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11 August 2015

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January 19, 2016

Abstract

The monetary authority loses the ability to implement the Taylor Rule at the zero lower bound. However, the promise to implement a Taylor Rule upon exit remains an effective policy instrument. We show that a Taylor Rule, with an optimally-chosen exit date and time varying inflation target, delivers fully optimal policy at the ZLB. Additionally, a Taylor Rule with only an optimally chosen exit date but a zero inflation target delivers almost all the welfare gains of optimal policy and is simpler to communicate.

JEL Classification: E63, E52, E5

Keywords: New-Keynesian Model, Inflation Target, Liquidity Trap

*The authors would like to thank Klaus Adam, Carl Walsh and seminar participants at Ryerson University, University of California - Santa Cruz, Louisiana State University and the National Institute of Public Finance and Policy, New Delhi, India for helpful comments on earlier versions of the paper.

1 Introduction

Once the nominal interest rate reaches the zero lower bound (ZLB), monetary policy looses the ability to stimulate the economy by further reducing the nominal interest rate. Yet, the monetary authority retains the ability to stimulate by promising a path for future interest rates which can raise expected inflation, thereby reducing the current real interest rate. Conventional monetary policy, defined as setting current and future short-term interest rates, retains a role at the ZLB when the monetary authority is willing to announce "forward guidance" for short-term rates.

In the standard New Keynesian model, monetary policy is characterized by a Taylor Rule, whereby the nominal interest rate is set to equal a target, comprised of the sum of targets for the real interest rate and inflation, and to respond strongly to deviations of inflation and output from their respective targets. Woodford (2003, p. 287) argues that when all shocks are to demand, a Taylor Rule with a time-varying interest rate target equal to the natural rate, implements optimal monetary policy. Setting the nominal interest rate equal to the natural rate assures that both the output gap and inflation are zero. The strong response of the interest rate to deviations of inflation and output from their targets eliminates sunspot equilibria, thereby assuring that the equilibrium is locally unique.

The monetary authority cannot set the nominal interest rate equal to the natural rate, as required by Woodford's implementation of optimal monetary policy with the Taylor Rule, when the natural rate is negative. We show that there is a Taylor-Rule policy for exiting the ZLB, which can implement optimal monetary policy at the ZLB. The monetary authority must make two changes to Woodford's Taylor Rule. First, it must announce the first date on which the Taylor Rule applies, an exit date, setting the nominal interest rate to zero until that date. Second, the monetary authority modifies the Taylor Rule with an inflation target which declines at a fixed rate after the exit date.[1] This Taylor-Rule exit policy differs from a "truncated" version of Woodford's Taylor Rule on two counts. First, exit is postponed beyond the date on which the natural rate first becomes positive; second, exit occurs at a non-zero inflation target.

We show that when the policy parameters are chosen optimally, commitment to the optimal Taylor-Rule exit policy implements optimal monetary policy at the ZLB. The

[1] There is empirical evidence supporting the hypothesis that actual monetary policy has operated with a time-varying inflation target in the Taylor Rule. Ireland (2007) argues that US inflation can be explained by a New Keynesian model with a Taylor Rule only if the inflation target is allowed to vary over time. Additionally, Kozicki and Tinsley (2001), Rudebusch and Wu (2004) and Gurkaynak, Sack and Swanson (2005) provide evidence of a time-varying short-run inflation target for the US.

postponed exit date provides stimulus since the interest rate will be kept at zero even after the natural rate becomes positive. The optimal inflation target is negative, allowing the monetary authority to smooth the squared deviations of the output gap and inflation, responsible for welfare, reducing the large early deviations at the expense of creating small negative deviations later. The welfare gains over a truncated Taylor Rule can be large. Using our benchmark parameter values, over a range of adverse shocks for which the initial natural interest rate varies between -0.58% to -4.97% at an annual rate, loss with the truncated Taylor Rule ranges from 2.5 to 7 times the loss under optimal policy. However, the optimal Taylor-Rule exit policy requires commitment to future deflation and recession, a requirement which could prove difficult politically. And the policy requires communicating both an exit time and the path for an inflation target upon exit, communication which could be complicated.

We also consider an alternative Taylor-Rule exit policy in which the monetary authority commits only to a particular exit time in the future, with this exit time chosen optimally, subject to a zero inflation target. We find that this T-only policy achieves almost all of the welfare gains of moving from a truncated Taylor Rule to the optimal Taylor-Rule exit policy. Additionally, communication of this policy is particularly simple, requiring announcement of the exit date, upon which the monetary authority will return to letting the nominal rate follow the natural rate. These results justify the US Federal Reserve policy of announcing that the nominal interest rate would be fixed near zero for a "considerable period" of time, without any additional announcement of future recession or deflation.

Our paper is related to other papers which address monetary policy at the ZLB. Adam and Billi (2006, 2007) and Nakov (2008) have analyzed optimal policy under discretion and under commitment when autoregressive demand shocks yield the possibility of the ZLB. They do not explicitly consider implementation, communication, or the Taylor Rule. Cochrane (2013) shows that the discretionary commitment to exit the ZLB with zero values for inflation and the output gap yields a unique equilibrium at the ZLB. But, he also argues that if the policy maker could commit to exit the ZLB at different values for inflation and the output gap, this could yield a preferable equilibrium during the ZLB. Krugman (1998), Eggertson and Woodford (2003), Adam and Billi (2006), and Nakov (2008) demonstrate that optimal monetary policy with commitment relies on an increase in inflationary expectations to leave the ZLB. Levin, Lopez-Salido, Nelson, and Yun (2009) argue that, when the shock sending the economy to the ZLB is large and persistent, the stimulus, which conventional monetary policy can provide at the ZLB, is not sufficient to prevent a sizeable recession.

These policies work within the confines of a simple New Keynesian model, in which the effects of monetary policy are transmitted through the real interest rate. Much of the literature on monetary policy in a liquidity trap expands policy to unconventional methods, which are effective to the extent that financial-market arbitrage is imperfect, that the monetary authority assumes risk on its balance sheet, and/or the quantity of money has an effect on the economy independent of its effect on the real interest rate. These policies are interesting and potentially useful, but the simple New Keynesian model is not complex enough to provide a role for them.[2] In a similar context, Williamson (2010) argues that there is no ZLB, in the sense that the monetary authority can always find some stimulative instrument. This instrument can be unconventional monetary policy, but we argue that it can also be a commitment to a Taylor-Rule exit policy.

Additionally, Christiano, Eichenbaum, and Rebelo (2009), Woodford (2011), Werning (2012), Erceg and Linde (2014), among others, have considered the implications of using fiscal policy when monetary policy loses its effectiveness.[3] Understanding the effectiveness of fiscal policy at the ZLB, together with its interactions with conventional and unconventional monetary policy is interesting and important, but is not the subject of this paper. Our focus is more narrow – what can the monetary authority do in the absence of fiscal cooperation in the stimulus effort?

This paper is organized as follows. Section 2 presents the simple New Keynesian model with a Taylor Rule for monetary policy. The next sections provide solutions under certainty. Section 3 provides the solution with commitment to the optimal Taylor-Rule exit policy. Section 4 provides the solution with commitment to optimal monetary policy, and Section 5 shows that, with parameter values optimally chosen, the Taylor-Rule exit policy implements optimal monetary policy. Section 6 solves numerically for optimal values of the exit time and inflation target upon exit for the optimal Taylor-Rule exit policy. Section 7 solves the model under the T-only policy. Section 8 extends the results to uncertainty and Section 9 concludes.

[2] Examples of unconventional monetary policy include Auerbach and Obstfeld (2004), Blinder (2000, 2010), Bernanke (2002), Bernanke and Reinhart (2004), Bernanke, Reinhart and Sack (2004), Clouse *et.al.* (2003) and Gurkaynak, Sack and Swanson (2004,2005).

[3] Some unconventional monetary policies are arguable fiscal policies.

2 Simple New Keynesian Model with Taylor Rule

Following Woodford (2003) and Walsh (2010), we represent the simple standard linearized New Keynesian model as an IS curve, derived from the Euler Equation of the representative agent, and a Phillips Curve, derived from a model of Calvo pricing (Calvo, 1983). The linearization is about an equilibrium with a long-run inflation rate of zero.[4]

$$y_t = E_t\left(y_{t+1}\right) - \sigma\left[i_t - r_t^n - E_t \pi_{t+1}\right] \tag{1}$$

$$\pi_t = \beta E_t\left(\pi_{t+1}\right) + \kappa y_t. \tag{2}$$

In these equations y_t denotes the output gap; inflation (π_t) is the deviation about a long-run value of zero; i_t denotes the nominal interest rate, with a long-run equilibrium value of $r = \frac{1-\beta}{\beta}$, where r is defined as the long-run real interest rate and r_t^n as the natural rate of interest; σ represents the intertemporal elasticity of substitution with $\sigma \geq 1$; κ represents the degree of price stickiness;[5] $\beta \in (0,1)$ denotes the discount factor. The natural rate of interest embodies the combination of the long-run natural rate together with shocks associated with preferences, technology, fiscal policy, etc. Following Woodford (2003, Chapter 4), we do not add an independent shock to inflation in the Phillips Curve.[6] This restricts the analysis to the case where monetary policy faces no trade-off between inflation and the output gap.

We assume that, if the economy has not recently experienced the zero lower bound, the monetary authority sets the nominal interest rate according to a Taylor Rule, given by

$$i_t = r_t^n + \pi_{t+1}^* + \phi_\pi\left(\pi_t - \pi_t^*\right) + \phi_y\left(y_t - y_t^*\right), \tag{3}$$

where π_t^* represents a potentially time-varying inflation target and y_t^* is the output target,[7]

[4] This does not require that the inflation rate be zero in the long run, only that it not be so far from zero to make the linearization inappropriate (Woodford 2003, p. 79).

[5] $\kappa = \frac{(1-s)(1-\beta s)}{s}\frac{\sigma^{-1}+\omega}{1+\omega\varepsilon}$, where $s \in (0,1)$ represents the fraction of randomly selected firms that cannot adjust their price optimally in a given period. Therefore, $s = 0 \Rightarrow \kappa \rightarrow \infty \Rightarrow$ complete flexibility and $s = 1 \Rightarrow \kappa = 0 \Rightarrow$ complete stickiness. Hence, $\kappa \in (0,\infty) \Rightarrow$ incomplete flexibility. $\omega > 0$ is the elasticity of firm's real marginal cost with respect to its own output, $\varepsilon > 0$ is the price elasticity of demand of the goods produced by monopolistic firms. See, Adam and Billi (2006) and Woodford (2003) for details.

[6] Adam and Billi (2006) demonstrate that calibrated supply shocks are not large enough to send the economy to the zero lower bound.

[7] This specification for target output follows Woodford (2003), p. 246. He sets target output equal to the solution of equation (2) with inflation set at target inflation. Ours differs because the target inflation can vary over time.

given by

$$y_t^* = \frac{\pi_t^* - \beta\pi_{t+1}^*}{\kappa}. \tag{4}$$

This Taylor Rule has two distinguishing characteristics. First, it allows a potentially time-varying inflation target. In periods for which the zero lower bound is distant history, the optimal value for the inflation target is zero, and we assume that the monetary authority chooses an inflation target of zero in these circumstances. Second, Woodford (2003) has shown that optimal policy requires allowing the nominal rate to vary with the natural rate, yielding a time-varying intercept. Since we allow a potentially time-varying inflation target, our intercept varies not only with the natural rate, but also with the inflation target.

The equilibrium solution for the output gap and inflation is independent of the values for φ_π and φ_y as long as they are large enough to assure two unstable roots.[8] Therefore, it is important to understand the role of these policy parameters. The promise to respond strongly to any sunspot shocks that raise inflation and/or output, in Cochrane's (2011) words, "to blow up the economy" in the event of sunspot shocks, serves to rule out sunspot equilibria and to assure a locally unique equilibrium. This requires that the monetary authority be completely transparent, communicating the intention to "blow up the economy" and that this threat be completely credible. This is because φ_π and φ_y do not show up in the equilibrium solution and therefore cannot be inferred from any observable evidence.[9]

The monetary authority can follow the Taylor Rule, described by equation (3), as long as it yields a positive nominal interest rate. Once the natural rate of interest falls below zero, the Taylor Rule becomes infeasible. We follow Jung, Teranishi, and Watanabe (2005) by assuming that a large adverse shock creates the ZLB. Additionally, the shock is autoregressive and vanishes at a fixed rate. Specifically, we assume that in period $t = 1$ a large adverse shock to the natural rate sends the nominal interest rate in the Taylor Rule to zero. The shock (υ) deteriorates at rate ξ such that

$$r_t^n = r^n + \sigma^{-1}\xi^{t-1}\upsilon.$$

[8] The criteria for two unstable roots is: $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_y > 0$.

