Mathematics vs. Statistics in tackling Environmental Economics uncertainty

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Abstract

In this paper the appropriate background in Mathematics and Statistics is considered in developing methods to investigate Risk Analysis problems associated with Environmental Economics uncertainty. New senses of uncertainty are introduced and a number of sources of uncertainty are discussed and presented. The causes of uncertainty are recognized helping to understand how they affect the adopted policies and how important their management is in any decision-making process. We show Mathematical Models formulate the problem and Statistical models offer possible solutions, restricting the underlying uncertainty, given the model and the error assumptions are correct. As uncertainty is always present we suggest ways on how to handle it.

Keywords: Uncertainty; Environmental Economics; Mathematics; Statistics.

JEL Codes: C02; C60; Q00; Q50.

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1. Introduction

Human activities have caused environmental damages and degradation and the internalization of externalities has been the main concern worldwide. At the same time, uncertainty and unconvincing scientific evidence of many biological and biophysical processes are present in many policy decisions. In such cases uncertainty cannot be disregarded or be ignored in any analysis. Uncertainty has to be taken into consideration in all decision-making agendas.

Recognizing the causes of uncertainty, realizing how they affect decision making and managing them as possible is crucial in any decision-making process. In Economics, a distinction is made between risk and uncertainty with risk referring to the probability of an outcome taking place being measurable, while uncertainty refers to the lack of information about outcomes and their probability and magnitude of taking place. Managing risk can be handled in a fairly simple way in a cost-benefit analysis setup while uncertainty requires the tackling of many issues together with the nature of uncertainty.

The easiest way to understand *uncertainty* is the situation for which we are not certain! That is there is limited knowledge, restricted information for the phenomenon under investigation and this justifies why eventually is related to Fisher's information, as well as with other information measures, involving probability density functions. As Fisher's information matrix is the inverse of the variance-covariance matrix, uncertainty is related to the degree of precision one variable is measured. When the uncertainty is such, with some levels of it causing undesired results (in politics, health, environment etc) – or even a significant loss (such as in Economics) we are referring to risk. To be more precise we should refer to Relative Risk (Halkos 2006, 2011), usually defined as the log(odds).

In Economics the early work of Knight (1921) clearly distinguished uncertainty and risk. Risk management is based on the fact that all processes and activities are controlled, so that to minimize the adverse effects of accidental loses on the organization under investigation. Let us underline that variability refers to how spread out a set of data is and how capable is to change. Similarly, in Statistics it is measured by dispersion (standard deviation). Variability has not to be confused with variation, actually meaning something different of the same type! In principle variability cannot be reduced (that is why it should be characterized as well as possible), while uncertainty can (enlarging the data set). This is not always easy to be succeeded with typical example being problems from Political Science (see the pioneering book on Political Statistics by Davis, 1954) or data applying in the extreme value theory (EVT) (see Gomes et. al., 2015). Actually the idea of analysing "risk" started investigating political conflicts, or even war (see the early work of Wright, 1942).

In this paper we search the appropriate background in Mathematics and/or Statistics, and the developed methods to investigate Risk Analysis problems associated with Environmental Economics uncertainty.

2. Background

When we refer to the term environment (to us) we have to clarify that we mean the complementary set to our self. So the environment to an industry B is simply B^c, the complementary set of B, that certainly includes physical and chemical forces. The set B^c can be considered that includes almost all Universe, besides B – the Universe (or Cosmos) is all the space and time and its content. The Mathematical Model (MM), which attempts to express the Universe (Cosmos) and its current behaviour, as well as its evolution over time, is known as Cosmological Model (CM). The most well known CM is the Λ-CDM (Lambda Cold Dark Matter) that insures Universe is flat in shape with only 0.4% margin of error (see among others Zeldovich and Novikov, 1983). This is why by Natural Environment we are restricted to Earth (recently) or part of Earth (as it was some years ago) for a certain time interval.

In principle we need to formalize uncertainty in Natural Resources and Environmental Economics problems to reflect the incomplete information available. A representation of uncertainty can be obtained by a global consideration of the involved uncertain quantities on the statements. In practice we know them (the quantities on the statements) only for a restrict number of values, only for a given set (S), subset of the real domain (D), i.e. $S \subseteq D$. Uncertainty analysis aims at exhausting/ spanning/ tracing this range S, so that to extract decision for D under some confidence.

Decision makers in Environmental Economics aspects are usually willing to depart from default assumptions and adopt a model to describe the problem under investigation. But what model: mathematical or statistical? The latter is flexible, considers (and accepts that exist) in principle errors, while the former is solid, strictly defined. The propagation of errors is also covered by Numerical Analysis, as far as the Mathematical point of view considering "errors". But the main difference, considering the theory of errors, between Mathematics and Statistics is the Normal distribution assumption adopted by Statistics. That is why we believe the Generalized Normal distribution with an extra (shape) parameter added to position and scale parameters is vital (see Kitsos and Tavoularis, 2009).

A number of different orientation approaches to represent the underlying uncertainty were considered. Namely:

- *Interval Mathematics* facing the imprecise measurement situation (see Broadwater et al., 1994; for a review Alefed and Mayer, 2000 and for the optimization problem Wolfe, 2000). The interesting point is the solution of simultaneous equations under the Interval Mathematics, for the environmental problem we discuss as an interval approach, i.e. working not with exact values x, but with the interval $[x', x'']$ might be useful. That is we try to include uncertainty in the solution of the problem, accepting more than one value for the "unknown".

- *Evidence Theory* which is defined through set theory (see Dempster, 1968); Shafer, 1976). Thus, a "distance" measure can be defined (see Jouselme and Maupin, 2012). The Quantitative Risk Analysis (QRA) is a particular branch of Risk Analysis (see Zarikas and Kitsos, 2015) investigating and assessing probability to "what can be wrong". Poor knowledge on the phenomenon under investigation, with Chernobyl accident being a typical example, provides obscure knowledge of the probability level that such "extreme' events are possible (Kitsos, 2005). The probability assigned is positive implying the event under consideration can take place but the Qualified Risk Analysis level is rather difficult to be estimated. That is why two new senses of uncertainty were introduced.
  - *Epistemic uncertainty* is the one due to lack of knowledge of quantities of the Environment (or the System, more general).
  - *Aleatory uncertainty* is the one associated with the Environment under consideration.

