Multi-Curve Discounting

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Abstract

This note considers the valuation of assets and liabilities on a balance sheet with liquidity risk. It introduces the multi-curve discounting (MCD) method, where the discount curve depends on the liquidity horizon of the asset. The difference between the value of an asset using OIS discounting and a discount curve referencing the liquidity horizon can be interpreted as a Funding Valuation Adjustment (FVA). We show that a simple model for liquidity risk implies MCD. The liquidity risk model formulation clarifies how a non-zero FVA occurs without violating the Modigliani-Miller theorem [6].

1 Introduction

During the crisis, the Libor-OIS spread increased from a few basis points to several hundred basis points at its peak. Since then the spreads have decreased to tens of basis points, but not back to pre-crisis levels.

This change in the market had a significant impact on the valuation of interest rate products. The practice of using multiple index or forecast curves by the major banks before the crisis became more relevant and more widespread. Also, the discount curve was changed. Pre-crisis the discount curve was based on Libor rates at most banks, during and after the crisis banks switched to OIS discounting [5, 3, 2].

OIS discounting values a derivative as if there is a collateral agreement with a daily settlement and without thresholds, initial margin or other complicating features. Since this is the closest one can get to eliminating credit risk, the resulting value is interpreted as risk-free value. A derivative without such a collateral agreement is exposed to credit risk leading to a so-called credit valuation adjustment (CVA) and debt valuation adjustment (DVA). In this paper, we will focus on the value without credit risk and assume that credit risk is separately accounted for in CVA and DVA.

Hence, using OIS discounting an incoming cash flow of 1 at time T has a present value

$$V_0 = DF_{\mathrm{OIS}}(T) = e^{-r_{\mathrm{OIS}}(T)T}. \tag{1.1}$$

where $r_{\mathrm{OIS}}(T)$ denotes the continuously compounded OIS rate at maturity T, and DF_{OIS} the OIS discountfactor.

Besides the OIS discounting curve, a bank builds multiple index curves [3, 2]. These curve reference different Libor rates. Typical rates are 1M, 3M, 6M, and 12M. We will denote by $r_{x\mathrm{M}}(T)$ the continuously compounded xM rate at maturity T. We will assume that a bank has some procedure to interpolate and extrapolate the index. For example, from the OIS, 1M, 3M, 6M, and 12M curves the bank can construct a 2M curve. Hence we assume that the rate $r_t(T)$ for any $t \geq$ 1day is available. In the case $t =$ 1day, the rate is an OIS rate: $r_{\mathrm{1day}}(T) = r_{\mathrm{OIS}}(T)$.

Multi-curve discounting comes into play when assets are not collateralized. The idea is that the term used for funding the asset depends on the liquidity of the asset. For example, a perfectly liquid asset (cash) can be funded on an overnight basis, but a corporate bond might need to be funded on e.g. a 3-month basis. The discount curve for this bond (excluding credit risk) should not be the OIS curve, as this would imply overnight funding. Instead, the 3-month curve is the appropriate discount curve in this case.

To generalize the above notions, we assume each asset has a well-defined liquidity horizon. In the BCBS standards "Minimum capital requirements for market risk" [11] the liquidity horizon is defined as the time required to exit or hedge a risk position without materially affecting market prices in stressed market conditions. Generalizing the previous example, the value of an incoming cash flow of 1 at time T with a liquidity horizon t_{liq} is

$$V_0 = DF_{t_{\text{liq}}}(T) = e^{-r_{t_{\text{liq}}}(T)T}. \tag{1.2}$$

Equation 1.2 can be viewed as the definition of the multi-curve discounting (MCD) method. It states that the discount curve used for discounting cash flows of an asset depends on the liquidity horizon of the asset.

The purpose of this paper is to motivate MCD as a valuation method for assets and liabilities (section 2) and to formulate a liquidity risk model that implies the MCD method (section 3). The benefit of such a formulation is that it clarifies the assumptions and conditions that lead to MCD, it links the MCD valuation to a risk, and it facilitates extension to the valuation of more complex products, such as derivatives (section 4).