[9] Cochrane (2011) emphasizes that at the optimal equilibrium, values for φ_π and φ_y do not affect the equilibrium. Woodford (2003, p. 288) makes the same point. If there were shocks to the Phillips Curve, or if the intercept to the Taylor Rule did not vary optimally, then we would have evidence on the values of φ_π and φ_y. However, we would not have evidence that the monetary authority would actually "blow up" the economy in the event of a sunspot shock.

where, $r^n = r = \frac{1-\beta}{\beta}$. In order to obtain analytical results, we continue to follow Jung et al (2005) and assume that there are no other shocks, restricting our solution to certainty. We extend the results to include uncertainty in the natural rate of interest in Section 9.

Nakov (2008) considered a "truncated" Taylor Rule, in which the monetary authority follows a Taylor Rule[10] whenever it implies a positive nominal interest rate and otherwise sets the nominal rate to zero. In this paper, we offer two alternative Taylor-Rule modifications. Both allow conventional monetary policy to retain stimulative effects at the ZLB, in contrast to the truncated Taylor Rule. In the first, the monetary authority commits to an exit date, whereupon it will begin to follow a Taylor rule with an announced inflation target, which declines at an announced rate. In the second, the monetary authority only commits to follow the Taylor Rule on an announced exit date.

3 Solution with Full Commitment to a Taylor-Rule Exit Policy

The Taylor-Rule exit policy requires that the monetary authority announce an exit policy whereby it promises to implement equation (3) with inflation target

$$\pi^*_{T+1+i} = \rho_\pi^i \pi^* \qquad i \geq 0,$$

on its chosen exit date $(T+1)$. The choice of the time-varying inflation target requires that the monetary authority choose two parameters, the inflation target on the exit date (π^*), and the rate at which it declines (ρ_π). Prior to the announced exit date, the nominal interest rate remains zero. The monetary authority must be able to fully commit.

We follow Jung et all (2005) and separate the solution into two phases, one after exiting the ZLB and the other before.

3.1 Solution on Exit Date from ZLB Forward

Substituting the interest rate from the Taylor Rule (3), and target output, from equation (4) using π_{t+1} from equation (2), into the demand equation (1) yields a two-equation system given by

$$y_{t+1} = \left[1 + \sigma\left(\phi_y + \frac{\kappa}{\beta}\right)\right] y_t + \sigma\left(\phi_\pi - \frac{1}{\beta}\right)\pi_t - \sigma\epsilon_{t+1}, \tag{5}$$

[10] Nakov's (2008) Taylor Rule does not have a time-varying intercept.

$$\pi_{t+1} = -\frac{\kappa}{\beta} y_t + \frac{1}{\beta} \pi_t, \tag{6}$$

where

$$\boldsymbol{\epsilon}_{t+1} = z\pi_t^* \qquad z = \phi_\pi - \rho_\pi + \frac{\phi_y}{\kappa}\left(1 - \beta\rho_\pi\right).$$

When ϕ_y and ϕ_π are chosen large enough to satisfy the Taylor Principle, as we assume here, both roots, denoted by γ_1 and γ_2, are larger than one. We solve forward, with both the output gap and inflation determined to eliminate the two unstable roots, yielding values for initial conditions upon exit as

$$y_{T+1} = \frac{\left(1 - \beta\rho_\pi\right)\sigma z}{\beta\left(\gamma_1 - \rho_\pi\right)\left(\gamma_2 - \rho_\pi\right)} \pi^*, \tag{7}$$

$$\pi_{T+1} = \frac{\kappa\sigma z}{\beta\left(\gamma_1 - \rho_\pi\right)\left(\gamma_2 - \rho_\pi\right)} \pi^*. \tag{8}$$

Note that y_{T+1} and π_{T+1} are related by

$$y_{T+1} = \frac{\left(1 - \beta\rho_\pi\right)}{\kappa} \pi_{T+1}. \tag{9}$$

Values for the output gap and inflation beyond the exit date are governed by the monetary authority's choices for the inflation target, π^* and the rate at which the target vanishes, ρ_π. We can write the solution either in terms of π^* and ρ_π, or, using equation (8), in terms of π_{T+1} and ρ_π. For $t \geq T+1$, the appendix shows that values are given by

$$y_t = \frac{\left(1 - \beta\rho_\pi\right)}{\beta\left(\gamma_1 - \rho_\pi\right)\left(\gamma_2 - \rho_\pi\right)} \sigma z \rho_\pi^{t-(T+1)} \pi^* = \frac{\left(1 - \beta\rho_\pi\right)}{\kappa} \rho_\pi^{t-(T+1)} \pi_{T+1}, \tag{10}$$

$$\pi_t = \frac{\kappa}{\beta\left(\gamma_1 - \rho_\pi\right)\left(\gamma_2 - \rho_\pi\right)} \sigma z \rho_\pi^{t-(T+1)} \pi^* = \rho_\pi^{t-(T+1)} \pi_{T+1}. \tag{11}$$

The nominal interest rate is set to achieve these values for the output gap and inflation. From equation (1), the nominal interest rate on the date of exit from the ZLB and beyond ($t \geq T+1$) is

$$i_t = r_t^n + \pi_{t+1} + \frac{1}{\sigma}\left(y_{t+1} - y_t\right), \qquad t \geq T+1.$$

3.2 Solution Prior to Exit ZLB

Equations (1) and (2), with the nominal interest rate set to zero, yield solutions for the output gap and inflation prior to exit. One root is less than one and one is greater. We denote the stable root by ω_1 and the unstable one by ω_2. The solutions are subject to the terminal conditions given by equations (7) and (8).

Equations (41) and (42) in the appendix contain solutions as

$$y_t = \tfrac{1}{\kappa(\omega_2-\omega_1)} \left[\left(\tfrac{1}{\omega_1}\right)^{T+1-t} (\omega_2 - \rho_\pi)(1-\beta\omega_1) + \left(\tfrac{1}{\omega_2}\right)^{T+1-t} (\rho_\pi - \omega_1)(1-\beta\omega_2) \right] \pi_{T+1} \\ + \tfrac{\sigma}{\beta(\omega_2-\omega_1)} \sum_{k=t}^{T} \left[\left(\tfrac{1}{\omega_1}\right)^{k+1-t} (1-\beta\omega_1) - \left(\tfrac{1}{\omega_2}\right)^{k+1-t} (1-\beta\omega_2) \right] r_k^n, \tag{12}$$

$$\pi_t = \tfrac{1}{(\omega_2-\omega_1)} \left[\left(\tfrac{1}{\omega_1}\right)^{T+1-t} (\omega_2 - \rho_\pi) + \left(\tfrac{1}{\omega_2}\right)^{T+1-t} (\rho_\pi - \omega_1) \right] \pi_{T+1} \\ + \tfrac{\sigma\kappa}{\beta(\omega_2-\omega_1)} \sum_{k=t}^{T} \left[\left(\tfrac{1}{\omega_1}\right)^{k+1-t} - \left(\tfrac{1}{\omega_2}\right)^{k+1-t} \right] r_k^n. \tag{13}$$

These equations illustrate how the Taylor-Rule exit policy affects the behavior of the output gap and inflation during the period of the ZLB. If we were truncating the Taylor Rule, then the only terms determining the output gap and inflation at the ZLB would be those with the natural rate of interest, while the natural rate is negative. For standard parameter values, the terms multiplying the natural rates are positive. Therefore, the negative natural rate terms yield negative effects.

The Taylor-Rule exit policy adds terms with positive natural rates up until the last period prior to the chosen exit date, providing a stimulative effect. The stimulus is greater the more natural rate terms are added, that is, the further into the future exit is postponed. The Taylor-Rule exit policy also adds a term with the value of inflation upon exit (π_{T+1}). The term multiplying π_{T+1} is positive and increasing in T. Therefore, a positive value of inflation upon exit also provides stimulus. From equation (8), the monetary authority chooses values for the inflation target (π^*) and the rate at which the inflation target vanishes (ρ_π), thereby choosing the value of inflation upon exit (π_{T+1}).

To gain insight on how optimal values for the policy parameters are determined under an optimal Taylor-Rule exit policy, we turn to the solution for fully optimal policy.

4 Solution under Optimal Policy

Under fully optimal policy the standard presentation has the monetary authority directly choose values for the output gap, inflation, and the nominal interest rate, subject to equations (1) and (2) and to the restriction that the nominal interest rate be positive, to maximize utility of the representative agent. We use Woodford's (2003) linear approximation to the utility function of the representative agent when equilibrium inflation is zero and the flexible-price value for output is efficient. The Lagrangian is given by

$$L_1 = \sum_{t=1}^{\infty} \beta^{t-1} \left\{ -\frac{1}{2} \left(\pi_t^2 + \lambda y_t^2 \right) - \phi_{1,t} \left[\sigma \left(i_t - r_t^n - \pi_{t+1} \right) - y_{t+1} + y_t \right] - \phi_{2,t} \left[\pi_t - \kappa y_t - \beta \pi_{t+1} \right] + \phi_{3,t} i_t \right\},$$

where the third restriction represents the inequality constraint on the nominal interest rate. First order conditions with respect to π_t, y_t, and i_t respectively are

$$\phi_{2,t} - \phi_{2,t-1} + \pi_t - \sigma \beta^{-1} \phi_{1,t-1} = 0, \tag{14}$$

$$\phi_{1,t} - \beta^{-1} \phi_{1,t-1} + \lambda y_t - \kappa \phi_{2,t} = 0, \tag{15}$$

$$-\sigma \phi_{1,t} + \phi_{3,t} = 0 \qquad \phi_{3,t} i_t \geq 0 \qquad \phi_{3,t} \geq 0 \qquad i_t \geq 0. \tag{16}$$

Equations (16) reveal that when the nominal interest rate is zero, in the period of the ZLB, that $\phi_{3,t}$ is weakly positive, implying that $\phi_{1,t}$ is weakly positive. In the period after exit from the ZLB, the nominal interest rate becomes positive, moving $\phi_{3,t}$ to zero, implying that $\phi_{1,t}$ is zero.

4.1 Solution for Output Gap and Inflation after Exit from ZLB ($t \geq T+2$)

Exit from the ZLB occurs in period $T+1$. After exit, $\phi_{1,t} = 0$ and $i_t \geq 0$. We begin the solution with period $T+2$ instead of period $T+1$, since $\phi_{1,T+1} = 0$, but its lag $\left(\phi_{1,T}\right)$ could be positive. The equations of the model become

$$y_{t+1} = y_t + \sigma \left(i_t - r_t^n - \pi_{t+1} \right), \tag{17}$$

$$\pi_{t+1} = -\frac{\kappa}{\beta} y_t + \frac{1}{\beta} \pi_t, \tag{18}$$

$$\phi_{2,t} - \phi_{2,t-1} + \pi_t = 0, \tag{19}$$

$$\lambda y_t - \kappa \phi_{2,t} = 0. \tag{20}$$

First difference equation (20) to yield

$$y_{t+1} - y_t = \frac{\kappa}{\lambda} \left(\phi_{2,t+1} - \phi_{2,t} \right).$$

Substitute from equation (19) to yield

$$y_{t+1} - y_t = -\frac{\kappa}{\lambda}\pi_{t+1} = \frac{\kappa}{\lambda}\left(\frac{\kappa}{\beta}y_t - \frac{1}{\beta}\pi_t\right). \tag{21}$$

Equations (18) and (21) can be solved to yield values for output and inflation in periods $T+2$ and beyond with initial values in period $T+1$.

One root exceeds unity and the other is less than unity. Letting ψ_2 be the smaller stable root, initial values for output and inflation must lie along the saddlepath, thereby eliminating the unstable root, and requiring

$$y_{T+1} = \frac{(1-\beta\psi_2)}{\kappa}\pi_{T+1} = \frac{\kappa\psi_2}{\lambda(1-\psi_2)}\pi_{T+1}, \tag{22}$$

where the second equality uses the characteristic equation for the system.[11] Solutions depend on the initial conditions, determined to assure stability after exit, and the stable root. Equations (45) and (46) in the appendix yield solutions for $t \geq T+1$ as

$$y_t = \left(\frac{1-\beta\psi_2}{\kappa}\right)\psi_2^{t-(T+1)}\pi_{T+1}, \tag{23}$$

$$\pi_t = \psi_2^{t-(T+1)}\pi_{T+1}. \tag{24}$$

The optimal values for T and π_{T+1} are unique and are provided by solution for the multipliers below in Section 6. These solutions provide guidance on how the monetary authority, operating the Taylor-Rule exit policy, should optimally choose policy parameters.