  The variation of the atmospheric conditions is an example for the latter, while the lack of experimental data is for the former. The variation of the estimated life of the "components" of a "system" is an aleatory uncertainty
- *Fuzzy Theory* is based on the analysis of vagueness of the involved variables, rather than the stochastic nature of them (see among others Klir and Yan, 1995). The Fuzzy Logic extends our current believes that an element x belongs or does not belong to the set Q from the (universal) set Ω, Q⊆Ω. In mathematical terms the binary system true-false or either 0 or 1 i.e. a sentence belongs to {0,1} is extended. The binary response is now the set [0,1]. That is the "characteristic function" is extended to "membership function" for the given fuzzy set Q ⊆ Ω as

$$M_Q = Q \subseteq \Omega \mapsto [0,1] : x \mapsto M_Q(x) \qquad (1)$$

The number $M_Q(x)$ in [0,1] declares the *degree of participation* of the element x in Q which belongs/participates in the fuzzy set Q of Ω. In particular:

$M_Q(x) = 1$ declares x belongs to Q

0 declares x does not belong to Q

q in (0,1) declares x belongs "partially", i.e. to some degree to Q.

We would like to stress that fuzziness and randomness are different terms approaching different lines of thought. Randomness concerns a well-defined event but it is uncertain if will take place or not. Fuzziness is referred to situation which is not well defined and can only be described, in a sufficient way, when it is known how we shall move between different classes.

- *Probabilistic Analysis* as appears in the pioneering work of Feller (1950) offers almost nothing to uncertainty, while Pfeiffer (1978), among others, associates uncertainty with Fishers information. Tan (1991) approaches the risk problem of cancer through a completely mathematical approach, while Bernado and Smith (1994) approach the Probabilistic Analysis from the Bayesian point of view. The Evidence Theory and the Fuzzy Theory introduce new terms, far from the classical theories, either measure theory or Bayesian. That is the probability and distribution function are not any more valid, but the details are beyond the target this paper. The pioneering work of Bliss (1934, 1939) addressed the Probability Theory to measure risk and uncertainty, and since then the statistical modelling was providing a theoretical background to calculate Relative Risks, rather, than probability levels. Under this line of though the Logit (and to a lesser extend) Probit models appear an "aesthetic appeal" in Risk Analysis especially for Cancer problems (see among others, Edler and Kitsos 2005).
- *Statistical Analysis* has been adopted as a Data Analysis tool for a number of different fields. When Risk is involved, related to uncertainty, a number of models have been applied and as far as the Cancer field concerns, Breslow and Day (1980) provide an extensive analysis. But more is needed for particular environmental problems based on

the statistical modelling as it was faced by Halkos (1996), Halkos and Kitsos (2005) and Halkos and Kitsou (2018). Adopting Logit method, a Statistical Analysis was presented by Halkos and Kitsos (2010, 2012) while Kitsos (2011) proved that Logit model remains invariant to linear transformations, which practically means that Logit model is valid (or just transformed) to areas with over polluted CO_2 and just polluted, provided the underlying source of pollution is the same.

- *Other Methods* were also discussed, for particular problems, either with Statistics or Mathematics orientation. Modelling extreme rainfall, Alves and Rosario (2015) adopted the extreme value theory – which models and measures events occurred, in principle, with a very small probability - and they evaluated the quartiles (even the 0.01 quartile) of monthly maximum rainfall. The acid rain problem, which involves environmental and economic analysis beyond the technical one, has been tackled through Linear Programming (Halkos 1993, 1994). Halkos (1996) provided results for abatement rates under certainty and uncertainty, which are compared with the Nash relative measures. As far as the Risk Analysis in Business, Zarikas and Kitsos (2015) worked under the reference class forecasting (RCF) adopting the tolerance regions rather than confidence regions.

The adopting modelling in Environmental Economics provide food for thought for the imposed dilemma: Mathematics or Statistics provide the appropriate background to solve an imposed Environmental Economics problem. From the above discussed methods is evident that both Mathematics and Statistics provide a "tool kit" for Environmental Economics analysts, who have to choose the appropriate one for the problem under consideration.

It is true that the Mathematical thinking is successfully adapted in a number of economic applications. Dynamic modelling is recently an attractive way to tackle a dynamic economic problem relying on Pontryagin's maximum principle with the main variables of the dynamic model differentiated into the state and control variables. The former is defined as a variable describing the state of the economic system transferred optimally from time zero (initial time) to the terminal time. Similarly, control variables may help this optimal transfer from initial to terminal time of the system's state (Halkos and Papageorgiou, 2016).

In principle the economic approach does not have the framework of Engineering, where the adoption of similar Mathematical Techniques seems a natural consequence (Pierre, 1986). Moreover Economics is not an Experimental Science. Mathematics in such cases formulates a physical phenomenon while in Environmental Economics helps in reducing the involved uncertainty. The problem becomes more crucial as in Environmental Economics a number of candidate and different models can be adopted, therefore there is neither the solid background of Engineering, nor that of the Economical fields mentioned already.

Moreover we would like to indicate that different approaches produce quite different measurements of probability, not the probability that everybody understands. That results to the Maximum Likelihood Estimator (MLE) assigned to a probability density function (pdf) it is not any more valid in Evidence Theory were the likelihood is assigned to sets. Under the Fuzzy Logic the probability function and the distribution function are replaced by the possibility function and the possibility distribution function.

This analysis (and comparison) is beyond the target of this paper, which focuses on trying to understand, handle and analyse uncertainty involved in Environmental Economics. Under the classical theory a Mathematical or a Statistical approach might be proved a useful tool in the hands of environmental economists and more generally policy makers to evaluate or reduce uncertainty.

3. Modelling and measuring uncertainty

Uncertainty is strongly related to the physical problem under investigation. There is an intrinsic relation between the underlying mechanistic problem and the sources of uncertainty. When an environmental system tends to produce pollution responses beyond the existing interval of observations (what we called set S in the beginning of section 2) i.e. within the range D-S, what are actually uncertain are the predictions, the extrapolations beyond the set S itself and not the responses themselves. Moreover the model, among various contestant models, adopted to approach the response of the environmental problem under investigation causes uncertainty.

