Forbidden zones and biases for the expectation of a random variable. Version 2

Harin, Alexander

Modern University for the Humanities

30 March 2018

Forbidden zones and biases for the expectation of a random variable. Version 2

Alexander Harin
aaharin@gmail.com
Modern University for the Humanities

A forbidden zones theorem is proven in the present article. If some non-zero lower bound exists for the variance of a random variable, whose support is located in a finite interval, then non-zero bounds or forbidden zones exist for its expectation near the boundaries of the interval.

The article is motivated by the need of a theoretical support for the practical analysis of the influence of a noise that was performed for the purposes of behavioral economics, utility and prospect theories, decision and social sciences and psychology.

The four main contributions of the present article are: the mathematical support, approach and model those are developed for this analysis and the successful uniform applications of the model in more than one domain.

In particular, the approach supposes that subjects decide as if there were some biases of the expectations.

Possible general consequences and applications of the theorem for a noise and biases of measurement data are preliminary considered.

JEL codes: C1; C02; D8; D81

Keywords: probability; variance; noise; bias; utility theory; prospect theory; behavioral economics; decision sciences; measurement;

Contents

1. Introduction

1.1. Functions, moments and bounds

Various bounds for functions and moments of random variables are considered in a number of works.

Continuous random variables on infinite interval are analyzed in Moriguti (1952). The expression for lower bounds on the *n*-th probability moments of any continuous distribution is obtained under the condition of finite variance.

Bounds for the probabilities and expectations of convex functions of discrete random variables with finite support are studied in Prékopa (1990).

Inequalities for the expectations of functions are studied in Prékopa (1992). These inequalities are based on information of the moments of discrete random variables.

A class of lower bounds on the expectation of a convex function using the first two moments of the random variable with a bounded support is considered in Dokov and Morton (2005).

Bounds on the exponential moments of $\min(y, X)$ and $X\,I\{X < y\}$ using the first two moments of the random variable X are considered in Pinelis (2011).

In the present article some information about the variance of a random variable is used to estimate bounds on its expectation.

1.2. Practical needs of consideration

1.2.1. Problems of probable and sure outcomes

A man is a key subject of economics and some other sciences. There are a number of problems concerned with the mathematical description of the behavior of a man. Examples of them are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, etc.

The essence of these problems consists in biases of preferences and choices of people (subjects) for the probable and sure outcomes in comparison with the predictions of the probability theory. These biases are maximal near the boundaries of the probability scale, that is, at high and low probabilities. Such problems are the well-known, basic and fundamental ones. They are the most important in behavioral economics in utility and prospect theories and also in decision sciences, social sciences and psychology.

The above basic problems are pointed out in a wealth of works.

For example, we see in Kahneman and Thaler (2006) p. 222:

> *"A long series of modern challenges to utility theory, starting with the paradoxes of Allais (1953) and Ellsberg (1961) and including framing effects, have demonstrated inconsistency in preferences"*

For example, we see in Kahneman and Tversky (1979) p. 265:

"PROBLEM1: Choose between

A:	*2,500 with probability*	*.33,*
	2,400 with probability	*.66,*
	0 with probability	*.01;*
B:	*2,400*	*with certainty.*
N=72	*[18]*	*[82]"*

For example, we see in Starmer and Sugden (1991) p. 974:

> *"… a choice between two lotteries R' (for "riskier") and S' (for "safer"). R' gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); S' gave £7.00 for sure."*

R' gives £10.00×0.2 + £7.00×0.75 = £7.25. S' gives £7.00×1 = £7.00. Here R' = £7.25 > S' = £7.00. The results are: 13 choices for R' and 27 choices for S'.

For example, we see in Barberis (2013) p.177 (after Gonzalez and Wu 1999) the median cash equivalents (in dollars) for the following non-mixed prospect:

Outcomes (0 or $100); Probability .90; Equivalent $63.

1.2.2. Problems of different domains

Moreover, an additional and, maybe, more hard problem is the inverse behavior of the people in different domains. For instance, there are a number of warrants of risk aversion in the domain of gains but risk seeking in the domain of losses (at the high probabilities).

For example, we see in Thaler (2016), p. 1582 (the **boldface** is my own):

> “*We observe a pattern that was frequently displayed: subjects were* ***risk averse*** *in the* ***domain of gains*** *but* ***risk seeking*** *in the* ***domain of losses***.

For example, we see in Kahneman and Tversky (1979) p. 268 Table 1:

“Problem 3:	*(4,000, .80)*	<	*(3,000).*
	[20]		*[80]*
Problem 3':	*(-4,000, .80)*	>	*(-3,000).*
	[92]		*[8]”*

For example, we see in Tversky and Kahneman (1992) p. 307 in Table 3 median cash equivalents (in dollars) for the following non-mixed prospects:

Outcomes (0 or $50); Probability .90; Equivalent $37.

Outcomes (0 or -$50); Probability .90; Equivalent -$39.

Outcomes (0 or $200); Probability .90; Equivalent $131.

Outcomes (0 or -$200); Probability .90; Equivalent -$155.

These and similar examples will be simplified and considered below in the next sections.

Note that subjects change their preferences and choices from aversion to seeking and vice versa not only when the domain are changed from gains to losses but from high to low probabilities as well. Such domains will be considered in future articles by means the approach and models proposed here.

The present article is motivated in large measure by the need of rigorous mathematical support for the already performed analysis of the influence of scattering and noisiness of data. The idea of the theorem considered here has explained, at least partially, the above problems (see, e.g., Harin 2012a, Harin 2012b, Harin 2015).

1.3. Two ways. Variance, expectation and forbidden zones

Many efforts were applied to explain the above basic problems of behavioral economics and other sciences.

One of possible ways to explain them is widely discussed, e.g., in Schoemaker and Hershey (1992), Hey and Orme (1994), Chay et al (2005), Butler and Loomes (2007). The essence of this way consists in a proper attention to uncertainty, imprecision, noise, incompleteness and other reasons that might cause dispersion, scattering and spread of data.

Another possible way to explain these problems is to consider the vicinities of the borders of the probability scale, e.g. at $p{\sim}1$. Steingrimsson and Luce (2007) and Aczél and Luce (2007) emphasized a fundamental question: whether Prelec's weighting function $W(p)$ (see Prelec, 1998) is equal to 1 at $p=1$.

In any case, one may suppose that a synthesis of the above two ways can be of interest. This idea of the synthesis turned out to be useful indeed. It has been successful to explain, at least partially, the underweighting of high and the overweighting of low probabilities, risk aversion, and some other problems (see, e.g., Harin 2012a, Harin 2012b and Harin 2015). There exist also works about experimental support of this synthesis (see, e.g., Harin 2014).

In the present article some information about the variance of a random variable which takes on values in a finite closed interval is used to estimate bounds on its expectation. It is proven that if there is a non-zero lower bound on the variance of the variable, then non-zero bounds or forbidden zones for its expectation exist near the boundaries of the interval.

The role of a noise, as a possible cause of these forbidden zones and their possible influence on results of measurements near the boundaries of intervals are preliminary considered as well.

Keeping in mind the above bounds on functions of random variables Prékopa (1990), Prékopa (1992), Dokov and Morton (2005) and Pinelis (2011), functions of the expectation of a random variable can be further investigated.

Due to the convenience of abbreviations and consonant with the usage in previous works, here the terms “bound” and “forbidden zones” will sometimes be referred to with the term "restriction," especially in mathematical expressions, using its first letter "r" or "R," for example "r_{Expect}" or "r_{μ}" or "R."

2. Theorem

2.1. Preliminaries

The practical need of the article is a discrete random variable taking the finite number of values. This corresponds to usual finite numbers of measurements in the behavioral economics. A general case will be considered here nevertheless.

Let us consider a probability space *(Ω, Æ, P)* and a random variable *X*, such that *Ω → R*. Suppose that the support of X is an interval $[a,b]: 0 < (b-a) < \infty$. Suppose that X can have a continuous part and a discrete part and at least one of these parts is not identically equal to zero.

Let us denote the possible discrete values of X as $\{x_k\}$, $k = 1,2,...,K$, where $K \geq 1$, and $a \leq x_k \leq b$, and the possible continuous values of X as $x \in [a,b]$.

Under the condition

$$\sum_{k=1}^{K} f_X(x_k) + \int_{-\infty}^{+\infty} f(x)dx = \sum_{k=1}^{K} f_X(x_k) + \int_{a}^{b} f(x)dx = 1,$$

let us consider the expectation of X

$$E[X] \equiv \sum_{k=1}^{K} x_k f_X(x_k) + \int_{a}^{b} xf(x)dx \equiv \mu,$$

its variance

$$E[X-\mu]^2 = \sum_{k=1}^{K} (x_k-\mu)^2 f_X(x_k) + \int_{a}^{b} x^2 f(x)dx \equiv \sigma^2$$

and possible interrelationships between them.

2.2. Conditions of the variance maximality

The maximal value of the variance of a random variable of any type is intuitively equal to the variance of the discrete random variable whose probability mass function has only two non-zero values located at the boundaries of the interval. This statement is nevertheless proven in lemmas in the Appendix.

Such a probability mass function can be represented by the two values $f_X(a) = (b-\mu)/(b-a)$ and $f_X(b) = (\mu-a)/(b-a)$. The following inequality is consequently true for the variance of the considered random variable X

$$E[X-\mu]^2 \leq (\mu-a)^2 \frac{b-\mu}{b-a} + (b-\mu)^2 \frac{\mu-a}{b-a} = (\mu-a)(b-\mu). \qquad (1)$$

2.3. Existence theorem

Theorem. Suppose a random variable X takes on values in an interval $[a, b]$, $0 < (b-a) < \infty$. If there is some minimal non-zero variance $\sigma^2_{Min} > 0 : E[X-\mu]2 \geq \sigma^2_{Min}$, then some non-zero bounds (restrictions) $r_\mu \equiv r_{Expect} \equiv r_{Restrict.Expect} > 0$ exist on its expectation $\mu \equiv E[X]$ near the boundaries of the interval $[a, b]$, that is

$$a < (a + r_\mu) \leq \mu \leq (b - r_\mu) < b. \quad (2).$$

Proof. It follows from (1) and the hypotheses of the theorem that

$$0 < \sigma^2_{Min} \leq E[X-\mu]^2 \leq (\mu - a)(b - \mu).$$

For the boundary a this leads to the inequalities $\sigma^2_{Min} \leq (\mu - a)(b - a)$ and

$$\mu \geq a + \frac{\sigma^2_{Min}}{b-a}. \quad (3).$$

For the boundary b the consideration is similar and gives the inequality

$$\mu \leq b - \frac{\sigma^2_{Min}}{b-a}. \quad (4).$$

So, if we consider the image I_μ of the values of the expectation μ, then we see that, if the minimal variance σ^2_{Min} is equal to zero in the above inequalities (3) and (4), this image coincides with the interval $[a, b]$. If the minimal variance σ^2_{Min} is more than zero, then I_μ is divided into the three zones.

These zones are the two forbidden zones $R_{\mu a}$ and $R_{\mu b}$ (or simply R_a and R_b) and the residual obtainable or open zone O_μ (or simply O). The forbidden zones are located near the boundaries of the interval $[a, b]$. They are restricted for the values of the expectation μ. The residual obtainable zone O is obtainable, open for the values of the expectation μ.

Denoting the bounds (restrictions r_μ) on the expectation μ as

$$r_\mu \equiv \frac{\sigma^2_{Min}}{b-a}, \quad (5)$$

and using (3) and (4), we obtain the generalized inequalities

$$a + r_\mu \leq \mu \leq b - r_\mu .$$

Therefore, if the inequalities $0 < (b-a) < \infty$ and $\sigma^2_{Min} > 0$ are true, then the non-zero bounds (restrictions) $r_\mu > 0$ exist, such that the inequalities (2)

$$a < (a + r_\mu) \leq \mu \leq (b - r_\mu) < b$$

are satisfied, which proves the theorem.

3. Consequences of the theorem. Examples

3.1. General consequences

3.1.1. Practical need. General implication. Mathematical support

The initial reason of the above theorem was to provide the mathematical support for the analysis of the practical experiments in behavioral economics.

Due to the need of financial incentives for subjects of the experiments and to the finiteness of financial possibilities of experimenter's teams, the numbers of experimental results are necessarily finite.

The theorem meets this practical need. It provides the mathematical support for the analysis of the above experiments. It proves the possibility of existence of the forbidden zones for the discrete random variables that take a limited number of values that were used in the above analysis. It determines also the conditions of their existence and their minimal width.

In addition to this particular practical value, the theorem proves that this result is true for any random variable. The examples below and earlier works (see, e.g., Harin 2012b) prove that the theorem supports the analysis more than in one domain, moreover.

This is, at least, a rare result for the considered problems.

3.1.2. Minimal variance. Data scattering. Noise

The theorem states that the factor which leads to the forbidden zones and determines their widths is the non-zero minimal variance. It is exactly the minimal variance, not the variance itself.

There can be a wealth of causes of this non-zero minimal variance. It can be caused evidently by any non-zero scattering and spread of data. The list of such causes is rather wide. It includes a noise, imprecision, errors, incompleteness, various types of uncertainty, etc. Such causes are considered in a lot of works, e.g., Schoemaker and Hershey (1992), Hey and Orme (1994), Chay et al (2005), Butler and Loomes (2007).

A noise can be one of usual sources of the non-zero minimal variance.

There are many types and subtypes of noise. A hypothetic task of determining of an exact relationship between a level of noise and a non-zero minimal variance of random variables can be a rather complicated one.

If, nevertheless, a noise leads to some non-zero minimal variance of the considered random variable, then such a noise leads evidently to the above non-zero forbidden zones. If a noise leads to some increasing of the value of this minimal variance then the value of these zones increase as well.

So the theorem can provide a new mathematical tool for description of the influence of at least some types of a noise near the boundaries of intervals.

3.2. Practical examples of existence

3.2.1. Practical example of existence. Ships and waves

Suppose the calm or mirror-like sea. Suppose a small rigid boat or any other small rigid floating body which is at rest in the mirror-like sea. Suppose that this boat or the body rests in the mirror-like sea right against (or be constantly touching) the moorage wall (which is also rigid).

As long as the sea is calm, the expectation of its side can touch the wall.

Suppose the heavy sea. Suppose a small rigid boat or any other small rigid floating body which oscillates on waves in the heavy sea. Suppose that this boat or the body oscillates on waves near the rigid moorage wall.

When the boat is oscillated by sea waves, then its side oscillates also (both up-down and left-right) and it can touch the wall only in the nearest extremity of the oscillations. Therefore, the expectation of the side cannot touch the wall (if the oscillations are non-zero). Therefore, the expectation of the side is biased from the wall.

So, one can say that, in the presence of the waves, a forbidden zone exists between the expectation of the side and the wall.

This forbidden zone biases and separates the expectation from the wall. The width of the forbidden zone is roughly about a half of the amplitude of the oscillations.

3.2.2. Practical examples of existence. Washing machine, drill, …

Suppose a washing machine that can vibrate when pressing bed linen. Suppose this washing machine near a rigid wall. Suppose an edgeless side of a drill or any other rigid body that can vibrate is located near a rigid surface or wall.

If the washing machine or the drill is at rest, then the expectation of its edgeless side can be located right against (be constantly touching) the wall.

If the washing machine or the drill vibrates, then the expectation of its edgeless side is biased and kept away from the wall due to its vibrations.

3.3. General example

3.3.1. Rigidness

The same is true for any other rigid body near any rigid surface or wall:

If the body is at rest, then the expectation of its side can be located right against the wall (be constantly touching the wall). If the body vibrates, then the expectation of its side is biased and kept away from the wall by the vibrations.

In other words, a forbidden zone arises between the rigid wall (surface) and the expectation of the side of the rigid body, when the body vibrates. The width of the forbidden zone is roughly about a half of the amplitude of the vibrations.

The above rigid boat near rigid moorage wall, rigid washing machine near rigid wall and rigid drill near rigid surface were the examples of a rigid body that can vibrate or oscillate near a rigid boundary (a rigid surface).

What do the conditions of “rigid” body and “rigid” boundary mean?

If either the body or the boundary or the both are not rigid, then the vibrations and oscillations can be suppressed partially or even totally. Hence the forbidden zone can be suppressed also.

3.3.2. Noise suppression. Sure outcomes

Vibrations, oscillations can be suppressed by some efforts. Such efforts can be, e.g., physical in the case of the physical vibrations of the body. A vibrating rigid body can be pressed by some drawing or pressing force exerted by some means. The suppressing means and their principles of action can be of different kinds, e.g., a flexible or inextensible cord, a pressure plate, etc. The forbidden zone can be suppressed either partially or even totally, depending on the parameters of the suppression and suppression means.

This suppression can correspond to the case of sure outcomes in behavioral economics, decision and social sciences and psychology.

Let us compare probable and sure outcomes and corresponding biases.

The term "sure" presumes usually that some efforts are applied to guarantee this sure outcome in comparison with the probable ones. This leads to some qualitative difference between these probable and sure outcomes. This qualitative difference can lead to some quantitative difference between the widths of the forbidden zones and hence the biases for the expectations of data for these probable and sure outcomes.

Due to the guaranteeing efforts, the width of the forbidden zones and hence the bias for sure outcomes can be less than the width and biases for the probable outcomes. The width for the sure outcomes can even be equal to zero, which means that the cause of the forbidden zones is too weak to overcome the guaranteeing efforts.

So, sure outcomes are guaranteed by some guaranteeing efforts. Due to these efforts, minimal variance σ^2_{Sure}, the forbidden zones and the bias for the sure outcomes can be suppressed and reduced.

The nature of these guaranteeing efforts can nevertheless vary for various cases. Therefore in the case of the sure outcomes, a consideration of the minimal variance σ^2_{Sure} and even of the forbidden zones can be more complicated than in the case of the probable outcomes.

3.4. Approach of biases

3.4.1. Preliminary considerations, suppositions and statements

First of all, the modern utility and prospect theories undoubtedly constitute a complex set of the data, rules, suppositions etc. In addition, the above problems of these theories have been analyzed many times by various teams of researchers but have not been adequately solved nevertheless. For example, Kahneman and Thaler (2006) noted (see p. 222):

> *"A long series of modern challenges to utility theory, starting with the paradoxes of Allais (1953) …, have demonstrated inconsistency in preferences"*

In other words, the problem that was revealed in 1953 was not adequately solved during more than a half of century (the available literature testifies that it was not adequately solved even in 2017).

All the circumstances and reasons lead to the deduction that an essential and elaborated contribution to the modern utility and prospect theories needs the elaborated work of a sufficient number of research teams. So it cannot be made by a single researcher and all the more by a single theorem and single article.

Therefore the leading principle of the approach should be "stage by stage and step by step." Consequently the approach that can be based on the proposed theorem and its consequences and can be proposed in the present single article should be only a preliminary stage for subsequent changes, modifications and refinements by some research teams.

So there is no sense and possibility for this single article to build a thorough and well-composed construction of rigorous statements proven by a wealth of experimental and theoretical works. So for such a preliminary stage it is sufficient to propose only the above theorem with its consequences and a collection of some suppositions, relationships and formulas.

Secondly, due to the theorem, the non-zero minimal variance of measurement data leads to the existence of the forbidden zones for the expectation of the data near the boundaries of the intervals of the data. These forbidden zones evidently lead to the biases of the expectations, at least right against the boundaries.

The above examples of this chapter evidently illustrate such forbidden zones. Similar examples are widespread and usual in the practical real life. Due to this prevalence, the subjects can keep in mind the feasibility of such forbidden zones and the biases of the expectations that can be caused by the zones. This can influence subjects' behavior and choices.

Due to all these considerations, the two main suppositions and some preliminary statements can be proposed for the approach:

The two main suppositions for the approach:

1. The subjects make their choices (at least to a considerable degree) as if there were some biases of the expectations of the outcomes.

(This supposition can be supported by the thought that such biases may be proposed and tested even from the purely formal point of view. The mathematical approach of the expectations biases is to explain not only the objective situations but mainly the subjective behavior and choices of subjects)

2. These biases (real biases or subjective reaction and choices of the subjects) can be explained (at least to a considerable degree) with the help of the theorem.

The preliminary statements:

Qualitative analysis. Only qualitative analysis will be performed.

Qualitative problems. Only qualitative problems will be considered.

Explanation. Only explanation of the existing problems will be given. No predictions will be made in the scope of this first stage of the approach.

Choices of subjects. The approach will explain mainly the subjective behavior and choices of subjects.

3.4.2. Denotations

I denote the expectations of the probable outcomes as $\mu_{Prob} \equiv \mu_{Probable}$ and of sure outcomes as μ_{Sure}.

The real measurement data represent the set of the choices of the subjects. Using this set, one can estimate the biases of the expectations of the data for the probable and sure outcomes that are required to obtain the data corresponding to these choices. I denote them as $\Delta_{Prob} \equiv \Delta_{Probable}$ and Δ_{Sure}.

Let us consider some abstract mode 1 and mode 2 of outcomes. Irrespective of these numbers, one of these modes corresponds to the probable outcomes (this may be either mode 1 or mode 2) and the other – to the sure ones. The corresponding expectations are μ_1 and μ_2 and the biases are Δ_1 and Δ_2.

One can introduce also the two more designations: a) the difference

$$d_\mu \equiv \mu_2 - \mu_1$$

between the expectations of the compared modes, b) the difference

$$d_{Choice} \equiv \Delta_2 - \Delta_1$$

that is required to obtain the data corresponding to the revealed choices.

The simplicity of the mathematical calculations and transformations allows to omit the most of intermediate mathematical manipulations.

3.4.3. Relationships

Let us consider some essential features of the examined situations and develop some mathematical relationships using these denotations.

1. Qualitative problems. The three qualitatively situations with different signs of the vectors of the expectations differences for the probable and sure outcomes can be separated: the expectation for the probable outcome can be more, less or equal to that for the sure ones. The three qualitatively different signs can be separated for choices: the subjects can be risk averse, risk seeking or neutral.

For the qualitative problems the signs for the choices of the subjects do not coincide with the signs of the expectations differences for the probable and sure outcomes.

That is when the difference of the expectations for the probable and sure outcomes is, e.g., positive, then the corresponding difference for subjects' choices is either negative or neutral.

The necessary and sufficient condition for the qualitative problems can be represented mathematically as

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_{\mu}. \quad (6)$$

That is: if the difference d_μ between the expectations of the compared modes is, for example, undoubtedly positive (the sign of d_μ is $sgn\ d_\mu > 0$), then the revealed choice of the subjects is such that the difference d_{Choice}, that is required to obtain the data corresponding to this choice, should be undoubtedly negative (the sign of d_{Choice} is $sgn\ d_{Choice} < 0$).

These qualitative types of the above problems are chosen as the examples that are usual in experiments (see, e.g., Kahneman and Tversky 1979, Starmer and Sugden 1991, Tversky and Kahneman 1992, Thaler 2016). They can manifest clear qualitative representations of the above problems and can be a background for some further generalizations.

2. Biases and differences. The necessary and sufficient condition (6) for the qualitative problems can be easily transformed to the necessary condition

$$|d_{Choice}| \geq |d_{\mu}| \qquad (7)$$

of existence of the qualitative problems. Due to $|\Delta_{\mu}| \geq 0$ and $d_{Choice} = \Delta_2 - \Delta_1$, this leads also to $|\Delta_2 - \Delta_1| \geq 0$ and the necessary condition

$$\Delta_2 \neq \Delta_1 \qquad (8)$$

of the existence of the qualitative changes. That is, to produce the qualitative changes, the biases of the modes 1 and 2 (of the probable and sure outcomes) should not be equal to each other. In other words, if the biases of the modes 1 and 2 are equal to each other, that is $\Delta_2 = \Delta_1$, then they cannot produce the qualitative changes.

This leads also to the condition

$$\text{both } \Delta_1 \neq 0 \text{ and } \Delta_2 \neq 0. \qquad (9)$$

That is the biases of the modes 1 and 2 should not be simultaneously equal to zero. If the both of the biases are equal to zero, then they also cannot indeed produce the qualitative changes. In other words, "zero leads to zero" or zero biases lead to zero qualitative change.

The last two particular conditions are natural within the scope of the approach but will be valuable if they will be proven beyond its scope. In this case they can support the necessity of the non-zero biases and of their mutual inequality.

The condition (9) of the impossibility leads to the general necessary condition of non-zero difference between the biases for the choices

$$|d_{Choice}| > 0 \qquad (10)$$

3. Origin of biases. Condition of explanation. The biases of the expectations may be introduced and considered purely formally. The question is not only whether these biases can explain the problems. The question is also whether these biases themselves can be explained by the theorem.

First of all, the theorem should be applicable. This condition is satisfied if

$$\sigma^2{}_{Min} > 0 .$$

Further let us denote the biases caused by the forbidden zones of the theorem by $\Delta_{Theorem}$ and the difference that can be explained by the theorem as $d_{Theorem}$. Then the conditions for the explanation can be represented as $\Delta_{Theorem} \approx \Delta_{Choice}$ and, therefore, $d_{Theorem} \approx d_{Choice}$, in the case when the forbidden zones of the theorem are the main source of the biases. If the forbidden zones of the theorem are one of the essential source of the biases, then the conditions for the explanation can be represented as $\Delta_{Theorem} = O(\Delta_{Choice})$ and, therefore, $d_{Theorem} = O(d_{Choice})$. So the conditions for the explanation can be represented as

$$d_{Theorem} \approx d_{Choise} \quad \text{or at least} \quad d_{Theorem} = O(d_{Choise}) . \tag{11}$$

The examples considered below prove that the theorem predicts the right signs of the difference and there is no need to state the concerned additional supposition.

The above considerations, suppositions and formulas may be used in more general situations as well. Let us consider a particular supposition.

4. Biases of sure outcomes. The above considerations of this sections about the noise suppression and sure outcomes lead to the deduction that the sure outcomes are guaranteed by some guaranteeing efforts. Due to these efforts, the bias for the sure outcomes can be suppressed and reduced. It can be moreover equal to zero.

In accordance with this deduction and the supposition (10) of the difference between biases for the choices, I suppose that the bias of the measurement data for the sure outcomes is equal to zero or, more generally, is strictly less than the bias for the probable outcomes. The application of the supposition of the difference between biases enables to deduce that the absolute value of the bias for the probable outcomes should be non-zero. This is also in correspondence with the condition (10) of non-zero difference between the biases for the choices.

This is supported by the examples considered below. They prove that the theorem predicts the true signs of the bias for the probable outcomes. So there is no need to state the concerned additional supposition.

These suppositions can be formulated as

$$|\Delta_{\mathrm{Pr}ob}| > 0 \quad \text{and} \quad |\Delta_{\mathrm{Pr}ob}| > |\Delta_{Sure}| \quad \text{and} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\mathrm{Pr}ob} . \qquad (12)$$

3.4.4. Summary of the main supposed relationships for the first stage of the approach

The above considerations and suppositions lead to the three groups of the supposed relationships:

The supposed relationships for the qualitative problems

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_{\mu} \quad \text{and} \quad |d_{Choice}| \geq |d_{\mu}| . \qquad (13)$$

The supposed relationships for the probable and sure outcomes

$$|\Delta_{\mathrm{Pr}ob}| > 0 \quad \text{and} \quad |\Delta_{\mathrm{Pr}ob}| > |\Delta_{Sure}| \quad \text{and} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\mathrm{Pr}ob} . \qquad (14)$$

The supposed relationships for the theorem and choices

$$\sigma^2{}_{Min} > 0 \quad \text{and} \quad d_{Theorem} \approx d_{Choise} \quad \text{or at least} \quad d_{Theorem} = O(d_{Choise}) . \qquad (15)$$

3.5. Qualitative models

Let us consider possible qualitative models for the analysis of the above problems in the scope of the approach of biases.

3.5.1 Theorem bound for the bias

Let us estimate the limits for the biases of the expectations with the help of the theorem.

Due to (5), the minimal value of the width of the forbidden zone (of the restriction r_μ) is

$$r_\mu = \frac{\sigma^2_{Min}}{b-a} \quad \text{and we have} \quad \frac{\sigma_{Min}}{b-a} = \sqrt{\frac{r_\mu}{b-a}}.$$

Due to

$$\frac{\sigma_{Max}}{b-a} \le \frac{1}{2} \quad \text{we have} \quad \frac{r_\mu}{b-a} \le \frac{1}{4}.$$

This is some rough estimate for the maximal width of the forbidden zone. More exact estimates will be given in next articles. In any case it is not more than *(b-a)/2*.

The bias of the expectation cannot be more than the width of the forbidden zone. The obtained estimate for the maximal width is therefore the estimate for the maximal bias. It should be noted that, for example, if one considers some normal distribution that is located near the boundary at the distance of three sigma from its expectation, then there is no need to use such an estimate.

Nevertheless this estimate of *0.25(b-a)* can be used as some secure upper bound for the bias. We can denote this secure upper bound as Δ_{Sequre} and write

$$\Delta_{Secure} \le \frac{b-a}{4}.$$

3.5.2 Certainty equivalents. Relative biases

Let us consider the real experimental data and normalize the values of the biases to the values of the gains/losses. These normalized values can represent the relative biases of the expectations or probabilities.

Let us consider the practical numerical examples of certainty equivalents.

For instance, we see in the above example of Barberis (2013):

The probable outcomes give $100*.9 = 90$. The median cash equivalent gives $63*1 = 63$. The expectations are

$$90 > 63$$

but the subjects manifest the equivalent choices. To provide the equivalent choices, the difference between the biases of the expectations for the probable and sure outcomes should be equal to $\Delta_{Prob} - \Delta_{Sure} = 27$. That is the bias for the probable outcome should not be less than $\Delta_{Prob} \geq 27$.

For instance, we see in the above examples of Tversky and Kahneman (1992):

1. Gain. The probable outcomes give $50*.9 = 45$. The median cash equivalent gives $37*1 = 37$. The expectations are

$$45 > 37,$$

but the subjects manifest the equivalent choices. The bias for the probable outcome should not be less than $\Delta_{Prob} \geq 8$.

Loss. The probable outcomes give $-50*.90 = -45$. The median cash equivalent gives $-39*1 = -39$. The expectations are

$$-45 < -39,$$

but the subjects manifest the equivalent choices. The bias for the probable outcome should not be less than $\Delta_{Prob} \geq -6$.

2. Gain. The probable outcomes give $200*.90 = 180$. The median cash equivalent gives $131*1 = 131$. The expectations are

$$180 > 131,$$

but the subjects manifest the equivalent choices. The bias for the probable outcome should not be less than $\Delta_{Prob} \geq 49$.

Loss. The probable outcomes give $-200*.90 = -180$. The median cash equivalent gives $-155*1 = -155$. The expectations are

$$-180 < -155,$$

but the subjects manifest the equivalent choices. The bias for the probable outcome should not be less than $\Delta_{Prob} \geq -35$.

Let us estimate the biases of the expectations for the probable outcomes in the scope of the approach.

The values of the considered biases differ essentially from each other. Let us normalize them to the values of the gain/loss. These normalized values can represent the relative biases of the expectations or the relative biases of the probabilities. So we obtain:

Barberis (2013): The relative bias is $\Delta_{Prob} \geq 30/100 = 0.3$.

Tversky and Kahneman (1992):

1. Gain. The relative bias is $\Delta_{Rel} \geq 8/50 = 0.16$.

Loss. The relative bias is $\Delta_{Rel} \geq -6/(-50) = 0.12$.

2. Gain. The relative bias is $\Delta_{Rel} \geq 49/200 = 0.245$.

Loss. The relative bias is $\Delta_{Rel} \geq -35/(-200) = 0.175$.

So sometimes the relative biases are comparable or even more than the above secure upper bound 0.25.

Therefore, and also from general and formal points of view, the following supposition can be stated:

"In general cases, along with the non-zero minimal variance of the measurement data, another source or sources of the biases can exist and cannot be excluded so far."

3.5.3. Basics of formal preliminary qualitative model

So, due to the theorem estimate of the secure upper bound for the bias and the experiments of certainty equivalents, the theorem does not guarantee that another source or sources of the biases can be excluded so far. Due to the law of the mean and the opposite signs of the biases predicted by the theorem near the opposite boundaries, the bias equals zero moreover in the middles of the intervals. Therefore, the more is distance from the boundary the less is the bias that is caused by the theorem. Therefore, and also from general and formal points of view, the following supposition can be stated:

"In general cases, along with the non-zero minimal variance of the data, another source or sources of the biases can exist and cannot be excluded so far."

So, for the above qualitative types of problems, the non-zero minimal variance of the measurement data can be considered as one of possible sources of the biases, but the possibility of another source or sources of the biases cannot be so far excluded and should be taken into account.

This formal preliminary qualitative model can be considered as a first step of this first stage of the approach and is to test the qualitative applicability of the model and approach to the simplest specific type of the above problems. The second step will be to test the quantitative limits of the model and approach.

Taking into account these considerations, the suppositions of the formal preliminary qualitative model can be formulated as follows:

Suppositions. The suppositions of the qualitative problems

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_{\mu} \quad \text{and} \quad |d_{Choice}| \geq |d_{\mu}|.$$

The suppositions for the probable and sure outcomes

$$|\Delta_{\Pr ob}| > 0 \quad \text{and} \quad |\Delta_{\Pr ob}| > |\Delta_{Sure}| \quad \text{and} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\Pr ob}.$$

The supposition for the theorem and choices

$$\sigma^2{}_{Min} > 0 \quad \text{and at least} \quad d_{Theorem} = O(d_{Choise}).$$

The suppositions of the types of the problems

$$|d_{Choice}| = |d_{\mu}|$$

for the problems of certainty equivalents and

$$|d_{Choice}| > |d_{\mu}|$$

for the other problems.

3.5.4. Trial examples of applications of formal preliminary qualitative model

Let us test the above examples of Section 1 by the formal preliminary qualitative model.

In the above citation from Kahneman and Tversky (1979) p. 265 the difference between the expectations is *2,500*.33 + 2,400*.66 - 2,400 = 2,400 - 2,400*.01 + 100*.33 - 2,400 = - 24 + 33 = 9.* The difference between the choices should be more than *9*. Let it be equal, for example, to *15*.

So the subjects decide if the resulting difference between the expectations was *15 – 9 = 6* in favor of the sure outcome.

The qualitative result is supported by the experiment. That is *82%* in favor of the sure outcome.

In the above citation from Starmer and Sugden (1991) p. 974 the difference between the expectations is *10.00*.2 + 7.00*.75 - 7.00 = 2.00 + 5.25 - 7.00 = +0.25*. The difference between the choices should be more than *0.25* and should be at least partially caused by a noise. Let it be equal, for example, to *0.4*.

So the subjects decide if the resulting difference between the expectations was *0.4 – 0.25 = 0.15* in favor of the sure outcome.

The qualitative result is supported by the experiment. That is *27/(13+27) = 27/40 = 87.5%* in favor of the sure outcome.

In the above citation from Barberis (2013) the difference between the expectations is *100*0.9 - 63 = 27*. The difference for the choices should be equal to *27* as well.

So the subjects decide if the resulting difference between the expectations was *27* in favor of the sure outcome. The qualitative result is supported by the experiment.

In the above citation from Tversky and Kahneman (1992) we can find:

1. Gain. The difference between the expectations is *50*0.9 - 37 = 8*. The difference for the choices should be equal to *8* as well.

So the subjects decide if the resulting difference between the expectations was *8*. This qualitative result is supported by the experiment.

Loss. The difference between the expectations is *-50*0.9 – (-39) = -6*. The difference for the choices should be equal to *-6* as well.

So the subjects decide if the resulting difference between the expectations was *-6*. This qualitative result is supported by the experiment.

2. Gain. The difference between the expectations is *200*.90 - 131 = 49*. The difference for the choices should be equal to *49* as well.

So the subjects decide if the resulting difference between the expectations was *49*. This qualitative result is supported by the experiment.

Loss. The difference between the expectations is *-200*.90 - (-155) = -35*. The difference for the choices should be equal to *-35* as well.

So the subjects decide if the resulting difference between the expectations was *-35*. This qualitative result is supported by the experiment.

In all the above examples the difference between the choices should be at least partially caused by the non-zero minimal variance of the data. These examples of applications of the formal preliminary qualitative model are trial because there is so far too little information about what part of the difference between the choices is caused by the non-zero minimal variance of the data.

3.5.5. Specific qualitative model

Let us consider the specific situation when $d_\mu = 0$. That is the expectations of the probable and sure outcomes are equal to each other, but the choices of the subjects are evidently biased to either probable or sure outcomes.

Due to the difference of the expectations is equal to zero, the difference for the choices should be either negative or positive.

This specific situation enables to avoid the constraints of the secure upper bound Δ_{Sequre} for the bias and to make the specific qualitative model less formal. The biases can be selected much less than Δ_{Sequre} and suppositions will be more simple.

Suppositions. So the suppositions of the specific qualitative model can be formulated as follows:

The suppositions for the differences for the biases of the choices and expectations

$$\operatorname{sgn} d_\mu = 0 \text{ and } d_{Max} \approx 0 \text{ and } \Delta_{Secure} >> |\Delta_{\Pr ob}| > 0.$$

The suppositions for the probable and sure outcomes

$$|\Delta_{\Pr ob}| > |\Delta_{Sure}| \text{ and } \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\Pr ob}.$$

The supposition for the theorem and choices

$$\sigma^2{}_{Min} > 0 \text{ and } d_{Theorem} \approx d_{Choise}.$$

This specific qualitative practical model can be considered as a first informal step of an explanation of the above problems. The model will be applied to practical numerical examples in the next chapter.

4. Applications of the theorem. Newness

4.1. Practical applications in behavioral economics and decision sciences

The idea of the considered forbidden zones was applied, e.g., in Harin (2012b). This work was devoted to the well-known problems of utility and prospect theories and was performed for the purposes of behavioral economics, decision and social sciences and psychology. Such problems were pointed out, e.g., in Kahneman and Thaler (2006).

In Harin (2012b), some examples of typical paradoxes were studied. The studied and similar paradoxes may concern problems such as the underweighting of high and the overweighting of low probabilities, risk aversion, etc.

The dispersion and noisiness of the initial data can lead to the forbidden zones for the expectations of these data. This should be taken into account when dealing with these kinds of problems. The described above forbidden zones explained, at least partially, the analyzed examples of paradoxes.

The concrete numerical examples of analysis and explanation of such problems by the proposed specific qualitative practical model will be considered below. To emphasize the uniformity of the proposed models, the parameters and analysis will be the same for the different domains.

The specific qualitative practical model is allowed to use small and convenient biases. It is convenient to consider integer numbers. The minimal non-zero integer for the bias for the sure outcome is *\$1*. Hence the minimal integer for the bias for the probable outcomes is *\$2*. Suppose that the parameters of the particular qualitative model for the gains are: the bias for the probable outcomes is equal to *\$2*, and for the sure outcome the bias is equal to *\$1* or to zero.

4.2. Practical numerical example. Gain

The above examples can be simplified to specific qualitative ones similar to Harin (2012b):

Imagine that you face the following pair of concurrent decisions.

Choose between:

A) A sure gain of $99.

B) 99% chance to gain $100 and 1% chance to gain or lose nothing.

4.2.1. Ideal case

In the ideal case, without taking into account the dispersion of the data, the expected values for the probable and sure outcomes are

$$\$99 \times 100\% = \$99 ,$$

$$\$100 \times 99\% = \$99 .$$

Here, the ideal expected values are exactly equal to each other

$$\$99 = \$99 .$$

Therefore the both outcomes should be equally preferable.

So in the ideal case, without taking into account the dispersion of the data, the probable and sure outcomes should be equally preferable.

4.2.2. Forbidden zones and biases

In the real case, one should take into account the dispersion of the data, some minimal non-zero variance caused by this dispersion and the forbidden zones caused by this variance. These forbidden zones can lead to the biases of the expectations, at least for the probable outcomes. Let us consider the case of the non-zero variance of the data, corresponding forbidden zones and biases.

Let the bias be equal to, say, $\Delta_{Prob} = \$2$ for the probable outcomes.

Let us consider the case when the bias for the sure outcome is equal to *$1*. We have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$1 = \$98,$$
$$\$100 \times 99\% - \Delta_{\mathrm{Pr}ob} = \$99 - \$2 = \$97.$$

Here, the probable expected value is biased more than the sure one and we have

$$\$98 > \$97.$$

Let us consider the case when the bias for the expectations of data for the sure outcome is equal to zero. We have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$0 = \$99,$$
$$\$100 \times 99\% - \Delta_{\mathrm{Pr}ob} = \$99 - \$2 = \$97.$$

Here, the probable expected value is biased but the sure expected value is not and we have

$$\$99 > \$97.$$

In all the cases, the probable expected value is biased more than the sure one. The bias decreases the advantage (preference) of the outcome. Therefore the probable gain is (due to the obvious difference between the expected values) less preferable than the sure one.

We see the clear and evident difference between the expected values and the corresponding salient and unequivocal preferences and choices.

So the theorem provides the mathematical support for the above analysis in the domain of gains.

So, the forbidden zones and their natural difference for probable and sure outcomes can predict the experimental fact that the subjects are risk averse in the domain of gains. They explain, at least qualitatively or partially, the analyzed example of Thaler (2016) and many other similar results.

The theorem provides the mathematical support for the analysis in the domain of gains.

4.3. Practical numerical example. Loss

The case of gains has been explained many times in a lot of ways. The uniform explanation for both gains and losses, without additional suppositions, such as in Kahneman and Tversky (1979), has not been recognized nevertheless by the author of the present article.

Let us consider the case of losses under the same suppositions as gains.

Imagine that you face the following pair of concurrent decisions. Choose between:

A) A sure loss of \$99.

B) 99% chance to loss \$100 and 1% chance to gain or lose nothing.

4.3.1. Ideal case

In the ideal case without the forbidden zones, the expected values for the probable and sure outcomes are

$$-\$99\times 100\% = -\$99,$$

$$-\$100\times 99\% = -\$99.$$

Here, the expected values are exactly equal to each other

$$-\$99 = -\$99.$$

Therefore the both outcomes should be equally preferable.

So in the ideal case, without taking into account the dispersion of the data, the probable and sure outcomes should be equally preferable.

4.3.2. Forbidden zones and biases

Let us consider the case of the forbidden zones and biases under the same suppositions as for the gains. That is for the same parameters of the models.

The forbidden zone biases the expectation from the boundary of the interval to its middle. The bias is subtracted from the absolute value for the both cases of gains and losses therefore. That is, due to the opposite signs of the values for gains and losses, the bias is subtracted from the expected values for the gains and added to the expected values for the losses. It should be emphasized that this is not a supposition but a rigorous conclusion. Therefore the applications of the specific qualitative model are naturally uniform for more than one domain.

The parameters of the specific qualitative model for the gains are: the bias for the probable outcomes is equal to *\$2*, and for the sure outcome to *\$1* or to zero.

Let us consider the case when the bias for the sure outcome is equal to *\$1*. We have

$$-\$99\times 100\% + \Delta_{Sure} = -\$99 + \$1 = -\$98\,,$$
$$-\$100\times 99\% + \Delta_{\Pr ob} = -\$99 + \$2 = -\$97\,.$$

Here, the probable expected value is biased more than the sure one and we have

$$-\$98 < -\$97\,.$$

Let us consider the case when the width of the forbidden zones for the expectations of data in the sure outcome is equal to zero. We have

$$-\$99\times 100\% + \Delta_{Sure} = -\$99 + \$0 = -\$99\,,$$
$$-\$100\times 99\% + \Delta_{\Pr ob} = -\$99 + \$2 = -\$97\,.$$

Here, the probable expected value is biased but the sure expected value is not and we have

$$-\$99 < -\$97\,.$$

In all the cases, the probable expected value is biased more than the sure one as in the case of gains, but here the bias increases the advantage (preference) of the outcome and the probable loss is (due to the obvious difference between the expected values) more preferable than the sure one.

We see the clear and evident difference between the expected values and the corresponding salient and unequivocal preferences and choices.

So the theorem can be naturally, uniformly and successfully applied in the domain of losses as well. Instead of the seeming simplicity of these applications, the author has not revealed such successful and uniform applications in more than one domain in the literature.

4.6. Practical application. Newness

So, the theorem provides the mathematical support for the explanation of the above problems in the domains of both gains and losses.

Due to, e.g., Harin (2012b), the forbidden zones and their natural difference for probable and sure outcomes can predict the experimental fact that the subjects are risk seeking in the domain of gains but risk seeking in the domain of losses. They explain, at least qualitatively or partially, the analyzed examples of Thaler (2016) and many other similar results.

The important feature is that, due to, e.g., Harin (2012b), the described forbidden zones can explain the problems and explain experimental results not only in the domains of the gains and losses. The important feature is also that this explanation is uniform in all the domains and need not additional suppositions. Hence the forbidden zones and their natural difference for probable and sure outcomes can qualitatively or, at least, partially predict the experimental facts and explain the problems in various domains.

The mathematical description of the above forbidden zones has been done in recent years. Unfortunately, these zones were not described in mathematics before.

The analysis of the literature, comments of comments of journals' editors and reviewers on similar articles and on the previous versions of the present article and more than 10-years experience of the editorship in NEP reports on utility and prospect theories (see Harin 2005-2017) allow to suppose the following.

The mathematical support for the above explanation, which is presented by the theorem and its consequences, is a new one.

Why did not this evident and widespread phenomenon be mathematically described before? The long absence of such a description can be explained by the evidence of its practical applications. That is these forbidden zones can be easily estimated as approximately a half of the amplitude of the oscillations and there is no need in more detailed analysis and calculation. The phenomena that are similar to the forbidden zones between ships boards and moorage wall, washing machines and walls, etc. are evident and are usually taken into account in the cases of their evident and essential influence.

The problems and paradoxes of the behavioral economics, utility and prospect theories are, probably, the first field where such phenomena are hidden by other details of experiments and hence are non-evident.

4.7. Possible applications. Noise. Biases of measurement data

4.5.1. Noise

Let us preliminary consider possible applications of the theorem to a noise.

If a noise leads to some non-zero minimal variance of the considered random variable, then this non-zero minimal variance and, consequently, this noise leads to the above non-zero forbidden zones for the expectation of this variable. If a noise leads to some increasing of the value of this minimal variance then the width of these forbidden zones increases also.

The presented theorem allows to make a step to developing of a possible new mathematical tool for description of the possible influence of noise near the boundaries of finite intervals. In particular, if a noise leads to a non-zero minimal variance $\sigma^2_{Min} : \sigma^2 > \sigma^2_{Min} > 0$ of a random variable, then the theorem predicts the forbidden zones having the width r_{Noise} which is not less than

$$r_{Noise} \geq \frac{\sigma^2_{Min}}{b-a}.$$

So, the presented theorem is the first preliminary step to a general mathematical description of the possible influence of noise near the boundaries of finite intervals.

4.5.2. Biases of measurement data

Let us preliminary consider possible applications of the theorem to possible biases of measurement data.

The considered forbidden zones can evidently lead to some biases in measurements. We can preliminary consider this a bit closer. Suppose some measurements are performed on a finite interval and its result is the expectation of the measurement data. Suppose some forbidden zones arise near the boundaries of the interval due to the minimal variance of the data.

The expectations of the data of the measurements cannot be indeed located inside the forbidden zones. They cannot be located closer to the boundaries of the interval than the width of the forbidden zone.

So the above forbidden zones can cause biases for the expectations of the data of measurements. The biases are directed from the boundaries to the middle of the interval. The biases have the opposite signs near the opposite boundaries of the interval. The absolute values of the biases decrease from the boundaries to the middle of the interval.

When the minimal variance of the data is equal to zero, then the expectations of the data of measurements can touch the boundaries of the interval. When the above forbidden zones are not taken into the consideration then the estimated results are also located closer to the boundaries than the real case. Hence the estimated results are biased in the comparison with the real ones.

Particular example of the biases. If the minimal variance of the data σ^2_{Min} is non-zero, that is if $\sigma^2 > \sigma^2_{Min} > 0$, then the theorem predicts that near the boundaries of intervals, the absolute value Δ_{Bias} of the biases is not less than

$$|\Delta_{Bias}| \geq \frac{\sigma^2_{Min}}{b-a}.$$

So, the presented theorem, its consequences and applications can be considered as the first preliminary step to a general mathematical description of the biases of measurement data near the boundaries of finite intervals.

5. Conclusions and discussion

The article can be concluded by the following five items:

1) Problems. There are well-known problems of behavioral economics (see, e.g., Hey and Orme 1994, Kahneman and Thaler 2006, Thaler 2016):

The choices of the subjects (people) don't correspond to the expectations of outcomes.

Some of the typical problems consist in comparison of sure and probable outcomes (see, e.g., Kahneman and Tversky 1979, Starmer and Sugden 1991, Barberis 2013, Thaler 2016). They are the most pronounced near the boundaries of intervals. For example, Thaler (2016) states (the **boldface** is my own):

"*We observe a pattern that was frequently displayed: subjects were **risk averse** in the domain of gains **but risk seeking** in the domain of losses.*"

The above problems can be represented in simplified and demonstrable form by the qualitative and specific qualitative problems that are considered in the present article similar to Harin (2012b). These specific qualitative problems are:

Domain of gains. Choose between:

A) A sure gain of *$99*.

B) *99%* chance to gain *$100* and *1%* chance to gain or lose nothing.

The expectations are

$$\$99 \times 100\% = \$99 = \$99 = \$100 \times 99\% .$$

Domain of losses. Choose between:

A) A sure loss of *-$99*.

B) *99%* chance to loss *-$100* and *1%* chance to gain or lose nothing.

The expectations are

$$-\$99 \times 100\% = -\$99 = -\$99 = -\$100 \times 99\% ,$$

The expected values are exactly equal to each other in the both domains. Nevertheless a wealth of experiments (see, e.g. Kahneman and Tversky 1979, Starmer and Sugden 1991, Thaler 2016) prove that the choices of the subjects are essentially biased. Moreover as is pointed out, e.g., in Thaler (2016), they are biased in the opposite directions for gains and losses. These are the well-known and fundamental problems that are usual in behavioral economics and other sciences.

2) Analysis of the problems. A new analysis of these problems was developed in recent years (see, e.g., Harin 2012a, Harin 2012b, Harin 2015). The analysis is founded on the idea of the forbidden zones studied here and enables at least qualitative explanation of these problems (see, e.g., Harin 2012b).

3) Mathematical support for the analysis. The forbidden zones theorem is proven in the present article. The theorem states that, for a finite interval $[a, b]$ under the condition of existence of some non-zero minimal variance $\sigma^2_{Min} : \sigma^2 \geq \sigma^2_{Min} > 0$, the expectation μ of the measurement data is separated from the boundaries of the interval by the forbidden zones

$$a + \frac{\sigma^2_{Min}}{b-a} \leq \mu \leq b - \frac{\sigma^2_{Min}}{b-a}.$$

In other words, the theorem proves the possibility of existence of the forbidden zones that were used in the above analysis. The forbidden zones can exist near the boundaries of the intervals of the measurement data. The theorem also determines the conditions of the existence of the zones and their minimal width.

4) Mathematical approach for the analysis. The mathematical approach of the biases is founded on the theorem and is to explain not only the objective situations but mainly the subjective behavior and choices of subjects.

The two main suppositions of the approach are:

1. The subjects make their choices (at least to a considerable degree) as if there were some expectations biases of the outcomes.

(This supposition can be supported by the thought that such biases may be proposed and tested even from the purely formal point of view)

2. These biases (real biases or subjective reaction and choices of the subjects) can be explained (at least to a considerable degree) with the help of the theorem.

The first stage of the approach is the qualitative mathematical explanation of the qualitative problems.

The main supposed relationships of the first stage of the approach can be accumulated into the three groups:

The supposed relationships (13) of the qualitative problems

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_{\mu} \quad \text{and} \quad |d_{Choice}| \geq |d_{\mu}|.$$

The supposed relationships (14) about the probable and sure outcomes

$$|\Delta_{\Pr ob}| > 0 \quad \text{and} \quad |\Delta_{\Pr ob}| > |\Delta_{Sure}| \quad \text{and} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\Pr ob}.$$

The supposed relationships (15) about the theorem and choices

$$\sigma^2_{Min} > 0 \quad \text{and} \quad d_{Theorem} \approx d_{Choise} \quad \text{or at least} \quad d_{Theorem} = O(d_{Choise}).$$

5) Mathematical qualitative models for the analysis. The specific qualitative mathematical model and the basics of the general qualitative mathematical models are developed here.

5.1) Specific qualitative model. The specific qualitative mathematical model is intended for the practical analysis of the above problems in the specific case when the expectations of the data for the probable and sure outcomes are exactly equal to each other. The model can be considered as the first step of the first stage of the approach.

The model implies the application of the forbidden zones theorem under the following two suppositions:

1. Probable outcomes. Due to relationships (14) of the approach, the bias

$$|\Delta_{\mathrm{Pr}obable}| > 0$$

should be finite but can be as small as possible. Therefore the minimal variance of the measurement data for the probable outcomes can be supposed to be equal to an arbitrary non-zero value that is as small as possible to be evidently explainable by a common noise and scattering of the data.

Due to the theorem, this supposition leads to the non-zero forbidden zones for the expectations of the data for the probable outcomes near the boundaries of the intervals and, consequently, to some small but non-zero biases of these expectations, at least right against the boundaries.

2. Sure outcomes. The bias for the sure outcomes is equal to zero or is strictly less than the bias for the probable outcomes.

Numerical examples. In the scope of the specific qualitative model, suppose that the biases of the expectations are equal, for example, to Δ_{Prob} = *$2* for the probable outcomes and Δ_{Sure} = *$1* for the sure outcomes. Then we have:

Gains. In the case of gains we have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$1 = \$98,$$
$$\$100 \times 99\% - \Delta_{Prob} = \$99 - \$2 = \$97.$$

The probable expected value is biased more than the sure one. The biases are directed from the boundary to the middle of the interval and, hence, decrease the modules of the values and the both values themselves. Therefore the biased sure expected value is more than the biased probable one

$$\$98 > \$97.$$

The sure gain is evidently more preferable than the probable one and this choice is supported by a wealth of experiments.

Losses. In the case of losses we have

$$-\$99 \times 100\% + \Delta_{Sure} = -\$99 + \$1 = -\$98,$$
$$-\$100 \times 99\% + \Delta_{Prob} = -\$99 + \$2 = -\$97.$$

The probable expected value is biased more than the sure one. The biases are directed from the boundary to the middle of the interval and, hence, reduce the modules of the values but, due to their negative signs, increase the both values. Therefore the biased sure expected value is less than the biased probable one

$$-\$98 < -\$97.$$

The probable loss is evidently more preferable than the sure one and this choice is supported by a wealth of experiments.

So, the qualitative model enables the qualitative analysis and qualitative explanation for the above problems in more than one domain.

5.2) Basics of general qualitative model. The basics of the general formal preliminary qualitative mathematical model are considered in the present article.

The suppositions of the model can be formulated as follows:

The suppositions of the qualitative problems

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_{\mu} \quad \text{and} \quad |d_{Choice}| \geq |d_{\mu}|.$$

The suppositions for the probable and sure outcomes

$$|\Delta_{\mathrm{Pr}ob}| > 0 \quad \text{and} \quad |\Delta_{\mathrm{Pr}ob}| > |\Delta_{Sure}| \quad \text{and} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\mathrm{Pr}ob}.$$

The supposition for the theorem and choices

$$\sigma^2{}_{Min} > 0 \quad \text{and at least} \quad d_{Theorem} = O(d_{Choise}).$$

The suppositions of the types of the problems are:

$$|d_{Choice}| = |d_{\mu}|$$

for the problems of certainty equivalents and

$$|d_{Choice}| > |d_{\mu}|$$

for the other problems.

Main contributions. The four main particular contributions of the present article are the mathematical support, approach and specific qualitative model for the above analysis and the successful uniform application of this model in more than one domain.

The author has not revealed in the literature such a natural, uniform and successful application of a model in more than one domain of the discussed problems. Hence, instead of seeming simplicity, the successful natural and uniform application of the specific qualitative model in more than one domain also belongs to the main contributions.

Possible additional contributions. Two more possible additional general contributions can be preliminary mentioned:

Possible general addition. Noise. In addition, possible general consequences and applications of the theorem for a noise are preliminary considered.

In particular, let us suppose that some type of noise leads to a non-zero minimal variance $\sigma^2_{Min} : \sigma^2 > \sigma^2_{Min} > 0$ of a random variable, then the theorem predicts the existence of the forbidden zones having the width r_{Noise} which is not less than

$$r_{Noise} \geq \frac{\sigma^2_{Min}}{b-a}.$$

The future goal of this consideration is a general mathematical description of the possible influence of noise near the boundaries of finite intervals.

Possible general addition. Biases. In addition, possible general consequences and applications of the theorem for biases of measurement data are preliminary considered.

In particular, if the minimal variance of the data σ^2_{Min} is non-zero, that is if $\sigma^2 > \sigma^2_{Min} > 0$, then the theorem predicts the biases of measurement data in general cases. The biases have the opposite signs near the opposite boundaries, are maximal near the boundaries and tend to zero in the middles of the intervals. Right against the boundaries of intervals, the absolute value Δ_{Bias} of the biases is not less than

$$|\Delta_{Bias}| \geq \frac{\sigma^2_{Min}}{b-a}.$$

The future goal of this consideration is a general mathematical description of the biases of measurement data that can be caused by the above forbidden zones.

Acknowledgements

The author wishes to say many words of gratitude to Professor A. A. Novosyolov for his long-term support and methodological tutorship.

References

Aczél, J., and D. R. Luce, "A behavioral condition for Prelec's weighting function on the positive line without assuming W(1)=1", *Journal of Mathematical Psychology*, 51 (2007), 126–129.

Barberis, N.C., 2013 "Thirty Years of Prospect Theory in Economics: A Review and Assessment," *Journal of Economic Perspective*, **27** (2013), 173-196.

Butler, David, and Graham Loomes, "Imprecision as an Account of the Preference Reversal Phenomenon," *American Economic Review*, 97 (2007), 277-297.

Chay, K., P. McEwan, and M. Urquiola, "The Central Role of Noise in Evaluating Interventions that Use Test Scores to Rank Schools", *American Economic Review*, 95 (2005), 1237-1258.

Dokov, S. P., Morton, D.P., 2005. Second-Order Lower Bounds on the Expectation of a Convex Function. *Math. Oper. Res.* **30**(3), 662–677.

Harin, A., 2012a, "Data dispersion in economics (I) – Possibility of restrictions," *Review of Economics & Finance*, 2 (2012), 59-70.

Harin, A., 2012b, "Data dispersion in economics (II) – Inevitability and Consequences of Restrictions," *Review of Economics & Finance*, 2 (2012), 24-36.

Harin, A. 2013, Data dispersion near the boundaries: can it partially explain the problems of decision and utility theories? Working Papers from HAL No. 00851022, 2013.

Harin, A., 2014, "The random-lottery incentive system. Can *p~1* experiments deductions be correct?" *16th conference on the Foundations of Utility and Risk*, 2014.

Harin, A., 2015. General bounds in economics and engineering at data dispersion and risk, *Proceedings of the Thirteenth International Scientific School* **13**, 105–117, in Modeling and Analysis of Safety and Risk in Complex Systems (Saint-Petersburg: IPME RAS).

Harin, A. 2017, Can forbidden zones for the expectation explain noise influence in behavioral economics and decision sciences? MPRA Paper No. 76240, 2017.

Hey, J., and C. Orme, "Investigating Generalizations of Expected Utility Theory Using Experimental Data," *Econometrica*, 62 (1994), 1291-1326.

Kahneman, D., and Thaler, R., 2006. Anomalies: Utility Maximization and Experienced Utility, *J Econ. Perspect.* **20**(1), 221–234.

Kahneman, D., and A. Tversky, "Prospect Theory: An Analysis of Decision under Risk," *Econometrica*, **47** (1979), 263-291.

Moriguti, S., "A lower bound for a probability moment of any absolutely continuous distribution with finite variance," *The Annals of Mathematical Statistics* **23**(2), 286–289.

Pinelis, I., 2011. Exact lower bounds on the exponential moments of truncated random variables, *J Appl. Probab.* **48**(2), 547–560.

Prékopa, A., 1990, The discrete moment problem and linear programming, *Discrete Appl. Math.* **27**(3), 235–254.

Prékopa, A., 1992. Inequalities on Expectations Based on the Knowledge of Multivariate Moments. Shaked M, Tong YL, eds., *Stochastic Inequalities*, 309–331, number 22 in Lecture Notes-Monograph Series (Institute of Mathematical Statistics).

Prelec, D., "The Probability Weighting Function," *Econometrica*, 66 (1998), 497-527.

Schoemaker, P., and J. Hershey, "Utility measurement: Signal, noise, and bias," *Organizational Behavior and Human Decision Processes*, 52 (1992), 397-424.

Starmer, C., & Sugden, R. (1991). Does the Random-Lottery Incentive System Elicit True Preferences? An Experimental Investigation. *American Economic Review*, 81: 971–78.

Steingrimsson, R., and R. D. Luce, "Empirical evaluation of a model of global psychophysical judgments: IV. Forms for the weighting function," *Journal of Mathematical Psychology*, **51** (2007), 29–44.

Thaler, R., 2016. Behavioral Economics: Past, Present, and Future, *American Economic Review.* **106**(7), 1577–1600.

Tversky, A. and D., Kahneman "Prospect Theory: Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty*, **5** (1992), 297-323.

Appendix. Lemmas of variance maximality conditions

Preliminaries

The initial particular need is the mathematical support for the analysis (see, e.g., Harin 2012a, Harin 2012b and Harin 2015) of the problems of behavioral economics. These problems take place for the discrete finite random variables. Nevertheless let us give the support for the general case.

In the general case, we have (see subsection 2.1)

$$E[X-\mu]^2 = \sum_{k=1}^{K} (x_k-\mu)^2 p(x_k) + \int_a^b (x-\mu)^2 f(x)dx .$$

under the condition that either the probability mass function or probability density function or alternatively both of them are not identically equal to zero, or

$$\sum_{k=1}^{K} p(x_k) + \int_a^b f(x)dx = 1 .$$

Pairs of values whose mean value coincides with the expectation of the random variable were used, e.g., in Harin (2013). More arbitrary choice of pairs of values was used in Harin (2017). Here every discrete and infinitesimal value will be divided into the pair of values in the following manner:

Let us divide every value $p(x_k)$ into the two values located at a and b

$$p(x_k)\frac{b-x_k}{b-a} \quad \text{and} \quad p(x_k)\frac{x_k-a}{b-a} .$$

The total value of these two parts is evidently equal to $p(x_k)$. The center of gravity of these two parts is evidently equal to x_k.

Let us divide every value of $f(x)$ into the two values located at a and b

$$f(x)\frac{b-x}{b-a} \quad \text{and} \quad f(x)\frac{x-a}{b-a} .$$

The total value of these two parts is evidently equal to $f(x)$. The center of gravity of these two parts is evidently equal to x.

Let us prove that the variances of the divided parts are not less than those of the initial parts.

A1. Lemma 1. Discrete part

Lemma 1. Discrete part lemma. If the support of a random variable X, is an interval $[a,b]: 0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2=\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx \equiv \sigma^2,$$

where p is the probability mass function of X, $a \le x_k \le b$, $k=1,2,...,K$, where $K \ge 1$ and $\mu \equiv E[X]$ and

$$\sum_{k=1}^{K} p(x_k) \ge 0,$$

then the inequality

$$\begin{aligned}&\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k) \ge \\ &\ge \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)\end{aligned} \quad . \qquad (16)$$

is true.

Proof. Let us find the difference between the transformed

$$\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)$$

and initial

$$\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)$$

expressions for the variance.

Let us consider separately the cases of $x_k \ge \mu$ and $x_k \le \mu$.

A.1.1. Case of $x_k \geq \mu$

If $x_k \geq \mu$, then the expression in the square brackets can be simplified

$$\left[(a-\mu)^2 \frac{b-x_k}{b-a} + (b-\mu)^2 \frac{x_k-a}{b-a} - (x_k-\mu)^2\right] \geq$$
$$\geq \left[(b-\mu)^2 \frac{x_k-a}{b-a} - (x_k-\mu)^2\right] = .$$
$$= (b-\mu)^2 \left[\frac{x_k-a}{b-a} - \left(\frac{x_k-\mu}{b-\mu}\right)^2\right]$$

Due to $x_k \leq b$ and

$$0 \leq \frac{x_k-\mu}{b-\mu} \leq 1,$$

it holds true that

$$\left(\frac{x_k-\mu}{b-\mu}\right)^2 \leq \frac{x_k-\mu}{b-\mu}$$

and

$$\frac{x_k-a}{b-a} - \left(\frac{x_k-\mu}{b-\mu}\right)^2 \geq \frac{x_k-a}{b-a} - \frac{x_k-\mu}{b-\mu}$$

and then

$$\frac{x_k-a}{b-a} - \frac{x_k-\mu}{b-\mu} \equiv \frac{(x_k-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)} - \frac{x_k-\mu}{b-\mu}.$$

Due to

$$0 \leq \frac{x_k-a}{b-a} \leq 1 \quad \text{and} \quad \mu-a \geq 0,$$

the inequality

$$\frac{(x_k-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)} \geq \frac{x_k-\mu}{b-\mu}$$

is true and

$$(b-\mu)^2 \left[\frac{x_k-a}{b-a} - \left(\frac{x_k-\mu}{b-\mu}\right)^2\right] \geq 0.$$

So in the case of $x_k \geq \mu$ the difference between the transformed and initial expressions for the variance is non-negative.

A.1.2. Case of $x_k \le \mu$

If $x_k \le \mu$, then

$$\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]=$$
$$=\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(\mu-x_k)^2\right]\ge$$
$$\ge\left[(\mu-a)^2\frac{b-x_k}{b-a}-(\mu-x_k)^2\right]=$$
$$=(\mu-a)^2\left[\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\right].$$

Due to

$$0\le\frac{\mu-x_k}{\mu-a}\le 1,$$

we have

$$\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\ge\frac{b-x_k}{b-a}-\frac{\mu-x_k}{\mu-a}.$$

Then

$$\frac{b-x_k}{b-a}-\frac{\mu-x_k}{\mu-a}\equiv\frac{(b-\mu)+(\mu-x_k)}{(b-\mu)+(\mu-a)}-\frac{\mu-x_k}{\mu-a}.$$

Due to

$$0\le\frac{\mu-x_k}{\mu-a}\le 1 \quad \text{and} \quad b-\mu\ge 0$$

we have

$$\frac{(b-\mu)+(\mu-x_k)}{(b-\mu)+(\mu-a)}\ge\frac{\mu-x_k}{\mu-a}$$

and

$$(\mu-a)^2\left[\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\right]\ge 0.$$

So in the case of $x_k \le \mu$ the difference between the transformed and initial expressions for the variance is non-negative.

A.1.3. Maximality

So the difference

$$(a-\mu)^2 p(x_k)\frac{b-x_k}{b-a}+(b-\mu)^2 p(x_k)\frac{x_k-a}{b-a}-(x_k-\mu)^2 p(x_k)=$$
$$=p(x_k)\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right].$$

is non-negative.

Let us calculate the difference between the transformed and initial expressions of the discrete part of the variance

$$E_{Discr.Transformed}[X-\mu]^2-E_{Discr.Initial}[X-\mu]^2=$$
$$=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)-\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)=.$$
$$=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]p(x_k)$$

Every member of a sum is non-negative, as in the above expression. Hence the total sum is non-negative as well.

So for the discrete case the variance is maximal when the probability mass function is concentrated at the boundaries of the interval.

$$E_{Discr.Transformed}[X-\mu]^2-E_{Discr.Initial}[X-\mu]^2=$$
$$=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)-\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)=.$$
$$=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]p(x_k)$$

If every member of a sum is non-negative, as in the above expression, then the total sum is non-negative as well.

A.1.4. Theorem of Huygens-Steiner

Also the expression

$$(a-\mu)^2(b-x_k)+(b-\mu)^2(x_k-a)\,.$$

can be identically rewritten to

$$[(x_k-a)+(\mu-x_k)]^2(b-x_k)+$$
$$+[(b-x_k)+(x_k-\mu)]^2(x_k-a)=$$

$$=[(x_k-a)^2+2(x_k-a)(\mu-x_k)+(\mu-x_k)^2](b-x_k)+$$

$$+[(b-x_k)^2+2(b-x_k)(x_k-\mu)+(x_k-\mu)^2](x_k-a)$$

and

$$[(x_k-a)^2+2(x_k-a)(\mu-x_k)+(\mu-x_k)^2](b-x_k)+$$
$$+[(b-x_k)^2+2(b-x_k)(x_k-\mu)+(x_k-\mu)^2](x_k-a)=$$

$$=(x_k-\mu)^2(b-a)+$$

$$+(x_k-a)^2(b-x_k)+(b-x_k)^2(x_k-a)+$$

$$+2(x_k-a)(b-x_k)[(\mu-x_k)+(x_k-\mu)]$$

This can be transformed to the expression

$$(x_k-\mu)^2+$$
$$(a-\mu)^2(b-x_k)+(b-\mu)^2(x_k-a)$$

that is analogous to the theorem of Huygens-Steiner (The general possibility of application of the Huygens-Steiner theorem was helpfully pointed out by one of the anonymous referees when the preceding version of the present article was refereed)

A2. Lemma 2. Continuous part

Let the probability density function is not identically equal to zero.

Lemma 2. Continuous part lemma. If the support of a random variable X, is an interval $[a,b]: 0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2 = \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k) + \int_a^b x^2 f(x)dx \equiv \sigma^2,$$

where f is the probability density function of X and $\mu \equiv E[X]$ and

$$\int_a^b f(x)dx \geq 0,$$

then the inequality

$$\int_a^b \left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx \geq \int_a^b (x-\mu)^2 f(x)dx. \qquad (18)$$

is true.

Proof. Let us find the difference between the transformed

$$\int_a^b \left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx$$

and initial

$$\int_a^b (x-\mu)^2 f(x)dx$$

expressions for the variance.

Let us consider separately the cases of $x \geq \mu$ and $x \leq \mu$.

A.2.1. Case of $x \geq \mu$

If $x_k \geq \mu$, then the difference can be simplified as

$$\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}-(x-\mu)^2\right]\geq$$
$$\geq\left[(b-\mu)^2\frac{x-a}{b-a}-(x-\mu)^2\right]=$$
$$=(b-\mu)^2\left[\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\right]$$

.

Due to $x \leq b$ and

$$0\leq\frac{x-\mu}{b-\mu}\leq 1,$$

it holds true that

$$\left(\frac{x-\mu}{b-\mu}\right)^2\leq\frac{x-\mu}{b-\mu}$$

and

$$\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\geq\frac{x-a}{b-a}-\frac{x-\mu}{b-\mu}$$

and then

$$\frac{x-a}{b-a}-\frac{x-\mu}{b-\mu}\equiv\frac{(x-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)}-\frac{x-\mu}{b-\mu}.$$

Due to

$$0\leq\frac{x-a}{b-a}\leq 1 \quad \text{and} \quad \mu-a\geq 0,$$

we have

$$\frac{(x-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)}\geq\frac{x-\mu}{b-\mu}.$$

and

$$(b-\mu)^2\left[\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\right]\geq 0.$$

A.2.2. Case of $x \le \mu$

If $x \le \mu$, then the difference can be simplified as

$$\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}-(\mu-x)^2\right]\ge$$
$$\ge\left[(\mu-a)^2\frac{b-x}{b-a}-(\mu-x)^2\right]=$$
$$=(\mu-a)^2\left[\frac{x-a}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\right].$$

Due to

$$0\le\frac{\mu-x}{\mu-a}\le 1,$$

we have

$$\frac{b-x}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\ge\frac{b-x}{b-a}-\frac{\mu-x}{\mu-a}.$$

Then

$$\frac{b-x}{b-a}-\frac{\mu-x}{\mu-a}\equiv\frac{(b-\mu)+(\mu-x)}{(b-\mu)+(\mu-a)}-\frac{\mu-x}{\mu-a}.$$

Due to

$$0\le\frac{\mu-x}{\mu-a}\le 1 \quad \text{and} \quad b-\mu\ge 0$$

we have

$$\frac{(b-\mu)+(\mu-x)}{(b-\mu)+(\mu-a)}\ge\frac{\mu-x}{\mu-a}$$

and

$$(\mu-a)^2\left[\frac{b-x}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\right]\ge 0.$$

A.2.3. Maximality

Let us calculate the difference between the transformed and initial expressions of the continuous part of the variance

$$E_{Contin.Transformed}[X-\mu]^2 - E_{Contin.Initial}[X-\mu]^2 =$$

$$= \int_a^b \left[(a-\mu)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} \right] f(x)dx - \int_a^b (x-\mu)^2 f(x)dx = .$$

$$= \int_a^b \left[(a-\mu)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} - (x-\mu)^2 \right] f(x)dx$$

If the integrand of an integral is non-negative for every point in the scope of the limits of integration as in the above expression, then the complete integral is non-negative as well. The difference is therefore non-negative.

So for the continuous case the variance is maximal when the probability density function is concentrated at the boundaries of the interval.

A3. Lemma 3. Mixed case

Let the probability mass function is not identically equal to zero.

Lemma 3. General mixed case lemma. If the support of a random variable X, is an interval $[a,b]: 0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2=\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx \equiv \sigma^2,$$

where p is the probability mass function of X, $a \le x_k \le b$, $k=1,2,...,K$, where $K \ge 1$ and f is the probability density function of X and $\mu \equiv E[X]$ and

$$\sum_{k=1}^{K} p(x_k)+\int_a^b f(x)dx=1,$$

then the inequality

$$\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)+$$
$$+\int_a^b\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx \ge ,$$
$$\ge \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx$$

is true.

Proof. The general mixed case is compiled from the discrete and continuous parts under the condition that at least one of them is not identically equal to zero. The conclusions concerned to these parts are true for their sum as well.

So in any case both for the probability mass function and/or probability density function and/or their mixed case, the variance is maximal when the probability mass function and/or probability density function are concentrated at the boundaries of the interval in the form of the probability mass function that has only the two values located at the boundaries of the interval.