4.2 Solution Prior to Exit the ZLB

The solution for y_t and π_t, prior to exiting the ZLB, is similar to that under the Taylor-Rule exit policy because, with the nominal interest rate set equal to zero, the dynamic behavior of the output gap and inflation is governed by identical equations. The only difference is that the relationship between output and inflation at $T+1$ is governed by ψ_2 in equations (22) instead of by ρ_π in equation (9). Solutions are given by

[11] The second expression is identical to that in Jung et al (2005).

$$
\begin{aligned}
y_t &= \frac{1}{\kappa\left(\omega_2-\omega_1\right)}\left[\left(\frac{1}{\omega_1}\right)^{T+1-t}\left(\omega_2-\psi_2\right)\left(1-\beta\omega_1\right)+\left(\frac{1}{\omega_2}\right)^{T+1-t}\left(\psi_2-\omega_1\right)\left(1-\beta\omega_2\right)\right]\pi_{T+1} \\
&\quad +\frac{\sigma}{\beta\left(\omega_2-\omega_1\right)}\left[\sum_{k=t}^{T}\left(\frac{1}{\omega_1}\right)^{k+1-t}\left(1-\beta\omega_1\right)-\left(\frac{1}{\omega_2}\right)^{k+1-t}\left(1-\beta\omega_2\right)\right]r_k^n, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
\pi_t &= \frac{1}{\left(\omega_2-\omega_1\right)}\left[\left(\frac{1}{\omega_1}\right)^{T+1-t}\left(\omega_2-\psi_2\right)+\left(\frac{1}{\omega_2}\right)^{T+1-t}\left(\psi_2-\omega_1\right)\right]\pi_{T+1} \\
&\quad +\frac{\sigma\kappa}{\beta\left(\omega_2-\omega_1\right)}\left[\sum_{k=t}^{T}\left(\frac{1}{\omega_1}\right)^{k+1-t}-\sum_{k=t}^{T}\left(\frac{1}{\omega_2}\right)^{k+1-t}\right]r_k^n. \qquad (26)
\end{aligned}
$$

These equations yield the same insights about how policy can affect the time paths of the output gap and inflation during the ZLB. Postponing exit time $(T+1)$ beyond the date on which the natural rate becomes positive adds terms with positive values of the natural rate, creating stimulus. The term multiplying inflation upon exit is positive and increasing in T. Therefore, a positive value of inflation upon exit (π_{T+1}) also provides stimulus.

5 Equivalence between Full Commitment to the Taylor-Rule Exit Policy and Optimal Policy

Theorem: If the monetary authority chooses its policy parameters, $T+1$, ρ_π, and π^*, optimally, then the Taylor Rule exit policy implements optimal monetary policy.

Proof: Solutions for the output gap and inflation before exit under the Taylor Rule, equations (12) and (13), are equivalent to those under optimal policy after exit, equations (25) and (26), if the monetary authority chooses $\rho_\pi = \psi_2$, chooses T equal to its optimal value, and chooses π^* to yield the optimal inflation rate upon exit, π_{T+1}. The last choice requires a value of the inflation target given by

$$\pi^* = \frac{\beta\left(\gamma_1-\rho_\pi\right)\left(\gamma_2-\rho_\pi\right)}{\kappa\sigma z}\pi_{T+1}.$$

Additionally, these choices imply that solutions for values of the output gap and inflation after exit under the Taylor Rule, given by equations (10) and (11), are identical to solutions after exit under optimal policy, given by equations (23) and (24).

Therefore, the monetary authority can implement optimal policy by postponing exit from the ZLB until the optimal exit time, choosing an inflation target in the Taylor Rule compatible with the optimal value of inflation upon exit, and allowing the target to vanish at a rate given by the value of the stable root with optimal policy after exit $(\rho_\pi = \psi_2)$.

Since agents are familiar with the Taylor Rule, and the addition of a time-varying inflation target is a small modification, the Taylor-Rule exit policy provides a way to implement and communicate optimal policy during and following a zero lower bound event. Full commitment to the Taylor-Rule exit policy is an optimal policy.

Optimal exit time and the optimal inflation target are determined by continuing to solve the optimal monetary policy problem for the multipliers.

6 Optimal Exit Time and Inflation Value

6.1 Analytical Solution

For $t \geq T+2$, equation (20), together with equation (23), yields a solution for $\phi_{2,t}$ given by

$$\phi_{2,t} = \frac{\lambda}{\kappa} y_t = \frac{\psi_2}{(1-\psi_2)} \psi_2^{t-(T+1)} \pi_{T+1}.$$

Therefore, the solution for $\phi_{2,T+2}$ is given by

$$\phi_{2,T+2} = \frac{\psi_2}{(1-\psi_2)} \psi_2 \pi_{T+1}. \tag{27}$$

We need solutions for $\phi_{2,T+1}$, $\phi_{2,T}$, and $\phi_{1,T}$. In period $T+1$, the period of exit, equations (14) and (15) with $\phi_{1,T+1} = 0$ yield

$$\phi_{2,T+1} - \phi_{2,T} + \pi_{T+1} - \sigma \beta^{-1} \phi_{1,T} = 0, \tag{28}$$

$$-\beta^{-1} \phi_{1,T} + \lambda y_{T+1} - \kappa \phi_{2,T+1} = 0. \tag{29}$$

In period $T+2$, these equations imply

$$\phi_{2,T+2} - \phi_{2,T+1} + \pi_{T+2} = 0,$$

$$\lambda y_{T+2} - \kappa \phi_{2,T+2} = 0.$$

Solving these equations, together with equation (22), yields

$$\phi_{1,T} = 0,$$

$$\phi_{2,T} = \frac{1}{(1-\psi_2)}\pi_{T+1}, \tag{30}$$

$$\phi_{2,T+1} = \frac{\psi_2}{(1-\psi_2)}\pi_{T+1} = \psi_2 \phi_{2,T}.$$

Solution for optimal values of π_{T+1} and T, requires solutions for the multipliers leading up to and including the exit period. The equations for the output gap and inflation for periods prior to exit ($t \leq T+1$) can be written in matrix notation as

$$Z_t = AZ_{t-1} - ar^n_{t-1}, \tag{31}$$

where

$$Z_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} \qquad A = \begin{bmatrix} 1 + \frac{\sigma\kappa}{\beta} & -\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \qquad a = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}.$$

A forward solution of the system to time $T+1$ yields

$$Z_t = \sum_{k=t}^{T} A^{-(k-t+1)} ar^n_k + A^{-(T-t+1)} Z_{T+1}.$$

From equations (22) and (30),

$$Z_{T+1} = W\Phi_T,$$

where

$$W = \begin{bmatrix} 0 & \frac{\kappa}{\lambda}\psi_2 \\ 0 & 1-\psi_2 \end{bmatrix} \qquad \Phi_T = \begin{bmatrix} \phi_{1,T} \\ \phi_{2,T} \end{bmatrix}. \tag{32}$$

Substituting, we can write the solution for Z_t as

$$Z_t = \sum_{k=t}^{T} A^{-(k-t+1)} ar^n_k + A^{-(T-t+1)} W\Phi_T. \tag{33}$$

Write the equations for the multipliers as

$$\Phi_t = C\Phi_{t-1} - DZ_t, \tag{34}$$

where

$$\Phi_t = \begin{bmatrix} \phi_{1,t} \\ \phi_{2,t} \end{bmatrix} \qquad C = \begin{bmatrix} \frac{1+\kappa\sigma}{\beta} & \kappa \\ \frac{\sigma}{\beta} & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \lambda & \kappa \\ 0 & 1 \end{bmatrix}. \tag{35}$$

Solve Φ_t forward to time T, imposing that initial values (period 0) of both multipliers are zero, to yield

$$\Phi_T = -\sum_{t=1}^{T} C^{T-t} DZ_t.$$

Substituting from equation (33), we have an equation in Φ_T, given by

$$\Phi_T = -\sum_{t=1}^{T} C^{T-t} D \left[\sum_{k=t}^{T} A^{-(k-t+1)} a r_k^n + A^{-(T-t+1)} W \Phi_T \right].$$

The solution for T is given by the value of T which solves

$$\begin{aligned} \Phi_T &= \begin{bmatrix} 0 \\ \phi_{2,T} \end{bmatrix} \\ &= -\left[I + \sum_{t=1}^{T} C^{T-t} D A^{-(T-t+1)} W \right]^{-1} \sum_{t=1}^{T} C^{T-t} D \left[\sum_{k=t}^{T} A^{-(k-t+1)} a r_k^n \right]. \end{aligned} \tag{36}$$

Given T from equation (36), the solution for π_{T+1} is given by equation (30) as $(1-\psi_2)\,\phi_{2,T}$.

6.2 Quantitative Solution

6.2.1 Benchmark Parameter Values

As a benchmark, we use the RBC parameterization from Adam and Billi (2006),

$$\sigma = 1,\ \beta = 0.99,\ \kappa = 0.057,\ \varphi_\pi = 1.5,\ \varphi_y = 0.5,\ \lambda = 0.0074.$$

All flow values are expressed at quarterly rates. The values for the elasticity of substitution and the discount factor are standard. The value of κ is consistent with 34% of firms adjusting their price each period when demand elasticity is 7.66 and the elasticity of firm marginal cost is 0.47.

6.2.2 Problems with Integer T

The numerical algorithm considers alternative values for the optimal T by computing values for $\Phi_{1,T}$ for successive values of T, beginning with T large enough for the nominal interest rate to be positive. In this range, $\phi_{1,T}$ is falling in T, eventually becoming negative as T continues to increase. Equation (36) requires that $\phi_{1,T} = 0$ at the optimum. However, since T increases discretely, with $\phi_{1,T}$ falling in T, for a given value for the shock, there is a value for T for which $\phi_{1,T} > 0$ and $\phi_{1,T+1} < 0$. We never actually observe a value of T for which $\phi_{1,T} = 0$ due to the integer constraint on T.

For each value of the shock, consider choosing T as the last date for which $\phi_{1,T}$ remains positive (theoretically, it is never negative). The value for the inflation target is determined by the value for ϕ_{2T}. Figure (1) plots values for $\phi_{1,T}$ and $\phi_{2,T}$, over a range of values for the initial shock, υ, where the value for T is calculated as suggested above.

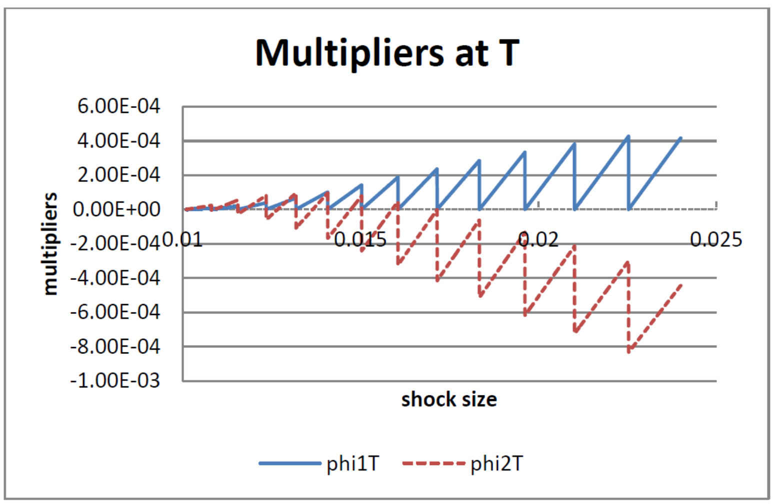

Figure: 1 Multipliers for Different Shocks

As the size of the shock increases, there is a range of values for the shock, for which exit time is fixed (not shown in graph) and both multipliers rise. As the size of the shock continues to increase, there is a critical value for the shock at which exit time rises by one unit and both multipliers fall discretely. As shock size continues to rise above each of these critical values, the size of both multipliers rises until the shock reaches another critical value. Therefore, both multipliers reach local minima at critical values of shock size for which exit time discretely rises.