Therefore the relation between data and response it might be proved as a source of uncertainty: are the involved data set of variables sufficient? Are all the variables actually needed to explain the underlying problem included? The model uncertainty needs a special consideration. The Heisenberg principle is certainly applied in environmental problems to estimate uncertainty, as Fisher's information is related to Uncertainty (see among others, Rehacek and Hradil, 2004). We cannot assign appropriate damage estimations in the large number of Chernobyl accidents gathering reliable data. We cannot provide a large number of atomic bomb experiments to estimate atomic bomb survivors and the environmental damages caused. But certainly there are epidemiological studies of industrial pollutants (Diggle and Richardson, 1993) while their cost is still unevaluated.

The 'usual pollution' levels have been studied more precise due to the industrial development although the kind of pollution has changed, we moved from smog to new types of pollution such as asbestos. Although it seems clear that if we would like to quantify the level of uncertainty this depends on time, we will avoid including time to our discussion. Moreover it is not clear that uncertainty is a decreasing function of time for all problems.

3.1 Modelling uncertainty

A typical example comes from cancer (not only influenced from environmental conditions): it is unclear if cancer is best described as a multistage or a multi-hit process (Kitsos, 2012). Moreover there are various models tackling the cancer problem. Therefore there is an uncertainty about the model's structure, strongly related to theoretical knowledge about the underlying phenomena. From an environmental economics point of view, Halkos and Kitsos (2005) worked with a number of model specifications estimating eventually the Benefit Area as the intersection of given marginal abatement (hereafter MAC) with marginal damage cost functions (hereafter MD).

Another source of model uncertainty is emerged from the imposed assumption: either coming from the statistical process involved or from the distributions used to describe errors or the uncertainty itself. This line of thought covers a statistical approach, while the mathematical point of view reflects uncertainty for the moment you obtained that particular model and the involved assumptions.

It has to be clear that model uncertainty is not 'lack of fit' (Draper and Smith, 1998), and therefore it is not the error itself. In any case either under Mathematical or under Statistical approaches researchers have to work with consistence in the line of thought they adopt: the assumed model is correctly specified.

Even under Bayesian or Decision Theory the model uncertainty plays an important role. From the Bayesian point of view, Bernando and Smith (1994) refer to:

i. M-closed case, i.e. to believe that one of the models is 'true' without the explicit knowledge of which one it is.
ii. M-complete case, i.e. to work 'as if' the models are compared to a reference (not necessary the unknown one) model.

iii. M-open case i.e. is based on the model comparison, in the absence of reference model.

From the Decision Theory point of view the ‘data are correct on average’ and there is a number of loss functions, researchers choose the appropriate one to quantify the distance between model predictions and given data. The choice of the model is crucial and this will be clarified for the Environmental Economics Uncertainly in section 4.

3.2 Measuring uncertainty

One of the problems associated with the Normal distribution is the “fat tails” one. There are cases were the assumed Normal distribution in tails “contains more probability” than the usual 0.05. This is true in a number of economic applications and certainly in some environmental problems where pollution affects the “tails” more than 0.05. That is a need for a generalization of Normal was a necessity. There are some attempts to generalize the well-known Normal distribution or Gaussian. But the γ- order Generalized Normal Distribution (γ-GND) emerged from a completely mathematical problem – Logarithm Sobolev Inequality (LSI), which provides a solid background for it. The generalisation of the well-known multivariate distribution is discussed by Kitsos and Tavoularis (2009), Toulias et al. (2014) and Halkos and Kitsou (2015).

One of the merits of γ-GND is that for γ=2 coincides with the typical multivariate Normal while for γ=1 corresponds to Uniform and γ tends to ±∞ coincides with the Laplace. As a measure of *uncertainty* the Shannon entropy is usually adopted. It can be proved that the Shannon entropy of a random variable $X \sim N^{p_\gamma}(\mu,\Sigma)$ is

$$H(X) = p\frac{\gamma-1}{\gamma} + \log\frac{\sqrt{\det\Sigma}}{c(p.\gamma)} \qquad (2)$$

As the H(X) can be considered as a measure of uncertainty, expression (2) provides a measure of uncertainty for a number of distributions, belonging to the family of distributions of γ-GND. It can be proved that from (2) and X coming from the p-variate Normal, $N^P(\mu, \Sigma)$

it is:
$$H_N(X) = \frac{1}{2}\log(2\pi e)^p \left|\det \Sigma\right| \tag{3}$$

While for the p-variate Laplace with mean μ and variance Σ is

$$H(X) = p + \log \frac{p!\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2}+1)} \sqrt{\left|\det \Sigma\right|} \tag{4}$$

It is easy to be proved that when p = 1, one dimension exists then

For the Uniform distribution U (μ-σ, μ+σ) the entropy is $H_U(X)=\log 2\sigma$

For the normal N (μ, σ^2) is $H_N(X) = \log \sqrt{2\pi e}\sigma$ (5)

And for the Laplace L(μ, σ) is $H_L(X)=1+\log 2\sigma$

It is clear that the entropy depends only on the variance-covariance matrix Σ or σ in case p=1. In practice this means that the uncertainty is irrelevant to mean value μ (of pollution to an industry, say) but depends on the standard deviation (the experimental error). The Uniform distribution can be adopted if it is assumed that pollution levels are (almost) the same around the area [a, b], while the Laplace when it is assumed a "sharp explosion" around the center and much lower far from it. Estimates of (5) can be obtained in practical situations.

