$. If $\frac{1}{\sigma} < 2$ then $f(\beta_{+\infty}) = 0 \Rightarrow f(\beta_{+\infty},T) = 1 \Rightarrow C(+\infty) \rightarrow \frac{(1+r)^{T-1}}{(1+r)^b} = +\infty$ if $T-1 > b$, a plausible assumption. If $\frac{1}{\sigma} > 2$ then $f(\beta_{+\infty},T) = +\infty$. $f(\beta_r,T) = \beta_r^{T-1} f(\beta_r^{-1},T)$. $f(\beta_{+\infty}^{-1},T) = 1 \Rightarrow f(\beta_{+\infty},T) \rightarrow \beta_{+\infty}^{T-1} \Rightarrow C(+\infty) \rightarrow y_t^{b^l}(1+r)^{-b}\frac{\beta_{+\infty}^{T-1}(1+r)^{T-1}}{\beta_{+\infty}^{T-1}} = +\infty$ if $T-1 > b$■

If the ratio of the retirement period to the total duration of the model is lower than the elasticity of inter-temporal substitution $\frac{1}{\sigma}$, and $T > b-1$, then there exists a steady state equilibrium solution for r, where $B(r) = 0$, and excess borrowing is a decreasing function of r around that solution.

(iii) *The derivative of the real interest rate r with respect to any parameter of the model x can be expressed by the following expression when loan market is in equilibrium:*

$$\frac{\partial r}{\partial x} \equiv r_x = -\frac{B_x}{B_r} = -\frac{Y_x - C_x}{Y_r - C_r} = -\frac{\log_x Y - \log_x C}{\log_r Y - \log_r C} \tag{B.10}$$

Proof: We use the partial derivatives of steady state excess borrowing with respect to r and to any parameter x, from the steady state version of expression (A.16), to formulate the derivative of the natural rate of interest with respect to parameter x (in particular an age milestone v^i), assuming equilibrium in the loan market in steady state, or $B^{b^l}(r,x) = 0$:

$$B(r,x) = 0 \Rightarrow r_x = \frac{dr}{dx}(r,x) = -\frac{\frac{\partial B}{\partial x}}{\frac{\partial B}{\partial r}}(r,x) \Leftrightarrow r_x = -\frac{B_x}{B_r}(r,x) \tag{B.11}$$

To inspect how the equilibrium real interest rate changes with respect to change of any exogenous parameter of the model x, we derive closed form expressions for $\frac{dr}{dx_i}(r, x_i, x_{-i})$, where r solves $B^{b^l}(r,x) = 0$ given $x \equiv (x_i, x_{-i}) \in \mathbb{R}^n$, where n is the number os exogenous parameters of the model, and $x_i \in V \equiv$ all exogenous parameters of the model, in particular the age milestones with the exception of s^l. We use the fact that excess borrowing must remain equal to zero after a change in x_i. In order words, we need to derive by how much the equilibrium real interest rate must change in order to compensate for the impact of a change in x^i on excess borrowing so that it remains constant, and equal zero given loan market equilibrium:

$$dB(r,x) = \frac{\partial B}{\partial v^i}dx^i + \frac{\partial B}{\partial r}dr = 0 \Leftrightarrow \tag{B.12}$$

$$r_{x^i}(r,x) = \frac{dr}{dx^i} = -\frac{\frac{\partial B}{\partial x_i}}{\frac{\partial B}{\partial r}} = -\frac{B_{x_i}}{B_r}(r,x), \text{ for } r \text{ such that } B(r,x) = 0 \tag{B.13}$$

Note that

$$B_x \equiv \frac{\partial B}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1+r_{gz}}{r-r_{gz}}\right)(Y - C) + \frac{1+r_{gz}}{r-r_{gz}}\left(\frac{\partial Y}{\partial x} - \frac{\partial C}{\partial x}\right) \tag{B.14}$$

In equilibrium $B = 0 \Rightarrow Y = C$, so the previous expression can be simplified to:

$$B_x = \frac{1+r_{gz}}{r-r_{gz}}\left(\frac{\partial Y}{\partial x} - \frac{\partial C}{\partial x}\right) = \frac{1+r_{gz}}{r-r_{gz}}Y\left(\frac{Y_x}{Y} - \frac{C}{Y}\frac{C_x}{C}\right) = \frac{1+r_{gz}}{r-r_{gz}}Y\left(\log_x Y - \log_x C\right) \tag{B.15}$$

$$\text{where } \log_x Y = \frac{\partial \log Y}{\partial x} = \frac{1}{Y}\frac{\partial Y}{\partial x} = \frac{Y_x}{Y} \Leftrightarrow Y_x = Y \log_x Y \tag{B.16}$$