Since $\phi_{1,T} > 0$, the optimal exit time is actually larger than T. If exit time were continuous, we could raise exit time just enough to get $\phi_{1,T} = 0$. This increase in exit time would also reduce $\phi_{2,T}$. The closer $\phi_{1,T}$ is to zero, the less we would need to raise a continuous value for the exit time above our choice of T to get $\phi_{1,T}$ to reach zero. Therefore, the optimal exit time, chosen by the above method, approaches the optimum without an integer constraint, as the value for $\phi_{1,T}$ approaches zero. Since we optimally want to raise T less at critical shock values, the integer constraint is least binding at these critical values. And the value for $\phi_{2,T}$ is also closest to its value without an integer constraint for these critical values of shocks. Comparing the optimal value of inflation upon exit, implied by values of $\phi_{2,T}$ in Figure (1) for any two discrete values of the shock, reveals that the inflation value could rise or fall as the shock size increases depending on how binding the integer constraint is for the particular set of shock sizes we have chosen. The integer constraint is affecting the solution, particularly the value for optimal inflation upon exit.[12]

We do not believe that the integer constraint actually constrains monetary policy. A binding integer constraint would mean that there are only four dates in the year on which the monetary authority could choose to raise the interest rate for the first time. This restriction does not appear realistic. Therefore, we want a solution for which the integer constraint is as close to non-binding as possible. The integer constraint is least binding at the critical values of shocks for which the multipliers reach local minima. We consider shock values which increase in increments of 1.0e-9, so that minimum values for $\phi_{1,T}$ get very close to zero, and restrict attention to the set of critical shock values (those for which the multipliers reach local minima).

With these restrictions, multipliers are the lower envelopes of the two seesaw lines in Figure 1. As shock size rises, $\phi_{1,T}$ remains very close to zero. In contrast, $\phi_{2,T}$ is negative for any shock size and is falling in shock size. This later result implies that optimal inflation upon exit is always negative and that it is decreasing in shock size. Failure to restrict attention to shock values for which the multipliers reach local minima yields a positive value for optimal inflation upon exit for some shock values. Positive inflation is compensating for the inability to raise exit time by something less than one discrete unit and therefore for having exit time too small relative to the optimal continuous value. Additional experimentation has revealed that the negative value for inflation upon exit is robust to persistence of the shock and to changes in other parameter values when we

[12] Were we to actually impose the integer constraint in the solution for optimal exit time, we would not get the solution we propose. The monetary authority could explicity use the inflation target to compensate for not raising the exit time sufficiently or for raising it too much due to the integer constraint.

restrict attention to shock sizes which yield lower envelopes for the multipliers.[13]

6.2.3 Optimal Exit Strategy

We present the optimal exit strategy in terms of the Taylor-Rule exit policy. All values for the output gap, inflation and the nominal interest rate along the adjustment path are identical to those for optimal policy. Our purpose in using the Taylor-Rule exit policy to present the results is to illustrate that communication can occur in terms of the Taylor Rule, augmented with the time-varying inflation target and an date for exit from the ZLB.

Consider the time paths for the output gap and inflation with the optimal Taylor-Rule exit policy after a particularly large adverse shock, $\upsilon = -0.02253508$ in period one, sending the natural rate to an annual rate of -4.97% . We set persistence high ($\xi = 0.90$) such that the natural rate that does not return to positive territory until period nine. The monetary authority optimally postpones raising the interest rate until period 14, fully five periods after the nominal rate has become positive. Optimal inflation in the exit period is negative and is given by $(1-\psi_2)\,\phi_{2T} = -0.0396\%$ at a quarterly rate. This requires an inflation target for the Taylor Rule given by equation (8), as

$$\pi^* = \frac{\beta\,(\gamma_1 - \psi_2)\,(\gamma_2 - \psi_2)}{\kappa\sigma z}\pi_{T+1} = -0.0703\%,$$

where the monetary authority has chosen $\rho_\pi = \psi_2$.

Figure 2 plots the time paths for the output gap, inflation, and the nominal interest rate, beginning with the initial shock. Values on the vertical axis are quarterly percentages expressed at annual rates, while values on the horizontal axis are quarters. The shock occurs in quarter 1. As a benchmark, we also plot the time path that a truncated Taylor Rule, with a zero inflation target and an intercept given by the natural rate, would deliver.

[13] We have reduced persistence to 0.80, considered values of σ between 0.16 and 6.25, and considered a lower value for κ equal to 0.24.

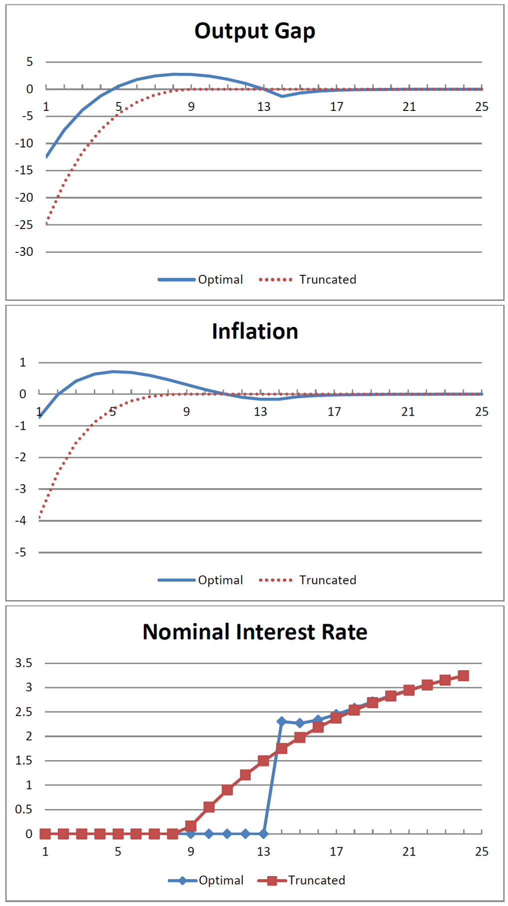

Figure: 2 Alternative Taylor Rules

Before continuing with the presentation of the Taylor-Rule exit policy, consider why the truncated Taylor Rule is a natural benchmark. The truncated Taylor Rule represents optimal policy when the monetary authority can commit only to follow a Taylor Rule, but not to an exit date or an inflation target. Essentially, the truncated Taylor Rule implements optimal discretionary policy.[14] Under this policy, the nominal interest rate is zero as long as the natural rate is negative. Once the natural rate becomes positive, the monetary authority optimally raises the nominal rate to the natural rate, thereby returning both the output gap and inflation to their optimal values of zero.

Under full commitment to the Taylor-Rule exit policy, the monetary authority promises

[14] As Cochrane (2011) argues, the Taylor Rule itself requires commitment to "blow up" the economy in the event of a sunspot shock, thereby assuring a locally unique equilibrium.

to postpone the exit date[15] and to exit with deflation which vanishes over time. The exit date and the inflation-target parameters are all chosen optimally. This policy provides considerable stimulus upon impact, all stemming from the postponed exit date. The later exit date implies that there are more periods for which the monetary authority could have raised the nominal interest rate, but has chosen not to. This raises inflationary expectations, raising output and inflation, compared to a truncated Taylor Rule.

In contrast, the negative inflation target upon exit reduces inflationary expectations and is contractionary. As the exit date nears, expectations of deflation actually cause a small recession coupled with deflation prior to the arrival of the exit date.[16] On the exit date, the monetary authority raises the nominal interest rate higher than the real rate to exacerbate the recession and deflation, which reach troughs at -0.16% and -1.34% respectively, at annual rates. Both remain small and quickly vanish over the next few quarters. The deflation and recession upon exit point to the importance of the ability to commit, not only to an exit date, but also to exit with deflation and recession.

The negative value of inflation upon exit runs counter to the notion that all means of monetary-policy stimulus should be employed at the ZLB, including postponing exit time and raising inflation upon exit. Walsh (2009), Levine et al (2009), and Cochrane (2013) all discuss the benefits of promising to exit the ZLB with positive inflation. Why does optimal policy require a negative inflation target which produces a small future recession with deflation? Loss is determined by discounted squared deviations. The large adverse shock itself creates large negative deviations, which vanish over time under a truncated Taylor Rule. With loss determined by discounted squared deviations, it is optimal to smooth these deviations over time, reducing the initial large and lightly-discounted deviations at the expense of creating new small and more heavily discounted deviations in the future. The postponed exit date reduces the magnitude of early deviations, while the negative inflation target creates small future deviations, creating some smoothing for deviations across time.

The loss under the truncated Taylor Rule is 4.85 times as great as the loss under the optimal Taylor-Rule exit policy. In general, relative loss is increasing in both the size of the shock and in its persistence. With high persistence, 0.90 in this example, and a range of initial shocks sending the natural rate of interest to values between -0.06 and -4.97 at annual rates, loss due to failure to commit ranges from about 2.5 to 7 times that under

[15] The postponed exit date is the feature of optimal monetary policy emphasized by Jung et al (2005).

[16] Postponing exit beyond the first date on which the natural rate of interest becomes positive achieves the overshooting of the inflation rate, necessary to reduce the real rate of interest. And, although inflation is negative in the exit period, it is positive on the first date for which the natural rate becomes positive.

commitment. When persistence is lower, for example 0.80, the range of excess loss is smaller, between 2.5 and 3.2 times that under commitment. These results highlight the relative importance of pursuing the optimal Taylor-Rule exit policy when the negative shock is large and highly persistent.

The need to commit to a future recession and deflation could pose a political problem to commitment, even though the magnitude of the recession and deflation are small.[17] Additionally, communication of the policy in terms of an exit date and a time-varying inflation target could be complicated. The forward guidance provided by the Federal Reserve on US monetary policy stresses that the nominal interest rate will remain zero for a "considerable period," but never states that once that period ends, that it will rise sufficiently to exacerbate or create a recession and deflation. What does the monetary authority lose in welfare if it commits to postpone the exit date from the ZLB beyond that using a truncated Taylor Rule, but not to deflation target upon exit, a policy we label "T-only"?

7 T-only Taylor-Rule Exit Policy

In this section, we investigate a policy in which the monetary authority chooses the exit date optimally, conditional upon a zero inflation target upon exit. Upon exit, the monetary authority returns to the Taylor-Rule with an inflation target of zero.[18] This policy is very much like the "forward guidance" for interest rates which the US Federal Reserve enacted in 2008, whereby they have promised to keep nominal interest rates near zero for "a considerable period."

We solve this problem numerically, choosing the value for the exit date $(T+1)$ which yields the highest welfare. We solve the optimization problem over a large grid of magnitudes for the shocks and observe that as shock size increases, welfare has a downward trend, but the fall is not monotonic. Specifically, when the integer value for T is optimal, welfare reaches a local maximum, and, as the shock size changes in both directions, T remains fixed and welfare falls. As the shock size changes from a value for which the

[17] Jeanne and Svensson (2007) are concerned with the ability to commit to positive inflation upon exit. Their solution, relying on the central bank's desire to maintain the value of their foreign currency reserves, does not work when the commitment is to deflation. However, over much of the period for which the natural rate is positive and the nominal rate is zero, optimal inflation is positive. The positive inflation after the natural rate becomes positive requires commitment, which could be supported by their mechanism. The subsequent deflation cannot be supported by their mechanism.

[18] Carlstrom, Fuerst, and Paustian (2012) analyze a similar policy in the same New Keynesian model without the initial adverse shock creating the ZLB.

optimal value of T is an integer, agents would like to chose a non-integer value for T, but cannot, implying lower welfare. Since we do not believe that the integer constraint is actually binding in the real world, we would like to consider results where the integer value for T is very close optimal. Therefore, we follow a strategy similar to that in the solution of the optimal Taylor-Rule exit policy. We identify a critical set of shocks associated with local maxima for welfare. With this set of shocks, welfare is falling in the size of the shock.

We want welfare comparisons under Taylor Rules with optimal policy and with T-only policy. This is problematical since the admissible shock values in the two cases differ. However, there are two instances in which admissible shock values are identical up to four decimal points. We compare these two sets of shocks. When the admissible shock with optimal policy is 0.011557 and that with T-only is .0115960, then T-only creates loss 20% larger than loss under optimal policy. The second admissible pair of shocks is 0.018340, .0183370 with loss 7% larger under T-only. These results imply that postponing the exit date achieves most of the gains of moving from the truncated Taylor Rule to full commitment to the optimal Taylor-Rule exit policy.