2 Valuing Assets and Liabilities on a Balance Sheet

In this section we motivate the MCD method by considering simple examples of asset and liability valuation with liquidity risk. In section 3 a more formal derivation of the MCD method follows.

2.1 Set up

Consider the simple balance sheet,

Asset	Funding
Cash	Equity

where the amount of "Cash" equals the amount of "Equity". Hence the "Asset" is completely funded by the "Funding".

We consider an asset from which the bank is to receive a single cash flow at 10 years. Furthermore, we assume that this cash flow is a fixed amount and the counterparty is non-defaultable. Also we assume that interest rates are deterministic. The result of all these assumptions is that the asset does not carry credit risk or market risk. In fact, we have constructed an assset that is only exposed to liquidity risk.

To illustrate the main concepts involved, assume that the funding market consists of just two products: a 1-year loan and a 10-year loan. The current rates are:

10Y: 1.5%
1Y: 0%

Furthermore, we assume that there is a forwards market that determines the market rates for 1Y forwards:

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1Y forwards	0.0%	0.3%	0.6%	0.9%	1.2%	1.5%	1.8%	2.1%	2.4%	2.7%

This curve gives an effective 10 year rate of 1.35%. In the notation of equation (1.2): $r_{1\mathrm{Y}}(10Y) = 1.35\%$.

We assume that these rates are risk-free, the parties involved are non-defautable. The bank can get funding from this market as well as invest in this market.

2.2 Value of the Asset

In the simple market described above there are two funding strategies available:
A) fund the asset by a 10-year loan
B) fund the asset by a 1-year loan and roll over the loan until maturity at 10 years.

Strategy A does not expose the bank to liquidity risk. The asset is term-funded and the bank will never need to liquidate it.

Strategy B does expose the bank to liquidity risk. It is possible that at a time when the bank needs to roll over the loan there is no party willing to provide the loan. The bank would need to liquidate the asset under "fire sale" conditions which may result in a loss if the asset is illiquid.

From the valuation of derivatives we have learned that the value is determined by the value of the hedge that removes all risks. The strategy that removes all risks in this case is clearly strategy A. Therefore it seems that the value of the asset is the value of the 10-year loan required to fund the asset. Hence if the pay out is 1 at 10 years the value would be

$$\frac{1}{(1+1.5\%)^{10}} = 0.86 \tag{2.1}$$

However, this reasoning is flawed. The value of the asset would be independent of the liquidity of the asset. Such a value leads to arbitrage opportunities; e.g. an illiquid asset can be swapped with a liquid asset (with the same specifications besides the liquidity). The flaw is that the funding strategy, although it removes all risks, is not a replication strategy. In fact, if the asset is perfectly liquid even in a stress event, it can be funded by a 1-year loan that is rolled over. In case that the loan cannnot be rolled over the asset can be liquidated without discount at the market value.

Hence the discount rate for valuation is:

illiquid asset: 1.5%

liquid asset: 1.35%

It is a consequence of the simple set up we have chosen, in particular with just two funding strategies available, that leads to this discrete result of either a discount rate of 1.5% or 1.35%. A more realistic set up would result in a continuous discount rate depending on the liquidity of the asset. Nevertheless in this set up it is useful to determine the distinction between liquid and illiquid assets.

The asset is illiquid when the expected loss (per year) due to liquidation in a stress event exceeds the extra funding cost of funding the asset for 10 year. Hence in this example we define the asset as illiquid when probability stress event (per year) (P_{stress}) times the liquidity discount > 15bp:

$$\text{illiquid: } P_{\text{stress}} \times (1 - LV_{\text{asset}}) > 15\text{bp}, \tag{2.2}$$

where LV_{asset} denotes the liquidation value, the value of the asset in a liquidation event as a fraction of its fair value: $1 - LV_{\text{asset}}$ is the liquidity discount.

The nice aspect of (2.2) is that it combines two aspects of liquidity risk. One aspect is the funding liquidity risk which is modelled by P_{stress} the other aspect is market liquidity risk which is modelled by $(1 - LV_{\text{asset}})$.