And from where we can directly derive a simplified expression for the partial derivative of excess borrowing with respect to the real interest rate, in equilibrium, given by[22]:

$$B_r = \frac{1+r_{gz}}{r-r_{gz}}Y\left(Log_r Y - Log_r C\right) = -\frac{1+r_{gz}}{r-r_{gz}}Y Log_r C \tag{B.17}$$

[22]Because in this model aggregate endowment is independent of the real interest rate, $Log_r Y = 0$

By combining expressions (B.15) and (B.18) an expression for r_x is given by:

$$r_x \equiv \frac{\partial r}{\partial x} = -\frac{Bx}{B_r} = \frac{\log_x Y - \log_x C}{\log_r C} \tag{B.18}$$

C Proposition 3: Real Interest Rate Derivatives w.r.t age parameters

For a sufficiently generic households' endowment path expressed by:

$$y_t^i > 0 \text{ for } i \in [m^l, m^h] \tag{C.1}$$

$$y_t^i = 0 \text{ for } i \in [b^l, m^l[\cup]m^h, d^L] \tag{C.2}$$

where time and age related productivity growth rates, for $i \in [m^l, m^h]$, respectively given by $1 + z_t^i = \frac{y_{t+1}^i}{y_t^i}$, and $1 + \rho_t^i = \frac{y_t^{i+1}}{y_t^i}$, steady state expressions for aggregate endowment and consumption are respectively given by:

$$Y_t^{b^l} = \frac{y_t^{m^l}}{(1+g)^b} f\left(\frac{1+\rho}{1+g}, m\right) \tag{C.3}$$

$$C_t^{b^l} = \mathbb{Y}_t^{b^l} \frac{f(\gamma, T)}{f(\beta_r, T)} \tag{C.4}$$

and $\mathbb{Y}_t^{b^l}$ given by:

$$\mathbb{Y}_t^{b^l} = \frac{y_t^{m^l}}{(1+r_z)^b} f\left(\frac{1+\rho}{1+r_z}, m\right) \tag{C.5}$$

where periods duration are defined by $b = m^l - b^l$, $m = o^l - m^l$, where $o^l = m^l + 1$. In addition, the partial derivative of excess borrowing with respect to the real interest rate $B_r(r, v) \equiv B_r$ is the denominator common to the derivatives of the real interest rate with respect to all exogenous parameters of the model, where $Log_r C$ can be expressed by:

$$Log_r C = Log_r \mathbb{Y} + Log_r f(\gamma, T) - Log_r f(\beta_r, T) \tag{C.6}$$

where the algebraic expressions of each one of the terms are given by:

$$Log_r \mathbb{Y} = \frac{1}{1+r}\left(-b + \frac{m}{(1+r_{z\rho})^m - 1} - \frac{1}{r_{z\rho}}\right) \tag{C.7}$$

$$Log_r f(\gamma_r, T) = \frac{1}{1+r}\left(\frac{1}{\sigma}\right)\left(T\frac{\gamma_r^T}{\gamma_r^T - 1} - \frac{\gamma_r}{\gamma_r - 1}\right) \tag{C.8}$$

$$Log_r f(\beta_r, T) = \frac{1}{1+r}\left(\frac{1-\sigma}{\sigma}\right)\left(T\frac{\beta_r^T}{\beta_r^T - 1} - \frac{\beta_r}{\beta_r - 1}\right) = 0 \text{ for } \sigma = 1 \tag{C.9}$$

where $1 + r_{z\rho} = \frac{1+r}{(1+z)(1+\rho)}$. Now that we have closed-form expression for aggregate consumption and endowment, we start by expressing the derivatives of excess borrowing with respect to the durations d of the main periods of the model, $b = m^l - b^l$, $m = o^l - m^l$ and $T = L - b^l + 1$, (the retirement period is here a dependent variable), for algebraic simplification, and because we can express the partial derivatives of B and r w.r.t. age milestones as a linear combination of durations due to the above linear relation between those two sets of parameters:

$$B_{b^l} = B_b b_{b^l} + B_m m_{b^l} + B_T T_{b^l} = -B_b + 0 - B_T = -B_b - B_T \tag{C.10}$$

$$B_{m^l} = B^b - B^m \tag{C.11}$$

$$B_{o^l} = B^m \tag{C.12}$$

$$B_L = B_T \tag{C.13}$$

Same relations valid for r_{v_i}.