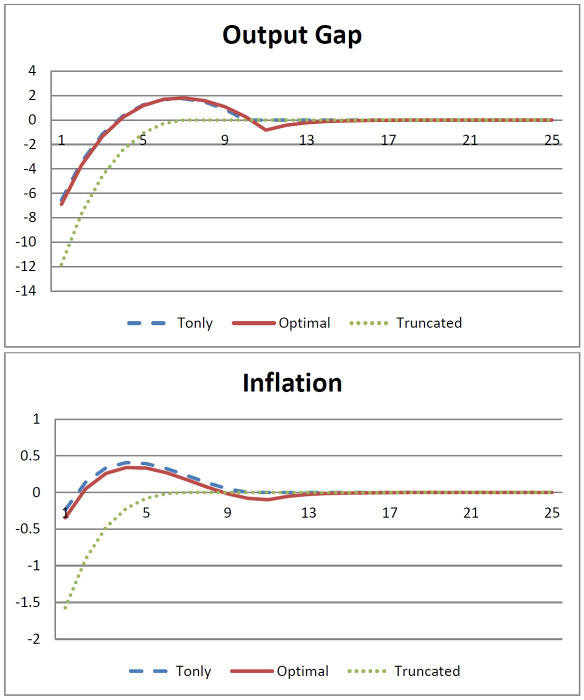

Figure: 3 Compare T-Only with Other Taylor Rules

We reinforce these insights by comparing time paths for the larger pair of shocks in Figure 3, shocks which initially send the natural rate to -3.30%.[19] This is a smaller shock than we considered in the previous section with correspondingly smaller adverse effects. With the T-only policy, exit occurs one period earlier than with optimal policy, in period 10 instead of in period 11. In the exit period, the nominal interest rate is set to equal the natural rate, and both the output gap and inflation return to zero. The time path for the output gap, leading up to the T-only exit period, is almost identical to that under optimal policy, with output slightly higher early, and slightly lower later. Inflation is uniformly higher under T-only than under optimal policy. T-only avoids the deflation and recession in the vicinity of the exit period.

These results seem to justify US Federal Reserve policy following the financial crisis. The Fed is likely to face political constraints in committing to future deflation and reces-

[19]We are comparing time paths for a slightly different shock value for the T-only policy. When we use the identical value, the difference is so small that there is no perceptable difference in results on the graph.

sion, but not in the timing for initially raising interest rates. We have shown that the optimal choice of exit time can achieve most of the gains of the optimal Taylor-Rule exit policy. Additionally, this extension of "nearly"optimal policy to the ZLB is a very simple extension of Woodford's optimal policy away from the ZLB. The monetary authority simply announces its date of exit from the ZLB and then returns to Woodford's optimal policy. Therefore, communication is relatively simple. The policy differs from optimal discretion in that the monetary authority can commit to keep the interest rate at the ZLB beyond the date on which the natural rate becomes positive.

8 Extension to Uncertainty

8.1 Analytical Solution under Uncertainty

We generalize our analysis to include stochastic shocks to the natural rate of interest. In Appendix 10.3, we show that equilibrium for the system under optimal policy with uncertainty is characterized by equations (1), (2), (14), (15), and (16).

Consider our stochastic specification for the natural rate of interest. Adam and Billi (2006) allow Markov shocks to the natural rate of interest and solve using value function iteration with the multipliers, φ_{1t} and φ_{2t}, as additional state variables. Our proof of equivalence between optimal policy and the fully optimal Taylor-Rule exit policy requires an analytical solution. This restriction limits the nature of the uncertainty we can introduce. The values for $\phi_{1,t}$ and $\phi_{2,t}$ at each value for t depend on the entire interest rate history since the period the system entered the ZLB. In the forward solution we use, the expectations must account for all possible realizations of the entire interest rate path until exit from the ZLB occurs. Introducing Markov shocks to the natural interest rate yields potential paths for the interest rate history, which expand too quickly, as time at the ZLB grows, to be practical. Therefore, we introduce a simpler form of uncertainty to the path of the natural interest rate.

We choose the nature of uncertainty to focus on the fact that the date, on which the natural rate of interest first becomes positive again, is unknown. We assume that there are three distinct paths for the real interest rate after the initial shock, determined by three distinct paths for υ. We refer to these alternative paths as interest rate regimes.

Prior to $t = \hat{t}$, the shock vanishes at rate ξ according to

$$\begin{aligned} \upsilon_t &= \xi^{t-1}\upsilon \qquad t \leq \hat{t}. \\ r_t^n &= r^n + \sigma^{-1}\upsilon_t, \quad t \leq \hat{t}. \end{aligned}$$

In period $\hat{t}+1$, persistence becomes either $\xi - \zeta$ or $\xi + \zeta$ with probability ϖ or remains ξ with probability $1 - 2\varpi$. Therefore, there are three paths for the natural rate, based on the shock to persistence given by

$$\begin{aligned} \upsilon_{1,t} &= (\xi - \zeta)^{t-\hat{t}}\, \xi^{\hat{t}-1}\upsilon \\ \upsilon_{2,t} &= \xi^{t-1}\upsilon \\ \upsilon_{3,t} &= (\xi + \zeta)^{t-\hat{t}}\, \xi^{\hat{t}-1}\upsilon \end{aligned}$$

with the natural interest rate given by

$$r_{i,t}^n = r^n + \sigma^{-1}\upsilon_{i,t}, \qquad\qquad t \geq \hat{t}+1 \qquad i \in \{1,2,3\}$$

In Appendix 10.3, we solve the equations for output and inflation with uncertainty forward, yielding solutions similar to those in the certainty case. The solution differs from that under certainty because agents' expectations of future output and inflation depend on expectations of the future interest rate, where the interest rate can follow one of three paths. The solution prior to exit from the ZLB depends upon exit times, T_i, $i \in (1,2,3)$ and on values for inflation upon exit in each interest-rate regime, with outcomes in each regime weighted by its probability. This system of equations and its solutions, as a function of exit times and the path of inflation upon exit, applies whether monetary policy is conducted according to the Taylor-Rule exit policy or optimal policy. With optimal monetary policy, additional equations for the multipliers, also solved in the appendix, yield solutions for optimal inflation upon exit and exit times.

When the monetary authority chooses the optimal values for exit time and for inflation upon exit, together with its rate of persistence, then the Taylor-Rule exit policy implements optimal monetary policy as before. Uncertainty does not invalidate the fundamental theorem that the Taylor-Rule exit policy implements optimal policy if exit times and inflation upon exit are chosen optimally.

8.2 Quantitative Solution under Uncertainty

In this section, we compare time paths under certainty and uncertainty, and consider the welfare cost of moving from the fully optimal Taylor-Rule exit policy to the T-only policy under uncertainty. The additional parameters we need are those for the change in the rate of persistence of the initial shock under each interest rate regime (ζ) and the probability of each interest rate regime (ϖ). We set $\zeta = 0.05$, and $\varpi = 0.25$. With these alternative rates of persistence of the initial shocks, the natural rate of interest first becomes positive in period 6 for regime 1 with low persistence, in period 8 with persistence equal to the benchmark value with certainty, and in period 13 with high persistence. Therefore, we consider interest rate regimes which return to positive half a year earlier than the benchmark and a year and a quarter later.

Both the optimal values for exit time and for inflation upon exit, conditional on obtaining the benchmark interest rate path, are identical to their values under certainty. Therefore, Figure (4) shows that the time paths for the output gap and inflation under certainty and uncertainty, conditional on realization of regime 2, are virtually identical, having only a slight difference prior to the realization of the uncertainty. The difference occurs because under uncertainty, the expected future path of interest rates replaces the actual path while the path is unknown.

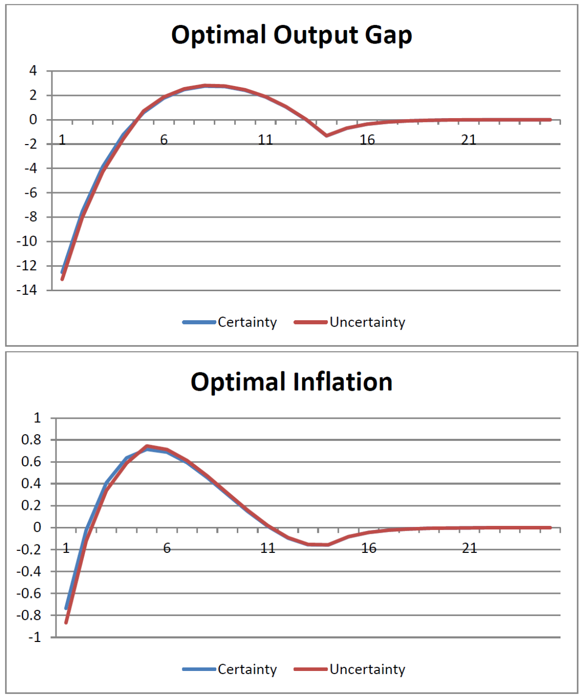

Figure 4: Time Paths under Certainty and Uncertainty

The very small effect of uncertainty on the actual path, conditional on realization of regime 2, occurs even though exit times and time paths are different under realizations of alternative interest rate paths. Figure (5) plots paths for the output gap and inflation with optimal policy in each of the three alternative interest rate regimes. Exit dates are different in each regime at 12, 14, and 19, and it is primarily these different exit dates that yield the different time paths. Exit values for inflation are negative and small with values of -0.170%, -0.158%, and -0.076%. The amount of uncertainty we have introduced creates an additional welfare loss of about 15%, implying that its welfare implications are relatively small.[20]

[20] Greater uncertainty would yield greater welfare losses.

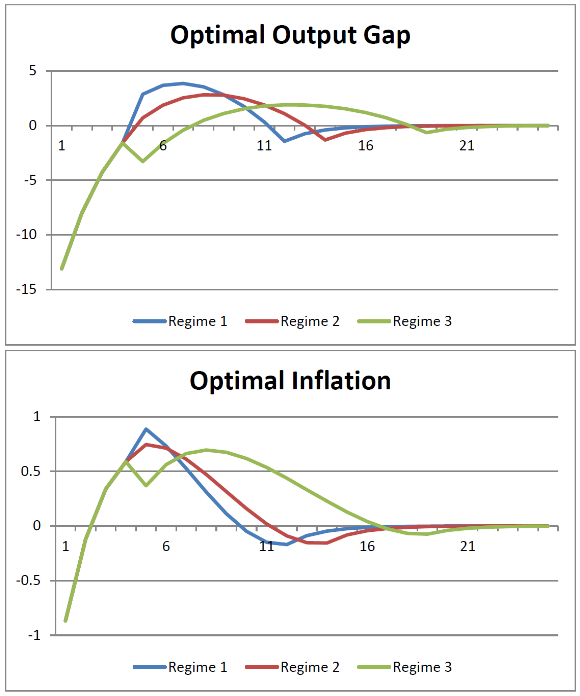

Figure 5: Time Paths under Alternative Interest Rate Regimes

Finally, consider the implications of the T-only policy, whereby the monetary authority chooses inflation equal to zero upon exit, but chooses an exit time for each of the three interest rate path realizations. We choose values for the exit time by searching over a grid of possible exit times and choosing the set of three which maximizes welfare. Exit times under T-only are slightly earlier, at values of (11,12,18) compared with exit times under optimal policy of (12,14,19). Loss under the T-only policy is about 10% higher than loss under the fully optimal policy.

As in our solution under certainty, our comparison of welfare loss under T-only and optimal policy is not entirely reliable since a real-world monetary authority would not have to respect the integer constraint in its choice of exit time. In each measure of welfare, choices could be distorted more or less by the integer constraint, reducing welfare, possibly by different amounts in the two measures. And it appears that the distortion for this value of the shock is important for the T-only policy because time paths under alternative interest rate regimes imply that the most contractionary regime is regime 2, when the most contractionary path for interest rates is regime 3. We let exit in regime 2 come one period earlier and find that now regime 2 is the least contractionary. Clearly, the integer constraint is preventing a result where time paths for regime 2 should lie between time paths for regimes 1 and 3.

Therefore, we consider an alternative value of the shock, -0.0183370, chosen in the section with certainty to minimize the distortion under the T-only policy. Loss under this alternative shock value with T-only policy is only 1.45% higher than loss under fully optimal policy. Therefore, using either shock, the order of magnitude of loss is small and similar to values we found in our example under certainty where we could choose shock values to minimize distortions. Additionally, the loss incurred by moving from optimal policy to T-only is tiny compared to a loss under discretion 10.65 times as large as under optimal policy.

