Default risk and equity value: forgotten factor or cultural revolution?

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Summary:

Default risk is the forgotten factor when it comes to equity valuation. And yet, in this article, we show that default risk has a bigger impact on equity values than it does on bond values.

Our work is based on a default intensity model that we extrapolate to equities. This model does not presuppose a particular method for estimating distance to default. As a result, unlike Merton structural models, which only apply to indebted companies, it can be used to assess default risk for any company.

Highlighting a default risk premium in the cost of capital calculation makes it possible to reconcile the CAPM with evaluation methods based on forecasts in the event of survival. At the same time, the CAPM and default risk can explain the vast majority of bond spreads.

The test consisting of estimating "physical" implied default probabilities and the share of systemic risk included in corporate euro bond spreads at end-2015 led us to detect the likely existence of excessive remuneration of investment grade bonds. This finding corroborates identical conclusions reached earlier by other researchers. This potential market anomaly could indicate a windfall for investors. Performing this test again at various points in the economic and financial cycle would help establish whether the bond market is serving a free lunch to investors not bound by regulatory reserve requirements.

Key words:

Cost of equity, credit risk, default risk, credit spread, default spread, default premium, systematic risk, cost of leverage, cost of default, APV, adjusted present value, reduced form model, debt beta, CAPM, Spread AAA, implied cost of capital, ex-ante equity risk premium, forecast bias, optimistic bias premium, recovery rate, probability of default conditional and non-conditional.

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Introduction

The importance of default risk

Default risk is the forgotten factor when it comes to routine equity valuation. Too often, once appraisers have chosen and calculated the parameters for cost of equity, they only consider default risk in terms of the cost of debt used to determine the weighted average cost of capital (WACC). Moreover, many calculations assume that the debt leverage on the calculation date will remain constant forever, and for the refinancing rate, they use the interest rate on the company's most recent loan – or worse, the apparent average rate on all of the company's debt, even though the loans were issued on different dates. So most appraisers believe that they are correctly allowing for debt-related risks, which include:

- the default risk in the cost of debt used to calculate the WACC; and
- the increase in systematic risk, generally taken from Hamada's equation, needed to calculate levered beta.

This practice is outdated.

- There are now actuarial techniques that make it possible to factor in forecast changes in the level of debt used to calculate the cost of equity to ensure that the risk generated by leverage is consistent with the discount rate. A description of those techniques is outside the scope of this article, but those looking for more information on the topic should consult the work of Pablo Fernandez[1].
- Systematic risk is only one component of the increased remuneration shareholders demand. Leverage definitely increases the dispersion of free cash flow for the shareholder, thus raising the stock's beta, but it also reduces the expected shareholder return owing to the probable loss resulting from the risk of failure. Default risk arises when the margin on variable costs doesn't cover fixed costs, and because leverage only reduces the former or increases the latter, it is an aggravating factor. Default risk pre-exists debt leverage and must be taken into account by the appraiser, regardless of whether it is amplified by debt.

In other words, the appraiser must demonstrate that he accounts for default risk, either i) by reducing the expected flows forecast in case of survival by subtracting the expected loss given default; or ii) by adding a default risk premium to the discount rate used to calculate the present value of flows in case of survival.

On average, the recovery rate for unsecured financial debt is significantly lower than 100% in the event of default. For shareholders, the average is close to zero. If, like John Hull (see below), we suppose that default spread is proportional to the loss given default, then the default premium is greatest in the case of a stock. Even if a loan's spread only includes a few dozen basis points to compensate for default risk, incorporating a multiple of that premium into the cost of equity calculation probably has a material influence on the stock value, given that it already materially affects bond values. Let us look at an example that illustrates the premium's significance.

[1] For example: Fernandez, Pablo, "Equivalence of Ten Different Methods for Valuing Companies by Cash Flow Discounting", EFMA 2004 Basel Meetings Paper, October 11, 2003, SSRN : https://ssrn.com/abstract=367161

Example: listed French SMEs and intermediate-sized companies

In a report[2] delivered to the Corporate Capital Market Observatory *(Observatoire du financement des entreprises par le marché)*, the research firm IDMidCaps looked at corporate failures that occurred between 2005 and 2013 in a sample of listed companies on the C and B compartments of Euronext (for issuers with market caps of up to €150m and €1bn respectively) and on Alternext, which became Euronext Growth in 2017.

Over those nine years, 71 companies failed and were placed in receivership. Of the 71:

- 61 (or 86%) were wound up at a loss to their shareholders;
- 3 (or 4%) were acquired under terms that amounted to a zero recovery rate for shareholders;
- 7 (or 10%) were still operating as of end-2013 under a reorganization plan, typically at the cost of financing deals that were highly dilutive for the former shareholders.

According to these data, an average of 1.7% of the companies operating on January 1 were placed in receivership each year[3].

SME & ETI sample (1)	2005	2006	2007	2008	2009	2010	2011	2012	2013	Average	Implied rating
Sample size	426	470	483	456	459	459	451	438	441	**454**	
Default rate	1.64%	0.85%	1.45%	1.75%	2.83%	1.53%	2.00%	2.05%	1.59%	**1.74%**	B+ to BB-

(1) Companies listed on compartments B, C and Alternext, excluding REITs, holding companies, shells companies, banks and insurance companies.

According to the data on corporate failures compiled by ratings agency Standard & Poor's that were available at the start of the research effort[4], this rate corresponds to the default frequency for companies rated B^+ to BB^-, or those in the "speculative" category.

If we split the sample into SMEs[5] and intermediate-sized companies (ETIs) and then examine the default statistics over one year, or the cumulative statistics over four, five and nine years – as shown in the table below – listed SMEs' default rates put them in the category B^+ to BB^-. The ETIs in the sample fare somewhat better, with rates that put them in the range of BB^- to BB, corresponding to the upper part of the non-investment grade category[6].

2 "Etude sur les défaillances et sur les performances 2005-2013 dans les PME – ETI cotées", http://www.pme-bourse.fr/publications/etudes-et-rapports.html

3 In this statistic, companies are included until they are delisted, which tends to overestimate the annual percentage of companies placed in receivership. But this does not seem to alter the ex post classification of this sample in the speculative category, which has a rating of BB to BB+.

4 See Poncet, Patrice, and Portait, Roland, Finance de marché, 4th ed. Dalloz, page 928.

5 The INSEE defines SMEs as companies with fewer than 250 employees and revenues of less than €50m, or total assets of less than €43m. The study's authors have assigned the remaining companies in the sample to the ETI category, even though some of them technically meet the INSEE definition of a large company (at least 5,000 employees and over €1.5bn in revenues, or total assets of over €2bn).

6 Remember that keeping companies in the sample and in the default category until they are delisted almost certainly tends to overestimate the default rate.

	SME		ETI	
Default rate	**Average**	**Implied rating**	**Average**	**Implied rating**
1 year average	2.7%	B+	1.3%	BB-
Cumulative average / 4 years*	11%	B+ to BB-	6%	BB- to BB
Cumulative average / 5 years*	14%	B+ to BB-	7%	BB- to BB
Cumulative / 9 ans*	24%	B+ to BB-	11%	BB- to BB

*Based on analysis of the data in the cited study.

Thus, the listed SMEs in the sample have a default rate double that of the ETIs, with respective cumulative rates over nine years of over 20% and 10%.

These default rates mean significant impact for shareholders' returns. According to research firm IDMidCaps, the sample's cumulative performance over the period, equally weighted and with dividends reinvested, was 88.5%. If we exclude failing companies, the performance would have been 106%. Thus, defaults erased 16.5% of the equity investor's capital gains.

Annual returns, (dividends reinvested)	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**Cumulative**
Total sample	4.6%	31.8%	-1.2%	-45.4%	48.5%	39.3%	-6.4%	4.9%	24.9%	88.7%
Excluding failing companies	4.9%	32.1%	-0.3%	-45.1%	50.5%	40.5%	-6.0%	8.6%	25.8%	106.0%

Source : IDMidCaps

The first lesson we can learn from this analysis is that small cap investors must be prepared for a significant mortality rate.

In this report, we will establish the link between probability of default and its remuneration using a default intensity model. In Chapter 1 we apply the model to debt securities, and in Chapter 2 we expand it to equities.

We then demonstrate that the effective shareholder return, in this case 88.5% over nine years, or 7.3% per year on average, is net of the defaults that occurred during the calculation period. This is why average historical returns calculated over long periods, which are typically used to calculate risk premia, provide a good estimate of expected shareholder returns, since companies go bankrupt every year. As a result, if this estimate is used to discount forecast cash flows net of probable loss given default, then that makes sense. On the other hand, if the forecasts are – as is generally the case – calculated assuming that the company survives, then it should be discounted at gross rate, i.e. before factoring in the impact of corporate failures in the market return. In the above example, the gross rate would be 106% over nine years, or 8.36% per year on average.

Because a forecast that assumes survival is biased in the sense that it is not a mathematical expectation, using historical returns to discount the value is fundamentally flawed. This is why it is necessary to distinguish between expected shareholder returns and cost of capital (discount rate). As a general rule, the former is lower than the latter.

1 Introduction to the default intensity model

The default model known as the default "intensity" model is an actuarial approach that lets us establish a functional link between default risk over a certain time period, the recovery rate in the event of default, and the default risk premium or default spread.

Unlike the so-called structural models that have sprung up in abundance and follow in the footsteps of Merton's groundbreaking work, the intensity approach does not require any presuppositions regarding the model used to estimate default risk. The structural model links the probability of default with the probability that the value of the operating asset will be less than or equal to the amount of financial debt due on a given date, basing the probability on the volatility of the issuer's stock price. Thus, under this approach, the default risk is merely a question of overindebtedness, which is unrealistically oversimplified[7]. Furthermore, these models posit a risk-neutral universe and a complete market for underlying assets, which is clearly not the case for operating assets (unlike listed securities). And using the risk-free rate as a central tendency parameter gives a risk-neutral probability that has no direct relationship with the real (or physical) probability of default. As a result, the appraiser is forced to recalibrate the parameters so that they reflect the real world[8], and this does not appear to provide a decisive advantage compared to conventional financial analysis of credit risk. As we noted in the introduction, default risk measures whether the margin on variable costs is enough to cover fixed costs, and there is no obvious link between this question and a company's equity value[9], which is influenced by market risk premium volatility in addition to earnings estimate revisions.

For these reasons, our research has focused on the default intensity model. We try to show how, under certain conditions, this approach makes it possible to factor in "real" probabilities of default, in the sense that the spread employed to remunerate risk is consistent with the expectation of default resulting from forecast cash flow models. And conversely, how knowing the required market spread makes it possible to deduce the implied probability of default the market expects. In the following section, we expand the model's application to equity valuation (Chapter 2) and test the model using market data (Chapter 3).

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7 Apart from estimating NAV volatility for a real estate asset or a holding company with a view to winding it up, this approach does not give us information about a company's ability to withstand temporary losses without compromising its ability to continue operating.

8 See, for example: Cooper, Ian A., and Davydenko, Sergei A., "Using Yield Spreads to Estimate Expected Returns on Debt and Equity", London Business School IFA Working Paper, EFA 2003 Annual Conference Paper No. 901, December 2003, SSRN: https://ssrn.com/abstract=387380

9 For a more in-depth discussion of using a stochastic model to gauge default risk resulting from the margin on variable costs not covering fixed costs, please consult: Clère, Roland, "After Modigliani, Miller and Hamada; A New Way to Estimate Cost of Capital?" (Après Modigliani, Miller et Hamada : une nouvelle façon d'estimer le coût du capital ?), November 23, 2016, SSRN: https://ssrn.com/abstract=2868702

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1.1 A formal expression of default risk using the intensity (or reduced form) model

This model is based on conventional functional analysis tools. Furthermore, we can establish the link between the formal expression of default risk and the Poisson distribution (Siméon Denis Poisson), which means this approach is similar to a jump diffusion model.

1.1.1 Conditional default probability

In the following analysis, we simplify matters by assuming that the company's survival, S, is incompatible with the occurrence of a default event, D. This approach is valid even if default does not necessarily lead to failure, since the issue of recovering payment in the event of default is handled later. Let us assume for our purposes that survival means "survival with no prior default event or implementation of a reorganization plan". In that case, according to the definition of two mutually incompatible or exclusive events:

$$p(S) = 1 - p(D) \qquad Equation\ \#1$$

If we know the company's survival probability (with no default) at date t, written as $p(S_t)$, then the probability of survival until the time horizon $t+1$ is equal to the product of the survival probability at date t and the probability of no default occurring in the intervening period $]t\,;t+1]$. We can deduce this probability from the probability that a default will occur during the period $]t\,;t+1]$. Written as $p(D_{t+1}|S_t)$, the probability of default during the period is considered conditional upon the prior survival of the company until date t:

$$p(S_{t+1}) = p(S_t) \times \big(1 - p(D_{t+1}|S_t)\big) \qquad \#2$$

To simplify the expression, we write S_t for the probability of survival to date t, and d_t for the conditional probability of default in the subsequent intervening period, Δ_t ; $]t\,;t+1]$.

$$S_{t+1} = S_t \times (1 - d_t) = S_t - d_t \times S_t \qquad \#3$$

If the conditional default probability is a constant d for every period with a duration equal to Δ_t, then the previous equation becomes:

$$S_t = S_{t-\Delta_t} \times (1 - d) \qquad \#4$$

$$where \quad \begin{cases} p\big(D_t \big| S_{t-\Delta_t}\big) = d & \forall\ t\ with\ \Delta_t\ constant \\ p(S_t) = S_t \end{cases}$$

As a result, when the conditional probability is constant regardless of the period Δ_t, the probability of survival until date $T = n \times \Delta_t$, written as S_T, is equal to:

$$S_T = S_{t=0} \times (1 - d)^n = (1 - d)^n \qquad \#5$$

1.1.2 The unconditional or cumulative probability of default

The unconditional probability of default in a given period $[0\,;t]$ results from the cumulative effect of conditional default probabilities in each of the smaller intervening periods that make up the larger period. Assuming that the period $[0\,;T]$ is divided into n time intervals Δt for which the conditional survival probability d_i is known, then according to equation #2, the probability of survival at date T is equal to:

$$S_T = \big(1 - p(D_1|S_0)\big) \times \big(1 - p(D_2|S_1)\big) \times \dots \times \big(1 - p(D_n|S_{n-1})\big) \qquad \#6$$

$$= (1 - d_0) \times (1 - d_1) \times \quad \dots \quad \times (1 - d_{n-1})$$

$$\Rightarrow\; S_T = \prod_{i=1}^{n} \big(1 - p(D_i|S_{i-1})\big) \qquad \#7$$

From which we can deduce D_T, the unconditional default probability at time horizon T:

$$\Rightarrow D_T = 1 - S_T = 1 - \prod_{i=1}^{n} (1 - d_{i-1}) \qquad \#8$$

If the conditional probabilities are equal to a constant, d, then:

$$D_t = 1 - (1 - d)^n = 1 - S_T \qquad \#9$$

Where S_T is definited by equation #5.

1.1.3 Marginal probability of default

The marginal probability of default ΔD_n is the increase in the unconditional probability of default during time interval Δt. Let us assign this interval a constant duration $\Delta t = \frac{T}{m}$, where $m \in \mathbb{N}$, and in positing $n \leq m$, we define the unconditional probability of default until date $t = n \times \Delta t$, $p(D_{n\times\Delta t})$, written as D_n:

$$\Delta D_n = D_n - D_{n-1} = (1 - S_n) - (1 - S_{n-1}) = S_{n-1} - S_n \qquad \#10$$

In the specific case where the conditional probabilities are equal to a constant, d, then:

$$\Delta D_n = \; (1 - d)^{n-1} - (1 - d)^n \qquad \#11$$

$$\Rightarrow \Delta D_n = \sum_{i=1}^{n-1} \mathrm{C}_{n-1}^{i} \times (-1)^i \; \times d^{i+1} \qquad \#12$$

1.1.4 Default probability and default intensity

Over time interval $]t\,; t + \Delta t]$ the marginal probability of default ΔD_t for an additional unit of time Δt, according to equations #3 and #10, is:

$$\Delta D_t = S_t - S_{t+\Delta t} = S_t \times p(D_{t+\Delta t}|S_t)$$

$$\Rightarrow \quad -\Delta D_t = S_{t+\Delta t} - S_t = -d_t \times S_t \qquad \#13$$

Let us assume that there is a function of t, expressed as $\lambda(t)$ and called the “default intensity function”, such that the conditional probability of default during a future time interval Δt is equal to the product of the intensity at date t and the duration of the time interval:

$$\lambda(t) \times \Delta t = p(D_{t+\Delta t}|S_t) = d_t \qquad \#14$$

In this case, equation #13 becomes:

$$S_{t+\Delta t} - S_t = -\lambda(t) \times \Delta t \times S_t \qquad \#15$$

Because the probability of survival S_t is also a function of t, equation #15 becomes:

$$\Delta S(t) = -\lambda(t) \times \Delta t \times S_t \qquad \#16$$

Therefore, approaching the limit when Δt tends towards zero, in the neighborhood of t:

$$dS = -\lambda \times dt \times S \qquad \#17$$

$$\frac{dS}{S} = -\lambda \times dt \qquad \#18$$

In the first term of the equation, we can see the differential of $ln(S)$, which gives:

$$\Leftrightarrow \ dln(S) = -\lambda \times dt \qquad \#18$$

$$\Leftrightarrow \int ln'\big(S(t)\big)dt = -\int \lambda(t)dt \qquad \#19$$

$$\Leftrightarrow \ ln\big(S(t)\big) = -\int \lambda(t)dt + C \qquad where \ \ C \ is \ a \ constant$$

$$\Leftrightarrow \ S(t) = e^{-\int \lambda(t)dt + C} \qquad \#20$$

This integral of the function $\lambda(t)$ is defined for the interval $[0\,;t]$. The probability of survival is equal to 1 when t is equal to zero. Conversely, it is reasonable to posit that the probability tends toward zero as t tends toward infinity. Furthermore, when it comes to a defined integral, we can ignore the constant C in equation #20:

$$\Longrightarrow \ S(t) = e^{-\int_0^t \lambda(x)dx} \qquad \#21$$

In cases where there is a primitive function $F(t)$ of $\lambda(t)$, assuming – to simplify matters – that it cancels itself out when $t = 0$, then the integral defined in equation #21 can be expressed as follows:

$$F(t) = |F(x)|_0^t = \int_0^t \lambda(x)dx \qquad \#22$$

According to mean value theorem, there is a mean value "$\bar{\lambda}$" of $\lambda(t)$ when x varies between 0 and t, such that:

$$\bar{\lambda} = \frac{1}{t}\int_0^t \lambda(x)dx \ = \frac{F(t)}{t} \qquad \#23$$

As a result, equation #21 becomes:

$$S(t) = \ e^{-F(t)} = e^{-\int_0^t \lambda(x)dx} = e^{-\bar{\lambda} \times t} \qquad \#24$$

If the intensity function $\lambda(t)$ is constant, equal to a positive real number λ, then:

$$\forall\, t \quad \lambda(t) = \lambda \quad \Rightarrow \quad F(t) = \int_0^t \lambda\, dx = \left|\lambda x + C\right|_0^t = \lambda \times t$$

In this case, the mean default intensity $\bar{\lambda}$, defined according to #23, is equal to this constant λ:

$$\bar{\lambda} = \lambda \quad and \quad \lambda(t) \times t = \lambda \times t$$

And equation #24 becomes:

$$S(t) = e^{-\lambda \times t} = \left(\frac{1}{e^{\lambda}}\right)^t \qquad \#25$$

1.2 Default and intensity functions deduced from transition matrices

1.2.1 Formulae deduced from unconditional probabilities of default

- In practice, if we know the unconditional probability of default D_t at time horizon t, we can deduce the mean intensity $\bar{\lambda}_t$ between 0 and t:

$$D_t = 1 - S_t = 1 - e^{-\bar{\lambda} t} = 1 - e^{-F(t)}$$

$$\Rightarrow e^{-F(t)} = 1 - D_t$$

$$\Rightarrow F(t) = -\ln(1 - D_t) \qquad \#26$$

$$\Rightarrow \bar{\lambda}_t = -\frac{\ln(1 - D_t)}{t} \qquad \#27$$

- The mean default intensity over a time interval Δt, $[t - \Delta t\,; t]$, which we express as $\bar{\lambda}_{t-\Delta t;t}$, is deduced from the mean intensity $\bar{\lambda}_{t-\Delta t}$ for the period $[0; t - \Delta t]$, and $\bar{\lambda}_t$ for the period $[0; t]$. According to equation #23:

$$\bar{\lambda}_t = \overline{\lambda(t)} = \frac{1}{t}\int_0^t \lambda(x) dx = \frac{F(t)}{t}$$

$$\Rightarrow \bar{\lambda}_{t-\Delta t;t} = \frac{1}{\Delta t}\int_{t-\Delta t}^{t} \lambda(x) dx = \frac{\left|F(t)\right|_{t-\Delta t}^{t}}{\Delta t}$$

According to equation #26:

$$\Rightarrow \bar{\lambda}_{t-\Delta t;t} = \frac{1}{\Delta t}\left(ln(1 - D_{t-\Delta t}) - ln(1 - D_t)\right) \qquad \#28$$

$$\Rightarrow \bar{\lambda}_{t-\Delta t;t} = \frac{1}{\Delta t}\left(ln(S_{t-\Delta t}) - ln(S_t)\right) \qquad \#29$$

Whereas:

$$\begin{cases} S_t = e^{-\bar{\lambda}_t \times t} \\ S_{t-\Delta t} = e^{-\bar{\lambda}_{t-\Delta t} \times (t-\Delta t)} \end{cases} \Rightarrow \begin{cases} \bar{\lambda}_t \times t = -ln(S_t) \\ \bar{\lambda}_{t-\Delta t} \times (t - \Delta t) = -ln(\ S_{t-\Delta t}) \end{cases}$$

In equation #29, if we replace $ln(S_{t-\Delta t})$ and $ln(\ S_{t-\Delta t})$ with their values in each of the preceding equations:

$$\Rightarrow \bar{\lambda}_{t-\Delta t;t} \times \Delta t = \bar{\lambda}_t \times t - \ \bar{\lambda}_{t-\Delta t} \times (t - \Delta t) \qquad \#30$$

- In addition, we can deduce the intensity function $\lambda(t)$ from the unconditional probability of default function $D(t)$:

$$F(t) = -\ln(1 - D_t)$$

Where $F(t)$ is the compound function of $F(u) = -\ln(u_D)$, with $u_D = 1 - D_t$

$$\Rightarrow \ F'(t) = \lambda(t) = \frac{D'(t)}{1 - D(t)} \qquad \#31$$

To clarify the importance of equation #31, we share the following quote by Gilbert Saporta[10]: "The intensity function $\lambda(t)$, depending upon which field it is applied in, may be referred to as the 'spot default rate', the 'hazard function', or even the 'mortality quotient'. For a life span T, where T is a continuous variable representing a duration, $\lambda(t)$ is understood as the probability of death immediately after t, assuming survival until $T = t$."

Indeed, equation #31 may be rewritten as follows in its differential expression:

$$\lambda(t) = \frac{\frac{dD_t}{dt}}{1 - D(t)} \ \Rightarrow \ \lambda(t).dt = \frac{dD_t}{1 - D(t)}$$

$\lambda(t).dt$ represents the conditional probability of default in the vicinity of t when Δt becomes infinitely small, dD_t is the marginal increase in the unconditional default function relative to the time it takes to reach the neighborhood of t, and $1 - D(t)$ is the survival function relative to time, until t.

Lastly, equation #31 can be written in its probabilistic form based on the distribution function $D(t) = p(T \leq t)$ and the mortality density function $D'(t) = p(t < T < t + dt)$, or even the survival function $p(T \geq t) = 1 - p(T \leq t)$:

$$\lambda(t) = \frac{p(t < T < t + dt)}{1 - p(T \leq t)} = \frac{p(t < T < t + dt)}{p(T \geq t)} = p(t < T < t + dt \mid T > t)$$

- The conditional probability of default $d_{\Delta t}$ over time period Δt equal to $[t - \Delta t; t]$ may be deduced from the survival probability $S(t) = 1 - D(t)$:

$$S_t = S_{t-\Delta t} \times (1 - d_{\Delta t})$$

$$d_{\Delta t} = 1 - \frac{S_t}{S_{t-\Delta t}} = 1 - \frac{e^{-F(t)}}{e^{-F(t-\Delta t)}} = 1 - \frac{e^{-\bar{\lambda}_t \times t}}{e^{-\bar{\lambda}_{t-\Delta t} \times (t-\Delta t)}} \qquad \#32$$

[10] Saporta, Gilbert, Probabilités, analyse des données et statistique, Éd. Technip, 3rd edition, pp. 19-20.

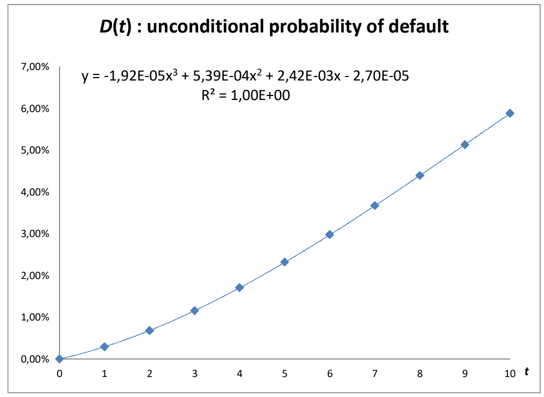
1.2.2 Example using an S&P transition matrix

The expectation of default as a function of time (expressed in years) for a given credit rating can be deduced from the one-year transition matrices published by rating agencies. For this example, we look at the category of BBB-rated companies[11].

- By interpolation, we can approximate the default probability function $D(t)$ with the following polynomial:

$$D(t) \cong -\frac{1.9}{10^5}t^3 + \frac{5.4}{10^4}t^2 + \frac{2.4}{10^3}t - \frac{2.7}{10^5}$$

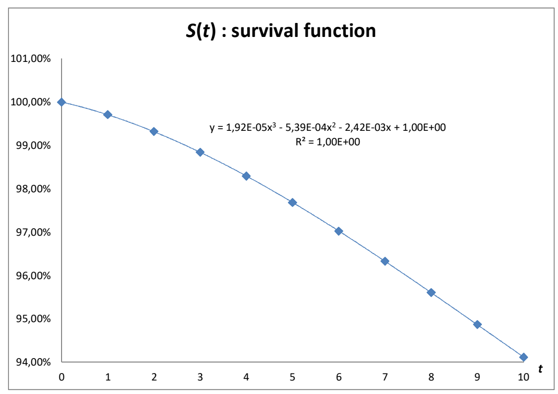
$$\Rightarrow D'(t) \cong -\frac{5.8}{10^5}t^2 + \frac{10.8}{10^4}t + \frac{2.4}{10^3}$$



We can easily deduce the survival function $S(t)$ from the default function $D(t)$:

$$S(t) = 1 - D(t)$$

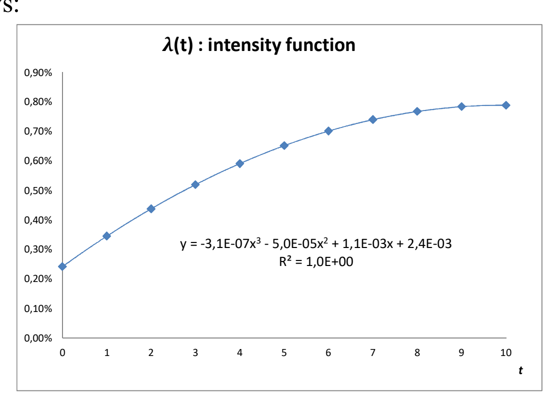
$$\Rightarrow S(t) \cong 1 + \frac{2.7}{10^5} + \frac{1.9}{10^5}t^3 - \frac{5.4}{10^4}t^2 - \frac{2.4}{10^3}t$$



- From equation #31, $\lambda(t) = \frac{D'(t)}{1-D(t)}$, we can deduce the values of the intensity function $\lambda(t)$. By polynomial interpolation, the equation is as follows:

$$\lambda(t) \cong -\frac{3.1}{10^7}t^3 - \frac{5.0}{10^5}t^2 + \frac{1.1}{10^3}t + \frac{2.4}{10^3}$$

A rising / (falling) default intensity indicates ageing or deterioration / (rejuvenation or recovery). In the absence of ageing or, conversely, rejuvenation, a constant intensity reflects a situation in which death results from random external causes with a constant intensity.



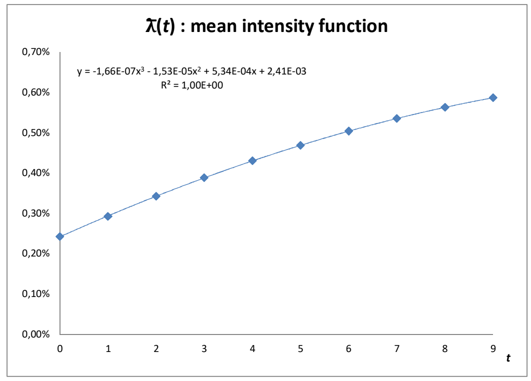
[11] For our initial approach, here we assume that the transition matrices are Markov chains, i.e. that the default probability in period t depends solely on the probability in the previous period, t-1. The one-year transition matrix underlying the default probabilities used here for subsequent years, created in 2005 by S&P, was taken from: Patrice Poncet and Roland Portait, Finance de marché, 4th ed. Dalloz, page 928.

- From equation #27 we can deduce the values of the mean intensity function $\bar{\lambda}_t$:

$$\bar{\lambda}_t = -\frac{\ln(1 - D_t)}{t}$$

By polynomial interpolation, the equation is as follows:

$$\bar{\lambda}_t \cong -\frac{1.66}{10^7}x^3 - \frac{1.53}{10^5}x^2 + \frac{5.34}{10^4}x + \frac{2.41}{10^3}$$

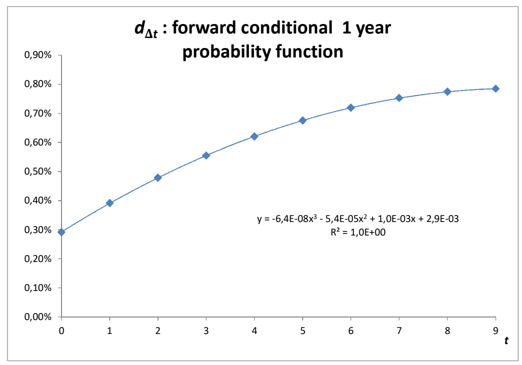


- From equation #32 we can deduce the conditional probability function $d_{\Delta t}$:

$$d_{\Delta t} = 1 - \frac{e^{-\bar{\lambda}_t \times t}}{e^{-\bar{\lambda}_{t-\Delta t} \times (t-\Delta t)}}$$

By interpolation, the equation for $d_{\Delta t}$, with Δt standing for one year, can be approached as follows:

$$d_{\Delta t} \cong -\frac{6.4}{10^8}t^3 - \frac{5.4}{10^5}t^2 + \frac{1.0}{10^3}t + \frac{2.9}{10^3}$$



1.3 The constant default intensity can be compared to a Poisson distribution

The probability distribution of a Poisson (Siméon Denis) random variable is characterized by a coefficient of intensity λ per unit of time, which makes such a process similar to the specific case noted above in which the default intensity function is constant.

Remember that according to the definition of a Poisson distribution, the number of occurrences in the discrete time intervals is independent and their average per unit of time is equal to λ, i.e. the coefficient of intensity[12].

It may seem absurd to assume independence and multiple occurrences with respect to the default risk of a single company, but in this case, these assumptions are purely secondary. Our real goal is to determine the probability that the first default will occur. In practice, the first default will be analyzed in a way that excludes the following occurrences.

Subsequently, if we use Y_θ to designate the Poisson random variable in the parameter λ ; $Y_\theta \rightarrow \mathcal{P}(\lambda)$, corresponding to the number of default occurrences per unit of time θ given ; $\theta = 1$, then the variable $Y_t = \frac{t}{\theta} Y_\theta$, which refers to the number of potential occurrences in the time interval ; $]t-1\,;t]$, such that $t/\theta = t$; follows a Poisson distribution with intensity $\lambda \times t$:

$$Y_t \rightarrow \mathcal{P}(\lambda \times t)$$

The probability of k defaults occurring during a time interval $[0\,;t]$ would thus be equal to:

$$p(Y_t = k) = e^{-\lambda t} \times \frac{(\lambda t)^k}{k!} \qquad \#33$$

As a result, the probability of no default occurring, i.e. the probability of survival during the time interval]0 ;t] is equal to:

$$S_t = p(Y_t = 0) = e^{-\lambda t} \times \frac{(\lambda t)^0}{0!} = \ e^{-\lambda} \qquad \#34$$

In equation #34, we find the expression for the probability of survival from equation #25, i.e. in cases where the default intensity function is a constant. This reflects a situation in which “death” is the result of random external causes with a constant intensity.

In other words, we could just as easily say that the default intensity function is a constant λ, or that the number of potential defaults per unit of time is a Poisson random variable with a parameter λ.

[12] For more detail, see Gilbert Saporta, Probabilités, analyse des données et statistique, 3rd edition, 2011, Éd. Technip, pp. 48-52

1.4 Bond default spread and real-world probability of default according to the intensity model

1.4.1 Breaking down the credit spread with real-world probabilities of default

The default risk gives rise to a foreseeable loss, whose expected value can be used as an investment target provided the asset is held in a portfolio of debt securities with the same risk profile. In this sense, the expected loss given default converges towards the expected value of the loss via a system of diversification resulting from the law of large numbers. The contractual return expected for a bond if the issuing company survives is thus not equal to its expected return. The spread between the contractual return if the company survives and the expected return is the default risk premium. This premium may be viewed as an adjustment to be made to the discount rate (or cost of capital) to reflect the fact that cash flows are slanted relative to their expected value.

The models typically used to estimate default risk assume a "risk-neutral" universe. That means that apart from the default premium, they do not take into account any remuneration for the return dispersion risk relative to the expected value. In other words, once adjusted for default risk, the cash flows are assumed to be certain and are discounted at the risk-free rate.

$$\hat{r} = \Pi_d + r_f \qquad \#35$$

Where $\hat{r}$ refers to the internal rate of return at which the bond price is equal to the present value of the theoretical contractual cash flows, i.e. coupon payments plus principal; Π_d refers to the default risk premium (or default spread); and r_f refers to the risk-free rate for an identical time to maturity.

There is no need to look for an economic justification for the risk-neutral approach. Rather, it is a generally accepted calculation practice that makes sense when valuing bond options, for which there is no need to estimate a risk premium[13].

Unlike the risk-neutral approach, we start from a real-world risk aversion context, which means that investors would require a risk premium when calculating their return, as in the CAPM. Bonds are risk assets and their prices and returns thus fluctuate constantly, both for reasons specific to the issuer's credit quality (in the form of rating changes) and for exogenous reasons – chiefly risk aversion as measured by movements in the overall market risk premium and variation in the risk-free rate. On top of the return dispersion risk premium, there may also be a liquidity premium and an incompressible residual premium[14] empirically observed as the difference between the yield to maturity of an AAA-rated corporate bond and that of a government bond with the same rating, denominated in the same currency:

$$\hat{r} = \Pi_d + \underbrace{\beta_d \Pi_R + \Pi_A + \Pi_L + r_f}_{E(r)} \qquad \#36$$

In which β_d and Π_R refer respectively to the bond's beta and the financial market risk premium in the CAPM sense, Π_A refers to the residual risk premium required of AAA-rated corporate bonds, and Π_L to the liquidity premium.

To keep matters simple at this point, before we develop these ideas more thoroughly, we will merely say that the default premium Π_d is an adjustment made to the expected return $E(r)$ of the risk asset made to account for the fact that forecast cash flows are not expected values, but rather conditional cash flows if

13 Cf. John Hull, Options futures, and other derivatives, Pearson, 8th edition, § 23.5

14 See section § 3.4, page 40.

the issuer survives. Thus, we make a distinction between i) **the cost of capital** $\hat{r}$, which is a discount rate used to calculate the present value of cash flows in case of survival, i.e. the contractual payments on a bond (coupons and principal), and ii) **the bond's expected return** $E(r)$, which is the discount rate applied to the expected value of cash flows, in keeping with the CAPM.

$$E(r) = \hat{r} - \Pi_d \qquad \#37$$

$$\Leftrightarrow \quad E(r) = \beta_d \Pi_R + \Pi_A + \Pi_L + r_f \qquad \#38$$

This being laid, there are two possible approaches to calculating the present value of the bond V_0 at the date t_0:

- The first consists in discounting the contractual payment flows, CF, i.e. those expected if the issuer survives, at the cost of capital $\hat{r}$:

$$V_0 = \sum \frac{CF_i}{(1+\hat{r})^i}$$

- The second consists in discounting the expected value of the cash flows, $E(CF)$, i.e. taking into account the probability of survival, the probability of default, and the recovery rate in the event of default (we explain these calculations below). The cash flows are discounted at a rate equal to the expected value of the bond's rate of return, $E(r)$:

$$V_0 = \sum \frac{E(CF_i)}{\left(1+E(r)\right)^i}$$

$$\Rightarrow \quad V_0 = \sum \frac{CF_i}{(1+\hat{r})^i} = \sum \frac{E(CF_i)}{\left(1+E(r)\right)^i} \qquad \#39$$

$$\text{Where: } \begin{cases} E(r) \leq \hat{r} \\ E(CF) \leq CF \end{cases}$$

1.4.2 Bond value as a function of default probability and recovery rate

1.4.2.1 Cash flow over one period

If we first consider the value V_0 of a bond with cash flow over only one period, with payment received at date T, with a probability of $(1-d)$ and – in the event of a default with probability d – a recovery rate equal to R expressed as a percentage of the amount due, then:

$$V_0 = \frac{(1-d).CF_T + d.R.CF_T}{\left(1+E(r)\right)^T} \qquad \#40$$

If we use V_f to refer to the bond's expected future value, i.e. V_0 capitalized at a rate equal to the expected return $E(r)$, which corresponds to the rate of return expected in case of survival $\hat{r}$ (the contractual rate of return), minus the default premium Π_d, then simplifying the expression by positing $E(r) = r$ gives us:

$$\Rightarrow \quad V_0 \times (1+r)^T = V_f = (1-d).CF_T + d.R.CF_T \qquad \#41$$

$$\Rightarrow V_f = CF_T - d.CF_T.(1-R) \qquad \#42$$

In the second part of the equation above, we can see the difference between the cash flow in case of survival CF_T and the loss given default; $P_t = CF_T.(1-R)$, weighted by the probability of default d.

$$\Rightarrow V_f = CF_T - d.P_T \qquad \#43$$

In other words, the expected future value V_f can be seen as:

- The average of the forecast cash flows given by the individual outcome scenarios weighted by the probability of each scenario, i.e. equation #41, which is the definition of the mathematical expectation of the future value;
- The conditional cash flows in case of survival CF_T, minus the probable loss given default, i.e. equation #43.

Comments:

In general, we consider that; $R \in [0;1]$, i.e. that the recovery rate cannot be below zero. This means ignoring cases where a lender is liable for improper financial support. Furthermore, we generally exclude cases where the recovery rate exceeds 1, which can happen when a lender becomes a shareholder in a capital increase paid for by writing off debt and then manages to right the company's situation and realize an enterprise value higher than the amount of the initial debt. These kinds of situations do indeed occur – how often depends on the bankruptcy and contract laws that apply.

In the standard scenario where $R \in [0;1]$, equation #42 can be rewritten as:

$$\Rightarrow V_f = CF_T\left(1 - d.(1-R)\right) \qquad \#44$$

$$\Rightarrow V_f = CF_T\, S_€ \qquad \#45$$

Where $S_€$ refers to the probability of survival adjusted for the rate of loss given default. When $R \in [0;1]$ (see above), the rate of loss given default is also between zero and one inclusive, which implies that the probability of survival for the flows expressed in the currency unit $S_€$ is generally greater than or equal to the survival probability S_t of the companies included in the loan portfolio.

$$R \in [0;1] \Rightarrow S_€ \geq S_T$$

- If we assume that the bond value is equal to 100%, which implies that the nominal yield is equal to the yield to maturity, then by definition:

$$\begin{cases} CF = (1+\hat{r})^T \\ V_0 = \dfrac{CF}{(1+\hat{r})^T} = 1 \end{cases} \qquad \#46$$

Replacing the CF in equation #44 with its value in the first part of equation #46, gives us:

$$V_0 = 1 = \frac{(1+\hat{r})^T.\left(1-d.(1-R)\right)}{(1+r)^T}$$

$$\Leftrightarrow \quad \frac{1+\hat{r}}{1+r} = \left(1 - d.(1-R)\right)^{-\frac{1}{T}}$$

From which we derive the expression for the default premium, Π_d, for a discrete, single period cash flow:

$$\hat{r} - r = \left(\left(1 - d.(1-R)\right)^{-\frac{1}{T}} - 1\right)(1+r) = \Pi_d \qquad \#47$$

Furthermore, replacing $1 - d.(1-R)$ with $S_€$ gives us:

$$\Pi_d = (1+r)\left(\frac{1}{\sqrt[T]{S_€}} - 1\right) \qquad \#48$$

- We can also express the default premium using log returns, i.e. in a continuous time framework, positing $r' = \ln(1+r)$ and $\hat{r}' = \ln(1+\hat{r})$:

$$\begin{cases} CF = e^{\hat{r}'T} \\ V_0 = e^{\hat{r}'T}\left(1 - d(1-R)\right)e^{-r'T} = 1 \end{cases} \qquad \#49$$

$$\Pi'_d = \hat{r}' - r' = -\frac{ln\left(1 - d(1-R)\right)}{T} = -\frac{ln(S_€)}{T} \qquad \#50$$

With: $\hat{r} = e^{(\Pi'_d + r')} - 1$

1.4.2.2 Case study: straight bond with maturity of over 1 year

When there are multiple forecast cash flows, the default risk of a straight bond cannot be treated like a basket of mutually independent zero-coupon bonds. A default prior to maturity jeopardizes all subsequent cash flows, which are therefore dependent upon the company surviving until the date preceding their payment. As a result, we cannot simply take the single period cash flow approach used in equation #41 and apply it to each forecast cash flow. On the other hand, equation #43 does work. In other words, we can value a bond by discounting at the “default risk-free” rate r the contractual future cash flows minus the potential losses given default calculated for each coupon payment date.

For example, the equation for a straight bond paying annual coupons and valued at V_0 on date $t = 0$ is as follows:

$$V_0 = \frac{CF_1}{1+\hat{r}} + \frac{CF_2}{(1+\hat{r})^2} \qquad \#51$$

Where $\hat{r}$ refers to the bond’s yield to maturity`.

In this case, V_0 is also equal to the value of the cash flows in case of survival minus the probability-weighted estimate of losses:

$$V_0 = \frac{CF_1 - d_1 P_1}{1+r} + \frac{CF_2 - d_2 P_2}{(1+r)^2} \qquad \#52$$

Where r refers to the bond’s expected return, i.e. the yield to maturity minus the default premium: $r = \hat{r} - \Pi_d$, in keeping with equation #37. And where, furthermore, d_1 refers to the unconditional default probability over the period $]0\,;t_1]$ and d_2, to the conditional default probability over the period $]t_1\,;t_2]$. This gives us:

$$\begin{cases} P_2 = CF_2(1-R) \\ P_1 = F_1(1-R) = \dfrac{CF_2(1-R)}{1+r} \end{cases} \quad \#53$$

Where $F_1(1-R)$ refers to the value at date t_1 in the event of default during the period $]0\,;t_1]$. This loss corresponds to the product of the expected value at that date, $F_1 = \frac{CF_2}{1+r}$, and the rate of loss in the event of default $(1-R)$.

More generally, if the recovery rate in the event of default is constant and equal to R:

$$V_0 = \sum_{1}^{T} \frac{CF_i}{(1+r)^i} - (1-R) \times \sum_{1}^{T} \frac{d_i F_i}{(1+r)^i} \quad \#54$$

Which gives us:

$$\begin{cases} F_i = \displaystyle\sum_{\theta=1}^{n} \frac{CF_{i+\theta}}{(1+r)^\theta} & if\ i < T \\ \qquad\qquad with \quad n = T - i & \\ F_i = CF_T & if\ i = T \end{cases} \quad \#55$$

Comments:

It is not hard to imagine the recovery rate R also being a function of time. For example, it may decline as the debt-financed assets depreciate. Conversely, it may increase if the debt is repaid more quickly than the assets depreciate. Furthermore, the recovery rate R may be treated as a random variable whose expected value is known at a future date T, but which presents a certain degree of uncertainty, as expressed by its coefficient of variation. That coefficient, in turn, may increase the further one gets from T. Certain authors have noted that R is influenced by the economic cycle[15], as the market value of assets declines in recessionary periods, which are correlated with defaults and generate an abundance of assets for sale.

1.4.3 Default probability as a function of default spread and recovery rate

Before making a calculation, we can get some indication of default probability by referring to the transition matrices published by rating agencies. However, these statistics are based on long observation periods, so they cover multiple economic cycles. So-called through the cycle matrices tend to smooth out the variations in default rate seen between the bottom and the top of the cycle. But someone forming an expectation based on their current position in the economic cycle may reach a very different conclusion than the long-term probability. In these cases, it makes sense to infer the expected default rates at a given date using bond spreads. That said, there are a number of classic pitfalls to this approach owing to the fact that it is impossible to know economic agents' expectations or interpret them with certainty:

- The general level of risk aversion must be estimated, which means using the CAPM to estimate the market risk premium for a portfolio comprising equities, bonds and derivatives;
- The recovery rate in the event of default is not independent of the economic cycle;
- Asset betas are hard to forecast;

[15] E.I. Altman, B. Brady, A. Resti, and A. Sironi, "The link between Default and recovery rates: Theory, Empirical Evidences and Implications", Journal of Business, 78, 6 (2005), pp. 2203-2228.

- Liquidity premia are not stable either;
- And so on.

Thus, in order to estimate default expectations, it is necessary to use models that are simplified and subjective.

1.4.3.1 Estimates using a straight bond, without default rate curves

If we refer to formula #54 and posit that it is possible to predict both the expected return r of the financial asset (risk premium, asset beta) and the recovery rate R, then the only unknown values are the annual default probabilities d_i:

$$V_0 = \sum_{1}^{T} \frac{CF_i}{(1+r)^i} - (1-R) \times \sum_{1}^{T} \frac{d_i F_i}{(1+r)^i} \qquad \#56$$

If no "conditional default rate curve" is available, then as a first approximation we can estimate the default probability at date T, corresponding to the bond's maturity, by assuming that the conditional default function is constant: $d(t) = d \quad \forall\, t$. In that case, equation #54 becomes:

$$\sum_{1}^{T} \frac{CF_i}{(1+r)^i} - V_0 = d \times (1-R) \times \sum_{1}^{T} \frac{F_i}{(1+r)^i} \qquad \#57$$

The sum of the first part of the equation is simply the hypothetical value of the bond without taking into account the default premium, i.e. the present value of the contractual cash flows discounted at a rate equal to the bond's expected return as defined in equations #36 and #38. If we label this hypothetical value with no default by time horizon i as $V_{f;i}$, then we get the following equation:

$$V_{f;0} - V_0 = d \times (1-R) \times \sum_{1}^{T} \frac{V_{f;i}}{(1+r)^i} \qquad \#58$$

Where "$V_{f;0} - V$" may be interpreted as the decrease in the risk-weighted bond's value relative to its hypothetical value if there were no risk of default.

From this, we can deduce the value of the conditional default probability, d:

$$d = \frac{V_{f;0} - V_0}{(1-R) \times \sum_1^T \frac{V_{f;i}}{(1+r)^i}} \qquad \#59$$

As an illustration, let's take the example of a three-year straight bond issued at par – hence its face value is equal to its market value – with yield to maturity identical to its nominal yield of 2.6%. Let us assume that we know the average beta for straight bonds with the same rating and the same time to maturity, giving us an estimated expected return r of 2.0%. Lastly, the estimated recovery rate is 40%:

- The hypothetical value of the bond barring default, $V_{f;0}$, would be equal to 101.73.

$$V_{f;0} = \frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + \frac{CF_3}{(1+r)^3}$$

$$V_{f;0} = \frac{2.6}{1.02} + \frac{2.6}{1.02^2} + \frac{102.6}{1.02^3} = 101.73$$

- The hypothetical value in one year, all else being equal, $V_{f;1}$, would be 101.16.

$$V_{f;1} = \frac{CF_2}{(1+r)^1} + \frac{CF_3}{(1+r)^2}$$

$$V_{f;1} = \frac{2.6}{1.02} + \frac{102.6}{1.02^2} = 101.16$$

- The hypothetical value in two years, $V_{f;2}$, would be 100.59.

$$V_{f;2} = \frac{CF_3}{(1+r)^1}$$

$$V_{f;2} = \frac{102.6}{1.02} = 100.59$$

- Lastly, the value in three years, $V_{f;3}$, when the bond repays 100 and pays the final coupon of 2.6, and there is no discounting, would be 102.6.
- The present value of future losses given default would thus be equal 175.53:

$$(1-R) \times \sum_{1}^{T} \frac{V_{f;i}}{(1+r)^i} = 0.6 \times \left(\frac{101.16}{1.02} + \frac{100.59}{1.02^2} + \frac{102.6}{1.02^3}\right) = 175.53$$

- The difference in value that results from taking default risk into account is equal to 1.73:

$$V_{f;0} - V_0 = 101.73 - 100 = 1.73$$

- With no additional data, if we assume that the conditional default probability in each of the three coming years is equal to a constant d, then according to equation #59, it comes to 0.9858%:

$$d = \frac{1.73}{175.53} = 0.9858 \times 10^{-2}$$

- In this case, the average default intensity is equal to a constant: $\bar{\lambda} \times \Delta t = d$, where $\Delta t = 1$. In keeping with equation #25, the survival function is equal to $S(t) = e^{-\bar{\lambda} \times t}$, from which we can deduce the cumulative default function: $D(t) = 1 - S(t)$:

		1 year: t = 1	2 years: t = 2	3 years: t = 3
S (t)	Survival function	99.0191%	98.0477%	97.0859%
D (t)	Cumulative default function	0.9809%	1.9523%	2.9141%

Thus, in this case, the estimated three-year cumulative default probability implied by the bond default spread would be 2.91%.

1.4.3.2 Approximation using the John Hull approach

In their work, John Hull et al. use the following formula to approximate the default spread, Π_d, in a risk-neutral world[16]:

$$\Pi_d = \bar{\lambda} \times (1 - R) \qquad \textit{equation } \#60$$

Where Π_d refers to the spread between the bond's yield to maturity and the risk-free rate (both expressed as log returns in continuous time). In our view, there is no reason not to extend this approach to the real world. That would mean that the default premium is equal to the difference between the logarithmic yield to maturity $\widehat{r'}$ and the logarithmic expected return r' (as defined earlier in equation #49).

In the example above, the simplified approach used in equation #57 gives us a default intensity of 0.9775%, materially identical to the value generated by a detailed calculation, i.e. 0.9858%:

$$\bar{\lambda} = \left(\ln\left(1 + \frac{2.6}{100}\right) - \ln(1 + \frac{2}{100})\right) \Big/ (1 - \frac{40}{100})$$

Conversely, if we take the average default intensity given by the example, 0.9858%, then applying formula #57 leads to a (log) default spread estimate of 0.5915%, compared with the 0.5865% given by our approach. The difference is not meaningful, at 0.8%.

$$\frac{0.5865}{100} = \ln\left(1 + \frac{2.6}{100}\right) - \ln(1 + \frac{2}{100})$$

Lastly, using the average intensity of 0.9775% resulting from the simplified approach in equation #25 gives us a cumulative three-year default probability of 2.89%, vs. 2.91%, which is again a minimal difference relative to the detailed approach.

As a general rule, in our experience this approximation produces satisfactory results for investment grade returns, but we start to see some significant differences for weaker ratings, below B. However, we do want to point out that the errors appear to be acceptable in a working context, considering the amount of subjectivity that goes into assigning the rating itself. This approximation formula is not only extremely simple to use, it is also valuable because it highlights the link between the default premium and its two key explanatory variables: average annual default intensity, $\bar{\lambda}$, and the percentage of loss given default, $(1 - R)$.

1.4.3.3 Estimating a default rate curve using default spreads for different maturities

It is possible to use the approach explained in section 1.2.3.1 to estimate the unconditional default rate of a one-year bond with a yield to maturity equal to the average for straight bonds with the same rating. In addition, because this calculation makes it possible to know the average default intensity at the first maturity, that information can be reused to calculate the default probability of two-year bonds with the same rating, which in turn can be used for subsequent maturities. Therefore, this process generates default intensity curves and cumulative default rate curves for each maturity and rating.

[16] Hull, John C., Predescu, Mirela, and White, Alan, "Bond Prices, Default Probabilities and Risk Premiums", March 9, 2005, SSRN: https://ssrn.com/abstract=2173148

2 From bond default spread to equity default premium

2.1 Bond/equity commonalities and extrapolating the default premium

The same reasoning we used for a straight corporate bond can easily be extended to the valuation of a fixed-rate perpetual bond, and thus to that of a stock:

- unlike a straight bond, a perpetual bond carries some uncertainty as to the exit value for the investor at the end of his investment horizon, which is obviously not limitless, even though the debt has no maturity. The exit value is contingent upon the risk-free rate, the overall risk premium, and the default risk on the exit date;

- the model used to value the stock adds in some additional uncertainty regarding the level of free cash flow to the shareholder (see below for definition), which is conditional upon the company earning a profit and respecting certain prudential constraints on its debt ratios.

The fundamental difference between stock and bond valuation is that the expected cash flows accruing to the shareholder are not contractual and are not capped, and are defined as estimates derived from the future earnings capacity of the issuer. As a result, the cash flow to the shareholder is intrinsically more variable than those accruing to bondholders. On the other hand, both actuarial models are based on cash flow forecasts that are – either by definition or de facto – conditional upon the issuer's survival:

- the bondholder will only receive the coupons and principal repayments used to calculate the yield to maturity of the bond, $\hat{r}$, if the issuer survives. As we have mentioned, on average, the expected value of the return for the investor, r, is lower than the yield to maturity owing to the possibility of the bond issuer defaulting;
- similarly, the forecast cash flow to the shareholder is either explicitly or implicitly linked to the company's survival. As a result, the cost of capital used to discount the conditional forecast cash flows, $\hat{r}$, is also higher than the stock's expected return, r, based on the cash flows adjusted for loss given default.

We invite readers with experience in valuing a company, either using management forecasts or outside forecasts, to ask themselves: have they ever used forecasts that were truly expected values, i.e. forecasts adjusted for loss given default? While probability weighted forecasts are by no means rare, those forecasts almost never take into account the mortality scenario.

It never occurs to people to discount a bond's contractual cash flows at a rate other than the market yield to maturity, $\hat{r}$, much less at the rate of r, equal to the expected return adjusted for the default risk. So it is odd that so many equity valuations use a rate that is assumed to correspond to the stock's expected return, r, to discount forecasts that are conditional upon the company's survival!

Remember that the default risk "reduces" the expected cash flow compared with the forecast in case of survival. The idiosyncratic nature of the default risk or its inclusion in systematic risk is a different question, one that involves how the beta is calculated. In this respect, using the Hamada formula to adjust the beta so that it accounts for debt leverage is solely intended to correct the systematic risk for the debt leverage. It does not take care of the need to adjust the expected cash flow calculation to reflect the risk of a corporate default[17], which exists regardless of whether or not the company has any debt. In other words,

[17] We refer readers interested in this question to the following article: Clère, Roland, "After Modigliani, Miller and Hamada; A New Way to Estimate Cost of Capital?" (Après Modigliani, Miller et Hamada: une nouvelle façon d'estimer le coût du capital?), November 23, 2016, available at SSRN: https://ssrn.com/abstract=2868702

we reject the too-widely-held assertion that default risk (or even forecasting risk) is adequately reflected in the beta. The CAPM on which the beta is based does account for the expected value of cash flows and not that of flows slanted because they do not factor in default risk (or excessive optimism)[18].

From the above, we can deduce the formula for **cost of equity**, i.e. the rate at which to discount free cash flow to the shareholder, as well as the stock's **expected return**, which is assumed to represent the investor's return on investment, i.e. the total of the instrument's increase in value plus its dividend yield.

$$\hat{r} = \Pi_d + \Pi_O + \underbrace{\beta_L \Pi_R + \Pi_L + r_f}_{r} \qquad \#61$$

As we can see, this equation is not fundamentally different from equation #36, which applies to a bond. What distinguishes it, however, is that it also includes a risk premium for excessive optimism or confidence, Π_O, a bias that from our standpoint is inseparable from any prediction made by human beings. As a reminder, an average risk premium for optimism bias can be estimated by comparing the forecasts made by analysts (or the companies themselves) against actual performance over multiple economic cycles[19].

Clarification regarding free cash flow to the shareholder:

The free cash flow that a company generates for its shareholders over a given time period (one financial year, for example), is defined as cash earnings[20] (or cash flow), minus investment net of proceeds from divesting fixed assets, plus/(minus) the reduction/(increase) in working capital and the increase/(reduction) in financial debt, and adjusted for earmarking a portion of the cash flow for prudential purposes, since there are few companies able to operate without a minimum of equity capital and cash in hand. As a result, to calculate the residual value using a DCF model, the presumed normalized cash flow cannot exceed the level of earnings (at least in a standalone approach).

$$CF_F = CF - Iv - \Delta WCR + \Delta D - \Delta SE$$

Where CF refers to the cash flow over the period, Iv to investment net of divestment proceeds, ΔWCR to the change in working capital requirement, ΔD to the change in financial debt, and ΔSE, to the prudential reallocation of some shareholders' equity to managing payments to the company's suppliers, clients, lenders, and employees.

[18] For a more in-depth look at this question: http://www.fairness-finance.com/fairness-finance/cms/en/2-34/fiche-n-3-compatibilite-avec-le-medaf.dhtml

[19] Same as the above note. Readers will find an abundance of literature on forecasting bias in financial analysis here: http://www.fairness-finance.com/fairness-finance/cms/en/4-25/liens-utiles-bibliographie-choisie.dhtml

[20] Earnings adjusted for non-cash items such as depreciation and amortization and allowance for provisions or impairments.

2.2 Example of a default probability inferred from the implied cost of equity

Here we present an example involving the steps used to calculate the estimated default probability of an issuer based on a stock's implied cost of capital[21]. In this example, the stock's implied cost of capital, $\hat{r}$, is the IRR at which the present value of forecast free cash flows to the shareholder, per share, is equal to the value of the stock, for example as observed on the stock market. To simplify matters, we assume that the cash flow forecasts are not tainted by optimism bias or excessive confidence, which means that Π_O in equation #61 is zero.

Let us assume that the explicit forecast horizon is limited to three years, the cash flows are received at the end of each year, and the residual value generated by selling the stock at the end of year three is equal to the value of a perpetual annuity with an annual cash flow growth rate in perpetuity of g; in this case, the implied cost of equity, $\hat{r}$, is the IRR at which:

$$V_0 = \frac{CF_1}{(1+\hat{r})^1} + \frac{CF_2}{(1+\hat{r})^2} + \frac{CF_3}{(1+\hat{r})^3} + \frac{\frac{CF_4}{\hat{r}-g}}{(1+\hat{r})^3} \qquad equation\ \#62$$

The final term is equal to the residual value according to a perpetual annuity formula, discounted over three years. In this term, CF_4 refers to the "normalized" cash flow expected in year 4, i.e. the cash flow at which the theoretical annual growth rate of g is compatible with the company's change in WCR, investment, change in shareholders' equity, and change in financial debt.

Comments on the rate of growth in perpetuity beyond the explicit forecast horizon:

Unlike the bond value calculations discussed earlier, the formula detailed above requires us to make an assumption about growth in perpetuity beyond the explicit forecast horizon, which is generally five years. Common sense dictates that the perpetuity rate should not exceed the forecast for nominal economic growth (real + inflation). In practice, appraisers tend to use rates that converge towards a floor that is a fraction of the economy's nominal medium-term growth rate. Over the explicit forecast period, however, annual free cash flow growth can easily exceed broader economic growth, and this is what financial analysts typically assume. If the explicit forecast horizon is shorter than five years, then the growth rate g used to calculate the residual value can naturally be higher than the normalized base rate. This simply reflects the fact that the forecaster is taking into account stronger growth between the final explicit forecast year and the usual convergence horizon of five years.

With those clarifications in mind, let us look at the example of an implied cost of capital $\hat{r}$ of 7.1% (no optimism bias premium), an expected return on the stock, r, equal to 5.6%, growth in perpetuity of g starting in year three of 3%, and the following projected cash flows per share in the event of survival: $CF_1 =$ €7.10; $CF_2 =$ €7.53; $CF_3 =$ €7.86; $CF_4 =$ €8.10.

On the calculation date $t = 0$, these assumptions give us a share value V_0 of €180.42.

$$V_0 = \frac{7.1}{\left(1+\frac{7.1}{100}\right)^1} + \frac{7.53}{\left(1+\frac{7.1}{100}\right)^2} + \frac{7.86}{\left(1+\frac{7.1}{100}\right)^3} + \frac{\frac{8.10}{\frac{7.1}{100}-\frac{3}{100}}}{\left(1+\frac{7.1}{100}\right)^3} = 180.42$$

[21] As above, this is the probability in a real-world setting and not a risk-neutral setting.

As before, with bonds, the value calculated above using a discount rate that includes a default premium, $\hat{r}$, can also be obtained by discounting cash flows adjusted for loss given default at a rate that does not include a default premium, r. Recalling the definitions we used for equation #58, we can express the market value of the stock as follows:

$$V_0 = V_{f;0} - d \times (1 - R) \times \sum_{1}^{T} \frac{V_{f;i}}{(1+r)^i}$$

Where $V_{f;i}$ refers to the hypothetical value of the stock at date i, if the conditional cash flows in the event of survival are discounted at the rate r, deflated to allow for the default premium.

☙

Clarification regarding the recovery rate for shareholders:

Estimating the recovery rate raises its own questions because it takes time to recover the funds and there is no public record cataloguing such transactions, so no recovery rate calculation will be completely above criticism or free of subjectivity. When possible, one way to approximate the recovery rate would be to compare the value of the receivable immediately after the default occurred with the amount actually due on that same date. The drawback to this approach is that the level of discount observed is a product of both the issuer's credit risk and the receivable's lack of liquidity. The second approach consists of taking an ex post survey of the amounts actually recovered and valuing them at the default date, discounting them to account for the recovery period. For this approach to work, one would need access to that information and a way to justify the discount rate used.

☙

With these limits in mind, the available studies put the expected recovery rate at around 40% for bank debt with no guaranty or senior unsecured bonds[22]. The recovery rate is lower for more subordinated debt, which means that by definition, the recovery rate for shareholders is lower than for all categories of debt holders. As a result, in the event of a failure, the expected recovery rate for shareholders must be very low in general, and particularly in cases where the company has financial debt. It may even be considered zero in the case of a court-ordered liquidation, or in the case of a leveraged recapitalization preceded by a capital reduction (known in French as a *coup d'accordéon*), which generally has the effect of wiping out the value of existing shareholders' shares. Lastly, experience shows that even if a company implements a reorganization plan, there is no way to be sure its share price won't tumble into penny stocks territory, at the expense of existing shareholders.

With this in mind, in the following discussion we assume that the value of the stock is so low following the default that it can safely be ignored, meaning a recovery rate of zero for shareholders. Based on this assumption, equation #58 leads to the following formula for the value of a stock adjusted for loss given default:

$$V_0 = V_{f;0} - d \times \sum_{1}^{T} \frac{V_{f;i}}{(1+r)^i} \qquad \textit{equation } \#63$$

[22] See Altman, Edward, and Vallore, Kishore, "Almost Everything You Wanted to Know about Recoveries on defaulted Bonds", Financial Analysts Journal, Nov/Dec 1996. Also, de Servigny, Arnaud and Zelenko, Ivan , Le risque de credit, 4th ed., Dunod, 2010, pp. 129 et seq.

- The hypothetical value at the time of issue if there is no default, $V_{f;0}$, would be equal to €187.93.

$$V_{f;0} = \frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + \frac{CF_3}{(1+r)^3} + \frac{\frac{CF_4}{\hat{r}-g}}{(1+r)^3}$$

$$V_{f;0} = \frac{7.10}{1.056} + \frac{7.58}{1.056^2} + \frac{7.86}{1.056^3} + \frac{\frac{8.10}{0.071-0.03}}{(1.056)^3} = 187.93$$

The discounted cash flows, as in the previous case of the bond, are cash flows in the event of survival. As a result, the residual value before discounting, i.e. 197.58, is obtained by capitalizing the final cash flow at the rate $\hat{r} - g$.

- The hypothetical value in one year all else being equal, $V_{f;1}$, would be €191.36.

$$V_{f;1} = \frac{CF_2}{(1+r)^1} + \frac{CF_3}{(1+r)^2} + \frac{\frac{CF_4}{\hat{r}-g}}{(1+r)^2} \Rightarrow V_{f;1} = \frac{7.58}{1.056} + \frac{7.86}{1.056^2} + \frac{\frac{8.10}{0.056-0.03}}{(1.056)^2} = 191.36$$

- The hypothetical value in two years, $V_{f;2}$, would be €194.55.

$$V_{f;2} = \frac{CF_3}{(1+r)^1} + \frac{\frac{CF_4}{\hat{r}-g}}{(1+r)^1} \Rightarrow V_{f;2} = \frac{7.86}{1.056} + \frac{\frac{8.10}{0.056-0.03}}{1.056} = 194.55$$

- Lastly, the value in three years, $V_{f;3}$, the amount expected to be received at the time of the sale, corresponds to the sum of the final explicit cash flow and the non-discounted value of the annuity, i.e. €205.44.

$$V_{f;3} = CF_3 + \frac{CF_4}{r-g} = 7.86 + \frac{8.10}{0.056-0.03} = 7.86 + 197.58 = 205.44$$

- The present value of future losses given default is thus equal to €530.13:

$$\sum_{1}^{T} \frac{V_{f;i}}{(1+r)^i} = \frac{196.36}{1.056} + \frac{194.55}{1.056^2} + \frac{205.44}{1.056^3} = 530.13$$

- The difference in value that results from factoring in the default risk is equal to €7.51:

$$V_{f;0} - V_0 = 187.93 - 180.42 = 7.51$$

- For lack of additional data, if we assume that the conditional default probabilities in each of the three coming years are equal to a constant d, then according to equation #59, the constant comes to 1.4166%:

$$d = \frac{7.51}{530.16} = 1.4166 \times 10^{-2}$$

- This is a case where the average default intensity is equal to a constant: $\bar{\lambda} \times \Delta t = d$, where $\Delta t = 1$. In accordance with equation #25, the survival function is equal to $S(t) = e^{-\bar{\lambda} \times t}$, from which we can deduce the cumulative default function: $D(t) = 1 - S(t)$:

		1 year: t = 1	2 years: t = 2	3 years: t = 3
S (t)	Survival function	98.5934%	97.2065%	95.8392%
D (t)	Cumulative default function	1.4066%	2.7935%	4.1608%

Thus, the default probability implied by the bond's default spread is estimated to be 4.16% over three years in this particular case.

2.3 Example of a default risk-adjusted cost of capital

In this example, we take the opposite tack from the one we followed in the previous example. We assume that the default risk related to the business plan is known and use it to deduce the appropriate discount rate for determining the value of the company's stock.

Let us assume that:

- we have the business plan of a biotech company whose work to develop medical treatments is at a relatively advanced stage, such that the last of the treatments to receive market approval will probably do so within the next five years;
- the forecast cash flow in year 6 is deemed to be a normal level;
- the forecast cash flows for each treatment are weighted for the probability that they will make it from the clinical stage to the next stage and on through to market approval. If each of the treatments is based on unique techniques and active compounds, then each project's success can be seen as a variable with a binomial distribution, independent of the other projects. Conversely, if all the projects use the same active ingredient and it has not been approved, a failure to win approval would mean the failure of all of the applications in development. We can imagine a spectrum of possible scenarios in which the projects' success is more or less correlated. But regardless of the degree of correlation, the probability of default would be associated with enough failures occurring that the company goes bankrupt. We shall assume that the likelihood of such a scenario over a five-year horizon is one in three.
- the average expected return of a stock in this sector is 5.60%, excluding the default premium, the liquidity premium, and any forecast bias;
- the cost of capital for mature companies, $\hat{r}$, is equal to 10%;
- the perpetuity growth rate, g, used to calculate a perpetual annuity at the end of year 5 is 2.5%;
- the forecasts for cash flows per share in the event of survival are as follows: $\mathrm{CF}_1 = -€24.62$; $\mathrm{CF}_2 = €0.00$; $\mathrm{CF}_3 = €6.00$; $\mathrm{CF}_4 = €12.50$; $\mathrm{CF}_5 = €6.00$; and $\mathrm{CF}_6 = €33.87$.

Given our assumption of a 1-in-3 default probability over five years, what default premium and what cost of capital can we use to value the company? What are the values per share that result from these assumptions?

As a first step, the answers to these questions can be obtained initially using equation # 63, in which V_0 is the only unknown variable, by deducing the average conditional probability of default from the expected cumulative probability of $33^{1/3}$%:

$$S_5 = e^{-\bar{d}\times 5} = \frac{2}{3} \quad \Rightarrow \quad \bar{d} = \frac{\ln(3) - \ln(2)}{5} = 8{,}11\ \%$$

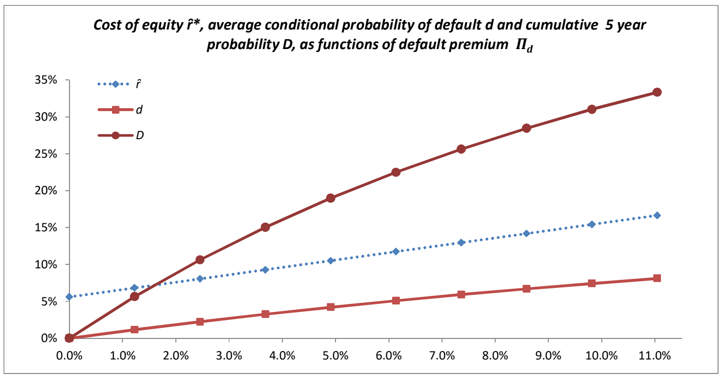
Knowing $\bar{d}$, we can deduce the value of V_0, namely 209.43:

$$V_0 = V_{f;0} - \bar{d} \times \sum_{1}^{T} \frac{V_{f;i}}{(1+r)^i} = 209{,}43$$

To solve this equation, we need to calculate the future losses given default, $V_{f;i}$, as in the second approach described below.

The second modus operandi is to proceed by successive approximations, by adding an increasingly large default premium to the expected return of 5.60% until we reach a 5-year probability of default that is basically equal to $33^{1/3}$%. For example, adding a default premium Π_d to the expected return r gives us a cost of capital over five years, $\hat{r}^*$, which determines: i) the market value of the stock, V_0, as well as ii) the value of the conditional default probability d, in accordance with equation #59. From that equation, we can deduce the average default intensity $\bar{\lambda}$ and the cumulative default probability D over five years.

The effect of the default premium's variation on the conditional default probability and on the cumulative five-year default probability is summed up in the chart and table below:



Π_d	$\hat{r}$	d	D
0.00%	5.60%	0.00%	0.00%
1.23%	6.83%	1.16%	5.65%
2.45%	8.05%	2.25%	10.63%
3.68%	9.28%	3.26%	15.04%
4.91%	10.51%	4.21%	18.97%
6.14%	11.74%	5.09%	22.48%
7.36%	12.96%	5.92%	25.63%
8.59%	14.19%	6.70%	28.46%
9.82%	15.42%	7.43%	31.02%
11.04%	16.64%	8.11%	33.33%

As shown above, taking into account a default premium of 11.04% leads to a cost of equity of 16.64%, which implies: i) an average conditional default probability of 8.11% and ii) a cumulative default probability of $33^{1/3}$% over a five-year horizon, the target level defined ex ante in our example.

We should emphasize that these results are specific to this particular business plan and that any change in the time horizon would change the level of default premium that matches the estimate for the default probability. For example, if the duration were longer, the initial negative cash flow would be larger in absolute terms, so the default premium needed to reach the target default probability of $33^{1/3}$% would be lower. Conversely, reducing the absolute value of the first cash flow would shorten the duration of the value, which would make it necessary to increase the default premium in order to reach the five-year cumulative default probability target.

Given the cost of capital $\hat{r}^*$ calculated using this method, i.e. 16.64%, the stock's value comes to €209.43.

$$V_0 = \frac{-26.00}{\left(1+\frac{16.64}{100}\right)^1} + \frac{0.00}{\left(1+\frac{16.64}{100}\right)^2} + \frac{6.00}{\left(1+\frac{16.64}{100}\right)^3} + \frac{12.50}{\left(1+\frac{16.64}{100}\right)^4} + \frac{26.00}{\left(1+\frac{16.64}{100}\right)^5} + \frac{\frac{33.87}{\frac{10.0}{100}-\frac{2.50}{100}}}{\left(1+\frac{16.64}{100}\right)^5} = 209.43$$

- The hypothetical value at the time of issue, assuming no default, $V_{f;0}$, would equal €354.24.

$$V_{f;0} = \frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + \frac{CF_3}{(1+r)^3} + \frac{CF_4}{(1+r)^4} + \frac{CF_5}{(1+r)^5} + \frac{\frac{CF_6}{\hat{r}-g}}{(1+r)^5}$$

$$V_{f;0} = \frac{-26.00}{1.056} + \frac{0.00}{1.056^2} + \frac{6.00}{1.056^3} + \frac{12.50}{1.056^4} + \frac{26.00}{1.056^5} + \frac{\frac{33.87}{0.1-0.025}}{1.056^5} = 354.24$$

- The hypothetical value in one year, all else being equal, $V_{f;1}$, would be €380.82.

$$V_{f;1} = \frac{CF_2}{(1+r)^1} + \frac{CF_3}{(1+r)^2} + \frac{CF_4}{(1+r)^3} + \frac{CF_5}{(1+r)^4} + \frac{\frac{CF_6}{\hat{r}-g}}{(1+r)^4}$$

$$V_{f;1} = \frac{0.00}{1.056^1} + \frac{6.00}{1.056^2} + \frac{12.50}{1.056^3} + \frac{26.00}{1.056^4} + \frac{\frac{33.87}{0.1-0.025}}{1.056^4} = 380.82$$

- The hypothetical value in two years, $V_{f;2}$, would be €402.14.

$$V_{f;2} = \frac{CF_3}{(1+r)^1} + \frac{CF_4}{(1+r)^2} + \frac{CF_5}{(1+r)^3} + \frac{\frac{CF_6}{\hat{r}-g}}{(1+r)^3}$$

$$V_{f;2} = \frac{6.00}{1.056^1} + \frac{12.50}{1.056^2} + \frac{26.00}{1.056^3} + \frac{\frac{33.87}{0.1-0.025}}{1.056^3} = 402.14$$

- The hypothetical value in three years, $V_{f;3}$, would be €418.66.

$$V_{f;3} = \frac{CF_4}{(1+r)^1} + \frac{CF_5}{(1+r)^2} + \frac{\frac{CF_6}{\hat{r}-g}}{(1+r)^2}$$

$$V_{f;3} = \frac{12.50}{1.056^1} + \frac{26.00}{1.056^2} + \frac{\frac{33.87}{0.1-0.025}}{1.056^2} = 418.66$$

- The hypothetical value in four years, $V_{f;4}$, would be €429.61.

$$V_{f;4} = \frac{CF_5}{(1+r)^1} + \frac{\frac{CF_6}{\hat{r}-g}}{(1+r)^1}$$

$$V_{f;4} = \frac{26.00}{1.056^1} + \frac{\frac{33.87}{0.1-0.025}}{1.056^1} = 429.61$$

- And lastly, the value in five years, $V_{f;5}$, the amount expected to be received at the time of the sale, corresponds to the sum of the final explicit cash flow and the non-discounted value of the annuity, i.e. €477.61.

$$V_{f;5} = CF_5 + \frac{CF_6}{\hat{r}-g} = 26 + \frac{33.87}{0.1-0.025} = 26 + 451.61 = 477.61$$

- The present value of future losses given default is thus equal to €1,785.96:

$$\sum_{1}^{T} \frac{V_{f;i}}{(1+r)^i} = \frac{380.82}{1.056} + \frac{402.14}{1.056^2} + \frac{418.66}{1.056^3} + \frac{429.61}{1.056^4} + \frac{477.61}{1.056^5} = 1{,}785.96$$

- The difference in value that results from factoring in the default risk is equal to €144.81:

$$V_{f;0} - V_0 = 354.24 - 209.43 = 144.81$$

- For lack of additional data, if we assume that the conditional default probabilities in each of the five coming years are equal to a constant d, then according to equation #59, the constant comes to 8.1083%:

$$d = \frac{144.81}{1.785.86} = 8.1083 \times 10^{-2}$$

- This is a case where the average default intensity is equal to a constant: $\bar{\lambda} \times \Delta t = d$, where $\Delta t = 1$. In accordance with equation #25, the survival function is equal to $S(t) = e^{-\bar{\lambda} \times t}$, from which we can deduce the cumulative default function: $D(t) = 1 - S(t)$:

		1 year: t = 1	2 years: t = 2	3 years: t = 3	4 years: t = 4	5 years: t = 5
S (t)	Survival function	92.2117%	85.0300%	78.4076%	72.3010%	66.6700%
D (t)	Cumulative default function	7.7883%	14.9700%	21.5924%	27.6990%	33.3300%

The five-year cumulative probability of default is thus equal to $33^{1/3}$%.

As a reminder, in keeping with equation #56 (adjusted for the lack of recovery and at the constant conditional default probability, d), we verify that the market value of the stock is equal to the present value of future cash flows adjusted for the probability-weighted losses given default, discounted at the rate r, excluding a default risk premium.

$$V_0 = \sum_{1}^{T} \frac{CF_i - dF_i}{(1+r)^i} \qquad equation\ \#64$$

According to this definition, the steps used to calculate the stock's value are broken down in the table below, in which the cash flows adjusted for probable loss given default, CF^*, are calculated and then discounted at the rate r, i.e. the expected rate of return (with no default premium):

		1 year: t = 1	2 years: t = 2	3 years: t = 3	4 years: t = 4	5 years: t = 5
d	Conditional probability of default	8.1083%	8.1083%	8.1083%	8.1083%	8.1083%
Vf_i	Future loss given default	-380.82	-402.14	-418.66	-429.61	-477.61
d x Vf_i	Weighted loss given default	-30.88	-32.61	-33.95	-34.83	-38.73
CF	CF in case of survival	-26.00	0.00	6.00	12.50	477.61
CF* = CF - d x Vf_i	**CF adjusted for probable loss given default**	**-56.88**	**-32.61**	**-27.95**	**-22.33**	**438.89**
$1/(1+r)^t$	Discount factor with expected r	94.6970%	89.6752%	84.9197%	80.4163%	76.1518%
VA{CF*}	Present value	-53.86	-29.24	-23.73	-17.96	334.22
$V_0 = \sum VA\{CF_i^*\}$	Market Value	209.43				

3 Testing the combined intensity model / CAPM on listed bonds

To test the intensity model combined with CAPM, we have just presented, we need to estimate the market risk premium. For consistency's sake, we think only one forward-looking model should be used. The theoretical basis of the CAPM is forward looking, and it will only be useful to an appraiser if it is based on investor expectations. To approximate those expectations, we use the only data available, to our knowledge, i.e. sell-side analysts' forecasts for listed companies. This approach assumes that these forecasts can be compiled into a marketplace consensus and, furthermore, that the consensus view is shared – at least partly accepted or mostly interpreted in the same way – by investors, the largest of whom have their own buy-side research departments. Based on these forecasts, we can determine the market's "implied cost of capital" and derive the CAPM risk premium.

Much has been written on sell-side analysts' forecasting bias. Their forecasts notably tend to trail movements in market value, so they present a lag relative to investors' aggregate expectations. In addition, analysts' expectations are tainted by excessive optimism: there is a systematic gap between financial forecasts and actual performances, and the gap grows larger as the forecast horizon gets longer. However, because these shortcomings are well documented and understood by the market and the forecasters themselves, the trick is to be able to measure the average gap with respect to the estimated value of the implied cost of capital in order to measure the optimism premium and subtract it from the cost of capital used to calculate the market's expected return.

Similarly, financial forecasts assume that the company will survive. As far as we know, there are no research firms using a methodology that adjusts forecasts to factor in the risk of loss given default. Furthermore, from a more practical standpoint, when companies are having such a tough time that they become penny stocks, analyst no longer cover them, so they do not factor meaningfully into the implied cost of capital calculation as we have framed it here.

Clearly, testing the intensity model using forward-looking data requires a general analytical framework that can be used to evaluate both stocks and bonds. The latter makes it possible to estimate the default risk that simultaneously affects both asset classes. A detailed description of such a complicated exercise exceeds the scope of this article. Thus, we will limit ourselves to using the results of the model developed by the firm Fairness Finance to calculate the risk premium. Nevertheless, we will give a brief description of the model's fundamentals[23].

We will then try to estimate the credit spreads on a given date as a function of rating and other explanatory variables. Lastly, using our estimates, we will calculate the market's implied default probabilities based on bond spreads according to i) the CAPM (to first eliminate the systematic risk) and ii) tthe intensity model (to then deduce probabilities from the residual default spreads).

[23] For more details, please consult the methodology notes at the following website: http://www.fairness-finance.com/fairness-finance/cms/en/2/methodologie.dhtml;jsessionid=1D87F09FE75A068FF806C97B5B5E4B37

3.1 Estimating the market risk premium

Research laboratory Fairness Finance performs a monthly estimate of the implied cost of capital in Europe and North America using a sample of more than 1,750 listed companies in Europe (as of end-September 2017) and more than 1,920 listed companies in the US and Canada. Its results are based on financial analyst forecasts, which are used to recreate five years of forecast cash flows for each of the companies in the two samples, making it possible to generate company-specific estimates for the IRR at which the present value of discounted cash flows is equal to the stock price on the calculation date. By aggregating these IRRs, weighted by market capitalization, we arrive at the implied cost of capital for the equity markets in Europe and North America. The market risk premium in the CAPM sense is then calculated by subtracting an optimism bias premium and a default risk premium from the IRR.

We estimate the average impact on the implied cost of capital from overestimating cash flows, or the optimism premium, Π_O, by comparing past forecasts with actual performances.

To calculate the default premium, Π_d, we measure the unlevered implied cost of capital, k_U, by estimating the companies' IRR, which we calculate by setting their enterprise value equal to the discounted value of their cash flows split into their three different components according to an APV approach[24]: unlevered cash flow, tax shield cash flow, and the negative cash flow corresponding to the cost of systematic risk[25], the last two of which are the result of debt. It should be noted that the implied unlevered cost of equity ($\mathrm{k_U}$) resulting from this work is still incorporates a default premium (partly due to financial leverage), as stock prices are not adjusted to neutralize the risk of default. Hence, the higher the risk of default, the larger Π_d will be in the observed implied cost of capital, as at this stage, the latter still includes the contribution of the financial debt at risk of default.

Once we have estimated this cost of capital, we can then calculate the default premium using the following equation:

$$k_U = r_f + \beta_U \Pi_R + \Pi_d + \Pi_O + \Pi_A \qquad \textit{equation}\ \#65$$

Where r_f refers to the risk-free rate, assumed to be equal to the average yield on 10-year government bonds rated at least AA[26] for each region (Europe and North America), β_U refers to the weighted average beta with zero debt for the companies in each of the two samples, and Π_R refers to the equity market risk premium in the CAPM sense. Π_A corresponds to the portion of the AAA corporate spread not explained by systematic risk or default risk (see below).

The market's average beta with zero debt is estimated using that of the companies in the sample, which are in turn inferred from the betas of the stocks calculated on the basis of three years of share price movements using Hamada's formula to approximate the effect of debt leverage on systematic risk:

$$\beta_L = \beta_U \left(1 + \frac{DN}{E}(1 - Tx)\right) \qquad \#66$$

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24 The estimate for k_U is calculated using an adjusted present value approach taken from the work of Pablo Fernandez, which explicitly singles out debt's tax shield and systemic risk components in the cash flows. For more detail, please read "Methodological note 5: the APV approach" on the website of Fairness Finance, http://www.fairness-finance.com/fairness-finance/cms/en/2-35/fiche-n-5-l-approche-apv.dhtml

25 According to P. Fernandez, this APV is equivalent to Hamada's formula.

26 For our eurozone calculations, the risk-free rate is assumed to be equal to the German government bond yield for the desired maturity.

Where β_L refers to the beta of the stock with debt leverage, DN is the average debt over three years, E is the average market capitalization for each company, and Tx is the estimated rate at which interest expense may be deducted.

Furthermore, in keeping with equation #36, the average spread of the companies in the sample is expressed as follows:

$$\overline{Sc} = \overline{\beta_D} \times \Pi_R + \Pi_{dD} + \Pi_A \qquad \#67$$

Where $\overline{Sc}$ refers to the weighted average credit spread that corresponds to the weighted average rating of the companies in the sample (determined using the methodology described below in section 3.2), $\overline{\beta_D}$ refers to the average beta of the debt, Π_{dD} refers to the default premium included in the spread, and Π_A refers to the residual average spread of the AAA-rated companies in our sample. In this example, we ignore the liquidity premium because the spread is estimated for large cap issuers.

Defining P as the average rate of loss experienced by lenders in the event of a default, we assume that it is 100% for shareholders. According to John Hull's approximation (see equation #60), this produces the following relationship between the average bond default premium and the average equity default premium:

$$\Pi_d = \frac{\Pi_{dD}}{P} \qquad \textit{equation } \#68$$

Where: $P = 1 - R$, and R refers to the recovery rate in the event of a default.

According to equations #65 to #68, we can finally deduce the CAPM risk premium, Π_R:

$$\Pi_R = \frac{\Pi_{EU} - \Pi_O - (Sc - R.\Pi_A)/P}{\beta_U - \frac{\beta_D}{P}} \qquad \#69$$

Where Π_{EU} is the market premium with zero debt, i.e. $k_U - r_f$.

Having estimated the CAPM risk premium, Π_R, we can use this value to calculate the equity default premium, as defined in equation #65.

From the above, we can infer the market's expected return, $E(k_L)$, from the implied cost of capital in accordance with the CAPM:

$$E(k_L) = \Pi_R + r_f$$

The implied cost of capital, k_L, is itself equal to the market's expected return (with leverage), $E(k_L)$, plus a premium for optimism bias and one for default risk:

$$k_L = E(k_L) + \Pi_O + \Pi_d$$

3.2 Ranking of corporate spreads at end-December 2015 in euros

As we discussed earlier, to estimate the market risk premium, we need an estimate for the market's weighted average spread. At 31 December 2015, the weighted average rating for issuers in our sample of listed stocks in Europe, as compiled by Fairness Finance, was BBB$^+$.

To estimate the average spread, first we compile a sample of straight bonds with fixed coupons, issued and listed in euros, excluding bonds with a remaining time to maturity of less than one year and those without a rating. The yield to maturity of each bond is supplied by the Thomson Reuters Eikon database, which also shows the risk-free fate for each maturity, as determined by the yield curve for straight fixed-rate German government bonds. To give us an element of volatility for each instrument, we selected bonds issued by companies that also have listed stocks. Lastly, we created two samples so that we could separate banking institutions from other companies in order to limit the financial sector to its weight in the equity index, i.e. around 10%, whereas it accounts for an overwhelming share of the European bond market. After this screening, as of end-December 2015, we arrived at a sample of 512 bonds[27] for the non-banking sector. We present an analysis of their spreads below.

For this exercise, we created a multiple linear regression model in which the spread, "*Sc*", is the unknown variable and in which the five explanatory variables are:

- the rating, "*N*", converted into a scale of 1 (D) to 23 (AAA), which is assumed to be negatively correlated with the spread. The ratings are taken from the agencies Sandard & Poor's, Moody's and Fitch Ratings;
- a measure of size, "*T*", which is the market capitalization in millions of euros, since a larger company is assumed to have more favorable financing conditions than a smaller one with a similar rating and amount of issuance;
- a measure of liquidity, "*L*", which is the amount of the bond issue in millions of euros, and which is assumed to have a negative correlation with the spread;
- a measure of the dispersion of the issuing company's market value, "σ", which is the annualized volatility of the stock price over a three-year period. This variable is used to reflect the challenges the market faces in forecasting the company's future performance and is thus assumed to have a positive correlation with the spread;
- the bond's remaining time to maturity, "*D*", since the risk of a rating change or a default increases with time and the bond's value becomes more sensitive to interest rate fluctuations. As a result, time to maturity is assumed to have a positive correlation with the spread.

The table below shows the breakdown of our sample according to company ratings and the average values for each sub-category's explanatory variables:

Sample selected for the calculation of the spread: average values	Subsets by rating class *							Total sample
	AAA	AA	A	BBB	BB	B	CCC	BBB
T (Market capitalization of the issuer in € m)	408,075	149,354	48,616	17,053	7,944	1,803	104	33,397
σ (annualized monthly volatility, 3 years)	24.80%	17.52%	24.26%	21.30%	27.71%	53.32%	31.91%	23.50%
D (time to maturity in years)	12.93	8.16	7.33	5.74	4.15	5.98	2.96	6.38
L (size of the issue in € m)	1750	963	867	588	645	322	160	568
Number of bonds	2	14	36	124	17	13	4	512

** Only the classes corresponding to whole ratings are presented, the total of 512 bond issues including the intermediate note classes.*

In this linear regression, the unknown variable and the explanatory variables produce the following equation:

$$\ln(Sc) = \alpha_0 + \alpha_1 N + \alpha_2 \ln(T) + \alpha_3 \ln(L) + \alpha_4 \sigma + \alpha_5 \ln(D) + \varepsilon$$

The coefficients of the variables are all meaningful (the liquidity measure is the least meaningful) and their influence on the spread bears out our assumptions. The table below summarizes our analysis of the explanatory strength of the measures used in the regression:

[27] Out of a total of 542 bonds, after excluding the 5% of cases with the largest standardized residuals in absolute terms, as calculated using the linear regression presented below.

Coefficients of determination			**Residuals analysis**			
Coefficient of determination R^2	86.93%		Kurtosis	-0.027		
Adjusted coefficient de determination R^2	86.80%		Skewness	0.419		
Standard error	0.260		Khi²	9.154	p(x<X²) =	3%
Number of observations	512		K-S	5.26%	p(x<K-S) =	1%

Regression coefficients	N	Ln(T)	Ln(L)	σ	Ln(D)	Constant
Coefficient	-0.1742	-0.0528	-0.0222	1.1728	0.4077	-1.8849
t	29.9	4.5	1.3	7.5	22.2	14.2
Probability (unilateral except the constant)	2.694E-114	4.575E-06	9.3%	1.653E-13	3.478E-77	1.099E-38
Variance inflation factor	1.5	1.2	2.2	2.4	1.1	n/a

Confidence interval of the mean E(Sc)*	E(ln(Sc))	E(Sc)	**Fisher test**	
Expected value of the mean	-4.20	1.50%	F	506
95% confidence interval	[-4.14 ; -4.26]	[1.41% ; 1.59%]	P value	6.3E-221

**By retaining the average values of the variables and in the idealized hypothesis of a perfect normal residual distribution*

Given the small number of AAA-rated bonds, we created a second ad hoc model that pools all of the AAA-rated bonds, regardless of sector, including those of issuers without a publicly quoted stock. In the second model, it is no longer possible to perform a regression of the spread as a function of stock price volatility, so we replace that variable with the bonds' own price volatility. However, that volatility is naturally dependent upon the time to maturity. As a result, we only use the portion of the volatility not attributable to maturity (r² of 86%) as our explanatory variable, which we refer to as "$\Delta\sigma_\delta$". We also factor in the difference between the issuer's rating and that of the bond, ΔN, since the spread of an AAA-rated bond is greater the weaker the issuer's rating. Lastly, as earlier, we use variables for duration ($ln(D)$) and liquidity ($ln(L)$). This approach gives us a sample of 42 AAA-rated bonds. The coefficients in the regression are all meaningful using a threshold of 5%, and the (adjusted) coefficient of determination comes to 70.8%.

Coefficients of determination			**Residuals analysis**			
Coefficient of determination R^2	72.38%		Kurtosis	-0.659		
Adjusted coefficient de determination R^2	70.83%		Skewness	0.368		
Standard error	0.316					
Number of observations	42		K-S	10.77%	p(x<K-S) =	1.5%

Regression coefficients	ΔN	$Ln(L)$	$ln(D)$	$\Delta\sigma_\delta$	*Cst*
Coefficient	-0.2243	-0.0541	0.2547	6.0786	-5.6036
t	6.4	1.7	4.5	1.7	25.7
Probability (unilateral except the constant)	1.021E-07	5.0%	3.183E-05	4.752E-02	1.641E-25
Variance inflation factor	1.1	1.1	1.0	1.0	na

Confidence interval of the mean E(Sc)*	E(ln(Sc))	E(Sc)	**Fisher test**	
Expected value of the mean	-5.44	0.43%	F	24.2
95% confidence interval	[-5,21 ; -5,67]	[0,34% ; 0,55%]	P value	6.6E-10

**By retaining the average values of the variables and in the idealized hypothesis of a perfect normal residual distribution*

3.3 Bond betas as a function of bond rating and time to maturity

As shown in equation #67, to estimate the default premium within the bond spread, we need to know the bond's beta. With this in mind, we created portfolios of identically rated bonds according to their remaining time to maturity. For example, bonds with the same rating and a time to maturity of between 1 and 2 years make up a portfolio with an average time to maturity of 1.5 years. Variations in the value of the (equally weighted) portfolio indices are then recorded each month. On the first of the month, we adjust the composition of each index as a function of bonds' remaining time to maturity and rating before measuring the index's movement. We then calculate the beta for each index relative to the equity market index using monthly price frequencies over a moving three-year time period.

Using the portfolio betas observed on December 31, 2015, we performed a regression of the betas (log) as a function of the portfolios' remaining time to maturity (log) and rating (log). The theoretical values of the betas as a function of time to maturity and the coefficients of determination of the regressions are detailed in the table below, with betas expressed as a percentage for clarity's sake:

$\boldsymbol{\beta}$	AAA	AA	A	BBB	BB	B	CCC
Time to maturity \ r^2	90%	90%	90%	90%	90%	90%	75%
1	1.3%	1.6%	2.2%	2.5%	5%	9%	21%
2	2.1%	2.6%	3.5%	4.0%	8%	15%	28%
3	2.8%	3.4%	4.6%	5.2%	11%	19%	33%
4	3.4%	4.1%	5.6%	6.3%	13%	23%	38%
5	3.9%	4.8%	6.5%	7.3%	15%	27%	43%
6	4.5%	5.4%	7.4%	8.3%	17%	31%	48%
7	4.9%	6.0%	8.2%	9.2%	19%	34%	52%
8	5.4%	6.5%	9.0%	10.1%	21%	37%	56%
9	5.9%	7.1%	9.7%	10.9%	22%	40%	60%
10	6.3%	7.6%	10.4%	11.7%	24%	43%	64%
15	8.3%	10.0%	13.7%	15.4%	31%	57%	81%
20	10.0%	12.1%	16.6%	18.7%	38%	69%	97%

For the CCC rating, which is at the limit for the model, we used the average of the previous log model and a linear model, (as the r^2 of 75% corresponds to the lower coefficient of the two approaches). Both approaches encompass the values actually observed for longer maturities.

3.4 The model's implied default probabilities: a market anomaly?

If we know the theoretical spread for corporate issuers (excluding banks), "*Sc*", and bond debt betas, "β_D", as a function of their rating, "*N*", and time to maturity, "*D*", we can deduce the bond's default spread, "Π_d", as well as the default probability "$E(D_t)$" implied by the time to maturity and rating.

To do so, we use the CAPM estimate of the eurozone equity market risk premium calculated by Fairness Finance, "Π_R". At end-2015, the premium came to 4.23% for the eurozone compared with the German 10-year risk-free rate. However, the risk premium derived from the difference between the required return on equities and the risk-free rate depends upon the investment horizon. If we assume that the required return on equities is not very sensitive to the time horizon[28], then the market risk premium will generally be a decreasing function of the investment duration, as the risk-free yield curve typically has an upward slope.

Furthermore, the variables measuring the bond's liquidity, the issuer's size and the stock price's volatility are equal to the average values for bonds in same rating category (see table detailing the sample used to calculate the spread).

Once we have estimated bond spreads and betas as a function of time to maturity, we can calculate the implied default premium by subtracting the portion resulting from systemic risk, using equation #67, if we ignore the residual risk premium for AAA-rated European corporates. The default probability is determined in the second step, by applying John Hull's approximation (see equation #60).

AAA-rated corporate bonds:

Time to maturity	β_d	Π_R	$\Pi_R \times \beta_d$	Π_A	$\Pi_R \times \beta_d + \Pi_A$	*Sc*	Π_d	S_t	$E(D_t)$	*S&P 2016**
1	1.3%	4.6%	0.06%	-	0.06%	0.26%	0.20%	99.67%	**0.33%**	0.00%
2	2.1%	4.6%	0.10%	-	0.10%	0.31%	0.21%	99.30%	**0.70%**	0.03%
3	2.8%	4.5%	0.13%	-	0.13%	0.34%	0.21%	98.93%	**1.07%**	0.14%
4	3.4%	4.4%	0.15%	-	0.15%	0.37%	0.22%	98.56%	**1.44%**	
5	3.9%	4.3%	0.17%	-	0.17%	0.39%	0.22%	98.19%	**1.81%**	0.41%
6	4.5%	4.2%	0.19%	-	0.19%	0.41%	0.22%	97.82%	**2.18%**	
7	4.9%	4.0%	0.20%	-	0.20%	0.42%	0.22%	97.44%	**2.56%**	0.67%
8	5.4%	3.9%	0.21%	-	0.21%	0.44%	0.22%	97.06%	**2.94%**	
9	5.9%	3.7%	0.22%	-	0.22%	0.45%	0.23%	96.59%	**3.41%**	
10	6.3%	3.7%	0.23%	-	0.23%	0.46%	0.23%	96.22%	**3.78%**	1.03%
15	8.3%	3.2%	0.27%	-	0.27%	0.51%	0.24%	94.10%	**5.90%**	1.46%
20	10.0%	3.0%	0.30%	-	0.30%	0.55%	0.25%	91.93%	**8.07%**	2.45%

* Calculation period 1981 - 2016. Source: S&P Global ratings, 2016 Annuel Global Corporate Default Study And Rating Transitions, available on the S & P website: https://www.spglobal.com/our-insights/2016-Annual-Global-Corporate-Default-Study-and-Rating-Transitions.html

As shown in the table above, this approach delivers a 10-year default premium of 0.23%, which implies a default probability of 3.8%. That seems excessive. In the transition matrices published by Standard & Poor's, for example, the rate comes to 1.03%[29] (adjusting for the proportion of bonds that have ceased to be rated 10 years, on average, after the bond was issued). As the coefficient of variation of the default rate calculated by S&P is 107%, a value of 3.8% would thus be 3.5 standard deviations away from the mean.

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28 The implied cost of capital is a single IRR that applies to all maturities. We do not have enough information to establish a yield curve for the equity cost of capital using this methodology.

29 Here we assume that the default rate for the population of bonds that have ceased to be rated is close enough to that of the bonds that are still rated. For more on this assumption and the reasons bonds are removed from coverage, please consult: Hamilton, David T., and Cantor, Richard, "Measuring Corporate Default Rates", Moody's, November 2006.

There are several hypotheses that could explain this apparent overestimation of default risk:

- First of all, the beta is tainted by a margin of error and is estimated using past correlation over three years, which may differ from investor expectations. For the default probability to be in line with the historical level of 1.03%, the expected beta would have to be 10.5%, which is 67% higher than the value used in our calculations;

- The expected average recovery rate in the event of default may be different than the 40% generally used in the literature. If the rate were 50% owing to the "blue chip" status of AAA listed companies, then the implied default rate would be 4.51% over ten years. Conversely, lowering the recovery rate would not be enough to move exactly in line with the historical statistic. Even if the recovery rate were zero, in this case the implied default rate would be 2.2% over 10 years, or double the long-term value;

- The risk-free rate may be different than the Bund rate. Companies of every nationality, including US companies, issue bonds in euros. However, the data available to us do not show a significant difference between European issuers and those whose corporate offices are outside of the eurozone. Furthermore, the risk-free rate is often confused with the interbank lending rate, which incorporates counterparty risk, resulting in a premium (vs. the Bund) that has peaked during crisis periods. When it comes to bond debt, this intermediation cost does not appear to be warranted;

- After consideration, we have opted for the hypothesis that says that there is an incompressible premium that cannot be explained either by systematic risk or by default risk. This premium for a lack of liquidity or adverse selection penalizes AAA-rated corporate bonds relative to government bonds with the same rating. The volume of publicly traded corporate bonds is small compared with the volume of bonds issued by governments. In addition, certain investors face regulatory constraints that encourage them to favor AAA-rated government bonds over corporate bonds. And lastly, tax treatment typically favors government bonds. In fact, in certain countries, such as the US[30], this phenomenon might explain much of the yield spread investors require between AAA-rated corporate bonds and government bonds with the same rating.

If we take this last hypothesis as our assumption and set the AAA premium such that the implied default probability is minimized for every maturity (1 year to 30 years), then "Π_A" represents approximately 74% of the difference between the AAA corporate spread, its systematic component[31], and the long-term default rate.

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30 Edwin, Elton, Gruber, Martin, Agrawal Deepak, and Mann, Christopher, "Explaining the rate spread on corporate bonds", Journal of Finance, 56, 2001, pp. 247-277

31 In this case we use a coefficient "α" common to all the maturities, such that: $\Pi_{A_t} = \alpha\left(Sc_t - \beta_t \Pi_{R_t}\right)$.

AAA-rated corporate bonds (probabilities factoring in the residual AAA premium):

Time to maturity	[illegible]$_d$	[illegible]	[illegible] x [illegible]$_d$	[illegible]	[illegible] x [illegible]$_d$ + [illegible]	Sc	[illegible]	S_t	$E(D_t)$	S&P 2016
1	1.3%	4.6%	0.06%	0.14%	0.21%	0.26%	0.05%	99.91%	**0.09%**	0.00%
2	2.1%	4.6%	0.10%	0.15%	0.25%	0.31%	0.06%	99.81%	**0.19%**	0.03%
3	2.8%	4.5%	0.13%	0.16%	0.28%	0.34%	0.06%	99.72%	**0.28%**	0.14%
4	3.4%	4.4%	0.15%	0.16%	0.31%	0.37%	0.06%	99.62%	**0.38%**	
5	3.9%	4.3%	0.17%	0.16%	0.33%	0.39%	0.06%	99.52%	**0.48%**	0.41%
6	4.5%	4.2%	0.19%	0.16%	0.35%	0.41%	0.06%	99.42%	**0.58%**	
7	4.9%	4.0%	0.20%	0.16%	0.36%	0.42%	0.06%	99.31%	**0.69%**	0.67%
8	5.4%	3.9%	0.21%	0.17%	0.38%	0.44%	0.06%	99.21%	**0.79%**	
9	5.9%	3.7%	0.22%	0.17%	0.39%	0.45%	0.06%	99.09%	**0.91%**	
10	6.3%	3.7%	0.23%	0.17%	0.40%	0.46%	0.06%	98.99%	**1.01%**	1.03%
15	8.3%	3.2%	0.27%	0.18%	0.45%	0.51%	0.06%	98.40%	**1.60%**	1.46%
20	10.0%	3.0%	0.30%	0.19%	0.49%	0.55%	0.07%	97.80%	**2.20%**	2.45%

As shown in the table above, the AAA premium thought to correspond to the part of the corporate bond spread left unexplained by either systematic risk (which represents a majority) or default risk (equal to the historical average) falls into a range of 14bp to 19bp inclusive at end-2015.

By setting the AAA premium this way, we can estimate the implied default probabilities for ratings A to CCC. As shown in the six tables that follow, according to this approach, the implied default probabilities converge towards the historical averages for ratings below BBB. However, for ratings between BBB and AA, the implied default probabilities appear to be materially above their historical averages.

AA-rated corporate bonds:

Time to maturity	[illegible]$_d$	[illegible]	[illegible] x [illegible]$_d$	[illegible]	[illegible] x [illegible]$_d$ + [illegible]	Sc	[illegible]	S_t	D_t	S&P 2016
1	1.6%	4.6%	0.07%	0.14%	0.22%	0.22%	0.00%	100.00%	**0.00%**	0.02%
2	2.6%	4.6%	0.12%	0.15%	0.27%	0.29%	0.02%	99.93%	**0.07%**	0.07%
3	3.4%	4.5%	0.15%	0.16%	0.31%	0.34%	0.03%	99.83%	**0.17%**	0.15%
4	4.1%	4.4%	0.18%	0.16%	0.34%	0.39%	0.05%	99.69%	**0.31%**	
5	4.8%	4.3%	0.20%	0.16%	0.37%	0.42%	0.06%	99.51%	**0.49%**	0.42%
6	5.4%	4.2%	0.22%	0.16%	0.39%	0.46%	0.07%	99.30%	**0.70%**	
7	6.0%	4.0%	0.24%	0.16%	0.41%	0.49%	0.08%	99.06%	**0.94%**	0.75%
8	6.5%	3.9%	0.26%	0.17%	0.42%	0.51%	0.09%	98.79%	**1.21%**	
9	7.1%	3.7%	0.26%	0.17%	0.43%	0.54%	0.10%	98.46%	**1.54%**	
10	7.6%	3.7%	0.28%	0.17%	0.45%	0.56%	0.11%	98.14%	**1.86%**	1.22%
15	10.0%	3.2%	0.32%	0.18%	0.50%	0.66%	0.16%	96.13%	**3.87%**	2.02%
20	12.1%	3.0%	0.36%	0.19%	0.55%	0.75%	0.20%	93.65%	**6.35%**	3.77%

A-rated corporate bonds:

Time to maturity	[illegible]$_d$	[illegible]	[illegible] x [illegible]$_d$	[illegible]	[illegible] x [illegible]$_d$ + [illegible]	Sc	[illegible]	S_t	D_t	S&P 2016
1	2.2%	4.6%	0.10%	0.14%	0.25%	0.43%	0.18%	99.70%	**0.3%**	0.06%
2	3.5%	4.6%	0.16%	0.15%	0.32%	0.57%	0.25%	99.17%	**0.8%**	0.16%
3	4.6%	4.5%	0.21%	0.16%	0.37%	0.67%	0.30%	98.51%	**1.5%**	0.30%
4	5.6%	4.4%	0.25%	0.16%	0.41%	0.75%	0.34%	97.74%	**2.3%**	
5	6.5%	4.3%	0.28%	0.16%	0.44%	0.82%	0.38%	96.88%	**3.12%**	0.72%
6	7.4%	4.2%	0.31%	0.16%	0.47%	0.89%	0.41%	95.95%	**4.0%**	
7	8.2%	4.0%	0.33%	0.16%	0.49%	0.94%	0.45%	94.93%	**5.1%**	1.35%
8	9.0%	3.9%	0.35%	0.17%	0.52%	1.00%	0.47%	93.86%	**6.1%**	
9	9.7%	3.7%	0.36%	0.17%	0.53%	1.05%	0.51%	92.67%	**7.3%**	
10	10.4%	3.7%	0.38%	0.17%	0.55%	1.09%	0.53%	91.51%	**8.5%**	2.51%
15	13.7%	3.2%	0.44%	0.18%	0.62%	1.29%	0.65%	84.96%	**15.0%**	5.07%
20	16.6%	3.0%	0.49%	0.19%	0.68%	1.45%	0.75%	77.83%	**22.2%**	8.66%

BBB-rated corporate bonds:

Time to maturity	[redacted]	[redacted]	[redacted] x [redacted]	[redacted]	[redacted] x [redacted] + [redacted]	Sc	[redacted]	S_t	D_t	S&P 2016
1	2.5%	4.6%	0.11%	0.14%	0.26%	0.74%	0.48%	99.20%	**0.80%**	0.19%
2	4.0%	4.6%	0.18%	0.15%	0.34%	0.98%	0.65%	97.87%	**2.1%**	0.59%
3	5.2%	4.5%	0.24%	0.16%	0.39%	1.16%	0.76%	96.25%	**3.7%**	1.09%
4	6.3%	4.4%	0.28%	0.16%	0.44%	1.31%	0.86%	94.42%	**5.6%**	
5	7.3%	4.3%	0.32%	0.16%	0.48%	1.43%	0.94%	92.43%	**7.6%**	2.60%
6	8.3%	4.2%	0.35%	0.16%	0.51%	1.54%	1.02%	90.30%	**9.7%**	
7	9.2%	4.0%	0.37%	0.16%	0.54%	1.64%	1.09%	88.06%	**11.9%**	4.48%
8	10.1%	3.9%	0.40%	0.17%	0.56%	1.73%	1.15%	85.74%	**14.3%**	
9	10.9%	3.7%	0.41%	0.17%	0.58%	1.82%	1.22%	83.29%	**16.7%**	
10	11.7%	3.7%	0.43%	0.17%	0.60%	1.89%	1.26%	81.01%	**19.0%**	7.83%
15	15.4%	3.2%	0.50%	0.18%	0.68%	2.24%	1.52%	68.36%	**31.6%**	15.88%
20	18.7%	3.0%	0.55%	0.19%	0.74%	2.52%	1.73%	56.26%	**43.7%**	23.12%

BB-rated corporate bonds:

Time to maturity	[redacted]	[redacted]	[redacted] x [redacted]	[redacted]	[redacted] x [redacted] + [redacted]	Sc	[redacted]	S_t	D_t	S&P 2016
1	5.1%	4.6%	0.23%	0.14%	0.38%	1.40%	1.02%	98.32%	**1.7%**	0.80%
2	8.1%	4.6%	0.37%	0.15%	0.53%	1.86%	1.32%	95.69%	**4.3%**	2.74%
3	10.7%	4.5%	0.48%	0.16%	0.64%	2.19%	1.54%	92.61%	**7.4%**	5.40%
4	12.9%	4.4%	0.57%	0.16%	0.73%	2.47%	1.71%	89.22%	**10.8%**	
5	15.0%	4.3%	0.65%	0.16%	0.81%	2.70%	1.86%	85.62%	**14.4%**	12.07%
6	17.0%	4.2%	0.71%	0.16%	0.87%	2.91%	2.00%	81.89%	**18.1%**	
7	18.8%	4.0%	0.76%	0.16%	0.93%	3.10%	2.13%	78.04%	**22.0%**	19.27%
8	20.6%	3.9%	0.81%	0.17%	0.98%	3.27%	2.24%	74.17%	**25.8%**	
9	22.3%	3.7%	0.83%	0.17%	1.00%	3.43%	2.36%	70.14%	**29.9%**	
10	24.0%	3.7%	0.88%	0.17%	1.05%	3.58%	2.46%	66.34%	**33.7%**	29.84%
15	31.5%	3.2%	1.02%	0.18%	1.20%	4.23%	2.92%	48.21%	**51.8%**	47.24%
20	38.2%	3.0%	1.13%	0.19%	1.32%	4.75%	3.29%	33.37%	**66.6%**	59.13%

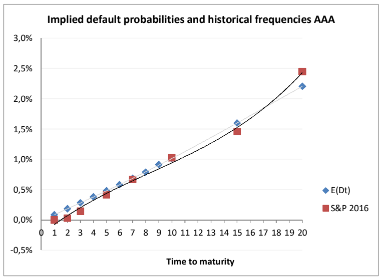
B-rated corporate bonds:

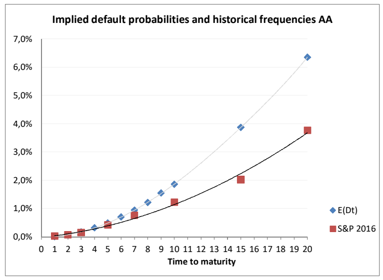
Time to maturity	[redacted]	[redacted]	[redacted] x [redacted]	[redacted]	[redacted] x [redacted] + [redacted]	Sc	[redacted]	S_t	D_t	S&P 2016
1	9.2%	4.6%	0.42%	0.14%	0.57%	3.50%	2.89%	95.30%	**4.7%**	4.28%
2	14.7%	4.6%	0.67%	0.15%	0.83%	4.65%	3.73%	88.30%	**11.7%**	10.95%
3	19.3%	4.5%	0.87%	0.16%	1.03%	5.48%	4.33%	80.54%	**19.5%**	18.16%
4	23.4%	4.4%	1.03%	0.16%	1.19%	6.17%	4.81%	72.57%	**27.4%**	
5	27.2%	4.3%	1.17%	0.16%	1.33%	6.75%	5.22%	64.73%	**35.3%**	32.31%
6	30.7%	4.2%	1.28%	0.16%	1.45%	7.28%	5.59%	57.20%	**42.8%**	
7	34.1%	4.0%	1.38%	0.16%	1.54%	7.75%	5.92%	50.11%	**49.9%**	45.53%
8	37.3%	3.9%	1.46%	0.17%	1.63%	8.18%	6.23%	43.58%	**56.4%**	
9	40.3%	3.7%	1.50%	0.17%	1.68%	8.58%	6.54%	37.48%	**62.5%**	
10	43.3%	3.7%	1.59%	0.17%	1.76%	8.96%	6.80%	32.19%	**67.8%**	59.55%
15	56.9%	3.2%	1.84%	0.18%	2.02%	10.57%	7.97%	13.63%	**86.4%**	76.99%
20	69.0%	3.0%	2.04%	0.19%	2.23%	11.89%	8.92%	5.11%	**94.9%**	81.46%

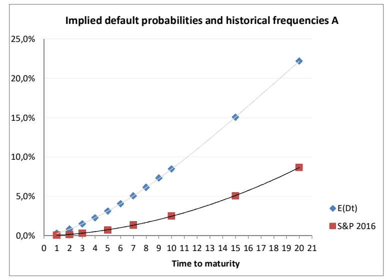
CCC-rated corporate bonds:

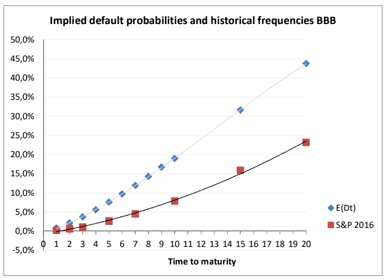
Time to maturity	[redacted]	[redacted]	[redacted] x [redacted]	[redacted]	[redacted] x [redacted] + [redacted]	Sc	[redacted]	S_t	D_t	S&P 2016
1	20.8%	4.6%	0.96%	0.14%	1.11%	5.43%	4.20%	93.24%	**6.8%**	31.65%
2	27.5%	4.6%	1.26%	0.15%	1.42%	7.20%	5.57%	83.07%	**16.9%**	46.77%
3	33.2%	4.5%	1.51%	0.16%	1.66%	8.50%	6.52%	72.17%	**27.8%**	57.62%
4	38.4%	4.4%	1.69%	0.16%	1.85%	9.55%	7.30%	61.47%	**38.5%**	
5	43.2%	4.3%	1.86%	0.16%	2.02%	10.46%	7.96%	51.52%	**48.5%**	71.68%
6	47.6%	4.2%	1.99%	0.16%	2.15%	11.27%	8.54%	42.55%	**57.4%**	
7	51.9%	4.0%	2.10%	0.16%	2.26%	12.00%	9.08%	34.66%	**65.3%**	77.78%
8	56.0%	3.9%	2.20%	0.17%	2.37%	12.67%	9.57%	27.93%	**72.1%**	
9	59.9%	3.7%	2.24%	0.17%	2.41%	13.29%	10.06%	22.12%	**77.9%**	
10	63.8%	3.7%	2.34%	0.17%	2.51%	13.88%	10.46%	17.49%	**82.5%**	84.31%
15	81.4%	3.2%	2.64%	0.18%	2.82%	16.37%	12.27%	4.65%	**95.3%**	91.44%
20	97.4%	3.0%	2.88%	0.19%	3.07%	18.41%	13.72%	1.03%	**99.0%**	93.76%

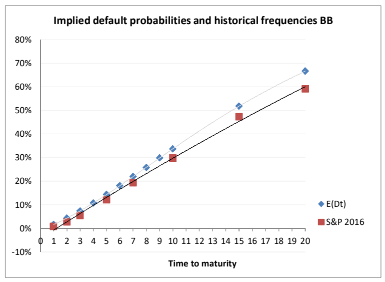
The implied default probabilities as a function of rating and remaining time to maturity can be depicted by the following charts. We note that the curves' concavity or convexity is matched by that of the historical curves:

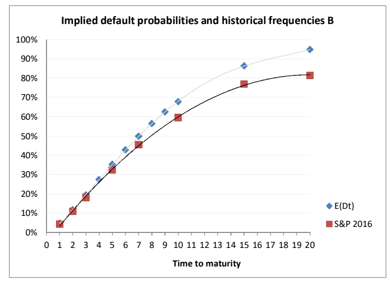


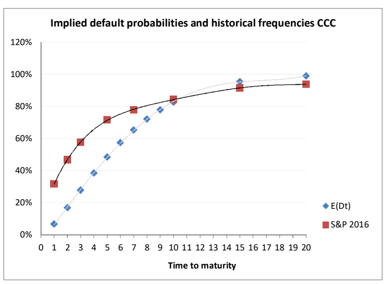












The observation of "physical" implied default probabilities that are higher than the historical default rates is a phenomenon that has already been noted in the work that Davidenko et al. performed in 2003 on investment grade bonds. As with that work, the spreads for speculative ratings, on the other hand, are smaller. Remember that these results were obtained by a structural-type model using a totally different approach that what we have used. Thus, the confirmation of the earlier result appears to indicate that investment grade bond spreads may be excessive relative to the systematic risk and default risk present in the instruments in question.

We have formulated a hypothesis that the residual premium on AAA bonds is the result of an adverse selection process, since corporate bonds are less liquid than government bonds, which are very similar but also tend to enjoy more favorable tax treatment in Europe and the US. This adverse selection premium is therefore assumed to apply to all of the rating categories. On the other hand, for other investment grade ratings, it is easy to imagine that there might be additional premia using the following mechanisms:

- on a rating scale of 1 for the weakest (D) to 23 for the strongest (AAA), we note that the rating is positively correlated to the size of the issuer and the size of the issue. As a result, spreads for a given rating partly reflect a size and liquidity effect. For example, for the BBB rating, according to our calculations, bonds whose size and liquidity are identical to those in the population of AAA bonds, the expected spread is 1.57% for a 10-year maturity (vs. 1.90%), bringing the implied default probability to 14.6%, compared with 19.2% previously using the average parameters for the BBB-rated class. As a result, even after correcting for size and liquidity effects, the implied default probability would still be double the historical default rate. This explanation does not seem to adequately explain the apparent excess remuneration of investment grade spreads at end-2015;

- Thus, we prefer the hypothesis which says that there is an additional spread on top of the AAA residual premium that is not explained by liquidity/size, nor by systemic risk, nor by default risk. This spread, between 60bp and 80bp inclusive for BBB-rated bonds with maturities of between 5 and 10 years, may result from banks' influence on pricing, since they originate and/or subscribe part of the issue themselves. Banks' intermediation margins must ensure that they earn the regulatory required return on the capital that they hold in reserve for their lending activities. Their profit motives and regulatory constraints clearly require them to demand an additional margin over and above just the remuneration for default risk and systematic risk. A comparison of bond spreads and intermediation margins in the US, furthermore, reveals that there is an additional, material banking spread of around 240bp[32] for bonds with comparable guarantees. We posit that such a spread also exists in Europe.

To test this hypothesis, let us assume that the BBB credit spread on a 10-year maturity, which we calculate to be 1.90%, includes a premium of 0.49%, and that the premium corresponds to an implied default probability of 7.8%, equal to the historical rate for senior unsecured debt, with a 40% expected rate of recovery. What spread would a bank need to finance this type of loan? To answer that question, we need to set a target level for bank return on equity, assuming that its expected ROE is the same as the expected return an investor would require of the bank's stock. To perform this calculation, we use the following parameters:

- a bank beta of around 1.25, which is the level observed, for example, by Fairness Finance for the year 2015 (based on 36 months of historical returns comprising a portfolio of equities issued by 39 listed European institutions);
- a German 10-year risk-free rate equal to 0.56% and a residual AAA premium of 0.17%;
- an equity market risk premium in the CAPM sense of 4.23% compared to the 10-year Bund yield.

The expected return on bank equity thus comes to 6.02%, excluding premia for optimism bias and for not taking the bank's own default risk into account.

Once the level of target ROE is set, we simply need to recreate the simplified income statement for a lending business with zero growth. Let us assume that the percentage of regulatory reserves relative to the outstanding level of intermediated lending (or bonds) held by the bank is 8% (6% to 10% inclusive). Then for a net return on equity of 6.02%, the bank would have to generate net profit margin of 48bp. As shown in the table below, by assuming an efficiency ratio of 60%, the bank spread relative to the 10-year Bund would be 3.25%, which is nearly 136bp higher than what the bond market would have required at end-2015. This gap would narrow to 95bp if we used a reserve ratio of 6%, and would expand to 177bp if we used a ratio of 10%.

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[32] Schwert, Michael, "Is Borrowing from Banks More Expensive than Borrowing from the Market?", Fisher College of Business Working Paper No. 2017-03-23, October 25, 2017, available at SSRN: https://ssrn.com/abstract=3059607

Targeted ROE and P&L

Outstanding loans	100.000
Equity	8.000
Indebtedness	92.000
Interest income	**3.814**
Interest expense	-0.920
Net interest income	**2.894**
Non interest expenses	-1.736
Efficiency ratio	60.0%
Earnings before loan losses	**1.158**
Cost of default **	-0.470
As a % of net interest income	-16.2%
Earnings before taxes	**0.688**
Taxes	-0.206
Net earnings	**0.481**
ROE	**6.02%**

*** 1/10th outstanding loans x default rate x (1- R)*

Target spread vs Bund

Contractual interest rate	3.81%
Risk-free rate (bund) rf	0.56%
Spread vs rf (bund)	**3.25%**
Contractual interest rate	3.81%
Expected loss as % of loans	-0.47%
Expected return before tax / loans	**3.34%**
Risk-free rate (bund) rf	0.56%
Expected spread net of default	**2.78%**
Risk-free rate (bund) rf	0.56%
Interbank spread vs rf	0.44%
Cost of debt	**1.00%**
Euribor 1 year as of 31/12/15	0.06%
Bund 1 year	-0.38%
Interbank spread	0.44%

Expected return on bank equity

Beta	1.25
Equity risk premium CAPM Π_R	4.23%
Risk-free rate 10 years (bund)	0.56%
Premium AAA Π_A	0.17%
$E(K_L)$	**6.02%**

Thus, if credit institutions were to subscribe a significant portion of the listed bonds, that would apparently be sufficient to explain the existence of a residual spread of 80bp relative to the bond market at end-2015[33] for BBB-rated bonds with a 10-year maturity (assuming that at that date, the 10-year default expectation is close to the long-term average). However, banks probably have a much smaller influence on non-investment grade spreads. That segment basically corresponds to the secondary market, unlike the investment grade segment, which corresponds to both the primary and secondary markets. It would be interesting to test this hypothesis by verifying that investment grade spreads usually provide additional remuneration over and above the default risk and systematic risk, whereas in the non-investment grade segment, spreads fluctuate more freely, since they don't necessarily cover the average levels of systematic and default risk.

In any case, if it could be confirmed that the investment grade segment is excessively remunerated, there would be a boon for individual investors and for any investor not subject to reserve requirements. The rapid development of crowdfunding platforms and the more widespread decline in banking intermediation may be clues to the existence of such a remuneration spread. The confirmation of the bond market's greater efficiency, both for the investor and for the issuer, would bolster the case that this financing channel's is useful as a substitute for bank intermediation as a way to avoid credit crunches and pave the way to recovery. This question is especially relevant now in the wake of the Great Recession of 2008-2009[34].

[33] If bank capital were inadequately remunerated by bond spreads, they could make up the difference with origination fees.

[34] Grjebine, Thomas, Szczerbowicz, Urszula, and Tripier, Fabien, "Corporate Debt Structure and Economic Recoveries", Banque de France Working Paper No. 646, October 2017, available at SSRN: https://ssrn.com/abstract=3057390

4 Conclusion

Our results corroborate those previously produced by the Merton model, but using an entirely different approach. Unlike so-called structural approaches, the one we use in this report is a direct descendant of the CAPM, meaning it operates in a risk-averse world. It is based on an intensity model that does not presuppose a particular method for estimating distance to default and which, for that reason, does not limit its scope to indebted companies. Furthermore, it constitutes a first concrete attempt to reconcile models for valuing stocks and bonds within the CAPM. It thus includes a forward-looking element to the extent that it relies on a model for estimating the implied cost of capital to calculate an expected risk premium. Fine-tuning the forward-looking element by adding a completely endogenous credit risk model, i.e. one also based on analyst forecasts, is a complementary line of inquiry, and one we began to explore in a previous article. That article used the cost of capital to tackle the question of forecasting risk[35], i.e. matching fixed costs with the margin on variable costs, which is the fundamental issue of default risk.

In order to deduce investors' expected default probabilities from credit spreads, we designed a broader model for calculating the implied cost of capital. Indeed, stocks and bonds are risk financial assets and we think that they need to be assessed jointly, in keeping with the CAPM. Valuing them separately is a purely theoretical exercise and highly artificial. Furthermore, keeping them separate appears to create market anomalies. If that proves to be the case, it would mean that investors are receiving a windfall. That may seem anachronistic in a modern financial economy, but it could be due to the fact that a regulated sector dominates credit in Europe. As a result, it would seem to be advantageous to continue investigating this market anomaly hypothesis, as a better understanding of it and the advantages that could be gained from it are both topics of research in and of themselves.

[35] Clère, Roland, "After Modigliani, Miller and Hamada; A New Way to Estimate Cost of Capital?" (Après Modigliani, Miller et Hamada: une nouvelle façon d'estimer le coût du capital?), November 23, 2016, available at SSRN: https://ssrn.com/abstract=2868702

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