Deriving $\log_d Y$*, where* $d \in \{b, m, T\}$*:*

$$\log_d Y = \log_d \left[\frac{y^{m^l}}{(1+g)^b} f\left(\frac{1+\rho}{1+g}, m\right)\right] \tag{C.14}$$

$$= -b_d \log(1+g) + \log_d f\left(\frac{1+\rho}{1+g}, m\right) \tag{C.15}$$

$$\text{then } \log_b Y = -log(1+g) \tag{C.16}$$

$$\log_m Y = \log_m f\left(\frac{1+\rho}{1+g}, m\right) = H\left(\frac{1+\rho}{1+g}, m\right) \tag{C.17}$$

$$\text{and } \log_T Y = 0 \tag{C.18}$$

where $H(x, m) = \frac{h(x^{-m})}{m}$, and $h(y) = \frac{\log y}{y-1} = \frac{\log y - \log 1}{y-1} > 0$, naturally[23]. Note that $H_x \equiv \frac{\partial H}{\partial x} > 0$. Proof $H_x > 0$:

[23] $h(y) > 0$ by applying the *Mean Value Theorem*

(i) First we prove that $h'(y) < 0$: $h'(y) = \frac{1-\frac{1}{y}-\log y}{(y-1)^2} < 0 \Leftrightarrow 1 - \frac{1}{y} - \log y = g(y) < 0$. $g'(y) = \frac{1-y}{y^2} \Rightarrow g'(y) > 0$ if $y < 1$, $g'(y) < 0$ if $y > 1$, and $g'(1) = 0$. So $g(y)$,and $h'(y)$ are always strictly negative for $y \neq 1$, with a maximum equal to 0 for $y = 1$.

(ii) $H_x(x, m) = \frac{1}{m} h'(x^{-m})(x^{-m})' = -x^{-m-1} h'(x^{-m}) > 0 \forall m \in \mathbb{Z}$ ■

Deriving $\log_d C$*:*

$$\log_d C = \log_d \left[\mathbb{Y} \frac{f(\gamma_r, T)}{f(\beta_r, T)} \right] \tag{C.19}$$

$$= -b_d \log(1 + r_z) + \log_d f\left(\frac{1+\rho}{1+r_z}, m\right) + \log_d f(\gamma_r, T) - \log_d f(\beta_r, T) \tag{C.20}$$

$$\text{then } \log_b C = -log(1 + r_z) \tag{C.21}$$

$$\log_m C = H\left(\frac{1+\rho}{1+r_z}, m\right) \tag{C.22}$$

$$\text{and } \log_T C = H(\gamma_r, T) - H(\beta_r, T) \tag{C.23}$$

Let $\triangle f(x, y) = \frac{f(x)-f(y)}{x-y}$,and note that $f' > 0 \Rightarrow \triangle f > 0$, directly from the *Mean Value Theorem*. For notation purposes let $H^m(x) \equiv H(x, m)$, $H^m_x \equiv H_x(x, m)$, and $\triangle H^m(x, y) = \frac{H^m(x)-H^m(y)}{x-y} > 0 \forall m \in \mathbb{Z}$, since $H^m_x > 0$. The derivatives of excess borrowing w.r.t. periods duration d are given by:

$$B_b = Y \frac{1+r_{gz}}{r-r_{gz}} (\log_b Y - \log_b C) = Y(1 + r_{gz}) \triangle \log(1 + r, 1 + r_{gz}) \tag{C.24}$$

$$= Y \beta_r \triangle \log(\gamma_r, \beta_r) > 0 \tag{C.25}$$

$$B_m = Y \frac{1+\rho}{1+r_z} \triangle H^m \left(\frac{1+\rho}{1+g}, \frac{1+\rho}{1+r_z}\right) > 0 \tag{C.26}$$

$$B_T = -Y \beta_r \triangle H^T(\gamma_r, \beta_r) < 0 \tag{C.27}$$

$$\tag{C.28}$$

Derivatives of Excess Borrowing w.r.t. age milestones:

$$B_{b^l} = -B_b - B_t = Y \beta_r \left[\triangle H^T(\gamma_r, \beta_r) - \triangle \log(\gamma_r, \beta_r)\right] \tag{C.29}$$

$$= -Y \beta_r \triangle H^{-T}(\gamma_r, \beta_r) < 0 \tag{C.30}$$

$$B_{m^l} = B_b - B_m = Y \frac{1+\rho}{1+r_z} \triangle H^{-m} \left(\frac{1+\rho}{1+g}, \frac{1+\rho}{1+r_z}\right) > 0 \tag{C.31}$$

$$B_{o^l} = B_m > 0 \tag{C.32}$$

$$B_L = B_T < 0 \tag{C.33}$$

Derivatives r with respect to durations:

$$r_b = \frac{\log_b Y - \log_b C}{\log_r C} = \frac{1}{\log_r C} \log\left(\frac{1+r}{1+r_{gz}}\right) = \frac{1}{\log_r C}\left(\log(\gamma_r) - \log(\beta_r)\right) > 0 \tag{C.34}$$

$$r_m = \frac{\log_m Y - \log_m C}{\log_r C} = \frac{1}{\log_r C}\left[H^T\left(\frac{1+\rho}{1+g}\right) - H^T\left(\frac{1+\rho}{1+r_z}\right)\right] > 0 \tag{C.35}$$

$$r_T = \frac{\log_T Y - \log_T C}{\log_r C} = -\frac{1}{\log_r C}\left[H^T(\gamma_r) - H^T(\beta_r)\right] < 0 \tag{C.36}$$

Derivatives of r with respect to age milestones, and their sign, assuming that $B_r < 0$:

$$r_{b^l} = -r_b - r_T = -\frac{1}{\log_r C}\left[H^{-T}(\gamma_r) - H^{-T}(\beta_r)\right] < 0 \tag{C.37}$$

$$r_{m^l} = r_b - r_m = \frac{1}{\log_r C}\left[H^{-m}\left(\frac{1+\rho}{1+g}\right) - H^{-m}\left(\frac{1+\rho}{1+r_z}\right)\right] > 0 \tag{C.38}$$

$$r_{o^l} = r_m = \frac{1}{\log_r C}\left[H^{m}\left(\frac{1+\rho}{1+g}\right) - H^{m}\left(\frac{1+\rho}{1+r_z}\right)\right] > 0 \tag{C.39}$$

$$r_L = r_T = -\frac{1}{\log_r C}\left[H^T(\gamma_r) - H^T(\beta_r)\right] < 0 \tag{C.40}$$

D OLG model with endogenous output and capital:

Household problem

The same household objective function, and the budget constraints where wages and capital are introduced have expressions given by:

$$\max_{c_{t+i}^{b^l+i}} E_t\left\{\sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i})\right\} \tag{D.1}$$

$$\text{s.t. } c_t^{b^l} = w_t^{b^l} l_t^{b^l} - a_t^{b^l} \tag{D.2}$$

$$c_{t+i}^{b^l+i} = w_{t+i}^{b^l+i} l_{t+i}^{b^l+i} + (1+r_{t+i-1})a_{t+i-1}^{b^l+i-1} - a_{t+i}^{b^l+i} \tag{D.3}$$

$$c_{t+T-1}^{o^h} = w_{t+T-1}^{o^h} l_{t+T-1}^{o^h} + (1+r_{t+T-2})a_{t+T-2}^{o^h-1} \tag{D.4}$$

where utility of consumption $U(c) = \frac{C^\sigma - 1}{1-\sigma}$, and elasticity of inter-temporal substitution $EIS = \frac{1}{\sigma}$. Assets $a_t = k_t - b_t$ are composed by capital k_t that households rent to firms, and loans to other households $-b_t$. The capital portion of assets is always positive but the loans to other households $-b_t$ can be positive or negative. In any case, agents are called borrowers when $a_t < 0$ and savers otherwise. While employed each household is given an exogenous annual labor endowment that is assumed to increase with work experience at a stable rate[24]. The expression for labor endowment at age i is given by:

$$l_t^{m^l+i} = l_t^{m^l}(1+\rho)^i \tag{D.5}$$

Without loss of generality we assume that the labor endowment in the beginning of the working period $l^{m^l} = 1$.