We also calculate exit times and time paths for alternative interest rate regimes using the alternative initial shock. Exit times for T-only are (8,9,12) in regimes (1,2,3) compared with exit times under optimal policy of (9,9,13). Therefore, exit times are earlier for the first and third regimes under T-only, but identical for the middle regime. We compare time paths for each interest rate regime for T-only and optimal policies in Figure (6).

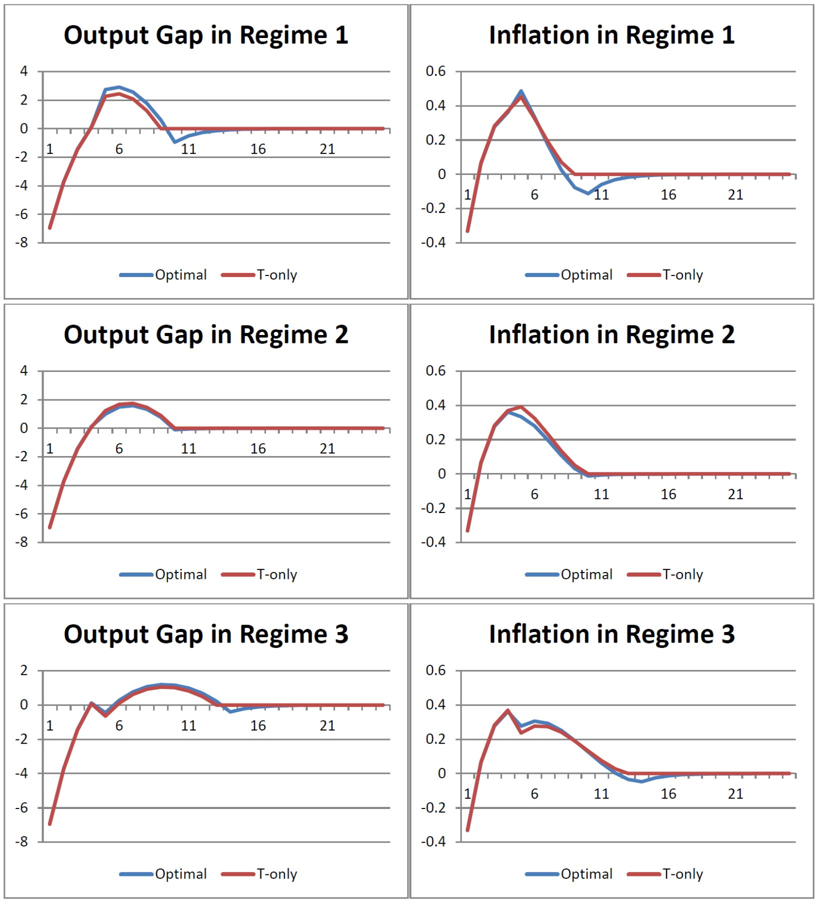

Figure 6: Time Paths under Optimal and T-only Policies

For all, the T-only policy eliminates the small negative effects around the exit date, required by the negative inflation upon exit in optimal policy. For regime 2, since exit dates are identical, paths are virtually identical. Differences for regimes 1 and 3 are also small. For optimal policy, the negative inflation target reduces stimulus, while for T-only, the earlier exit date provides the stimulus reduction.

9 Conclusion

Our first result is theoretical. We prove analytically that a Taylor Rule exit policy, with an optimally-chosen value for the exit date and for the time path of inflation upon exit, implements optimal monetary policy at the zero lower bound. This implies that implementation of optimal monetary policy at the ZLB requires focus on three parameters:

exit time, the inflation target upon exit, and the rate of decline of the inflation target.

Our second result is quantitative. We find that the welfare cost of moving from optimal policy to a T-only policy is small. Therefore, the monetary authority can implement a policy very close to optimal by announcing only an exit date from the ZLB. This result implies that of the three parameters necessary to implement optimal policy, the exit date is the most important.

We derive our results under certainty and show that they continue to hold when we introduce uncertainty. These results justify the policy by the Federal Reserve of promising that interest rates will remain low for a "considerable period."

10 Appendix

10.1 Solution under Taylor Rule Policy

10.1.1 Solution from Exit Date Forward

We can write this system of equations given by (5) and (6) in matrix notation as

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\sigma\left(\phi_y + \frac{\kappa}{\beta}\right) & \sigma\left(\phi_\pi - \frac{1}{\beta}\right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} - \begin{bmatrix} \sigma\epsilon_{t+1} \\ 0 \end{bmatrix}.$$

The eigenvalues are given by

$$\gamma = \frac{1+\frac{1}{\beta}+\sigma\left(\phi_y+\frac{\kappa}{\beta}\right) \pm \left[\left(1+\frac{1}{\beta}+\sigma\left(\phi_y+\frac{\kappa}{\beta}\right)\right)^2 - 4\left(\frac{1}{\beta}\right)\left(1+\sigma\left(\phi_y+\kappa\phi_\pi\right)\right)\right]^{\frac{1}{2}}}{2},$$

where ϕ_y and ϕ_π are chosen such that both eigenvalues exceed unity. Decomposing the system into eigenvalues and eigenvectors yields

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = E\Gamma E^{-1} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} - \begin{bmatrix} \sigma\epsilon_{t+1} \\ 0 \end{bmatrix},$$

where

$$E = \begin{bmatrix} \frac{1-\beta\gamma_1}{\kappa} & \frac{1-\beta\gamma_2}{\kappa} \\ 1 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \qquad E^{-1} = \frac{\kappa}{\beta\left(\gamma_2-\gamma_1\right)} \begin{bmatrix} 1 & -\frac{1-\beta\gamma_2}{\kappa} \\ -1 & \frac{1-\beta\gamma_1}{\kappa} \end{bmatrix},$$

with

$$\gamma_1\gamma_2 = \frac{1}{\beta}\left(1+\sigma\left(\phi_y+\kappa\phi_\pi\right)\right).$$

Pre-multiplying by E^{-1} yields

$$E^{-1}\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \Gamma E^{-1}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} - E^{-1}\begin{bmatrix} \sigma\epsilon_{t+1} \\ 0 \end{bmatrix},$$

where

$$E^{-1}\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} y'_{t+1} \\ \pi'_{t+1} \end{bmatrix} \qquad E^{-1}\begin{bmatrix} \sigma\epsilon_{t+1} \\ 0 \end{bmatrix} = \frac{\kappa\sigma}{\beta\left(\gamma_2-\gamma_1\right)}\begin{bmatrix} \epsilon_{t+1} \\ -\epsilon_{t+1} \end{bmatrix}.$$

Substituting yields

$$\begin{bmatrix} y'_{t+1} \\ \pi'_{t+1} \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}\begin{bmatrix} y'_t \\ \pi'_t \end{bmatrix} + \begin{bmatrix} -\frac{\kappa\sigma\epsilon_{t+1}}{\beta(\gamma_2-\gamma_1)} \\ \frac{\kappa\sigma\epsilon_{t+1}}{\beta(\gamma_2-\gamma_1)} \end{bmatrix}.$$

Since both roots exceed unity, we solve each equation forward to yield

$$y'_t = \frac{\kappa\sigma}{\beta\left(\gamma_2-\gamma_1\right)}\sum_{i=1}^{\infty}\left(\frac{1}{\gamma_1}\right)^i\epsilon_{t+i},$$

$$\pi'_t = \frac{-\kappa\sigma}{\beta\left(\gamma_2-\gamma_1\right)}\sum_{i=1}^{\infty}\left(\frac{1}{\gamma_2}\right)^i\epsilon_{t+i}.$$

We are interested in the value of the variables in the period of exit from the ZLB, that is in period $T+1$.

$$y'_{T+1} = \frac{\kappa\sigma}{\beta\left(\gamma_2-\gamma_1\right)}\sum_{i=1}^{\infty}\left(\frac{1}{\gamma_1}\right)^i\epsilon_{T+1+i},$$

$$\pi'_{T+1} = \frac{-\kappa\sigma}{\beta\left(\gamma_2-\gamma_1\right)}\sum_{i=1}^{\infty}\left(\frac{1}{\gamma_2}\right)^i\epsilon_{T+1+i}.$$

To do the summations, write the expressions for the $\epsilon' s$ as

$$\epsilon_{T+2} = \left[\left(\phi_\pi+\frac{\phi_y}{\kappa}\right)\pi^*_{T+1} - \left(1+\frac{\beta\phi_y}{\kappa}\right)\pi^*_{T+2}\right] = \left[\phi_\pi+\frac{\phi_y}{\kappa} - \left(1+\frac{\beta\phi_y}{\kappa}\right)\rho_\pi\right]\pi^* = z\pi^*,$$

$$
\begin{aligned}
\epsilon_{T+2+i} &= \left[\left(\phi_\pi + \frac{\phi_y}{\kappa}\right)\pi^*_{T+1+i} - \left(1 + \frac{\beta\phi_y}{\kappa}\right)\pi^*_{T+2+i}\right] = \left[\phi_\pi + \frac{\phi_y}{\kappa} - \left(1 + \frac{\beta\phi_y}{\kappa}\right)\rho_\pi\right]\rho_\pi^i \pi^* \\
&= z\rho_\pi^i \pi^*.
\end{aligned}
$$

Substituting, the sums can be expressed as

$$
\sum_{i=1}^{\infty}\left(\frac{1}{\gamma_1}\right)^i \epsilon_{T+1+i} = \frac{z}{\gamma_1 - \rho_\pi}\pi^*,
$$

$$
\sum_{i=1}^{\infty}\left(\frac{1}{\gamma_2}\right)^i \epsilon_{T+1+i} = \frac{z}{\gamma_2 - \rho_\pi}\pi^*.
$$

This allows us to write the solution for the transformed variables as

$$
y'_{T+1} = \frac{\kappa\sigma\pi^*}{\beta\left(\gamma_2 - \gamma_1\right)}\left[\frac{z}{\gamma_1 - \rho_\pi}\right],
$$

$$
\pi'_{T+1} = \frac{-\kappa\sigma\pi^*}{\beta\left(\gamma_2 - \gamma_1\right)}\left[\frac{z}{\gamma_2 - \rho_\pi}\right].
$$

To solve for the original variables, multiply by the matrix E,

$$
\begin{bmatrix} y_{T+1} \\ \pi_{T+1} \end{bmatrix} = \begin{bmatrix} \frac{1-\beta\gamma_1}{\kappa} & \frac{1-\beta\gamma_2}{\kappa} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y'_{T+1} \\ \pi'_{T+1} \end{bmatrix},
$$

yielding

$$
y_{T+1} = \frac{(1-\beta\rho_\pi)}{\beta\left(\gamma_1 - \rho_\pi\right)\left(\gamma_2 - \rho_\pi\right)}\sigma z\pi^*, \tag{37}
$$

$$
\pi_{T+1} = \frac{\kappa}{\beta\left(\gamma_1 - \rho_\pi\right)\left(\gamma_2 - \rho_\pi\right)}\sigma z\pi^*. \tag{38}
$$

Note that at $T+1$, the output gap is proportional to inflation according to

$$
y_{T+1} = \frac{(1-\beta\rho_\pi)}{\kappa}\pi_{T+1}.
$$

These values give us terminal conditions for the solution prior to exit.

Since there is only single stable root, provided by the rate at which the inflation target vanishes, values beyond $T+1$ are given by

$$
y_t = \frac{(1-\beta\rho_\pi)}{\beta\left(\gamma_1 - \rho_\pi\right)\left(\gamma_2 - \rho_\pi\right)}\sigma z\rho_\pi^{t-(T+1)}\pi^* = \frac{(1-\beta\rho_\pi)}{\kappa}\rho_\pi^{t-(T+1)}\pi_{T+1}, \tag{39}
$$

$$\pi_t = \frac{\kappa}{\beta(\gamma_1 - \rho_\pi)(\gamma_2 - \rho_\pi)} \sigma z \rho_\pi^{t-(T+1)} \pi^* = \rho_\pi^{t-(T+1)} \pi_{T+1}. \tag{40}$$

Substituting from the equations for output and inflation after exit, equations (39) and (40), yields the behavior of the interest rate after exit

$$\begin{aligned} i_t &= r_t^n + \left[1 + \frac{(1-\beta\rho_\pi)(\rho_\pi - 1)}{\rho_\pi \sigma \kappa}\right] \pi_{T+1} \rho_\pi^{t-T-1} \\ &= r_t^n + \left[1 + \frac{(1-\beta\rho_\pi)(\rho_\pi - 1)}{\rho_\pi \sigma \kappa}\right] \left[\frac{\kappa \sigma z}{\beta(\gamma_1 - \rho_\pi)(\gamma_2 - \rho_\pi)}\right] \pi^* \rho_\pi^{t-T-1}. \end{aligned}$$

10.1.2 Solution Prior to Exit ZLB

Equations (1) and (2) with the nominal interest rate set to zero can be written as,

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\sigma\kappa}{\beta} & -\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} - \begin{bmatrix} \sigma \\ 0 \end{bmatrix} r_t^n.$$

The roots of the system are given by

$$\omega = \frac{\frac{1+\sigma\kappa}{\beta} + 1 \pm \left[\left(1 + \frac{1+\sigma\kappa}{\beta}\right)^2 - 4\left(\frac{1}{\beta}\right)\right]^{\frac{1}{2}}}{2},$$

implying that one root is larger than unity and one is smaller. Let $\omega_1 > 1$, be the unstable root.

We solve the system subject to the terminal conditions given by equations (37) and (38). Using eigenvalues and eigenvectors, we can express the system as

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = F \Omega F^{-1} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} - \begin{bmatrix} \sigma r_t^n \\ 0 \end{bmatrix},$$

where

$$F = \begin{bmatrix} \frac{1-\beta\omega_1}{\kappa} & \frac{1-\beta\omega_2}{\kappa} \\ 1 & 1 \end{bmatrix} \qquad \Omega = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \qquad F^{-1} = \frac{\kappa}{\beta(\omega_2 - \omega_1)} \begin{bmatrix} 1 & -\frac{1-\beta\omega_2}{\kappa} \\ -1 & \frac{1-\beta\omega_1}{\kappa} \end{bmatrix},$$

with

$$\omega_1 \omega_2 = \frac{1}{\beta}.$$

Pre-multiplying by F^{-1} yields

$$F^{-1}\left[\begin{array}{c} y_{t+1} \\ \pi_{t+1} \end{array}\right] = \Omega F^{-1}\left[\begin{array}{c} y_t \\ \pi_t \end{array}\right] - F^{-1}\left[\begin{array}{c} \sigma r_t^n \\ 0 \end{array}\right],$$

where

$$F^{-1}\left[\begin{array}{c} y_{t+1} \\ \pi_{t+1} \end{array}\right] = \left[\begin{array}{c} y_{t+1}^{'} \\ \pi_{t+1}^{'} \end{array}\right] \qquad F^{-1}\left[\begin{array}{c} \sigma r_t^n \\ 0 \end{array}\right] = \frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)}\left[\begin{array}{c} r_t^n \\ -r_t^n \end{array}\right].$$

Substituting yields

$$\left[\begin{array}{c} y_{t+1}^{'} \\ \pi_{t+1}^{'} \end{array}\right] = \left[\begin{array}{cc} \omega_1 & 0 \\ 0 & \omega_2 \end{array}\right]\left[\begin{array}{c} y_t^{'} \\ \pi_t^{'} \end{array}\right] + \left[\begin{array}{c} -\frac{\kappa\sigma r_t^n}{\beta(\omega_2-\omega_1)} \\ \frac{\kappa\sigma r_t^n}{\beta(\omega_2-\omega_1)} \end{array}\right].$$

Solve each equation forward to period $T+1$, yielding

$$y_t^{'} = \left(\frac{1}{\omega_1}\right)^{T+1-t} y_{T+1}^{'} + \sum_{k=t}^{T}\left(\frac{1}{\omega_1}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n,$$

$$\pi_t^{'} = \left(\frac{1}{\omega_2}\right)^{T+1-t} \pi_{T+1}^{'} - \sum_{k=t}^{T}\left(\frac{1}{\omega_2}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n.$$

To solve for the original variables, we pre-multiply by the matrix F,

$$\left[\begin{array}{c} y_t \\ \pi_t \end{array}\right] = \left[\begin{array}{cc} \frac{1-\beta\omega_1}{\kappa} & \frac{1-\beta\omega_2}{\kappa} \\ 1 & 1 \end{array}\right]\left[\begin{array}{c} y_t^{'} \\ \pi_t^{'} \end{array}\right],$$

yielding

$$\begin{aligned} y_t \;=\; & \frac{1-\beta\omega_1}{\kappa}\left[\left(\frac{1}{\omega_1}\right)^{T+1-t} y_{T+1}^{'} + \sum_{k=t}^{T}\left(\frac{1}{\omega_1}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n\right] \\ & + \frac{1-\beta\omega_2}{\kappa}\left[\left(\frac{1}{\omega_2}\right)^{T+1-t} \pi_{T+1}^{'} - \sum_{k=t}^{T}\left(\frac{1}{\omega_2}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n\right], \end{aligned}$$

$$\pi_t = \left[\left(\frac{1}{\omega_1}\right)^{T+1-t} y'_{T+1} + \sum_{k=t}^{T}\left(\frac{1}{\omega_1}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n\right] + \left[\left(\frac{1}{\omega_2}\right)^{T+1-t} \pi'_{T+1} - \sum_{k=t}^{T}\left(\frac{1}{\omega_2}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n\right].$$

We transform the y'_{T+1} and π'_{T+1} into original variables using the terminal condition

$$y_{T+1} = \frac{(1-\beta\rho_\pi)}{\kappa}\pi_{T+1},$$

and

$$\begin{bmatrix} y'_{T+1} \\ \pi'_{T+1} \end{bmatrix} = F^{-1}\begin{bmatrix} y_{T+1} \\ \pi_{T+1} \end{bmatrix} = \frac{\kappa}{\beta\left(\omega_2-\omega_1\right)}\begin{bmatrix} 1 & -\frac{1-\beta\omega_2}{\kappa} \\ -1 & \frac{1-\beta\omega_1}{\kappa} \end{bmatrix}\begin{bmatrix} y_{T+1} \\ \pi_{T+1} \end{bmatrix},$$

to yield

$$y'_{T+1} = \left[\frac{\omega_2-\rho_\pi}{\omega_2-\omega_1}\right]\pi_{T+1},$$

$$\pi'_{T+1} = \left[\frac{\rho_\pi-\omega_1}{\omega_2-\omega_1}\right]\pi_{T+1}.$$

Substituting into the solutions for y_t and π_t yields

$$y_t = \frac{1-\beta\omega_1}{\kappa}\left[\left(\frac{1}{\omega_1}\right)^{T+1-t}\left[\frac{\omega_2-\rho_\pi}{\omega_2-\omega_1}\right]\pi_{T+1} + \sum_{k=t}^{T}\left(\frac{1}{\omega_1}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n\right] + \frac{1-\beta\omega_2}{\kappa}\left[\left(\frac{1}{\omega_2}\right)^{T+1-t}\left[\frac{\rho_\pi-\omega_1}{\omega_2-\omega_1}\right]\pi_{T+1} - \sum_{k=t}^{T}\left(\frac{1}{\omega_2}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n\right], \quad (41)$$

$$\pi_t = \left[\left(\frac{1}{\omega_1}\right)^{T+1-t}\left[\frac{\omega_2-\rho_\pi}{\omega_2-\omega_1}\right]\pi_{T+1} + \sum_{k=t}^{T}\left(\frac{1}{\omega_1}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n\right] + \left[\left(\frac{1}{\omega_2}\right)^{T+1-t}\left[\frac{\rho_\pi-\omega_1}{\omega_2-\omega_1}\right]\pi_{T+1} - \sum_{k=t}^{T}\left(\frac{1}{\omega_2}\right)^{k+1-t}\frac{\kappa\sigma}{\beta\left(\omega_2-\omega_1\right)} r_k^n\right]. \quad (42)$$

10.2 Optimal Policy

10.2.1 After Exit ZLB

In matrix form, equations (18) and (21) can be written as

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\frac{\kappa^2}{\beta\lambda} & -\frac{\kappa}{\beta\lambda} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix},$$

with eigenvalues

$$\psi = \frac{1+\frac{\kappa^2+\lambda}{\beta\lambda} \pm \sqrt{\left(1+\frac{\kappa^2+\lambda}{\beta\lambda}\right)^2 - \frac{4}{\beta}}}{2},$$

implying that one stable and one unstable root. Decomposing the system using eigenvalues and eigenvectors yields

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = G\Psi G^{-1} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix},$$

with

$$G = \begin{bmatrix} \frac{1-\beta\psi_1}{\kappa} & \frac{1-\beta\psi_2}{\kappa} \\ 1 & 1 \end{bmatrix} \qquad \Psi = \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \qquad G^{-1} = \frac{\kappa}{\beta\left(\psi_2-\psi_1\right)} \begin{bmatrix} 1 & -\frac{1-\beta\psi_2}{\kappa} \\ -1 & \frac{1-\beta\psi_1}{\kappa} \end{bmatrix},$$

and

$$\psi_1\psi_2 = \frac{1}{\beta}.$$

Pre-multiply by G^{-1} to yield

$$\begin{aligned} G^{-1}\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} &= \Psi G^{-1}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} \\ \begin{bmatrix} y'_{t+1} \\ \pi'_{t+1} \end{bmatrix} &= \Psi \begin{bmatrix} y'_t \\ \pi'_t \end{bmatrix}. \end{aligned}$$

These two differential equations in y'_t and π'_t can be solved forward to yield

$$y'_t = \psi_1^{t-(T+1)} y'_{T+1},$$

$$\pi'_t = \psi_2^{t-(T+1)} \pi'_{T+1}.$$

Letting the unstable root be given by ψ_1, the system is explosive unless $y'_{T+1} = 0$. Therefore we set $y^{'}_{T+1} = 0$. Transforming back into original variables yields

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = G \begin{bmatrix} \psi_1^{t-(T+1)} y^{'}_{T+1} \\ \psi_2^{t-(T+1)} \pi^{'}_{T+1} \end{bmatrix} = \begin{bmatrix} \frac{1-\beta\psi_1}{\kappa} & \frac{1-\beta\psi_2}{\kappa} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \psi_2^{t-(T+1)} \pi^{'}_{T+1} \end{bmatrix}.$$

The two equations become

$$y_t = \frac{1-\beta\psi_2}{\kappa} \psi_2^{t-(T+1)} \pi^{'}_{T+1}, \tag{43}$$

$$\pi_t = \psi_2^{t-(T+1)} \pi^{'}_{T+1}. \tag{44}$$

We transform the $\pi^{'}_{T+1}$ back into original variables using

$$\begin{bmatrix} y^{'}_{T+1} \\ \pi^{'}_{T+1} \end{bmatrix} = G^{-1} \begin{bmatrix} y_{T+1} \\ \pi_{T+1} \end{bmatrix} = \frac{\kappa}{\beta\left(\psi_2 - \psi_1\right)} \begin{bmatrix} 1 & -\frac{1-\beta\psi_2}{\kappa} \\ -1 & \frac{1-\beta\psi_1}{\kappa} \end{bmatrix} \begin{bmatrix} y_{T+1} \\ \pi_{T+1} \end{bmatrix}.$$

Therefore,

$$y^{'}_{T+1} = \frac{\kappa}{\beta\left(\psi_2 - \psi_1\right)} \left[y_{T+1} - \frac{1-\beta\psi_2}{\kappa} \pi_{T+1} \right],$$

$$\pi^{'}_{T+1} = \frac{\kappa}{\beta\left(\psi_2 - \psi_1\right)} \left[-y_{T+1} + \frac{1-\beta\psi_1}{\kappa} \pi_{T+1} \right].$$

Setting $y^{'}_{T+1} = 0$, as previously assumes, assures that the system does not explode. This yields a relation between exit-period values of output and inflation given by

$$y_{T+1} = \frac{1-\beta\psi_2}{\kappa} \pi_{T+1}.$$

Substituting yields

$$\pi^{'}_{T+1} = \frac{\kappa}{\beta\left(\psi_2 - \psi_1\right)} \left[-\left(\frac{1-\beta\psi_2}{\kappa} \pi_{T+1} \right) + \frac{1-\beta\psi_1}{\kappa} \pi_{T+1} \right] = \pi_{T+1}.$$

Substituting into equations (43) and (44), the solutions for output and inflation for $t \geq T+1$ become

$$y_t = \left(\frac{1-\beta\psi_2}{\kappa} \right) \psi_2^{t-(T+1)} \pi_{T+1} \tag{45}$$

$$\pi_t = \psi_2^{t-(T+1)} \pi_{T+1} \tag{46}$$

10.3 Optimal Policy under Uncertainty

Consider the solution for optimal monetary policy. We repeat the Lagrangian, but reinstate expectations to yield

$$L_1 = E_1 \sum_{t=1}^{\infty} \beta^t \left\{ -\frac{1}{2}\left(\pi_t^2 + \lambda y_t^2\right) - \phi_{1,t}\left[\sigma\left(i_t - r_t^n - \pi_{t+1}\right) - y_{t+1} + y_t\right] - \phi_{2,t}\left[\pi_t - \kappa y_t - \beta\pi_{t+1}\right] + \phi_{3,t} i_t \right\},$$

Since the constraints are forward-looking, some terms involve period t Lagrange multipliers $\left(\phi_{1,t}, \phi_{2,t}\right)$ multiplied by period $t+1$ choice variables $\left(\pi_{t+1},\ y_{t+1}\right)$. Following Adam and Billi (2006), relabel the Lagrange multipliers for these terms as $\varphi_{i,t+1}$ $(i = 1, 2)$, and add transition equations $\varphi_{i,t+1} = \phi_{i,t}$. Collect all terms dated t and add $\varphi_{2,1}\pi_1 + \varphi_{1,1}\left(\frac{\sigma\pi_1 + y_1}{\beta}\right)$, defining $\varphi_{i,1} = 0$. This delivers the following Lagrangrian.

$$\begin{aligned} L_1 &= E_1 \sum_{t=1}^{\infty} \beta^t \left\{ -\frac{1}{2}\left(\pi_t^2 + \lambda y_t^2\right) - \phi_{1,t}\left[\sigma\left(i_t - r_t^n\right) + y_t\right] + \varphi_{1,t}\beta^{-1}\left(\sigma\pi_t + y_t\right) - \phi_{2,t}\left[\pi_t - \kappa y_t\right] + \varphi_{2,t}\pi_t \right\} \\ &\quad + \sum_{t=1}^{\infty} \beta^t \phi_{3,t} i_t. \end{aligned}$$

Note the Lagrangian is expressed as the infinite sum of period t objective functions, which depend only on variables in period t. At each t, the monetary authority has full information on the period-t objective function and chooses π_t, y_t, and i_t, to maximize that objective function, yielding first order conditions

$$-\phi_{2,t} + \varphi_{2,t} - \pi_t + \sigma\beta^{-1}\varphi_{1,t} = 0,$$

$$-\phi_{1,t} + \beta^{-1}\varphi_{1,t} - \lambda y_t + \kappa\phi_{2,t} = 0,$$

$$-\sigma\phi_{1,t} + \phi_{3,t} = 0 \qquad \phi_{3,t} i_t \geq 0 \qquad \phi_{3,t} \geq 0 \qquad i_t \geq 0.$$

Substituting for the $\varphi_{i,t}$ yields equations (14) and (15). This implies that choices made in period t do depend on lagged values, and therefore on past stochastic disturbances and choices. However, note that these equations are purely backward-looking, containing no expectations. The values of the multipliers do respond to past values and to stochastic changes in contemporaneous output and inflation.

We add the equations for output and inflation, equations (1) and (2). Together with equation (16), these five equations constitute a system in the two multipliers, the nominal interest rate, and in output and inflation and their expectations.

Equation (31) combines equations (1) and (2) using matrix notation. A forward solution of equation (31), allowing the possibility of each of the three interest rate regimes, for $t \leq \hat{t}$, yields

$$\begin{aligned} Z_t &= E_t Z_t = \left[\sum_{k=t}^{T_1} A^{-(k-t+1)} a r_{1,k}^n + A^{-(T_1-t+1)} Z_{T_1+1} \right] \varpi \qquad (47) \\ &+ \left[\sum_{k=t}^{T_3} A^{-(k-t+1)} a r_{3,k}^n + A^{-(T_3-t+1)} Z_{T_3+1} \right] \varpi \\ &+ \left[\sum_{k=t}^{T_2} A^{-(k-t+1)} a r_{2,k}^n + A^{-(T_2-t+1)} Z_{T_2+1} \right] (1 - 2\varpi) \\ &= R(1, t, T_1) \varpi + R(2, t, T_2)(1 - 2\varpi) + R(3, t, T_3) \varpi \\ &+ A^{-(T_1-t+1)} Z_{T_1+1} \varpi + A^{-(T_2-t+1)} Z_{T_2+1} (1 - 2\varpi) + A^{-(T_3-t+1)} Z_{T_3+1} \varpi. \end{aligned}$$

where

$$R(i, t, T_i) = \sum_{k=t}^{T_i} A^{-(k-t+1)} a r_{i,k}^n.$$

To obtain optimal values for exit times and inflation upon exit, we continue to solve the system. We center the system around the middle value for the shock and write the expression for Z_t for $t \leq \hat{t}$ as

$$\begin{aligned} Z_t &= R(2, t, T_2) + A^{-(T_2-t+1)} Z_{T_2+1} \\ &+ \varpi \left[R(1, t, T_1) + R(3, t, T_3) - 2R(2, t, T_2) \right] \\ &+ \varpi \left[A^{-(T_1-t+1)} Z_{T_1+1} + A^{-(T_3-t+1)} Z_{T_3+1} - 2A^{-(T_2-t+1)} Z_{T_2+1} \right]. \end{aligned}$$

There are three time paths of Z for $T_i \geq t > \hat{t}$, each conditional on the realization of the interest rate regime. For $i \in (1, 2, 3)$, the time path for Z for $t > \hat{t}$ is given by

$$\begin{aligned} Z_{i,t} &= E_t Z_{1,t} = \sum_{k=t}^{T_i} A^{-(k-t+1)} a r_{i,k}^n + A^{-(T_i-t+1)} Z_{T_i+1} \\ &= R(i, t, T_i) + A^{-(T_i-t+1)} Z_{T_i+1}. \end{aligned}$$

From equations (22) and (30),

$$Z_{T_i+1} = W \Phi_{Ti},$$

with W given by equation (32) and the set of multipliers for each interest rate regime by

$$\Phi_{T_i} = \begin{bmatrix} \phi_{1,T_i} \\ \phi_{2,T_i} \end{bmatrix}.$$

Substituting, the values for the $Z_t's$, for $t \leq \hat{t}$, are given by

$$\begin{aligned} Z_t \;\; = \;\; & R(2,t,T_2) + A^{-(T_2-t+1)} W \Phi_{T_2} \\ & + \varpi \left[R(1,t,T_1) + R(3,t,T_3) - 2R(2,t,T_2) \right] \\ & + \varpi \left[A^{-(T_1-t+1)} W \Phi_{T_1} + A^{-(T_3-t+1)} W \Phi_{T_3} - 2A^{-(T_2-t+1)} W \Phi_{T_2} \right]. \end{aligned}$$

and for $t > \hat{t}$ are

$$Z_{i,t} = E_t Z_{i,t} = R(i,t,T_i) + A^{-(T_i-t+1)} W \Phi_{T_i}.$$

Write the equations for the multipliers, using equation (34) and recognizing that there is a set of equations for each interest rate regime, as

$$\Phi_{i,t} = C \Phi_{i,t-1} - D Z_{i,t},$$

where

$$\Phi_{i,t} = \begin{bmatrix} \phi_{1_i,t} \\ \phi_{2_i,t} \end{bmatrix}$$

and C and D are given by equation (35) as before. Actual values of the multipliers are chosen as a function of current and past values, so that we are solving for actual values of the multipliers not expectations. Solve Φ_t forward to time T, imposing that initial values (period 0) of both multipliers are zero, to yield

$$\Phi_{T_i} = -\sum_{t=1}^{\hat{t}} C^{T_i-t} D Z_t - \sum_{t=\hat{t}+1}^{T_i} C^{T_i-t} D Z_{i,t}.$$

We center the system around the middle value for the shock, yielding.

$$\Phi_{T_2} = -\sum_{t=1}^{\hat{t}} C^{T_2-t} D Z_t - \sum_{t=\hat{t}+1}^{T_2} C^{T_2-t} D Z_{2,t}$$

$$\Phi_{T_1} = C^{T_1-T_2} \Phi_{T_2} + \sum_{t=\hat{t}+1}^{T_2} C^{T_2-t} D Z_{2,t} - \sum_{t=\hat{t}+1}^{T_1} C^{T_1-t} D Z_{1,t}$$

$$\Phi_{T_3} = C^{T_3-T_2}\Phi_{T_2} + \sum_{t=\hat{t}+1}^{T_2} C^{T_2-t}DZ_{2,t} - \sum_{t=\hat{t}+1}^{T_3} C^{T_3-t}DZ_{3,t}.$$

Substituting the $Z's$ into Φ_{T_2} yields

$$\begin{aligned}
\Phi_{T_2} &= -\sum_{t=1}^{T_2} C^{T_2-t}D\left[R(2,t,T_2) + A^{-(T_2-t+1)}W\Phi_{T_2}\right] \\
&\quad -\varpi\sum_{t=1}^{\hat{t}} C^{T_2-t}D\left[R(1,t,T_1) + R(3,t,T_3) - 2R(2,t,T_2)\right] \\
&\quad -\varpi\sum_{t=1}^{\hat{t}} C^{T_2-t}D\left[A^{-(T_1-t+1)}W\Phi_{T_1} + A^{-(T_3-t+1)}W\Phi_{T_3} - 2A^{-(T_2-t+1)}W\Phi_{T_2}\right]
\end{aligned}$$

Solving for Φ_{T_2} as a function of differences yields

$$\Phi_{T_2} = -\left[I + \sum_{t=1}^{T_2} C^{T_2-t}DA^{-(T_2-t+1)}W\right]^{-1}\left[\sum_{t=1}^{T_2} C^{T_2-t}DR(2,t,T_2) + \Delta\varpi\right]$$

where

$$\begin{aligned}
\Delta &= -\sum_{t=1}^{\hat{t}} C^{T_2-t}D\left[R(1,t,T_1) + R(3,t,T_3) - 2R(2,t,T_2)\right] \\
&\quad -\sum_{t=1}^{\hat{t}} C^{T_2-t}D\left[A^{-(T_1-t+1)}W\Phi_{T_1} + A^{-(T_3-t+1)}W\Phi_{T_3} - 2A^{-(T_2-t+1)}W\Phi_{T_2}\right]
\end{aligned}$$

Solving for the multipliers associated with alternative interest rate paths ($i \in (1,3)$) as a function of the multipliers associated with regime 2 yields

$$\begin{aligned}
\Phi_{T_i} &= C^{T_i-T_2}\Phi_{T_2} + \sum_{t=\hat{t}+1}^{T_2} C^{T_2-t}D\left[R(2,t,T_2) + A^{-(T_2-t+1)}W\Phi_{T_2}\right] \\
&\quad -\sum_{t=\hat{t}+1}^{T_i} C^{T_i-t}D\left[R(1,t,T_i) + A^{-(T_i-t+1)}W\Phi_{T_i}\right]
\end{aligned}$$

Solving for Φ_{T_1} yields

$$\begin{aligned}\Phi_{T_1} \;=\;& \left[I + \sum_{t=\hat{t}+1}^{T_1} C^{T_1-t} D A^{-(T_1-t+1)} W\right]^{-1} \\ & \times \left[C^{T_1-T_2}\left(\Phi_{T_2} + \sum_{t=1}^{T_2} C^{T_2-t} D \left[R(2,t,T_2) + A^{-(T_2-t+1)} W \Phi_{T_2}\right]\right) - \sum_{t=\hat{t}+1}^{T_1} C^{T_1-t} D R(1,t,T_1)\right]\end{aligned}$$

We solve numerically for the values for Φ_{T_i} by first solving for Φ_{T_2}, setting $\Delta = 0$. We use this value to solve for Φ_{T_1} and Φ_{T_3}. Next, we use the values for Φ_{T_1} and Φ_{T_3} to solve for Δ. We resolve for Φ_{T_2} using the value for Δ. We resolve for Φ_{T_1} and Φ_{T_3} using the new value for Φ_{T_2} and continue to iterate the process until it converges.[21]

[21] We offer no proof that convergence will occur, but in our examples, it occurs very quickly, in about five iterations.

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