More specifically, as an example let us consider an analysis of the Total Pollution Cost (TPC) and provide food for thought of the involved uncertainty despite the extension and the accurate and sophisticated, so to speak, mathematical evaluations. The easiest way, as far as the mathematical calculations are concerned, despite its unrealistic character, is to assume that the stochastic "pollutant" variable X is uniformly distributed in the interval $\left[\frac{1}{2}-\delta, \frac{1}{2}+\delta\right]$, say, equivalently TPC is derived from the Uniform $U(\frac{1}{2}-\delta, \frac{1}{2}+\delta)$ implying a uniform density function for X of the form

$$f(X) = \frac{1}{2\delta} \quad for \quad X \in \left[\frac{1}{2}-\delta, \frac{1}{2}+\delta\right] \tag{6}$$

From the definition of the expected value the pollution related t- social cost for the linear tax

equals to $$\mathrm{E}\left[TPC_{t_l}\right]=\int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} TPC\ f(x)dx \tag{7}$$

It holds that (Halkos and Kitsou, 2015):

$$\mathrm{E}\left[(\kappa\mathrm{X}+\lambda)^2\right]=(\frac{\gamma}{\gamma-1})^{2\frac{\gamma-1}{\gamma}}\frac{\Gamma(3\frac{\gamma-1}{\gamma})}{\Gamma(\frac{\gamma-1}{\gamma})}(\kappa\delta)^2+\kappa\mu(\kappa\mu+2\lambda)+\lambda^2$$

$$Var((\kappa\mathrm{X}+\lambda^2))=(\frac{\gamma}{\gamma-1})^{4\frac{\gamma-1}{\gamma}}(\kappa\delta)^4\left[\frac{\Gamma(5\frac{\gamma-1}{\gamma})}{\Gamma(\frac{\gamma-1}{\gamma})}-4\frac{\Gamma^2(3\frac{\gamma-1}{\gamma})}{\Gamma^2(\frac{\gamma-1}{\gamma})}\right]-(\kappa\mu)^3(\kappa\mu+4\lambda)$$

$$+2(\kappa\delta)^2\left[2\lambda^2-(\kappa\mu)^2-2\kappa\lambda\mu\right](\frac{\gamma}{\gamma-1})\frac{2\gamma-1}{\gamma}\frac{\Gamma(3\frac{\gamma-1}{\gamma})}{\Gamma(\frac{\gamma-1}{\gamma})}$$

With different values of κ and λ a number of calculations for the corresponding TPC can be obtained. From the evaluated expectations it obviously holds that the quantity $\mathrm{E}\left[TPC_{t_l,\gamma}\right]$ in the case of Uniform distribution is less than the corresponding Normal distribution, which is less that the corresponding Laplace distribution. That is (Halkos and Kitsou, 2015):

$$\mathrm{E}\left[TPC_{t_l,1}\right]<\mathrm{E}\left[TPC_{t_l,2}\right]<\mathrm{E}\left[TPC_{t_l,\pm\infty}\right]$$

$$Var^U(TPC)<Var^N(TPC)<Var^L(TPC)$$

Now, recall that we have with γ=1 (the case of uniform) the expected value is less than in the case of γ=2 (the case of normal) and flatter compared to the other two cases. Similarly the results for the comparison between γ=2 (Normal) and γ=±∞ (the case of Laplace) show that Laplace is sharper among them. Thus estimates of Shannon entropy can be obtained.

As has been shown in Halkos and Kitsou (2015) if $X \sim N_\gamma(\mu, \sigma^2)$ then Proposition 1 for the expected value and the variance of TPC=(κX+λ)^2 are evaluated as functions of κ, λ, γ. With different values of κ and λ a number of calculations for the corresponding TPC can be obtained. Any general form of TPC=(κX+λ)^2 is presenting the appropriate area for TPC.

An extension of the calculation of expected value is needed as it can be either normal with the known tails or a "sharp" one around the 'center' with 'heavy tails', a Laplace distribution among others. Therefore the γ-order generalized Normal distribution was adopted. The expected value of TPC can be evaluated and it can be seen that that the distribution is not only the Uniform but the $N_\gamma(\mu, \sigma^2)$. Figure 1a represents the univariate γ-order generalized Normal distribution for various values of γ: γ=2 (normal distribution), γ=5, γ=10, γ=100, while Figure 1b, represents the bivariate 10-order generalized Normal $KT_{10}^2(0, I_2)$ with mean 0 and covariance matrix $\Sigma = I_2$. Alike, Figure 2 represents the relationship between Uniform, Normal and Laplace.

A new generalized entropy type measure of information $\mathbf{J}_\alpha(X)$, defined by Kitsos and Tavoularis (2009), a function of density function f(x), is

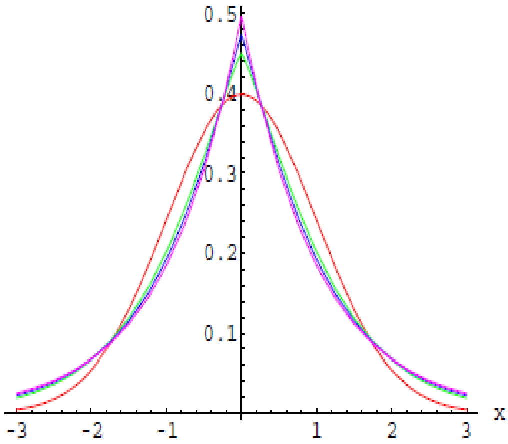
$$\mathbf{J}_\alpha(X) = \int_{\mathbb{R}^p} f(x) \left| \nabla \ln f(x) \right|^\alpha dx \tag{8}$$

For α=2, the measure of information $\mathbf{J}_2(X)$ is the Fisher's information measure

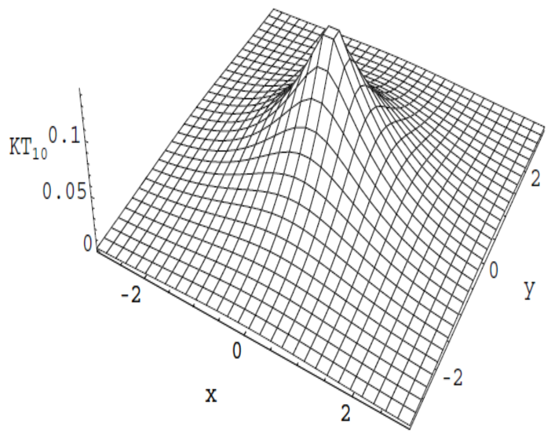
$$\mathbf{J}(X) = \int_{\mathbb{R}^p} f(x) \left[\frac{\nabla f(x)}{f(x)} \right]^2 dx = 4 \int_{\mathbb{R}^p} \left| \nabla \left(\sqrt{f(x)} \right) \right|^2 dx$$

i.e. $\mathbf{J}_2(X) = \mathbf{J}(X)$. That is, $\mathbf{J}_\alpha(X)$ is a generalized Fisher's entropy type information measure, and as the entropy, it is a function of density.

Figure 1: The univariate γ-order generalized Normals $KT_{\gamma}^{1}(0,1)$ for $\gamma = 2,5,10,100$ (**1a**) and the bivariate 10-order generalized Normal $KT_{10}^{2}(0,1)$ (**1b**)

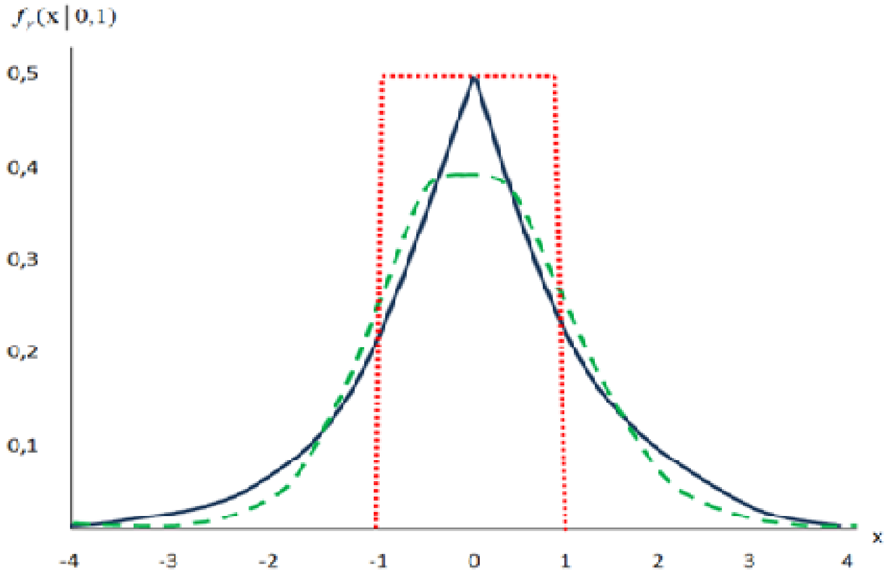


1a



1b

Figure 2: Graphical presentation of the relationship between Uniform, Normal and Laplace



Source: Halkos and Kitsou (2018)

Recall that the Shannon entropy H(X) is defined as $\mathbf{H}(X) = -\int_{\mathbb{R}^p} f(x)\ln f(x)dx$ (see Kitsos and Toulias, 2010]. The entropy power N(X) is defined through H(X) as

$$\mathbf{N}(X) = \frac{1}{2\pi e} e^{\frac{2}{p}\mathrm{H}(X)}$$

The definition of the entropy power of a random variable X was introduced by Shannon (1948) as the independent and identically distributed components of a *p*-dimensional white Gaussian random variable with entropy **H**(X).

The generalized entropy power $\mathbf{N}_\alpha(X)$ is of the form

$$\mathbf{N}_\alpha(X) = M_\alpha e^{\frac{2}{p}\mathrm{H}(X)},$$

with the normalizing factor being the appropriate generalization of $(2\pi e)^{-1}$, i.e.

$$M_\alpha = \left(\frac{\alpha-1}{\alpha e}\right)^{\alpha-1} \pi^{-\frac{\alpha}{2}} \left[\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(p\frac{\alpha-1}{\alpha}+1\right)}\right]^{\frac{a}{p}} = M_a(\xi_\alpha^p) \tag{9}$$

is still the power of the white Gaussian noise with the same entropy. Trivially, with α=2, the definition in (6) is reduced to the entropy power, i.e. $N_2(X)=N(X)$. In turn, the quantity

$$\xi_\gamma^p = \frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(p\frac{\gamma-1}{\gamma}+1\right)}$$

appears very often when we define various normalizing factors, under this line of thought.

Theorem 1: For the variance of X, Var(X) and the generalizing Fisher's entropy type information measure $J_\alpha(X)$, it holds $\left[\frac{2\pi e}{p}\mathrm{Var}(X)\right]^{1/2}\left[\frac{1}{p}M_\alpha \mathbf{J}_\alpha(X)\right]^{1/\alpha} \geq 1$ with M_α as in (8).

Corollary 1: When α=2 then $\mathrm{Var}(X)\mathbf{J}_2(X) \geq p^2$, and the Gramer-Rao inequality holds.

4. Environmental Economics Modeling

An environmental economics system, like any economic system, needs a compact description to study the effect of the differently involved components or to make prediction for the system under investigation through a Mathematical model. That is we translate the Environmental Economics problem into a Mathematical one. The first question arisen is if such a model exists. In limited cases a true functional relationship between a response and a variable (usually considered as independent) exists, mainly in Natural Science problems.

These functional models are not always available, and even if exist, the range of the involved variables are not always controllable. So we restrict the domain, to obtain Control models. The imposed assumptions and relationships constitute the Mathematical Model (MM), with typical examples in Economics and Statistics being the population growth models, the spread of technological innovations, etc. In principle a MM is referred to one response, needs the existing underlying mechanism, translating it into Mathematics and is based on clear and accurate definition of the problem (see for an approach to Mathematical Economics, among others, Chiang and Wainwright, 2005).

Needless to say there is a significant difference between economic models and engineering. In Engineering the underlying mechanism is (solid and experimentally verified) known and mathematically well interpreted (Pierre, 1986). Still, in Environmental Economics we are obliged to adopt the *calibration* procedure (see Halkos and Kitsos, 2005). This innovating approach helps us to tackle and solve the problem, overpassing calculation difficulties but at the same time creates a source of unexpected uncertainty. Although there is an optimal design approach for calibration (see Kitsos, 2002; Halkos, 1994; Hutton and Halkos, 1995) the non-experimental character of Environmental Economics, as well as that the Environmental conditions are unstable, calibration feeds the system with extra uncertainty (Halkos, 1996).

Working in these lines, Halkos and Papageorgiou (2008, 2016) presented the essentials of optimal control theory with reference to differential game as a theoretic analogue to optimal control. They discuss the Pontryagin's Maximum Principle as the main tool of analysis in open loop information structure for environmental models and the Hamilton–Jacobi–Bellman equation as tool for any closed loop informational structure.