Regarding households' asset composition $a_t = k_t - b_t$, we can rewrite the budget constraints in terms of loans and capital:

$$c_{t+i}^{b^l+i} = w_{t+i}^{b^l+i} l_{t+i}^{b^l+i} + \left[(1-\delta)k_{t+i-1}^{b^l+i-1} + r_{t+i-1}^k k_{t+i-1}^{b^l+i-1} - (1+r_{t+i-1})b_{t+i-1}^{b^l+i-1}\right] - \left[k_{t+i}^{b^l+i} - b_{t+i}^{b^l+i}\right] \tag{D.6}$$

From the first order conditions of k_t and b_t the *No-Arbitrage Condition (NAC)* is given by:

$$r_t^k = r_t + \delta \tag{D.7}$$

Equilibrium expressions for consumption and asset supply are derived as in section 2. From households' budget constraints with capital we derive a present value household budget constraint, now given by:

$$\mathbb{C}_t^{b^l+i} = \mathbb{W}_t^{b^l+i} + (1+r_{t-1})a_{t-1}^{b^l+i-1} \tag{D.8}$$

$$\text{where } \mathbb{W}_t^{b^l+i} = \mathbb{E}_t \sum_{j=0}^{m^h-i} \frac{w_{t+j} l_{t+j}^{b^l+j}}{\prod_{k=0}^{j-1}(1+r_{t+k})} \tag{D.9}$$

which means that the present value of consumption from age i until life expectancy L^e is equal to the present value of labor income during the same period plus the assets and respective interests from the previous age $i-1$. In the model with capital endowment y_t^i is replaced by labor income $w_t l_t^i$. From the same Euler equation (D.10) we find also the same expression for consumption of an household at time t and age b^l+i as a function of its consumption present value until L^e:

$$\mathbb{E}_t \frac{c_{t+1}^{b^l+i+1}}{1+r_t} = \beta_{r_t} c_t^{b^l+i} \Rightarrow \tag{D.10}$$

$$\Rightarrow \mathbb{C}_t^{b^l+i} = c_t^{b^l+i} f(\beta_{r_t}, T-i) \tag{D.11}$$

where $f(\beta_{r_t}, T-i) = \mathbb{E}_t \sum_{j=0}^{T-i-1} \prod_{k=1}^{j} \beta_{r_{t+k-1}}$. The equilibrium expressions for household

[24] We use this assumption to ensure algebra tractability.

consumption and assets have the same expressions as in section 2 where lending to other households $-b_t^i$ that can be positive or negative is replaced by assets a_t^i. For $i \neq 0$:

$$c_t^{b^l+i} = \frac{1}{f(\beta_{r_t}, T-i)} \left[\mathbb{W}_t^i + (1+r_{t-1})a_{t-1}^{i-1}\right] \tag{D.12}$$

$$a_t^{b^l+i} = w_t l_t^{b^l+i} - c_t^{b^l+i} + (1+r_{t-1})a_{t-1}^{b^l+i-1} \tag{D.13}$$

In the beginning of model's duration period, at age b^l, or for $i=0$:

$$c_t^{b^l} = \frac{\mathbb{W}_t^{b^l}}{f(\beta_r, T)} \tag{D.14}$$

$$a_t^{b^l} = w_t l_t^{b^l} - c_t^{b^l} \tag{D.15}$$

Firms problem:

Firms are perfectly competitive and take prices are given. They hire labor and rent capital to maximize profits. Their profit maximization problem is given by:

$$\max_{y_t, k_t} y_t - w_t l_t - r_t^k k_t \tag{D.16}$$

$$\text{s.t. } y_t = Z_t l_t^{\alpha} k_t^{1-\alpha} \text{ ,where } Z_t = (1+z)Z_{t-1} \tag{D.17}$$

Demand for labor and capital in equilibrium is respectively given by:

$$l_t = \alpha \frac{y_t}{w_t} \tag{D.18}$$

$$k_t = (1-\alpha)\frac{y_t}{r_t^k} \tag{D.19}$$

By replacing the previous demand expressions in the production function we derive equilibrium expressions for output and wages at time t:

$$y_t = Z_t l_t^{\alpha} k_t^{1-\alpha} = l_t Z_t^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{r_t^k}\right)^{\frac{1-\alpha}{\alpha}} \tag{D.20}$$

$$w_t = \alpha \frac{y_t}{l_t} = \alpha Z_t^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{r_t^k}\right)^{\frac{1-\alpha}{\alpha}} \tag{D.21}$$