2.3 Value of Liabilities

In the valuation of liabilities we can distinguish between so-called stable funding and unstable funding. In the context of the example, stable funding is funding that is certain to be available for 10 years. Unstable funding can be partly withdrawn after each year.

To determine the appropriate discount rate for stable funding we note that it can be hedged by investing in a 10 year loan. The combination of 10-year loan and stable funding is risk-free (by the definition of stable funding above). This results in a discount rate of 1.5% for stable funding.

The valuation of unstable funding is more interesting. Consider a simple case where each year there is a probability $P_{\text{withdrawal}} = 20\%$ of the funding can be withdrawn. The withdrawal can happen only once, hence it is certan that 80% of the funding is available for 10 years.

There are two strategies to consider:

A) 100% investment in a 10-year loan.

B) 80% investment in a 10-year loan and 20% investment in a 1-year loan.

The choice of the appropriate strategy is determined by the probability of withdrawal and liquidation loss of the 10-year loan. Strategy A is optimal when:

$$P_{\text{withdrawal}} \times (1 - LV_{\text{10y loan}}) > 20\% \times 15\text{bp} \tag{2.3}$$

Here the l.h.s. represents the expected loss (per year) from liquidating the loan and the r.h.s. represents the extra hedging costs for removing this risk (the difference in funding costs of strategy A and B).

If strategy A is optimal the appropriate discount rate is 1.5%, if strategy B is optimal the discount rate is 1.47%.

3 Liquidity Risk Model

In this section we formalize the intuition developed through the examples in section 2.

There are many definitions of liquidity risk. However since we are interested in the valuation of assets, we can limit the definition to the risk that affects the pay-off or return of an asset. Following [7] we define liquidity risk as the risk for an event to occur that forces the bank to liquidate some of its assets. We will call such an event a liquidity stress event (LSE).

If a bank needs to liquidate some assets in an LSE, these assets will be sold at a liquidity discount. We introduced a liquidation value LV that represents the fraction of the fair value at which an asset can be liquidated. The fraction $1 - LV$ is the liquidity discount (as a fraction of the fair value) and can be interpreted as the loss due to liquidation.

The liquidation value depends on how the asset is funded. In the model developed here, the funding can be specified by a single funding term t_{fun} (see [9] for a discussion on more general funding compositions). The funding is continuously rolled over so that the funding term is fixed until an LSE hits the bank. At that time funding cannot be rolled over anymore. When the funding term is larger than the liquidity horizon of the asset, the asset can be liquidated at fair value. When the funding term is shorter than the liquidity horizon, the asset needs to be liquidated at a discount. This liquidity discount is set at a fixed 100% to generate MCD. Hence, the liquidation value is

$$LV(t_{\text{fun}}) = I_{t_{\text{liq}} \leq t_{\text{fun}}} . \tag{3.1}$$

The inclusion of these events into the valuation of assets is facilitated by defining an effective pay-off. For a bullet loan that pays 1 at maturity T the effective pay-off reads

$$\text{Effective Pay-Off} = \begin{cases} 1 & \text{at time } T & \text{if } \tau + t_{\text{fun}} \geq T \\ V_{\text{rf}}(\tau + t_{\text{fun}}) LV(t_{\text{fun}}) & \text{at time } \tau + t_{\text{fun}} & \text{if } \tau + t_{\text{fun}} < T \end{cases} \tag{3.2}$$

Here τ denotes the time of the occurrence of the LSE. We will model τ as a random number that is distributed according to an exponential distribution with intensity λ

$$\tau \sim \lambda e^{-\lambda \tau} . \tag{3.3}$$

Furthermore the risk-free value at time t of the bullet loan is

$$V_{\text{rf}}(t) = DF_{\text{OIS}}(t, T) = e^{-r_{\text{OIS}}(T-t)} . \tag{3.4}$$

A funding-term dependent value is obtained by requiring that the return on the asset equals the return on the liability in expectation [9]. The result is

$$\begin{aligned} V_0(t_{\text{fun}}) &= e^{-r_{\text{fun}} T} \mathbb{P}(\tau + t_{\text{fun}} \geq T) \\ &\quad + \mathbb{E}\left[e^{-r_{\text{fun}}(\tau + t_{\text{fun}})} V_{\text{rf}}(\tau + t_{\text{fun}}) LV(t_{\text{fun}}) I_{\tau + t_{\text{fun}} < T} \right] , \end{aligned} \tag{3.5}$$

where we have introduced the shorthand $r_{\text{fun}} = r_{t_{\text{fun}}}$. The expectation value in above expression can be calculated analytically. Expanding the result for $V_0(t_{\text{fun}})$ in small $(r_{\text{fun}} - r_{\text{ON}})T$ and λT to first order allows the value to be decomposed in the risk-free value minus funding costs and liquidity costs as shown in appendix A

$$V_0(t_{\text{fun}}) = V_{\text{rf}}(0) - FC - LC . \tag{3.6}$$

Here the funding costs are given by

$$FC = (r_{\text{fun}} - r_{\text{OIS}}) T e^{-r_{\text{OIS}} T} , \tag{3.7}$$

and the liquidity costs equal the expected liquidation losses

$$LC = \lambda (T - t_{\text{fun}}) [1 - LV(t_{\text{fun}})] e^{-r_{\text{OIS}} T} . \tag{3.8}$$

As in [8] and [9] the optimal funding term is defined as the funding term that optimizes the value of the asset. In the model developed here to generate MCD, the solution is simply

$$t_{\text{fun}} = t_{\text{liq}} \tag{3.9}$$

under the conditions

$$\begin{aligned} &1) \;\; \lambda (T - t_{\text{liq}}) > (r_{\text{liq}} - r_{\text{OIS}}) T \\ &2) \;\; r_{t_2} > r_{t_1} \text{ if } t_2 > t_1 \end{aligned} \tag{3.10}$$

The first condition essentially states that the liquidity costs of any funding term shorter than t_{liq} (and thus having full liquidation losses when an LSE occurs) are higher than the funding costs of funding the liquidity horizon. Short dated assets may violate this condition when the maturity is before or close to the liquidity horizon. The second condition simply states that LIBOR-OIS basis spreads increase with the LIBOR tenor (at all maturities). The solution t_{fun} is the shortest funding term for which the expected liquidation losses vanish: $LC = 0$.

Hence under the conditions (3.10) the value of the asset is

$$V_0 = V_0(t_{\text{liq}}) \tag{3.11}$$

$$= V_{\text{rf}}(0) - (r_{\text{liq}} - r_{\text{OIS}})Te^{-r_{\text{OIS}}T}\,, \tag{3.12}$$

which is equal to the MCD equation (1.2) up to order first order in $(r_{\text{liq}} - r_{\text{OIS}})T$ (the order to which we have calculated V). Also, we can define the Funding Valuation Adjustment (FVA) as the funding costs when the funding term is optimal, which gives

$$\text{FVA} = (r_{\text{liq}} - r_{\text{OIS}})Te^{-r_{\text{OIS}}T}\,. \tag{3.13}$$

This non-zero FVA is a result of the higher cost of funding long(er) term to reduce liquidity risk.

There has been a fierce debate around FVA [4]. An important argument against (a non-zero) FVA is based on the Modigliani-Miller theorem [6]. This theorem states that the value of assets should be independent of the leverage of the balance sheet. Therefore, it seems to follow that the value of assets is independent of the funding costs. However, a critical assumption of the theorem is that the return (or more generally the pay-off) of the assets is the same for two balance sheets with different leverage. However, in the model presented here, liquid and illiquid assets have a different effective pay-off. Therefore, the Modigliani-Miller theorem does not prohibit different funding costs for these.

This completes the introduction of the simple liquidity risk model that generates MCD. The model shows how different discount curves (and a non-zero FVA) follow from valuing assets including liquidity risk.

3.1 Example of loan valuation

As an example consider the valuation of a bullet loan.

Figure 1 shows the curves that we will use in this example. This a somewhat stylized representation of the market in September 2011. Figure 1 shows the OIS, 1M, 3M, 6M and 12M curves. These curves have been bootstrapped

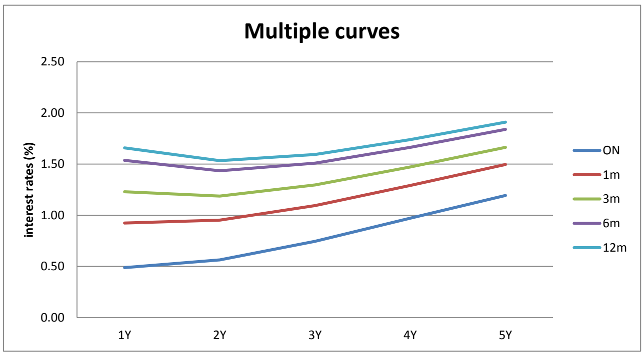

Figure 1: Multiple interest rate curves with different tenors.

maturity	r_{ON} (%)	r_{1Y} (%)	s_F (%)	FVA (%)
1Y	0.49	1.66	1.17	1.16
2Y	0.56	1.53	0.97	1.92
3Y	0.75	1.59	0.84	2.48
4Y	0.97	1.74	0.77	2.95
5Y	1.19	1.91	0.72	3.37

Table 1: Funding Valuation Adjustment for a bullet loan at different maturities. The funding spread has been defined has $s_F = r_{\mathrm{liq}} - r_{\mathrm{OIS}}$.

from (collateralized) swap quotes and other interest rate instruments, see e.g. [3, 2].

The other input needed for valuation is the liquidity horizon. In this example, we use a liquidity horizon of 1 year in this example.

Table 1 shows the results for bullet loans with different maturities. We note that the FVA can be significant. Since the spreads decreases with increasing maturity the FVA is not (exactly) linear in the maturity. Finally, we stress that the FVA here does not reflect the creditworthiness of the bank, but instead the illiquidity of the loan.

3.2 Relation to "A Model for the Valuation of Assets with Liquidity Risk"

It might be useful to discuss the relation to another model we have introduced in [9]. That model uses a more complicated, and possibly more realistic, liquidation value as a function of the funding term, instead of the simple step function in (3.1). The optimal funding term is in that case a result of an optimization procedure. The funding costs increase with the funding term and the expected liquidation loss decreases with increasing funding term. The optimal funding term is determined by minimizing the sum of expected liquidation losses and funding costs. The optimal funding term determines the reference rate of the discount factor. Since the expected liquidation loss is nonzero for the optimal funding term this model generates an additional spread.

This model might seem quite different since changes in the funding spreads lead to different optimal funding terms. However it is important to realize that after calibration als the expected liquidation loss changes. When there is a parallel shift in basis spreads, we expect that after calibration the optimal funding term is not changed (much)[1]. Therefore, the MCD method may not be too different from the model in [9].

4 Derivatives valuation

In this section, we consider the valuation of derivatives with the MCD method. To be specific, we focus on swaps.

The most straightforward way to apply the MCD method to swaps is to treat each leg as a loan and value each cash flow according to (1.2). Although this would give consistent valuation framework, it does not capture the right economics. From a liquidity risk perspective, the two legs of the swap cannot be separated. In particular, it is not possible to liquidate one leg of the swap. Hence, discounting the cash flows of both legs with a discount curve referencing the liquidity horizon is not the correct approach.

Therefore, we aim to adapt the liquidity risk model of the previous section to the valuation of swaps. The valuation of derivatives in this model is more complicated for three reasons:

1. a derivative may switch from an asset to a liability and vice versa depending on its market value.

[1] The precise transfromations under which the optimal funding term is invariant depends on the details of the calibration.

$V^{\text{receiver}}_{\text{uncoll}}$	$V^{\text{payer}}_{\text{coll}}$
posted collateral	$\text{Funding}_{\text{posted coll}}$
LB	$\text{Funding}_{\text{LB}}$

$V^{\text{receiver}}_{\text{coll}}$	$V^{\text{payer}}_{\text{uncoll}}$
$\text{Assets}_{\text{received coll}}$	received collateral
LB	$\text{Funding}_{\text{LB}}$

Figure 2: Part of the balance sheet for an uncollateraized receiver swap with MtM> 0, its collateralized hedge, the collateral position, its funding and the liquidity buffer. Similarly for an uncollateralized payer swap with MtM< 0.

2. the Mark-to-market (MtM) of a derivative fluctuates.

3. derivatives should be considered as part of their funding set, as discussed in [1].

We start with the first two complications and consider a single uncollateralized swap. We assume that both the bank and the counterparty are risk-free so that we can ignore any credit valuation adjustments. The funding valuation adjustment is defined as the difference between the uncollateralized and collateralized swap

$$V_{\text{uncoll}} - V_{\text{coll}} = -FVA\,. \tag{4.1}$$

Here the collateralized swap is assumed to have an idealized collateral agreement: cash collateral, daily margining, no thresholds, and no initial margin.

Consider the case where the MtM of the uncollateralized swap is positive. The swap is hedged with a collateralized swap, and collateral is posted. This collateral amount equals the MtM of the uncollateralized swap (under above idealized conditions). Similar as with the loan in the previous section the MtM can be funded with a term equal to the liquidity horizon of the swap. However, this is not sufficient. After an LSE occurs the MtM of the uncollateralized swap may increase, leading to a collateral outflow from the collateralized hedge, this collateral outflow cannot be funded and would lead to liquidation. Therefore, a liquidity buffer is required to absorb changes in MtM.

Similarly, if the MtM of the uncollateralized swap is negative. Then collateral is received. However if all received collateral is invested with a term equal to the liquidity horizon, an increase in the value of the MtM of the uncollateralized swap during an LSE leads to a collateral outflow and liquidation. Therefore, in this case, a liquidity buffer is also required.

Assume that a receiver swap has a positive MtM (and the corresponding payer swap a negative MtM). The positive and negative MtM case are illustrated in Fig 2. The balance sheets in Fig 2 illustrate the hedged positions. LB denotes the liquidity buffer, which consists of cash and earns the ON rate.

The rate received/paid on the collateral is also ON. Funding is obtained in the interbank market at a term equal to the liquidity horizon. Similarly the assets denote investment in the interbank market at the same term. For both we use LIBOR rates. Hence the cost at any point in time is

$$\text{cost} = (V_{\text{uncoll}} + LB)(\tilde{r}_{\text{liq}} - \tilde{r}_{\text{OIS}})\,. \tag{4.2}$$

where $\tilde{r}_{\text{liq}}$ denotes the instantaneous interest rate of the curve with index t_{liq}:

$$\int_0^t \tilde{r}_{\text{liq}}(t')dt' = r_{\text{liq}}(t) \tag{4.3}$$

The resulting FVA is

$$FVA = \mathbb{E}\left[\int (V_{\text{uncoll}} + LB)(\tilde{r}_{\text{liq}} - \tilde{r}_{\text{OIS}})e^{-\int_0^t \tilde{r}_{\text{OIS}}dt'}dt\right]\,. \tag{4.4}$$

The result for the FVA (4.4) breaks the asset-liability symmetry. The FVA on a (funding set consisting of a single) receiver swap and a payer swap are not exactly opposite. The cost of the liquidity buffer breaks this symmetry.

The reason for this symmetry-breaking is that our liquidity risk model contains events (LSEs) when it is not possible to convert assets into cash, but no events when it is not possible to convert cash into assets. To restore symmetry such events (when it is not possible to convert cash into assets) should be added. However, we believe that a model without these events better reflects reality.

4.1 Estimating the size of the liquidity buffer

The size of the liquidity buffer is determined by minimizing the sum of the buffer costs and expected loss from liquidating the position. In the previous section, for the bullet loan, this loss was assumed to equal the fair value of the loan to arrive at the MCD method. This suggests two options for a swap, either the liquidation loss equals

1. the risk-free value of the swap, or
2. the risk-free value of the incoming cash flows only.

The first option may be a good model when the fair value of the swap is large, and we have used it in [9]. But it runs into problems when the fair value is zero since this choice implies that then the liquidation loss would be zero. This is unrealistic since a collateral outflow still needs to be funded somehow, for instance by liquidating other assets on the balance sheet. Therefore, we use here the second option. Hence we use as liquidation loss

$$LL = (1 - LV(t_{\text{fun}}))PV_{\text{incoming cash flows}} \tag{4.5}$$

Note that this in combination with the liquidation value (3.1) is the most conservative choice possible.

The sum of the buffer costs and expected loss from liquidating the position that we want to minimize is given by

$$LB(t)(\tilde{r}_{\text{liq}} - \tilde{r}_{\text{OIS}}) + p(t)LL(t) \tag{4.6}$$

Here $p(t)$ denotes the probability that the liquidity buffer is insufficient, and the position needs to be liquidated:

$$p(t) = \mathbb{P}\left[V_{\text{uncoll}}(t + t') - V_{\text{uncoll}}(t) > LB(t) \text{ for any } t' \in [0, t_{\text{liq}}[\ \right] \tag{4.7}$$

Since the liquidity buffer will not be sufficient to avoid any loss we can define liquidity costs (LC) to account for the expected liquidation losses

$$LC = \mathbb{E}\left[\int_0^T p(t)LL(t)e^{-r_{\text{OIS}}t}dt\right]. \tag{4.8}$$

For loans the LC equal zero, since the model assumes that a funding term equal (or larger than) the liquidity horizon avoids any liquidation losses.

4.2 Example valuation of an FX forward

To illustrate the model for derivatives, we will consider an (uncollateralized) FX forward. We assume a maturity of 1 year and a strike (K) equal to 1. The bank receives domestic and pays foreign at maturity. Hence, the pay-off is $K - S_T$. The (domestic) rates are deterministic and given by Figure 1. For simplicity, we set the forward equal to the spot. Furthermore, we assume the spot follows a geometric Brownian motion with a volatility of 10%.

The results for the FVA and liquidity costs for different spot values are summarized in table 2. We would like to highlight two points:

- The FVA can become negative as we see from the case $S = 1.2$. This happens when the funding benefit from the collateral received on the hedge exceeds the average costs of the liquidity buffer.

- For the at-the-money forward this FVA is larger than the FVA that other authors find (see e.g. [10]). The reason is that their FVA is a

S/K	LB	FVA (in bp)	LC (in bp)	total (in bp)
0.8	19%	38.7	1.6	40.3
0.9	22%	28.7	1.8	30.5
1.0	24%	18.6	2.0	20.7
1.1	26%	8.6	2.2	10.8
1.2	28%	-1.5	2.5	0.9

Table 2: Liquidity buffer and FVA for an FX forward. The column LB contains the size of the liquidity buffer at time zero. The FVA and liquidity costs (LC) and their sum are denoted in basis points.

result of default risk of the bank and the higher funding costs. When the funding costs and funding benefit are (approximately) equal, the resulting FVA is (approximately) zero. In our approach based on liquidity risk, there is still a significant FVA due to the cost of the liquidity buffer.

Finally, in practice, FVA would not be calculated for a single derivative, but for a funding set of (uncollateralized) derivatives, see [1] for a discussion on funding sets. The extension of the above approach requires the value V_{uncoll} in (4.4) and further to be interpreted as the value of the funding set. Similarly, LL needs to be interpreted as the liquidation loss on the funding set.

5 Summary

We have introduced the multi-curve discounting (MCD) method. This method discounts the cash flows of assets with the curve that references the liquidity horizon of the asset. We have shown that a simple liquidity risk model generates this valuation method. The liquidity risk model formulation clarifies how a non-zero FVA can be generated without violating the Modigliani-Miller theorem [6].

The application of the liquidity risk model to derivatives required the introduction of a liquidity buffer due to the volatility of the MtM of derivatives. The contribution of the liquidity buffer to the value violates the asset-liability symmetry. In the liquidity risk model, this is caused by the absence of events where cash cannot be converted into assets. We have argued this is a proper model of reality.

Finally, it should be noted that the liquidity risk model described here is extremely simple by design. It is probably the simplest model that shows some of the interesting features: generation of a non-zero FVA from liquidity risk and the need for a liquidity buffer. Various extensions and generalizations can be considered, especially regarding the liquidation value (3.1) and liquidation loss (4.5), to make the model more realistic. This is left for future work.

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I would like to thank Menno Dobber and Rogier Swierstra for useful discussions.

A Derivation of (3.6)

Recall the expression for the value of a loan (3.5)

$$
\begin{aligned}
V_0(t_{\text{fun}}) &= e^{-r_{\text{fun}}T}\mathbb{P}(\tau + t_{\text{fun}} \geq T) \\
&\quad + \mathbb{E}\left[e^{-r_{\text{fun}}(\tau+t_{\text{fun}})}V_{\text{rf}}(\tau + t_{\text{fun}})LV(t_{\text{fun}})I_{\tau+t_{\text{fun}}<T}\right] \,, \qquad \text{(A.1)}
\end{aligned}
$$

From this expression we will derive the result (3.6). We will assume that $T > t_{\text{fun}}$.

The probability that $\tau+t_{\text{fun}} \geq T$ follows from the exponential distribution for τ

$$
\mathbb{P}(\tau + t_{\text{fun}} \geq T) = e^{-\lambda(T-t_{\text{fun}})} \qquad \text{(A.2)}
$$

The second term in (A.1) requires the calculation of an expectation. Note that $LV(t_{\text{fun}})$ is independent of τ. Hence the relevant expectation value that needs to be calculated is

$$
\begin{aligned}
\mathbb{E}\left[e^{-r_{\text{fun}}(\tau+t_{\text{fun}})}V_{\text{rf}}(\tau + t_{\text{fun}})I_{\tau+t_{\text{fun}}<T}\right] &= \int_0^{T-t_{\text{fun}}} e^{-r_{\text{fun}}(t'+t_{\text{fun}})}e^{-r_{\text{OIS}}(T-t'-t_{\text{fun}})}\lambda e^{-\lambda t'}dt' \\
&= e^{-r_{\text{fun}}t_{\text{fun}}}e^{-r_{\text{OIS}}(T-t_{\text{fun}})}\frac{\lambda}{\lambda + r_{\text{fun}} - r_{\text{OIS}}} \\
&\quad \times \left[1 - e^{(\lambda+r_{\text{fun}}-r_{\text{OIS}})(T-t_{\text{fun}})}\right] \qquad \text{(A.3)}
\end{aligned}
$$

Combining the results (A.2) and (A.3) gives the result for the value

$$
\begin{aligned}
V_0(t_{\text{fun}}) &= e^{-r_{\text{fun}}T}e^{-\lambda(T-t_{\text{fun}})} \\
&\quad + \frac{\lambda}{\lambda + r_{\text{fun}} - r_{\text{OIS}}}LV(t_{\text{fun}})e^{-r_{\text{fun}}t_{\text{fun}}}e^{-r_{\text{OIS}}(T-t_{\text{fun}})} \\
&\quad \times \left[1 - e^{(\lambda+r_{\text{fun}}-r_{\text{OIS}})(T-t_{\text{fun}})}\right] \qquad \text{(A.4)}
\end{aligned}
$$

A Taylor expansion in λT and $(r_{\text{fun}} - r_{\text{OIS}})T$ gives to first order

$$
V_0(t_{\text{fun}}) = e^{-r_{\text{OIS}}T}\left[1 - (r_{\text{fun}} - r_{\text{OIS}})T - \lambda(T - t_{\text{fun}})\left[1 - LV(t_{\text{fun}})\right]\right] \,. \qquad \text{(A.5)}
$$

This is the desired result (3.6).

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