Halkos (1992, 1994) discusses uncertainty in the damage cost function in a game theoretic set-up. The damage is defined as a function of depositions and takes the form

$$Q_i=Q_i(D_i) \quad \forall i \ =1,\ldots,27 \tag{10}$$

where $Q_i(.)$ is an increasing function of D_i. The total cost from a given level of pollutants emissions for country i is represented as,

$$C_i = \text{cost of abatement} + \text{damage cost} \tag{11}$$

And assuming damage costs are quadratic in deposition then:

$$C_i = A_i^2 + \beta_{1i} D_i + \beta_{2i} D_i^2 \tag{12}$$

In this way the total cost is minimized when

$$\beta_{2i} = (A_i/d_{ii}D_i) - (\beta_{1i}/2D_i) \tag{13}$$

and this is the information available to "calibrate" damage functions assuming national authorities perform as Nash partners in a non-cooperative game. If we set $\beta_{2i}=0$, then $\beta_{1i}=(2A_i/d_{ii})$, and total cost in the optimum is

$$C_i = A_i^2 + (2A_iD_i/d_{ii}) \tag{14}$$

Restrict β_{1i} to zero and calibrate β_{2i} as $\beta_{2i}=(A_i/d_{ii}D_i)$ yielding total costs of

$$C_i = A_i^2 + (A_iD_i/d_{ii}) \tag{15}$$

This assumption halves the implied damage costs at the optimum; the positive second derivative means that the benefits from reductions in depositions will also be less than implied by a linear damage function, while the costs of additional depositions will be greater. This, obviously, indicates the importance of damage cost functions uncertainty.

The situation is not the same when a Statistical Model (SM) is adopted. The main difference is that, in principle, a Linear Model is assumed to approach "reasonably" the existent data. To measure how well the, assumed correct, linear (usually regression) model, "fits the data", there is a number of well-known Statistical indexes. We shall not investigate the Statistical model as in the *mathematical* approach of McCullagh (2002), but we shall try to clarify how a regression model (Draper and Smith, 1998) works: Let n be the observed values of the input variable X. Each observed value of X, x_i , i=1,2,…n determines a commutative distribution function (cdf) F_i , i=1, 2,…n. From this cumulative density function (cdf) a random sample *of size one*, each time, is selected and denoted, usually, y_i, i=1,2,…,n. Thus the observed data are (x_i , y_i), i=1,2,…,n.

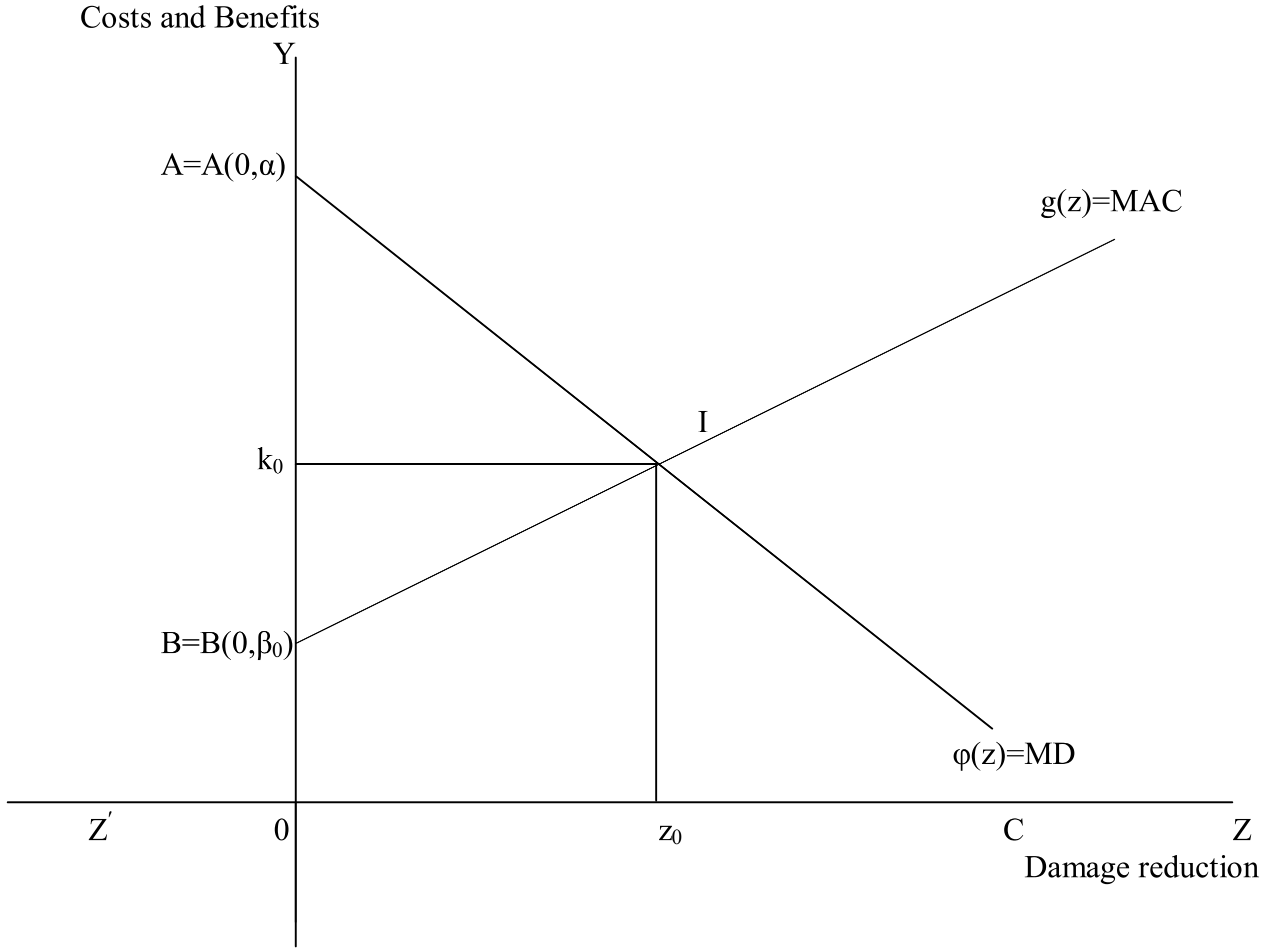
The possibility of different Mathematical models has been extensively discussed by Halkos and Kitsos (2005), when the evaluation of the Benefit Area (BA) is considered. Recall that Economic theory suggests that the optimal pollution level occurs when the marginal damage cost equals the marginal abatement cost. Consider the typical situation of the optimal pollution level as in Figure 3. The curves g(z) and φ(z) denote a country or a province or a municipality area's abatement and damage costs functions respectively. The point of their intersection I = I(z_o , k_o) represents the optimal level of pollution.

For modelling MAC three cases are considered : MAC=g(z)= linear [$\beta_0+\beta_1 z$, $\beta_1 \neq 0$]; quadratic [$\beta_0+\beta_1 z+\beta_2 z^2$, $\beta_2 > 0$]; exponential [$\beta_0 e^{\beta_1 z}$,$\beta_1 \neq 0$]. Also for modelling MD three cases were considered: MD=φ(z)=$\alpha+\beta z_o$ linear, or quadratic as $\mathrm{MB} = \varphi(z) = \alpha z^2 + \beta z + \gamma, \quad a \neq 0$ and $MD = \varphi(z) = \theta_0 \mathrm{e}^{\theta_1 z}$, $\beta_0 > 0,\ \theta_0 > \beta_0 > 0$. We believe that these three cases cover the majority of the real life problems.

In Figure 3 bellow it is assumed that the linear curves, MAC and MD, have an intersection (Mathematically this might not be true, when $\beta_o > \alpha$), and therefore the area of the region AIB, created by these curves, is what is known as Benefit Area (Kneese, 1972, among others). We shall denote hereafter by BA = (AIB). As it discussed below the benefit area (BA), is evaluated, as BA = (ABI) = ($AIz_0 0$) - ($BIz_0 0$)

with the areas represented in Figure 3.

Figure 3: Graphical presentation of the optimal level of Pollution



Halkos and Kitsos (2005) considered the abatement cost function MAC=g(z), as a continuous function g(.) and the marginal damage function MD=φ(z), as a continuous function φ(.), a number of cases were extensively discussed, and analytical results were imposed for the existence of the optimal pollution level. The BA was analytically evaluated in all the possible cases.

The possible mathematical cases discussed and the corresponding evaluated BA were:

Case 1: MD and MAC functions are both linear

$$BA_{LL} = (ABI) = \frac{(AB)(Ik_0)}{2} = \frac{(\alpha - \beta_0)(0z_0)}{2} = \frac{(\alpha - \beta_0)^2}{2(\beta_1 - \beta)} \tag{16}$$

Case 2: MD linear and MAC quadratic functions

$$BA_{LQ} = \frac{\alpha + g(z_0)}{2} z_0 - G(z_0) \ , \ with \ G(z) = \beta_0 z + \beta_1 \frac{z^2}{2} + \beta_2 \frac{z^3}{3} \tag{17}$$

Case 3: MD linear and MAC exponential functions

$$BA_{LE} = (AIz_0 0) - \int_0^{z_0} g(z)dz = \frac{\alpha + g(z_0)}{2} z_0 - \left[G(z_0) - G(0)\right] \tag{18}$$

$$With \ \ G(z_0) - G(0) = \int_0^{z_0} \beta_0 e^{\beta_1 z} dz = \frac{\beta_0}{\beta_1} \int_0^{z_0} e^{\beta_1 z} d(\beta_1 z) = \frac{\beta_0}{\beta_1}(e^{\beta_1 z_0} - 1)$$

Case 4: MD quadratic and MAC linear functions

$$\mathrm{BA}_{QL} = (\mathrm{ABI}) = \int_0^{z_0} (\varphi(z) - g(z))dz = \int_0^{z_0} (az^2 + (\beta - \beta_1)z + (\gamma - \beta_0))dz$$

$$= \left[\alpha \frac{z^3}{3} + (\beta - \beta_1)\frac{z^2}{2} + (\gamma - \beta_0)z\right]_0^{z_0} = \alpha \frac{z_0^3}{3} + (\beta - \beta_1)\frac{z_0^2}{2} + (\gamma - \beta_0)z_0 \tag{19}$$

Case 5: MD and MAC functions both quadratic

$$\mathrm{BA}_{QQ} = \int_0^{z_0} (\varphi(z) - g(z))dz = (\alpha - \beta_2)\frac{z_0^3}{3} + (\beta - \beta_1)\frac{z_0^2}{2} + (\gamma - \beta_0)z_0 \tag{20}$$

Case 6: MD quadratic and MAC exponential functions

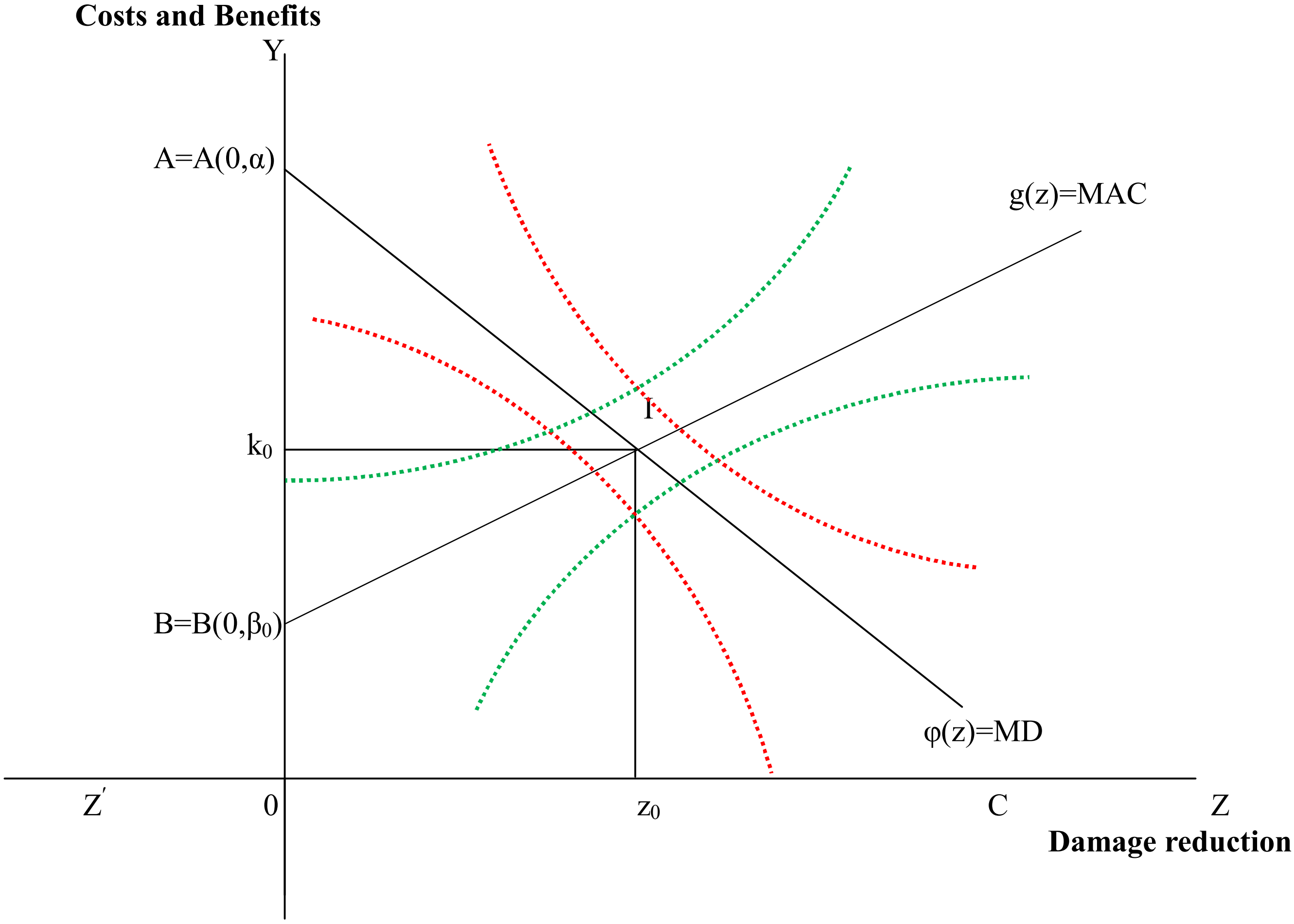
$$\mathrm{BA}_{EQ} = \int_0^{z_0} (\varphi(z) - g(z))dz = \alpha \frac{z_0^3}{3} + \beta \frac{z_0^2}{2} + \gamma z_0 - \frac{\beta_0}{\beta_1}\left(e^{\beta_1 z_0} - 1\right) \tag{21}$$

Case 7: MD and MAC both exponential functions

$$\mathrm{BA}_{EE} = \int_0^{z_0} (\varphi(z) - g(z))dz = \frac{\beta_0}{\beta_1}\left(e^{\beta_1 z_0} - 1\right) - \frac{\theta_0}{\theta_1}\left(e^{\theta_1 z_0} - 1\right) \tag{22}$$

In Case 3 point z_o, the optimal level of reduction pollution numerically (through Numerical iteration schemes) while in all cases was analytically evaluated. That is the optimal restriction of damages level, z_0, in the exponential case only approximately can be evaluated. That is the cause the corresponding optimal cost or benefit level can only approximated evaluated too.

Figure 4: Graphical presentation of the optimal level of pollution



It is clear that in all the above cases, the intersection of φ(z) and g(z) is fulfilled, i.e. $g(z_0)=\varphi(z_0)$, when z_0 is the optimal restriction in damages, for the case under consideration. In principle this is true for any MAC and MD. For Cases 2-7 the evaluated BA is a function of the corresponding z_o, the optimal level of pollution reduction, which has been evaluated explicitly in each case. Halkos and Kitsou (2015) considered the cases were BA is not possible, Mathematically, to be evaluated while Halkos and Kitsou (2018) considered the whole problem for a rather global and theoretical approach.

From the above, it is obvious that in Environmental Economics problems uncertainty has to be appropriately modelled. Needless to say all models serve a theoretical need and there is a certain target in each case to be adopted. The second point is how the assumed models will be evaluated. Halkos and Kitsos (2005) worked using regression analysis of the current abatement level. Facing the problem of limited or not existing data they moved to apply calibrating methods.

5. Discussion

A number of sources of uncertainty were discussed and presented. A fundamental source of model uncertainty is emerged from the imposed assumption either coming from the statistical process involved and from the distributions used to describe errors or the uncertainty itself as shown in section 3.1. The assumption for the assumed distribution has been enlarged, considering a broader family of distributions, the γ-order Generalized Normal, and the appropriate measures of uncertainty, mainly based on Shannon entropy were presented. Shannon entropy although so well working for engineering applications, does not have been applied widely in Environmental Economics – we shall encourage to be adopted and applied at least in the cases discussed above.

Despite the probability level contribution, which results to a variety of possible assumptions, depending on the scale parameter γ, the discussion on the model selection might be considerable.

The fact that the estimated, estMAC and estMD say, offer a source of error from the real MAC and MD, give the possibility to reduce it, considering the confidence intervals (L, U) and (l, u) for the MAC and MD respectively, as shown in Figure 4. Therefore there is the possibility of 4 intersections, plus the initial, so eventually it is possible to have 5 Benefit Areas for each case considered as above. The uncertainty we tried to reduce is still there, but under measurement! Nevertheless yet it is unknown which of the 5 BA is closer to the true one, so roughly speaking there is a 1/5 chance to choose the right one. We strongly believe we have to evaluate them: choose the model with some uncertainty, estimate the parameters with OLS to reduce error and evaluate the possible Benefit Areas.

Figures 3 and 4 provide evidence for the above analysis of Mathematics vs Statistics in coping with uncertainty in Environmental Economics. Specifically, Figure 3 provides due to solid MM ONE Benefit Area, while Figure 4 discusses the possibility of FIVE BA and a confidence interval for z_o, the optimal level of pollution reduction, as well as for k_o, the optimal cost level. That is MM formulates the problem and SM offers possible solutions, restricting the underlying uncertainty, provided the error assumptions is correct, as well as the model. When more complicated models are assumed (Halkos and Kitsou, 2018) it is clear that the evaluation of the Benefit Area is more complicated, that one might believe. Moreover *uncertainty* is always present but we now have insights on where and how we can handle it.

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