Equilibrium in the asset market

Same loan market equilibrium condition:

$$\forall t, L_t^T = B_t = 0 \Leftrightarrow A_t = K_t \tag{D.22}$$

$$\text{where } A_t^T = \frac{1}{N_t^{b^l}} \sum_{i=0}^{T-1} N_t^{b^l+i} a_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{N_t^{b^l+i}}{N_t^{b^l}} a_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{a_t^{b^l+i}}{\prod_{j=0}^{i-1}(1+g_{t-j})} \tag{D.23}$$

$$K_t = (1-\alpha)\frac{Y_t}{r_t^k} = \frac{1-\alpha}{\alpha}\frac{W_t}{r_t+\delta} \tag{D.24}$$

$$W_t = w_t L_t = \frac{w_t l_t^{m^l}}{\prod_{i=0}^{b-1}(1+g_{t-i})} \sum_{i=0}^{m-1} \prod_{j=0}^{i-1} \frac{1+\rho^{b^l+j}}{1+g_{t-j}} \tag{D.25}$$

An equilibrium is defined in the same way as in section 2, as a set of processes $\{c_t^i, a_t^i, r_t\}$ that solve (D.2),(D.3),(D.4),(D.10), and (D.22) $\forall i$, given an exogenous process for $\{g_t, b_t^l, m_t^l, o_t^l, L_t\}$.

Relevant expressions in steady state

Previous expressions for steady state equilibrium are algebraically more tractable. Aggregate borrowing is continuous and differentiable for $r \in]-\delta, +\infty[$:

$$B_t^{b^l} = \begin{cases} \frac{1+r_{gz}}{r-r_{gz}}(W_t^{b^l} - C_t^{b^l}) + K_t^{b^l} & \text{for} \quad r \neq r_{gz} \\ \\ -(1+r_{gz})w_t(r_{gz})\frac{\partial \frac{C^{b^l}}{w_t}}{\partial r}(r_{gz}) + K_t^{b^l} & \text{for} \quad r = r_{gz} \end{cases} \tag{D.26}$$

Where $1 + r_{gz} = (1+g)(1+z_\alpha)$, and $1 + z_\alpha = (1+z)^{\frac{1}{\alpha}}$. Aggregate steady state expressions of consumption , labor income and capital, normalized to the population size of the model's youngest generation and the total factor productivity at time t are given by:

$$C_t^{b^l} = w_t l_t^{m^l} \left(\frac{1}{1+r_z}\right)^b f\left(\frac{1+\rho}{1+r_z}, m\right) \frac{f(\gamma, T)}{f(\beta_r, T)} \tag{D.27}$$

$$W_t^{b^l} = w_t l_t^{m^l} \left(\frac{1}{1+g}\right)^b f\left(\frac{1+\rho}{1+g}, m\right) \tag{D.28}$$

$$K_t^{b^l} = \frac{1-\alpha}{\alpha}\frac{W_t^{b^l}}{r+\delta} = w_t l_t^{m^l} \left(\frac{1}{1+g}\right)^b f\left(\frac{1+\rho}{1+g}, m\right) \frac{1-\alpha}{\alpha}\frac{1}{r+\delta} \tag{D.29}$$

Deriving aggregate consumption with capital

Aggregate consumption with capital is derived in the same way as in section 3, only replacing the present value of the endowment path $\mathbb{Y}_t^{b^l}$ by the present value of labor income path $\mathbb{W}_t^{b^l}$:

$$C_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i}}{\prod_{j=0}^{i-1}(1+g_{t-j})} \tag{D.30}$$

Note that in steady state the consumption of and agent of age i at time t is given by the following expression:

$$c_t^{b^l+i} = c_{t-i}^{b^l}\left[\beta_r(1+r)\right]^i = c_t^{b^l}\left[\frac{\beta_r(1+r)}{1+z_\alpha}\right]^i = c_t^{b^l}\gamma^i \tag{D.31}$$

