Sion's minimax theorem and Nash equilibrium of symmetric three-players zero-sum game

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Abstract

About a symmetric three-players zero-sum game we will show the following results.

1. A modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy are proved by the existence of a symmetric Nash equilibrium.
2. The existence of a symmetric Nash equilibrium is proved by the modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy.

Thus, they are equivalent. If a zero-sum game is asymmetric, maximin strategies and minimax strategies of players do not correspond to Nash equilibrium strategies. If it is symmetric, the maximin strategies and the minimax strategies constitute a Nash equilibrium. However, without the coincidence of the maximin strategy and the minimax strategy there may exist an asymmetric equilibrium in a symmetric three-players zero-sum game.

Keywords: three-players zero-sum game, Nash equilibrium, Sion's minimax theorem

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1 Introduction

We consider the relation between Sion's minimax theorem for a continuous function and the existence of Nash equilibrium in a symmetric three-players zero-sum game. We will show the following results.

1. A modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy are proved by the existence of a symmetric Nash equilibrium.

2. The existence of a symmetric Nash equilibrium is proved by the modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy.

Thus, they are equivalent. However, without the coincidence of the maximin strategy and the minimax strategy there may exist an asymmetric equilibrium in a symmetric three-players zero-sum game.

An example of such a game is a relative profit maximization game in a Cournot oligopoly. Suppose that there are three firms, A, B and C in an oligopolistic industry. Let $\bar{\pi}_A$, $\bar{\pi}_B$ and $\bar{\pi}_C$ be the absolute profits of the firms. Then, their relative profits are

$$\pi_A = \bar{\pi}_A - \frac{1}{2}(\bar{\pi}_B + \bar{\pi}_C),\ \pi_B = \bar{\pi}_B - \frac{1}{2}(\bar{\pi}_A + \bar{\pi}_C),\ \pi_C = \bar{\pi}_C - \frac{1}{2}(\bar{\pi}_B + \bar{\pi}_C).$$

We see

$$\pi_A + \pi_B + \pi_C = \bar{\pi}_A + \bar{\pi}_B + \bar{\pi}_C - (\bar{\pi}_A + \bar{\pi}_B + \bar{\pi}_C) = 0.$$

Thus, the relative profit maximization game in a Cournot oligopoly is a zero-sum game[1]. If the oligopoly is asymmetric because the demand function is not symmetric (in a case of differentiated goods) or firms have different cost functions (in both homogeneous and differentiated goods cases), maximin strategies and minimax strategies of firms do not correspond to Nash equilibrium strategies. However, if the demand function is symmetric and the firms have the same cost function, the maximin strategies and the minimax strategies constitute a Nash equilibrium.

2 The model and Sion's minimax theorem

Consider a symmetric three-players zero-sum game. There are three players, A, B and C. The strategic variables for Players A, B and C are, respectively, s_A, s_B, s_C, and $(s_A, s_B, s_C) \in S_A \times S_B \times S_C$. S_A, S_B and S_C are convex and compact sets in linear topological spaces. The payoff function of each player is $u_i(s_A, s_B, s_C),\ i = A, B, C$. We assume

> u_A, u_B and u_C are continuous on $S_A \times S_B \times S_C$, quasi-concave on S_i for each $s_j \in S_j,\ j \neq i$, and quasi-convex on S_j for $j \neq i$ for each $s_i \in S_i,\ i = A, B, C$.

[1]About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

Symmetry of a game means that the payoff functions of the players are symmetric, and in the payoff function of each Player i, Players j and k, $j, k \neq i$, are interchangeable. If the game is symmetric and zero-sum, we have

$$u_A(s_A, s_B, s_C) + u_B(s_A, s_B, s_C) + u_C(s_A, s_B, s_C) = 0, \tag{1}$$

for given (s_A, s_B, s_C). Also S_A, S_B and S_C are identical. Denote them by S.

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \longrightarrow \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

We follow the description of this theorem in Kindler (2005).

When s_C is given, $u_A(s_A, s_B, s_C)$ is a function of s_A and s_B. We can apply Lemma 1 to such a situation, and get the following equation

$$\max_{s_A \in S_A} \min_{s_B \in S_B} u_A(s_A, s_B, s_C) = \min_{s_B \in S_B} \max_{s_A \in S_A} u_A(s_A, s_B, s_C) \text{ given } s_C. \tag{2}$$

Now we assume

Assumption 1.

$$\arg \max_{s_A \in S_A} \min_{s_B \in S_B} u_A(s_A, s_B, s_C) = \arg \min_{s_B \in S_B} \max_{s_A \in S_A} u_A(s_A, s_B, s_C),$$

that is, the maximin strategy and the minimax strategy for Player A coincide given s_C.

In (2) and this assumption Players A, B and C are mutually interchangeable.

Let s be a value of s_C. Consider the following function;

$$s \longrightarrow \arg \max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s).$$

Since u_A is continuous and S is compact, this function is also continuous. Thus, by the Glicksberg fixed point theorem (Glicksberg (1952)) there exists a fixed point. Denote it by $\tilde{s}$. $\tilde{s}$ satisfies

$$\arg \max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, \tilde{s}) = \tilde{s}.$$

Based on Assumption 1 we present a modified version of Sion's minimax theorem.

Lemma 2 (Modified version of Sion's minimax theorem). *Let S_A and S_B be non-void convex and compact subsets of two linear topological spaces, and let $u_A : S_A \times S_B \longrightarrow \mathbb{R}$ given s_C be a function that is continuous on $S_A \times S_B$, quasi-concave on S_A and quasi-convex on S_B. Then, under Assumption 1 there exists $s_C = s$ such that*

$$\max_{s_A \in S_A} \min_{s_B \in S_B} u_A(s_A, s_B, s) = \min_{s_B \in S_B} \max_{s_A \in S_A} u_A(s_A, s_B, s).$$

and

$$\arg\max_{s_A \in S_A} \min_{s_B \in S_B} u_A(s_A, s_B, s) = \arg\min_{s_B \in S_B} \max_{s_A \in S_A} u_A(s_A, s_B, s) = s.$$

We assume that $\arg\max_{s_A \in S_A} \min_{s_B \in S_B} u_A(s_A, s_B, s_C)$ and $\arg\min_{s_B \in S_B} \max_{s_A \in S_A} u_A(s_A, s_B, s_C)$ are unique, that is, single-valued. By the maximum theorem they are continuous in s_C. Also, throughout this paper we assume that the maximin strategy and the minimax strategy of players in any situation are unique, and the best response of players in any situation is unique.

3 The main results

Consider a Nash equilibrium of a symmetric three-players zero-sum game. Let s_A^*, s_B^*, s_C^* be the values of s_A, s_B, s_C which, respectively, maximize u_A given s_B and s_C, maximize u_B given s_A and s_C, maximize u_C given s_A and s_B, in a neighborhood around (s_A^*, s_B^*, s_C^*) in $S_A \times S_B \times S_C = S^3$. Then,

$$u_A(s_A^*, s_B^*, s_C^*) \geq u_A(s_A, s_B^*, s_C^*) \text{ for all } s_A \neq s_A^*,$$

$$u_B(s_A^*, s_B^*, s_C^*) \geq u_B(s_A^*, s_B, s_C^*) \text{ for all } s_B \neq s_B^*,$$

$$u_C(s_A^*, s_B^*, s_C^*) \geq u_C(s_A^*, s_B^*, s_C) \text{ for all } s_C \neq s_C^*.$$

If the Nash equilibrium is symmetric,s_A^*, s_B^* and s_C^* are equal at equilibria. Then, $u_A(s_A^*, s_B^*, s_C^*)$, $u_B(s_A^*, s_B^*, s_C^*)$ and $u_C(s_A^*, s_B^*, s_C^*)$ are equal, and by the property of zero-sum game they are zero.

We show the following theorem.

Theorem 1. *The existence of Nash equilibrium in a symmetric three-players zero-sum game implies the modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy at the symmetric Nash equilibrium.*

Proof. 1. Let $(s_A^*, s_B^*, s_C^*) = (s^*, s^*, s^*)$ be a symmetric Nash equilibrium of a three-players zero-sum game. Then,

$$u_A(s^*, s^*, s^*) = \max_{s_A \in S} u_A(s_A, s^*, s^*) \geq u_A(s_A, s^*, s^*). \tag{3}$$

Since the game is zero-sum,

$$u_A(s_A, s^*, s^*) + u_B(s_A, s^*, s^*) + u_C(s_A, s^*, s^*) = 0.$$

By symmetry of the game

$$u_A(s_A, s^*, s^*) + 2u_B(s_A, s^*, s^*) = 0.$$

This means

$$u_A(s_A, s^*, s^*) = -2u_B(s_A, s^*, s^*).$$

This equation holds for any s_A. Thus,

$$\arg\max_{s_A \in S} u_A(s_A, s^*, s^*) = \arg\min_{s_A \in S} u_B(s_A, s^*, s^*) = s^*.$$

By the assumption of the uniqueness of the best responses, they are unique. By symmetry of the game,

$$\arg\max_{s_A \in S} u_A(s_A, s^*, s^*) = \arg\min_{s_B \in S} u_A(s^*, s_B, s^*) = s^*.$$

Therefore,

$$u_A(s^*, s^*, s^*) = \min_{s_B \in S} u_A(s^*, s_B, s^*) \le u_A(s^*, s_B, s^*).$$

With (3), we get

$$\max_{s_A \in S} u_A(s_A, s^*, s^*) = u_A(s^*, s^*, s^*) = \min_{s_B \in S} u_A(s^*, s_B, s^*).$$

This means

$$\begin{aligned} \min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, s^*) &\le \max_{s_A \in S} u_A(s_A, s^*, s^*) \\ &= \min_{s_B \in S} u_A(s^*, s_B, s^*) \le \max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s^*). \end{aligned} \tag{4}$$

On the other hand, since

$$\min_{s_B \in S} u_A(s_A, s_B, s^*) \le u_A(s_A, s_B, s^*),$$

we have

$$\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s^*) \le \max_{s_A \in S} u_A(s_A, s_B, s^*).$$

This inequality holds for any s_B. Thus,

$$\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s^*) \le \min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, s^*).$$

With (4), we obtain

$$\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s^*) = \min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, s^*). \tag{5}$$

(4) implies

$$\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s^*) = \max_{s_A \in S} u_A(s_A, s^*, s^*),$$

and

$$\min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, s^*) = \min_{s_B \in S} u_A(s^*, s_B, s^*).$$

From

$$\min_{s_B \in S} u_A(s_A, s_B, s^*) \leq u_A(s_A, s^*, s^*),$$

and

$$\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s^*) = \max_{s_A \in S} u_A(s_A, s^*, s^*),$$

we have

$$\arg\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s^*) = \arg\max_{s_A \in S} u_A(s_A, s^*, s^*) = s^*.$$

Also, from

$$\max_{s_A \in S} u_A(s_A, s_B, s^*) \geq u_A(s^*, s_B, s^*),$$

and

$$\min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, s^*) = \min_{s_B \in S} u_A(s^*, s_B, s^*),$$

we get

$$\arg\min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, s^*) = \arg\min_{s_B \in S} u_A(s^*, s_B, s^*) = s^*.$$

Therefore,

$$\arg\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s^*) = \arg\min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, s^*) = s^*. \tag{6}$$

(5) and (6) are equivalent to Lemma 2. This result holds for Player B and Player C as well as Player A. □

Next we show the following theorem.

Theorem 2. *Under Assumption 1, the modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy imply the existence of a symmetric Nash equilibrium.*

Proof. Let $\tilde{s}$ be a value of s_C such that

$$\tilde{s} = \arg\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, \tilde{s}) = \arg\min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, \tilde{s}).$$

Then, from Lemma 2 we have

$$\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, \tilde{s}) = \min_{s_B \in S} u_A(\tilde{s}, s_B, \tilde{s}) = \min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, \tilde{s}) = \max_{s_A \in S} u_A(s_A, \tilde{s}, \tilde{s}).$$

Since

$$u_A(\tilde{s}, s_B, \tilde{s}) \le \max_{s_A \in S} u_A(s_A, s_B, \tilde{s}),$$

and

$$\min_{s_B \in S} u_A(\tilde{s}, s_B, \tilde{s}) = \min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, \tilde{s}),$$

we get

$$\arg\min_{s_B \in S} u_A(\tilde{s}, s_B, \tilde{s}) = \arg\min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, \tilde{s}) = \tilde{s}.$$

Also, since

$$u_A(s_A, \tilde{s}, \tilde{s}) \ge \min_{s_B \in S} u_A(s_A, s_B, \tilde{s}),$$

and

$$\max_{s_A \in S} u_A(s_A, \tilde{s}, \tilde{s}) = \max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, \tilde{s}),$$

we obtain

$$\arg\max_{s_A \in S} u_A(s_A, \tilde{s}, \tilde{s}) = \arg\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, \tilde{s}) = \tilde{s}.$$

Therefore,

$$u_A(\tilde{s}, s_B, \tilde{s}) \ge u_A(\tilde{s}, \tilde{s}, \tilde{s}) \ge u_A(s_A, \tilde{s}, \tilde{s}),$$

and so $(s_A, s_B, s_C) = (\tilde{s}, \tilde{s}, \tilde{s})$ is a symmetric Nash equilibrium of a three-players zero-sum game. □

4 Note on the case where Assumption 1 is not assumed.

Let $s_C = s$, and define

$$\bar{s} = \arg\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, s),$$

$$s^1 = \arg\min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, s).$$

Let $\bar{s}$ be the fixed point of the following function;

$$s \longrightarrow \bar{s}(s).$$

Then, by (2)

$$\max_{s_A \in S} \min_{s_B \in S} u_A(s_A, s_B, \bar{s}) = \min_{s_B \in S} u_A(\bar{s}, s_B, \bar{s}) = \min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, \bar{s}).$$

Since

$$\max_{s_A \in S} u_A(s_A, s_B, \bar{s}) \ge u_A(\bar{s}, s_B, \bar{s}),$$

and

$$\min_{s_B \in S} \max_{s_A \in S} u_A(s_A, s_B, \bar{s}) = \min_{s_B \in S} u_A(\bar{s}, s_B, \bar{s}),$$

we have

$$\arg\min_{s_B\in S}\max_{s_A\in S} u_A(s_A, s_B, \bar{s}) = \arg\min_{s_B\in S} u_A(\bar{s}, s_B, \bar{s}) = s^1.$$

Then,

$$\min_{s_B\in S}\max_{s_A\in S} u_A(s_A, s_B, \bar{s}) = \max_{s_A\in S} u_A(s_A, s^1, \bar{s}).$$

Since

$$\min_{s_B\in S} u_A(s_A, s_B, \bar{s}) \le u_A(s_A, s^1, \bar{s}),$$

and

$$\max_{s_A\in S}\min_{s_B\in S} u_A(s_A, s_B, \bar{s}) = \max_{s_A\in S} u_A(s_A, s^1, \bar{s}),$$

we have

$$\arg\max_{s_A\in S}\min_{s_B\in S} u_A(s_A, s_B, \bar{s}) = \arg\max_{s_A\in S} u_A(s_A, s^1, \bar{s}) = \bar{s}. \tag{7}$$

Because the game is symmetric and zero-sum,

$$u_A(\bar{s}, s_B, \bar{s}) + u_B(\bar{s}, s_B, \bar{s}) + u_C(\bar{s}, s_B, \bar{s}) = 0,$$

implies

$$2u_A(\bar{s}, s_B, \bar{s}) + u_B(\bar{s}, s_B, \bar{s}) = 0.$$

Thus,

$$2u_A(\bar{s}, s_B, \bar{s}) = -u_B(\bar{s}, s_B, \bar{s}),$$

and so

$$\arg\min_{s_B\in S} u_A(\bar{s}, s_B, \bar{s}) = \arg\max_{s_B\in S} u_B(\bar{s}, s_B, \bar{s}) = s^1. \tag{8}$$

Therefore, if $s^1 \neq \bar{s}$, there may exist an asymmetric Nash equilibrium denoted as follows.

$$(\bar{s}, s^1, \bar{s})$$

In which only $s_B = s^1$.

5 Concluding Remark

In this paper we have shown that a modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy is equivalent to the existence of a symmetric Nash equilibrium in a symmetric three-players zero-sum game. We want to extend this result to more general multi-players zero-sum game.

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