Note that $c_{t-i}^{b^l} = \frac{\mathbb{W}_{t-i}^{b^l}}{f(\beta_r,T)} = \frac{1}{(1+z_\alpha)^i}\frac{\mathbb{W}_t^{b^l}}{f(\beta_r,T)} = \frac{c_t^{b^l}}{(1+z_\alpha)^i}$. Then:

$$C_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i}}{(1+g)^i} = c_t^{b^l}\sum_{i=0}^{T-1}\left[\frac{\beta_r(1+r)}{(1+g)(1+z_\alpha)}\right]^i = c_t^{b^l} f(\gamma,T) = \mathbb{W}_t^{b^l}\frac{f(\gamma,T)}{f(\beta_r,T)} \tag{D.32}$$

where $\gamma = \frac{\beta_r(1+r)}{(1+g)(1+z_\alpha)}$. And where $\mathbb{W}_t^{b^l}$ is the steady state expression for the present value of an household labor income path at time t:

$$\mathbb{W}_t^{b^l} = \left(\frac{1+z_\alpha}{1+r}\right)^b \sum_{i=0}^{m-1}\frac{w_t^{m^l+i} l_t^{m^l+i}}{(1+r)^i} = w_t^{m^l} l_t^{m^l}\left(\frac{1}{1+r_z}\right)^b f\left(\frac{1+\rho}{1+r_z}, m\right) \tag{D.33}$$

where $1+r_z = \frac{1+r}{1+z_\alpha}$.

Continuity of excess borrowing when $r = r_{gz}$

If $r = r_{gz}$ then $\beta_r = \gamma$, and $1 + r_z = 1 + g$. Consequently, and directly from expressions (D.27) and (D.28), if $r = r_{gz}$ then $C^{b^l}(r) = W^{b^l}(r)$. $\frac{C^{b^l}}{w_t}(r_{gz}) = \frac{W^{b^l}}{w_t}$, where $\frac{W^{b^l}}{w_t} = l_t^{m^l}\left(\frac{1}{1+g}\right)^b f\left(\frac{1+\rho}{1+g}, m\right)$ is independent of the real interest rate r. Aggregate borrowing can be alternatively expressed, for $r \neq r_{gz}$ as:

$$B_t^{b^l} = \frac{1+r_{gz}}{r-r_{gz}}(W_t^{b^l} - C_t^{b^l}) + K_t^{b^l} \tag{D.34}$$

$$= K_t^{b^l} - (1+r_{gz})w_t(r_{gz})\frac{\frac{C^{b^l}}{w_t}(r) - \frac{C^{b^l}}{w_t}(r_{gz})}{r-r_{gz}} \tag{D.35}$$

Since $\frac{C^{b^l}}{w_t}(r) = l_t^{m^l}\left(\frac{1}{1+r_z}\right)^b f\left(\frac{1+\rho}{1+r_z}, m\right)\frac{f(\gamma,T)}{f(\beta_r,T)}$ is differentiable in $r \in]-\delta, +\infty[$ the following limit exists:

$$\lim_{r \to r_{gz}} B^{b^l}(r) = K^{b^l}(r_{gz}) - (1 + r_{gz}) w_t(r_{gz}) \lim_{r \to r_{gz}} \frac{\frac{C^{b^l}}{w_t}(r) - \frac{C^{b^l}}{w_t}(r_{gz})}{r - r_{gz}} \tag{D.36}$$

$$= K^{b^l}(r_{gz}) - (1 + r_{gz}) w_t(r_{gz}) \frac{\partial \frac{C^{b^l}}{w_t}}{\partial r}(r_{gz}) \tag{D.37}$$

The steady state natural rate of interest r_n solves $B(r_n) = 0$. The properties of $B(r)$ and the derivatives of the natural rate of interest with respect to age milestones are derived using the same methodology as in section 3.

The third point of proposition 3 has the following adjusted version:

(ii) Existence and properties of a solution B=0: $B(r, x) = 0$ *has at least one solution, if the duration of the initial no-endowment borrowing period*$\equiv b$ *is strictly lower than* $T - 1$. *Moreover,* $\frac{\partial B}{\partial r}(r) < 0$, *at least for one solution* r *solving* $B(r) = 0$.

Proof: We prove that excess borrowing tends to $+\infty$ when r tends to $-\delta^{+}$, and to $-\infty$ when r tends to $+\infty$ if $b < T - 1$. Then, because of the continuity and differentiability of excess borrowing there exists at least one solution r such that $B(r) = 0$, where $B'(r) \geq 0$: