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8 April 2018

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April 8, 2018

Abstract

In many decisions under uncertainty, there are technological constraints on the acts an agent can perform and on the events she can observe. To model this, we assume that the set $\mathcal{S}$ of possible states of the world and the set $\mathcal{X}$ of possible outcomes each have a topological structure. The only feasible acts are continuous functions from $\mathcal{S}$ to $\mathcal{X}$, and the only observable events are regular open subsets of $\mathcal{S}$. We axiomatically characterize Subjective Expected Utility (SEU) representations of conditional preferences over acts, involving a continuous utility function on $\mathcal{X}$ (unique up to positive affine transformations), and a unique Borel probability measure on $\mathcal{S}$, along with an auxiliary apparatus called a *liminal structure*, which describes the agent's imperfect perception of events. We also give other SEU representations, which use residual probability charges or compactifications of the state space.
Keywords: Subjective expected utility; topological space; technological feasibility; continuous utility; regular open set; Borel measure.
JEL classification: D81.

Natura non facit saltum. —Linnaeus

1 Introduction

Economic decisions under uncertainty often face technological constraints. Consider a farmer who must plant crops in the early spring, without knowing the meteorological conditions for the rest of the year. The crop yields of his various planting strategies are thus uncertain at the moment of choice. But slight variations of the meteorological conditions will only result in slight variations in yields. The constraints of the agricultural technology imply that the only strategies available to the farmer are those where crop yields depend *continuously* on the unpredictable meteorological conditions.

*We thank Jean Baccelli, Jean-Marc Tallon and Peter Wakker for their detailed comments. We also thank Sophie Bade, Denis Bouyssou, Alain Chateauneuf, Eric Danan, Franz Dietrich, Adam Dominiak, Jürgen Eichberger, Spyros Galanis, Ani Guerdjikova, Simon Grant, Andrew Mackenzie, Michael Mandler, Efe Ok, Emre Ozdenoren, Michael Richter, Klaus Ritzberger and Jingyi Xue for many helpful suggestions.

[†]THEMA, Université de Cergy-Pontoise, supported by Labex MME-DII (ANR11-LBX-0023-01).

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Continuity constraints manifest in many other decision problems under uncertainty; in particular, they arise in most economic activities which depend upon natural resource extraction, weather conditions, or any other interaction with unpredictable features of the natural environment. They also arise in medical decisions, where the uncertainty concerns the patient's medical condition, and the outcome is her prognosis. Anthropogenic climate change generates a plethora of such decision problems; there is uncertainty about the values of many parameters in climate models, which leads to uncertainty about the response of weather patterns (e.g. temperature, rainfall, floods, droughts) to rising CO_2 concentrations. There is also uncertainty about the social and economic impact of these weather patterns, as well as the proposed policies to reduce CO_2 emissions. Finally, there is hour-to-hour uncertainty about the electricity output of solar and wind-power facilities. But in all these cases, the outcome varies continuously with the unknown variables.

Continuity constraints also arise in financial decisions. For example, the income stream arising from an individual's investment in education or a firm's investment in physical capital is a continuous function of future economic conditions. The value of a portfolio is a linear combination of the values of its constituent assets. In most financial derivatives (e.g. futures, options), the payoff for both buyer and seller is a continuous function of the price of the underlying assets. In most insurance contracts, the indemnity is a continuous function of the loss. Finally, the future real value of a savings instrument is a continuous function of future real interest rates. In these examples, continuity restrictions can be interpreted as a kind of market incompleteness.

Feasibility considerations constrain not only the possible actions, but also the information available to agents. The limitations of her measurement technology restrict the events that an agent can observe and use to update her beliefs. Think of a measurement instrument as a device which converts each state of the world into a "signal". Such devices typically have three properties. First, although the set of possible states of the world may be infinite (even a continuum), the set of possible signals is finite. (Even if the set of signals is itself a continuum, a human user can only discriminate finitely many distinct values.) Second, the output is usually *continuous*: a small variation in the underlying state of the world leads to a small (or even no) variation in the signal generated. Third, the output is often *ambiguous*: the same state of the world could generate more than one signal — especially if the state lies on the boundary between the domains of two different signals. For example, consider a digital thermometer which displays temperature to the nearest degree Celsius. If the temperature is in the interval $(23.5, 24.5)$, then the thermometer will say "24" (assuming no measurement error). If the temperature is in $(24.5, 25.5)$, then the thermometer will say "25". But what if the temperature is exactly $24.5°$ C? In this case, the thermometer could say either "24" or "25". In this paper, we will suppose that agents receive information through devices of this form.

The Subjective Expected Utility (SEU) model is the standard paradigm to describe decision-making under uncertainty. A key feature of the classic axiomatic foundations of Savage (1954) is that the agent has preferences over *all* possible functions from states into outcomes, and can condition on *all* subsets of the state space. This makes sense in decision problems where the state space has a *discrete* topology (e.g. bets on coin flips, urn experiments, sports games, or Arrow-Debreu economies). One *could* apply the Savage approach to technologically constrained decision problems, but this would require the agent to rank infeasible acts and condition on unobservable events; this would undermine both the normative and the descriptive content of the preference relation, the axioms, and the resulting SEU representation. At a normative level, it is ill-conceived and possibly misleading to formulate preferences over infeasible acts, or to condition on unobservable events; hence we should be reluctant to apply the Savage axioms to such preferences. These axioms are supposed to impose some "internal consistency" on preferences. But why should preferences over *feasible* acts be consistent with, and sometimes even determined by, preferences over *infeasible* acts? At a descriptive level, it is impossible by definition to observe an agent's preference over infeasible acts, or her preferences conditional on unobservable events. Such acts and events might still play a role in thought experiments. But since they are impossible to properly incentivize, we question the empirical meaning of such preferences and their relevance to the elicitation of utility and beliefs. For these reasons, technological constraints make it desirable to depart from the Savage framework and restrict preferences to feasible acts and observable events.

This paper studies decision-making under uncertainty with technological constraints, and axiomatically characterizes SEU representations of conditional preferences in such an environment. The consequences of the decision range over a topological space of *outcomes*; these may be crop yields, health status, production levels, income streams, or consumption bundles. The underlying uncertainty is represented by a topological space of *states of the world*; this encodes all the meteorological, physiological, geophysical, or financial variables on which the outcome (continuously) depends. The feasible alternatives are given by a set of continuous functions, or *acts*, from the state space onto the outcome space; these could be production plans, medical interventions, climate policies, financial portfolios, or insurance contracts. We suppose that the agent acquires information about the state through measurement devices like the digital thermometer described above. We will present three different models of such devices; one in terms of multifunctions (Section 2), one in terms of almost-everywhere equivalence (Section 5), and one in terms of stochastic functions (Section 6). But in all three models, a measurement device determines a partition of the statespace into *regular open subsets*. These are the "observable events" for the agent; they are the basic unit of information that can be obtained by measuring meteorological conditions or performing medical tests. We will suppose that the agent can form *conditional preferences* over acts after observing such an event. We will show that these conditional preferences satisfy certain axioms if and only if they can admit an SEU representation.

Not *every* decision under uncertainty exhibits these sorts of topological constraints. But many do, often in important practical contexts. Does the axiomatization of SEU depend upon *ignoring* such constraints? Our results show that it does not. But these constraints do create some technical difficulties. For example, Savage's axioms (e.g. the *Sure Thing Principle*) depend on the ability to splice any two acts on any bipartition of the state space. Furthermore, Savage obtains the subjective probability measure and utility function by restricting preferences to two-valued acts and finitely-valued acts respectively. But both spliced acts and finitely-valued acts are typically discontinuous, and hence inadmissible in our framework. Thus, we must depart from Savage, and use a very different axiomatization.

Despite these obstacles, we obtain several SEU representations. In these representations, utility is a continuous function; thus, similar outcomes yield similar utility levels. This makes our representations particularly relevant to applications in economics and finance, which usually take continuity for granted (Gollier, 2001). Utility is is unique up to positive affine transformations. But the representation of beliefs depends on the topology of the state space. Our first two representations (Theorems 1 and 2) are "classical": beliefs take the form of a probability measure (called a *residual probability charge*), and are updated via Bayes rule as the agent acquires more information. Theorem 1 applies to *compact* state spaces, while Theorem 2 applies to *Baire* spaces. We then introduce the more informative *liminal* SEU representations (Theorems 3 and 4); in this case, beliefs consist not only of a Borel probability measure, but also a *liminal structure*, with which the agent compensates for her informational constraints. These structures provide dynamically consistent, consequentialist updating rules for the Borel probability measures that generalize the classical Bayes rule. Assuming only a *locally* compact state space, we obtain *compactification* SEU representations (Theorems 5 and 6); here, beliefs are given by a Borel probability measure and a liminal structure on a compactification of the state space.

This paper is the second in a series of three papers on similar themes. The first of these (Pivato and Vergopoulos, 2018a) developed an SEU representation for conditional preferences with *imperfect perception*. Although that paper used a topological framework similar to the present paper, its focused more on the constraints on the agent's *information*, and made no specific assumptions about the topology of the state space.

The present paper uses a different model of imperfect perception than the one in Pivato and Vergopoulos (2018a) (although both involve regular open sets). It allows agents to acquire more information, and uses stronger topological assumptions, but in exchange, it obtains more detailed, concrete and informative representation theorems. The third paper (Pivato and Vergopoulos, 2018b) extends these results to *piecewise* continuous acts.

The paper is organized as follows. Section 2 introduces our model of imperfect perception, while Section 3 introduces our model of conditional preferences under uncertainty. Section 4 introduces the six axioms used in all our results. Section 5 presents our first two SEU representations, which assume the state space is either compact Hausdorff or Baire, and which represent the agent's beliefs with a *residual probability charge* (which assigns probability zero to all meager sets). Sections 6 presents the next two SEU representations, which assume the state space is either compact Hausdorff or normal; in this case, the agent's beliefs are represented by a Borel probability measure and a *liminal structure*, which describes how the agent copes with imperfect perception. Section 7 provides similar SEU representations for *locally compact* state spaces, via their compactifications. Section 8 reviews prior literature. All the proofs are in the Appendix.

2 Regular sets and imperfect perception

We will first informally summarize the main ideas of this section. Let $g : \mathcal{S} \longrightarrow \mathbb{R}$ be a function representing some physical quantity (e.g. temperature). Suppose the agent measures this quantity using some device. The device cannot literally display a "real number" as output; it can only display a finite number of digits of precision. (Even if it *could* display an infinite number of digits, the agent couldn't read them.) To be concrete, suppose $g : \mathcal{S} \longrightarrow [-50, 50]$ represents a temperature which can range from $-50°$C to $50°$C. Suppose the agent uses a digital thermometer, which displays temperature to the nearest tenth of a degree. We could represent this by a function $f : \mathcal{S} \longrightarrow \{-500, -499, \ldots, 499\}$, where $f(s) = n$ if $\frac{n}{10} < g(s) < \frac{n+1}{10}$. Equivalently, we could represent it by a partition of $\mathcal{S}$ into a thousand subsets, namely the preimage sets $f^{-1}\{-500\}, f^{-1}\{-499\}, \ldots, f^{-1}\{499\}$.

Of course, we must make a choice about the value of $f(s)$ when $g(s) = \frac{n}{10}$ for some integer n. In such a "knife-edge" case, we could define $f(s) = n$ or $f(s) = n - 1$; both choices are equally reasonable. Typically, models make this choice in an arbitrary way. But it is often more realistic to suppose that f takes *both* values in a knife-edge case, because the output of the device is *indeterminate* —it could go either way. In this case, f is no longer a function, but a *multifunction*. Likewise, the preimages $f^{-1}\{-500\}, \ldots, f^{-1}\{499\}$ are no longer disjoint, so they do not define a partition, but rather, a *covering* of $\mathcal{S}$.

This leads to a more general and realistic model of information acquisition, in which measurement devices are represented as *multifunctions* (or equivalently, *coverings*) rather than ordinary functions (or equivalently, partitions). The greater the preponderance of multivalued outputs in a multifunction (i.e. the more overlaps in the corresponding covering), the less informative the corresponding measurement device is. Ordinary functions are just *single-valued* multifunctions, and partitions are coverings by *disjoint* sets; these correspond to the *most informative* measurement devices. Thus, it might seem obvious that an agent should always acquire information using functions/partitions. But in the topological setting of this paper, these may not be available. Physical quantities (like temperature) correspond to *continuous* functions on the state space, and the devices which measure them correspond to *upper hemicontinuous* multifunctions, which are *never* single-valued (unless $\mathcal{S}$ is disconnected). A maximally informative measurement device corresponds to a *maximally decisive* multifunction —in effect, a multifunction which is "as single-valued as possible", while remaining upper hemicontinuous. As we shall soon see, such a multifunction defines a covering of the state space by closed sets that overlap only on their boundaries. Their interiors are regular open sets, and determine a *regular partition* of $\mathcal{S}$.

Regular sets and partitions. A subset $\mathcal{R} \subseteq \mathcal{S}$ is ***regular*** if $\mathcal{R} = \mathrm{int}[\mathrm{clos}(\mathcal{R})]$.[1] Let $\mathfrak{R}(\mathcal{S})$ be the collection of all regular subsets of $\mathcal{S}$. For any $\mathcal{Q}, \mathcal{R} \in \mathfrak{R}(\mathcal{S})$, we define $\mathcal{Q} \vee \mathcal{R} := \mathrm{int}[\mathrm{clos}(\mathcal{Q} \cup \mathcal{R})]$. This is the smallest regular set containing both $\mathcal{Q}$ and $\mathcal{R}$. For example, if $\mathcal{S} = \mathbb{R}$, then $(0, 1) \vee (1, 2) = (0, 2)$. For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, let $\neg\mathcal{R}$ be the interior of $\mathcal{S} \setminus \mathcal{R}$ —another regular subset. The set $\mathfrak{R}(\mathcal{S})$ is a Boolean algebra under the operations $\vee$, $\cap$, and $\neg$ (Fremlin, 2004, §314P-314Q).

[1] Sometimes this is called a ***regular open*** subset.

A *regular partition* of $\mathcal{S}$ is a collection $\mathfrak{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots \mathcal{P}_N\}$ where $\mathcal{P}_1, \ldots, \mathcal{P}_N \subset \mathcal{S}$ are disjoint regular sets such that $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_N = \mathcal{S}$ —equivalently, such that $\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_N$ is dense in $\mathcal{S}$. For example, suppose $\mathcal{S} = \mathbb{R}^2$. For all $\mathbf{e} \in \{(1,1), (1,-1), (-1,1), (-1,-1)\}$, let $\mathcal{P}_{\mathbf{e}} := \{(x,y) \in \mathbb{R}^2;\ e_1\, x > 0 \text{ and } e_2\, y > 0\}$. Then $\{\mathcal{P}_{1,1}, \mathcal{P}_{1,-1}, \mathcal{P}_{-1,1}, \mathcal{P}_{-1,-1}\}$ is a regular partition of $\mathbb{R}^2$. We will use regular partitions to represent an agent's imperfect perception of her environment. To explain this, we will first make an interesting connection between regular partitions and upper hemicontinuous multifunctions.

Multifunctions. Let $\mathcal{S}$ and $\mathcal{N}$ be sets. Let $\wp(\mathcal{N})$ be the power set of $\mathcal{N}$. A *multifunction* (or *correspondence*) from $\mathcal{S}$ to sN is a function $f : \mathcal{S} \longrightarrow \wp(\mathcal{N})$ such that $f(s) \neq \emptyset$ for all $s \in \mathcal{S}$. We indicate this by writing "$f : \mathcal{S} \rightrightarrows \mathcal{N}$." Any function can be regarded as a multifunction in the obvious way. Conversely, if f is a multifunction and $f(s)$ is a singleton for all $s \in \mathcal{S}$, then we regard f as an ordinary function from $\mathcal{S}$ to $\mathcal{N}$, in the obvious way.

For example, let $\{\mathcal{K}_1, \ldots, \mathcal{K}_N\}$ be a *covering* of $\mathcal{S}$ —that is, a collection of subsets of $\mathcal{S}$, such that $\mathcal{S} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N$. Let $\mathcal{N} = [1 \ldots N]$, and define $f : \mathcal{S} \rightrightarrows \mathcal{N}$ by $f(s) := \{n \in \mathcal{N};\ s \in \mathcal{K}_n\}$; then f is a multifunction from $\mathcal{S}$ to $\mathcal{N}$. Conversely, *any* multifunction from $\mathcal{S}$ to $\mathcal{N}$ can be obtained from a covering of $\mathcal{S}$ in this way. Note that f is a *function* if and only if $\mathcal{K}_1, \ldots, \mathcal{K}_N$ do not overlap —that is, if and only if $\{\mathcal{K}_1, \ldots, \mathcal{K}_N\}$ is a *partition* of $\mathcal{S}$. As this example shows, a multifunction into a finite set can be thought of as a "generalized partition", where overlaps are allowed.

Let $\mathcal{S}$ and $\mathcal{N}$ be topological spaces, with $\mathcal{N}$ compact. The *graph* of a multifunction $f : \mathcal{S} \rightrightarrows \mathcal{N}$ is the set $\{(s,n);\ \ s \in \mathcal{S} \text{ and } n \in f(s)\}$. We say f is *upper hemicontinuous* (UHC) if this graph is a closed subset of $\mathcal{S} \times \mathcal{N}$. If $f : \mathcal{S} \longrightarrow \mathcal{N}$ is a function, and we regard f as a multifunction, then f is UHC if and only if it is continuous. So upper hemicontinuity generalizes continuity. Suppose $\mathcal{N} = [1 \ldots N]$ with the discrete topology, and $f : \mathcal{S} \rightrightarrows \mathcal{N}$ arises from a covering $\{\mathcal{K}_1, \ldots, \mathcal{K}_N\}$ as in the previous paragraph. Then f is UHC if and only if $\mathcal{K}_1, \ldots, \mathcal{K}_N$ are *closed* subsets of $\mathcal{S}$.

Let $f, g : \mathcal{S} \rightrightarrows \mathcal{N}$ be two multifunctions. We will say that g is *more decisive* than f if $g(s) \subseteq f(s)$ for all $s \in \mathcal{S}$. We indicate this by writing "$g \subseteq f$"; we write"$g \subset f$" if $g \subseteq f$ and $g \neq f$. Let $\mathcal{F}$ be a class of multifunctions, and let $f \in \mathcal{F}$. We say f is *maximally decisive* in $\mathcal{F}$ if there does not exist any $g \in \mathcal{F}$ such that $g \subset f$.

Multifunctions vs. regular partitions. There is a natural correspondence between the regular partitions of a topological space $\mathcal{S}$ and the maximally decisive UHC multifunctions from $\mathcal{S}$ into finite sets. To be more precise, for any $N \in \mathbb{N}$, let $\mathrm{RgPrt}(\mathcal{S}, N)$ be the set of all N-element regular partitions of $\mathcal{S}$ (indexed by $[1 \ldots N]$). Meanwhile, let $\mathrm{UHC}(\mathcal{S}, N)$ be the set of all UHC multifunctions from $\mathcal{S}$ to $[1 \ldots N]$ (with the discrete topology), and let $\mathrm{UHC}^*(\mathcal{S}, N)$ be the set of maximally decisive elements of $\mathrm{UHC}(\mathcal{S}, N)$. The next result says that there is a canonical bijection between $\mathrm{RgPrt}(\mathcal{S}, N)$ and $\mathrm{UHC}^*(\mathcal{S}, N)$.

Proposition 1 *Let $\mathcal{S}$ be a topological space and let $N \in \mathbb{N}$.*

(a) *Let $f \in \mathrm{UHC}(\mathcal{S}, N)$, and let $\{\mathcal{K}_1, \ldots, \mathcal{K}_N\}$ be the corresponding covering of $\mathcal{S}$ by closed subsets. For all $n \in [1 \ldots N]$, let $\mathcal{R}_n := \mathrm{int}(\mathcal{K}_n)$. If f is maximally decisive in*

$\mathrm{UHC}(\mathcal{S}, N)$*, then* $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\} \in \mathrm{RgPrt}(\mathcal{S}, N)$*.*

(b) *Let* $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\} \in \mathrm{RgPrt}(\mathcal{S}, N)$*. For all* $n \in [1 \ldots N]$*, let* $\mathcal{K}_n := \mathrm{clos}(\mathcal{R}_n)$*; then* $\{\mathcal{K}_1, \ldots, \mathcal{K}_N\}$ *is a covering of* $\mathcal{S}$ *. If* $f : \mathcal{S} \rightrightarrows [1 \ldots N]$ *is the corresponding multifunction; then* f *is maximally decisive in* $\mathrm{UHC}(\mathcal{S}, N)$*.*

(c) *The mappings* $\mathrm{UHC}^*(\mathcal{S}, N) \longrightarrow \mathrm{RgPrt}(\mathcal{S}, N)$ *and* $\mathrm{RgPrt}(\mathcal{S}, N) \longrightarrow \mathrm{UHC}^*(\mathcal{S}, N)$ *described in parts* (a) *and* (b) *are bijections, and inverses to one another.*

Imperfect perception. We will interpret multifunctions as models of *imperfect perception*. A multifunction $f : \mathcal{S} \rightrightarrows [1 \ldots N]$ represents an *observation* that the agent could make to learn something about the state of nature —say, using some instrument or device. If f was an arbitrary multifunction, then arbitrarily small perturbations of the state could cause wild variations in the output; this would make the device essentially useless in real life. *Upper hemicontinuity* simply means that the measurement device is insensitive to small perturbations. The fact that f can be multivalued indicates that, for some states, the observation is *ambiguous* —for example, the device may produce one of several outputs in arbitrary and unpredictable way. Clearly, the agent would like to minimize this kind of erratic behaviour. We will suppose the favorable case where any maximally decisive measurement is available. Therefore, the agent only considers these measurements.

For example, recall the function $g : \mathcal{S} \longrightarrow [-50, 50]$ representing temperature, which the agent measures using a digital thermometer, displaying temperature to the nearest tenth of a degree Celsius. This defines a multifunction $f : \mathcal{S} \rightrightarrows \{-500, -499, \ldots, 499\}$, where $n \in f(s)$ if $n/10 \leq g(s) \leq (n+1)/10$. If g is *continuous*, then it is easily verified that f is upper hemicontinuous. If g is also *open*, then it is easily verified that f is maximally decisive; in this case, f determines a regular partition of $\mathcal{S}$ into 1000 regular subsets, corresponding to the 1000 distinct intervals of 0.1 degrees Celsius between $-50°$C and $50°$C. To say that g is *continuous* is just to say that arbitrarily small perturbations of the state do not result in wild temperature changes. To say that g is *open* is just to say that at any state, any small variation in temperature can be obtained by a small perturbation of the state. Both of these are plausible assumptions.

This example illustrates the general case; almost any physical quantity of interest can be represented as a continuous function from the state space into some interval of real numbers; in most cases, this function will also be open. Thus, any measurement of this quantity by a digital measurement device can be described by an upper hemicontinuous multifunction from $\mathcal{S}$ into a finite set; in most cases, this multifunction will also be maximally decisive, and hence, yield a regular partition of $\mathcal{S}$.

It might seem that this analysis does not apply to purely *analog* measurement devices, but this is an misconception. Human eyes cannot perceive the output of an analog measurement device with infinite precision. So for all intents and purposes, their output might as well be digital. For example, a human eye looking at an (analog) mercury thermometer probably cannot discriminate any temperature difference smaller than 0.5 degrees Celsius; the resulting partition of the state space is even cruder than the one obtained from the digital thermometer described above.

Regular partitions (or equivalently, maximally decisive UHC multifunctions) arise frequently in economics as the output of *parametrized optimization* problems. Let $\mathcal{S}$ be a topological space, let $\mathcal{N}$ be a finite set, and let $u : \mathcal{S}\times\mathcal{N}\longrightarrow\mathbb{R}$ be continuous. For all $s \in \mathcal{S}$, let $f(s) := \{m \in \mathcal{N};\ \ u(s,m) \geq u(s,n)$ for all $n \in \mathcal{N}\}$. This describes a parametrized optimization problem, where $\mathcal{S}$ is the parameter space, $\mathcal{N}$ is the feasible set, and for any $s \in \mathcal{S}$, $f(s)$ is the set of optimal solutions given parameter value s. It is easy to see that f is UHC —but it might not be maximally decisive. So, suppose $\mathcal{S}$ is an open subset of $\mathbb{R}^N$, for all for all distinct $n, m \in \mathcal{N}$, define $v_{n,m} : \mathcal{S}\longrightarrow\mathbb{R}$ by $v_{n,m}(s) := u(s,n) - u(s,m)$. Suppose that $v_{n,m}$ is differentiable (which is automatically true if u is differentiable in the $\mathcal{S}$ coordinate), and furthermore, suppose that the gradient $\nabla v_{n,m}$ is everywhere nonzero. Then it is easily verified that f is maximally decisive. An observer monitoring the agent's optimization behaviour can indirectly learn about the parameter s via f, even if she cannot directly observe s. The information the observer can thereby acquire corresponds to a regular partition of the parameter space $\mathcal{S}$.

An important example arises in Bayesian games. Consider an M-player normal form game, with (finite) strategy sets $\mathcal{A}_1, \ldots, \mathcal{A}_M$. Let $\mathcal{T}_1$ be an open subset of $\mathbb{R}^N$, representing the space of possible *types* of Player 1. Suppose Player 1 has a type-dependent payoff function $u : \mathcal{T}_1 \times \mathcal{A}_1 \times \cdots \times \mathcal{A}_M\longrightarrow\mathbb{R}$, differentiable in the $\mathcal{T}_1$ coordinate. Let $\boldsymbol{\Delta}_{-1} := \Delta(\mathcal{A}_2) \times \cdots \times \Delta(\mathcal{A}_M)$ be the space of mixed strategy profiles for all the other players, and let $\mathcal{S} := \mathcal{T}_1 \times \boldsymbol{\Delta}_{-1}$. For any $t \in \mathcal{T}_1$, $\boldsymbol{\delta} \in \boldsymbol{\Delta}_{-1}$ and $a \in \mathcal{A}_1$, let $u(a, t, \boldsymbol{\delta})$ be Player 1's *expected utility*, defined in the obvious way; this yields a continuous function $u : \mathcal{A}_1\times\mathcal{S}\longrightarrow\mathbb{R}$. Thus, Player 1's *best response correspondence* is a UHC multifunction $f : \mathcal{S} \rightrightarrows \mathcal{A}_1$. In particular, for any $\boldsymbol{\delta} \in \boldsymbol{\Delta}_{-1}$, we get a UHC multifunction $f_{\boldsymbol{\delta}} : \mathcal{T}_1 \rightrightarrows \mathcal{A}_1$. For any $a, b \in \mathcal{A}_1$, define $v^{\boldsymbol{\delta}}_{a,b} : \mathcal{T}_1\longrightarrow\mathbb{R}$ by $v^{\boldsymbol{\delta}}_{a,b}(t) := u(a, t, \boldsymbol{\delta}) - u(b, t, \boldsymbol{\delta})$. This function is differentiable. If $\nabla v_{a,b}$ is everywhere nonzero, then $f_{\boldsymbol{\delta}}$ is maximally decisive.

Other players can learn about Player 1's type by observing her best responses. The information they can thereby acquire is exactly like the digital measurement devices we considered earlier: it corresponds to a regular partition of Player 1's type space.

Regular partitions also appear in mathematics (e.g. Voronoi partitions), statistical physics (e.g. phase diagrams), and even international relations (e.g. territorial boundaries). In light of Proposition 1 and these examples, we will suppose that an agent's perception of her environment is described by some regular partition $\mathfrak{P}$ of the state space. When she "observes" the state using $\mathfrak{P}$, the outcome of that observation is a regular set. Thus, regular sets will be the basic unit of information available to agents in our model.

3 Acts and preferences

Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces. Elements of $\mathcal{S}$ are called *states of the world* and describe the various possible resolutions of uncertainty. Elements of $\mathcal{X}$ are called *outcomes* and represent the various possible consequences of decisions. We will assume $\mathcal{X}$ is connected.

Acts. Like Savage, we will suppose that the agent can choose from a menu of *acts*, where each act is a function from the state space onto the outcome space. This function describes the outcome that would result from the choice of this act at each possible state of the world. Unlike Savage, we will assume only *continuous* acts are feasible.

Recall that a subset $\mathcal{Y} \subseteq \mathcal{X}$ is ***relatively compact*** if its closure $\mathrm{clos}(\mathcal{Y})$ is compact. (It follows that any continuous, real-valued function on $\mathcal{X}$ is bounded when restricted to $\mathcal{Y}$.) For example, if $\mathcal{X}$ is a metric space, then $\mathcal{Y}$ is relatively compact if and only if $\mathcal{Y}$ is a bounded subset of $\mathcal{X}$. A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is ***bounded*** if its image $\alpha(\mathcal{S})$ is relatively compact in $\mathcal{X}$. If $\mathcal{X}$ is a metric space, then this agrees with the usual definition of "bounded". But this definition makes sense even if $\mathcal{X}$ is nonmetrizable. Let $\mathcal{C}(\mathcal{S}, \mathcal{X})$ be the set of all continuous functions from $\mathcal{S}$ into $\mathcal{X}$, and let $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$ be the set of all *bounded* continuous functions from $\mathcal{S}$ into $\mathcal{X}$. Unlike Savage, we will assume only *bounded* acts are feasible. Meanwhile, our SEU representations will have potentially *unbounded* utility functions, whereas Savage's utility functions were bounded.[2]

There may be additional feasibility restrictions on acts, beyond continuity and boundedness. Thus, we introduce an exogenously given subset $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$; this is the set of *feasible acts*. If technological constraints only entail continuity and boundedness, then $\mathcal{A} = \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. But in general, $\mathcal{A}$ could be much smaller. For instance, in some economic models, it may be appropriate to define $\mathcal{A}$ to be the set of all infinitely differentiable functions from $\mathcal{S}$ to $\mathcal{X}$. However, the collection $\mathcal{A}$ cannot be *too* small; it must be large enough to satisfy structural condition (Rch) below, and must contain all constant acts; these represents *riskless* alternatives. The inclusion of such acts in $\mathcal{A}$ means that we can risklessly obtain any outcome by a feasible act.

Conditional preference structures. Savage (1954) started from a preference order on the set of unconditional acts. He then obtained conditional preferences via axiom P2 (the *Sure Thing Principle*). Axiom P2 assumes that, for any two feasible acts α and β, and any event $\mathcal{B}$, the "spliced" act $\alpha_{\mathcal{R}}\beta$ (which is equal to α on $\mathcal{R}$ and to β on the complement $\mathcal{R}^{\complement}$) is also feasible. But such "spliced" acts are often discontinuous, hence, inadmissible in our framework. So instead of defining conditional preferences *implicitly* via P2, we must assume they exist explicitly. But we will only assume that these preferences can rank *feasible acts*, and we only assume preferences conditional on *observable* events. Thus, in terms of its primitive behavioral data, our model is not directly comparable to the Savage (1954) theory: while Savage assumed a *single* preference order on the universal domain of acts, our approach relies on a *collection* of preference orders on a more restrictive domain. But compared to other conditional versions of SEU (e.g. Ghirardato, 2002), our approach requires less data, both in terms of the number of preference orders and their domain.

For any regular subset $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, and any act $\alpha \in \mathcal{A}$, let $\alpha_{\upharpoonright\mathcal{R}}$ denote the restriction of α to a function on $\mathcal{R}$. Let $\mathcal{A}(\mathcal{R}) := \{\alpha_{\upharpoonright\mathcal{R}};\ \alpha \in \mathcal{A}\}$ be the set of acts conditional upon $\mathcal{R}$.

[2] We need bounded acts for the same reason that Savage needs a bounded utility function u: to make the *composite* function $u \circ \alpha$ bounded so that its expected value can be computed without any technical complications. In a companion paper (Pivato and Vergopoulos, 2018a), we introduce a strengthening of our axiomatic framework which makes the utility function u bounded, thereby allowing us to consider *unbounded* acts. The axiomatic framework of the present paper could likewise be modified to allow for unbounded acts. But for reasons of space, we will not develop these variants here.

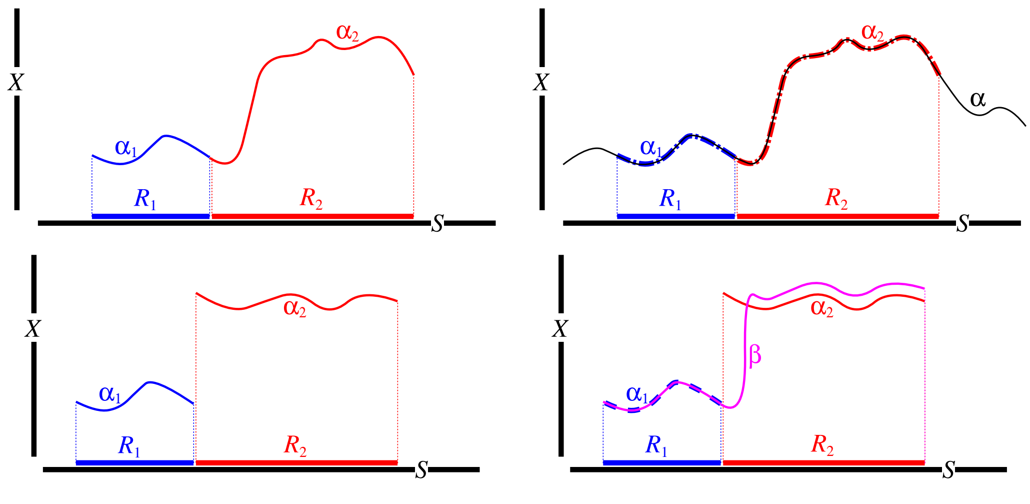

Figure 1: *Top row.* α_1 is compatible with α_2. *Bottom row.* The richness condition.

Even when $\mathcal{A} = \mathcal{C}(\mathcal{S}, \mathcal{X})$, the set $\mathcal{A}(\mathcal{R})$ will typically *not* be $\mathcal{C}(\mathcal{R}, \mathcal{X})$; not every continuous function on $\mathcal{R}$ extends to a continuous function on $\mathcal{S}$. For all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, let $\succeq_{\mathcal{R}}$ be a preference order on $\mathcal{A}(\mathcal{R})$. We interpret $\succeq_{\mathcal{R}}$ as the *conditional preferences* over $\mathcal{A}(\mathcal{R})$ of an agent who has observed the event $\mathcal{R}$. We will therefore refer to the system $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ as a *conditional preference structure*; this will be the primitive data of the model. Our goal is to axiomatically characterize an SEU representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$.

The richness condition. As already noted, the restriction to continuous acts means that we cannot rely on "spliced" acts the way that Savage did. Instead, we will require the set $\mathcal{A}$ of feasible acts to satisfy a "Richness" condition with respect to the conditional preference structure $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$. Let $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}(\mathcal{S})$ be disjoint regular subsets of $\mathcal{S}$. For any $\alpha_1 \in \mathcal{A}(\mathcal{R}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{R}_2)$, say that α_1 and α_2 are ***compatible*** if there is some $\alpha \in \mathcal{A}$ with $\alpha_{\upharpoonright\mathcal{R}_1} = \alpha_1$ and $\alpha_{\upharpoonright\mathcal{R}_2} = \alpha_2$. We need $\mathcal{A}$ to satisfy the following condition:

(Rch) For any disjoint regular subsets $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}(\mathcal{S})$, and any $\alpha_1 \in \mathcal{A}(\mathcal{R}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{R}_2)$, there is an act $\beta_2 \in \mathcal{A}(\mathcal{R}_2)$ which is compatible with α_1, such that $\alpha_2 \approx_{\mathcal{R}_2} \beta_2$.

Thus, the values of an act on a regular set $\mathcal{R}_1$ do not restrict the indifference class of that act conditional upon the disjoint regular set $\mathcal{R}_2$, in spite of the continuity requirement on feasible acts. If there is a "gap" between $\mathcal{R}_1$ and $\mathcal{R}_2$ in $\mathcal{S}$, then (Rch) is not very restrictive; often, *every* element of $\mathcal{A}(\mathcal{R}_2)$ is compatible with α_1. The nontrivial case of (Rch) is when $\mathcal{R}_1$ and $\mathcal{R}_2$ are "touching" –e.g. when $\mathcal{R}_1 = \neg\mathcal{R}_2$. In particular, (Rch) provides a weak version of Savage's act splicing: For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, and any $\alpha, \beta \in \mathcal{A}$, there is some $\gamma \in \mathcal{A}$ that is equal to α on $\mathcal{R}$ and *indifferent* to $\beta_{\upharpoonright\neg\mathcal{R}}$ conditional on $\neg\mathcal{R}$. (Rch) is also similar to *solvability*, a condition often used in axiomatizations of additive utility.

$\mathcal{A}$ need not contain *all* bounded continuous functions from $\mathcal{S}$ to $\mathcal{X}$, as long as it satisfies (Rch) and contains all constant acts. For example, suppose $\mathcal{S}$ and $\mathcal{X}$ are differentiable manifolds (e.g. open subsets of Euclidean spaces $\mathbb{R}^N$ and $\mathbb{R}^M$, for some $N, M \geq 1$), and let $\mathcal{A}$ be the set of all *differentiable* functions from $\mathcal{S}$ to $\mathcal{C}$; then a conditional preference structure on $\mathcal{A}$ can easily satisfy (Rch) along with our other axioms.[3] Alternatively, let $\mathcal{S}$ and $\mathcal{X}$ be metric spaces, let $c \in (0,1]$, and let $\mathcal{A}$ be the set of all *c-Hölder-continuous* functions from $\mathcal{S}$ to $\mathcal{X}$; then (Rch) is easily satisfied. Or, let $\mathcal{S}$ be a bounded interval in $\mathbb{R}$, let $\mathcal{X}$ be a path-connected metric space, and let $\mathcal{A}$ be the set of all continuous functions from $\mathcal{S}$ into $\mathcal{X}$ having *bounded variation*; then again (Rch) is easily satisfied. But if $\mathcal{S}$ and $\mathcal{X}$ are open subsets of Euclidean spaces, and $\mathcal{A}$ is a set of *analytic* functions from $\mathcal{S}$ to $\mathcal{X}$ (e.g. polynomials), then a conditional preference structure on $\mathcal{A}$ *cannot* satisfy (Rch).

4 Axioms

Throughout the paper, we will assume that each order $\succeq_{\mathcal{R}}$ in the conditional preference structure $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is *complete* (for any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$, at least one of $\alpha \succeq_{\mathcal{R}} \beta$ or $\beta \succeq_{\mathcal{R}} \alpha$ holds), *transitive* (for any $\alpha, \beta, \gamma \in \mathcal{A}(\mathcal{R})$, if $\alpha \succeq_{\mathcal{R}} \beta$ and $\beta \succeq_{\mathcal{R}} \gamma$, then $\alpha \succeq_{\mathcal{R}} \gamma$), and *nontrivial* (there exist $\alpha, \beta \in \mathcal{A}(\mathcal{R})$ such that $\alpha \succ_{\mathcal{R}} \beta$).

These assumptions are more natural in our framework than in Savage's: they only require a transitive ordering on *feasible* acts, not on all logically possible acts. To understand the interplay between feasibility and transitivity, consider a case where an agent observes event $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, and must choose between two feasible acts α and γ in $\mathcal{A}(\mathcal{R})$. Say momentarily that she has preferences over unfeasible acts, and that there is an unfeasible act β such that $\alpha \succeq_{\mathcal{R}} \beta$ and $\beta \succeq_{\mathcal{R}} \gamma$. A blind application of transitivity would yield $\alpha \succeq_{\mathcal{R}} \gamma$. But the unfeasibility of β undermines the meaningfulness of both rankings $\alpha \succeq_{\mathcal{R}} \beta$ and $\beta \succeq_{\mathcal{R}} \gamma$. Why should these two rankings influence the choice between α and γ? By restricting preferences to feasible acts, we eliminate such spurious influences.

We will now introduce the six axioms which appear in all of our main results. The same axioms appear in the companion paper Pivato and Vergopoulos (2018a); we refer the reader to this paper for more explanation and motivation of these axioms.

The separability axioms. Additive separability over disjoint events is a characteristic feature of SEU theories. In a Savage framework, it is captured by P2. In Ghirardato's (2002) model of conditional preferences, it is captured by the axiom of Dynamic Consistency. It also plays a central role in Hammond's (1988) derivation of SEU maximization on decision trees. Our first axiom captures separability through a version of Dynamic Consistency that only applies to regular partitions of a regular event.

(Sep) For any event $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, any disjoint events $\mathcal{D}, \mathcal{E} \in \mathfrak{R}(\mathcal{S})$ such that $\mathcal{D} \vee \mathcal{E} = \mathcal{R}$, and any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$ with $\alpha_{\upharpoonright\mathcal{D}} \approx_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$, we have $\alpha \succeq_{\mathcal{R}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \succeq_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$.

It is easy to see that the logical equivalence in Axiom (Sep) also holds for indifference and for strict preference: for any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$ with $\alpha_{\upharpoonright\mathcal{D}} \approx_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$, we have:

(i) $\alpha \succ_{\mathcal{R}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \succ_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$; and

[3] The same is true if $\mathcal{A}$ is the set of N-times differentiable functions, for any $N \in [2 \ldots \infty]$.

(ii) $\alpha \approx_{\mathcal{R}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \approx_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$.

Statement (i) means that *no event in* $\mathfrak{R}(\mathcal{S})$ *is null*. Thus, any SEU representation must give nonzero probability to all events in $\mathfrak{R}(\mathcal{S})$. Conversely, statement (ii) says that *the boundary of any event in* $\mathfrak{R}(\mathcal{S})$ *is null*: the behaviour of α and β on that small part of $\mathcal{R}$ that is not covered by $\mathcal{D} \cup \mathcal{E}$ is irrelevant for decisions conditional on $\mathcal{R}$. This seems to suggest that the SEU representation must give *zero probability* to the boundary of any regular set. But $\mathcal{A}$ is a set of *continuous* functions; thus, the behaviour of α and β on the open sets $\mathcal{D}$ and $\mathcal{E}$ entirely determines their behaviour on the common boundary $\partial\mathcal{D} \cap \partial\mathcal{E}$. Thus, statement (ii) does not mean that we *ignore* the behaviour of α and β on $\partial\mathcal{D} \cap \partial\mathcal{E}$, as if $\partial\mathcal{D} \cap \partial\mathcal{E}$ had zero probability; it just means that we have *already* implicitly accounted for this behaviour in our rankings of $\alpha_{\upharpoonright\mathcal{D}}$ versus $\beta_{\upharpoonright\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}}$ versus $\beta_{\upharpoonright\mathcal{E}}$. This implicit account will become explicit with the "liminal structures" of Sections 6 and 7.

If $\mathcal{D}, \mathcal{E} \in \mathfrak{R}(\mathcal{S})$ are disjoint and $\mathcal{R} = \mathcal{D} \vee \mathcal{E}$, then Axiom (Sep) says that the $\succeq_{\mathcal{R}}$-ranking of two acts $\alpha, \beta \in \mathcal{A}(\mathcal{R})$ is partly determined by the $\succeq_{\mathcal{D}}$-ranking of $\alpha_{\upharpoonright\mathcal{D}}$ versus $\beta_{\upharpoonright\mathcal{D}}$ and the $\succeq_{\mathcal{E}}$-ranking of $\alpha_{\upharpoonright\mathcal{E}}$ versus $\beta_{\upharpoonright\mathcal{E}}$. The next axiom says that this dependency is continuous.

(CCP) (*Continuity in conditional preferences*) Let $\mathcal{R} = \mathcal{D} \vee \mathcal{E}$ as in axiom (Sep). Let $\underline{\beta}, \alpha, \overline{\beta} \in \mathcal{A}(\mathcal{R})$ be three acts with $\underline{\beta} \prec_{\mathcal{R}} \alpha \prec_{\mathcal{R}} \overline{\beta}$. Then there exist $\underline{\delta}, \overline{\delta} \in \mathcal{A}(\mathcal{D})$ and $\underline{\epsilon}, \overline{\epsilon} \in \mathcal{A}(\mathcal{E})$, with $\underline{\delta} \prec_{\mathcal{D}} \alpha_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{E}} \alpha_{\upharpoonright\mathcal{E}} \prec_{\mathcal{E}} \overline{\epsilon}$ such that, for any $\alpha' \in \mathcal{A}(\mathcal{R})$, if $\underline{\delta} \prec_{\mathcal{D}} \alpha'_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{E}} \alpha'_{\upharpoonright\mathcal{E}} \prec_{\mathcal{E}} \overline{\epsilon}$ then $\underline{\beta} \prec_{\mathcal{R}} \alpha' \prec_{\mathcal{R}} \overline{\beta}$.

The intuition here is that a small variation in $\alpha_{\upharpoonright\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}}$ (relative to the order topologies on $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\mathcal{E})$) should not affect the $\succeq_{\mathcal{R}}$- ranking of α versus $\underline{\beta}$ and $\overline{\beta}$.

Continuity of *ex post* preferences. For any $x \in \mathcal{X}$, let κ^x be the constant x-valued act on $\mathcal{S}$. Let $\mathcal{K} := \{\kappa^x; \ x \in \mathcal{X}\}$. We have assumed $\mathcal{K} \subseteq \mathcal{A}$, so the preference order $\succeq_{\mathcal{S}}$, restricted to $\mathcal{K}$, induces a preference order $\succeq_{\text{xp}}$ on $\mathcal{X}$ as follows: for any $x, y \in \mathcal{X}$,

$$\Big(x \succeq_{\text{xp}} y\Big) \quad \iff \quad \Big(\kappa^x \succeq_{\mathcal{S}} \kappa^y\Big). \tag{1}$$

$\succeq_{\text{xp}}$ describes the *ex post* preferences of the agent on $\mathcal{X}$ when there is no uncertainty. The next axiom says that these preferences are compatible with the underlying topology on $\mathcal{X}$.

(C) The *ex post* order $\succeq_{\text{xp}}$ is continuous in the topology on $\mathcal{X}$. That is: for all $x \in \mathcal{X}$, the contour sets $\{y \in \mathcal{X}; \ y \succeq_{\text{xp}} x\}$ and $\{y \in \mathcal{X}; \ y \preceq_{\text{xp}} x\}$ are closed subsets of $\mathcal{X}$.

Certainty equivalents. For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and $x \in \mathcal{X}$, let $\kappa^x_{\mathcal{R}} := (\kappa^x)_{\upharpoonright\mathcal{R}}$; this is the constant x-valued act, conditional on $\mathcal{R}$. Given an act $\alpha \in \mathcal{A}(\mathcal{R})$, we say x is a *certainty equivalent* for α on $\mathcal{R}$ if $\kappa^x_{\mathcal{R}} \approx_{\mathcal{R}} \alpha$. The next axiom is a mild richness condition on $\mathcal{X}$.

(CEq) For any event $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, any act $\alpha \in \mathcal{A}(\mathcal{R})$ has a certainty equivalent on $\mathcal{R}$.

Axiom (CEq) may appear somewhat implausible. But it is a logical consequence of the following axiom of "constant continuity" which may seem more natural.

(CC) For any event $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and any act $\alpha \in \mathcal{A}(\mathcal{R})$, the sets $\{x \in \mathcal{X};\ \kappa^x_{\mathcal{R}} \succeq_{\mathcal{R}} \alpha\}$ and $\{x \in \mathcal{X};\ \alpha \succeq_{\mathcal{R}} \kappa^x_{\mathcal{R}}\}$ are closed in $\mathcal{X}$.

If $\mathcal{X}$ is connected and $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, then (CC) is equivalent to the conjunction of (C) and (CEq). So we could state our results with a single axiom (CC) in place of (CEq) and (C).

The statewise dominance axiom. Our next axiom imposes some consistency between the agent's conditional preference structure and her *ex post* preferences. It says that the agent always prefers a statewise dominating act.

(Dom) For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$, if $\alpha(b) \succeq_{\mathrm{xp}} \beta(b)$ for all $b \in \mathcal{R}$, then $\alpha \succeq_{\mathcal{R}} \beta$. Furthermore, if $\alpha(b) \succ_{\mathrm{xp}} \beta(b)$ for all $b \in \mathcal{R}$, then $\alpha \succ_{\mathcal{R}} \beta$.

Recall that if the agent "observes" the event $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, this does not mean the true state lies in $\mathcal{R}$ —it only lies in the *closure* of $\mathcal{R}$. But this does not undermine the first part of (Dom): since α and β are continuous functions, they have unique extensions to the closure of $\mathcal{R}$, and these extensions preserve weak statewise dominance. Thus, weak statewise dominance over $\mathcal{R}$ implies weak statewise dominance over all states that remain possible: those in the closure of $\mathcal{R}$. Of course, the extensions of α and β might not preserve *strict* dominance. So the second part of (Dom) requires some sub-event of $\mathcal{R}$ to be non-null. But as we have already observed, (Sep) implies that *all* events in $\mathfrak{R}(\mathcal{S})$ are non-null.

(Dom) appears similar to (Sep), and thus to Savage's axiom P2. The difference is that (Sep) applies to regular partitions, while (Dom) applies to partitions into singleton sets, which, in general, are *not* regular. Thus, (Dom) cannot be obtained as a special case of (Sep). Axiom (Dom) is also related to Savage's axioms P3 and P7. Axiom P3 requires the ranking of outcomes to be independent of the events that yield the outcomes. (Dom) entails a similar form of state independence: it implies that $\succeq_{\mathcal{S}}$ can be replaced by $\succeq_{\mathcal{R}}$ for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, in formula (1). Thus, the *ex post* preference orders obtained from different conditional preference orders must agree with one other. Finally, to see how (Dom) and P7 overlap, consider the special case of (Dom) where one of α or β is a constant act.

Tradeoff consistency. Our last axiom is a version of the *Cardinal Coordinate Independence* axiom used in Wakker's (1988) axiomatization of SEU. We need some preliminary definitions. Let $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, and let $\mathcal{Q} := \neg\mathcal{R}$. Consider an outcome $x \in \mathcal{X}$ and an act $\alpha \in \mathcal{A}(\mathcal{Q})$. Structural condition (Rch) yields an act $(x_{\mathcal{R}}\alpha) \in \mathcal{A}$ with two properties:

$$\textbf{(B1)}\quad (x_{\mathcal{R}}\alpha)_{\upharpoonright\mathcal{R}} \approx_{\mathcal{R}} \kappa^x_{\mathcal{R}}, \qquad \text{and} \qquad \textbf{(B2)}\quad (x_{\mathcal{R}}\alpha)_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \alpha.$$

We will call $(x_{\mathcal{R}}\alpha)$ an (x, α)-*bet* for $\mathcal{R}$; if $\mathcal{R}$ obtains, this bet is indifferent to the outcome x, while it is indifferent to α conditional on the complement of $\mathcal{R}$. Note that $(x_{\mathcal{R}}\alpha)$ is not uniquely defined by (B1) and (B2). But if $(x_{\mathcal{R}}\alpha)$ and $(x_{\mathcal{R}}\alpha)'$ are two acts satisfying (B1) and (B2), then axiom (Sep) implies that $(x_{\mathcal{R}}\alpha) \approx_{\mathcal{S}} (x_{\mathcal{R}}\alpha)'$.

Fix now four outcomes $x, y, v, w \in \mathcal{X}$, and a regular subset $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. Let $\mathcal{Q} := \neg\mathcal{R}$. We write $(x \overset{\mathcal{R}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{R}}{\rightsquigarrow} w)$ if there exist $\alpha, \beta \in \mathcal{A}(\mathcal{Q})$, an (x, α)-bet $(x_{\mathcal{R}}\alpha) \in \mathcal{A}$, a

(y,β)-bet $(y_{\mathcal{R}}\beta) \in \mathcal{A}$, a (v,α)-bet $(v_{\mathcal{R}}\alpha) \in \mathcal{A}$ and a (w,β)-bet $(w_{\mathcal{R}}\beta) \in \mathcal{A}$ such that $(x_{\mathcal{R}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{R}}\beta)$ while $(v_{\mathcal{R}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{R}}\beta)$. By the remark in the previous paragraph, this implies that for *any* such bets $(x_{\mathcal{R}}\alpha), (y_{\mathcal{R}}\beta), (v_{\mathcal{R}}\alpha), (w_{\mathcal{R}}\beta) \in \mathcal{A}$, we have $(x_{\mathcal{R}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{R}}\beta)$ and $(v_{\mathcal{R}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{R}}\beta)$.

If $(x_{\mathcal{R}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{R}}\beta)$, then the "gain" obtained by changing x to y on $\mathcal{R}$ is at least enough to compensate for the "loss" incurred by changing α to β on $\mathcal{Q}$. In contrast, if $(v_{\mathcal{R}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{R}}\beta)$, then the gain obtained by changing v to w on $\mathcal{R}$ is at *most* enough to compensate for the loss incurred by changing α to β on $\mathcal{Q}$. Together, these two observations imply that the gain obtained from changing x to y on $\mathcal{R}$ is at least as large as the gain from changing v to w on $\mathcal{R}$; hence the notation $(x \overset{\mathcal{R}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{R}}{\rightsquigarrow} w)$. If $\succeq_{\mathcal{S}}$ has an SEU representation with utility function u, then $(x \overset{\mathcal{R}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{R}}{\rightsquigarrow} w)$ means that $u(y) - u(x) \geq u(w) - u(v)$.

Conversely, we write $(x \overset{\mathcal{R}}{\rightsquigarrow} y) \prec (v \overset{\mathcal{R}}{\rightsquigarrow} w)$ if there exist $\gamma, \delta \in \mathcal{A}(\mathcal{Q})$, an (x,γ)-bet $(x_{\mathcal{R}}\gamma) \in \mathcal{A}$, a (y,δ)-bet $(y_{\mathcal{R}}\delta) \in \mathcal{A}$, a (v,γ)-bet $(v_{\mathcal{R}}\gamma) \in \mathcal{A}$ and a (w,δ)-bet $(w_{\mathcal{R}}\delta) \in \mathcal{A}$ such that $(x_{\mathcal{R}}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{R}}\delta)$ while $(v_{\mathcal{R}}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{R}}\delta)$. Again, this implies that $(x_{\mathcal{R}}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{R}}\delta)$ and $(v_{\mathcal{R}}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{R}}\delta)$ for *any* such bets $(x_{\mathcal{R}}\gamma), (y_{\mathcal{R}}\delta), (v_{\mathcal{R}}\gamma), (w_{\mathcal{R}}\delta) \in \mathcal{A}$. If $\succeq_{\mathcal{S}}$ had an SEU representation, then this means that $u(y) - u(x) < u(w) - u(v)$. Here is our final axiom:

(TC) For any two regular subsets $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}(\mathcal{S})$, there are no $x, y, v, w \in \mathcal{X}$ such that $(x \overset{\mathcal{R}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{R}_1}{\rightsquigarrow} w)$ while $(x \overset{\mathcal{R}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{R}_2}{\rightsquigarrow} w)$.

5 SEU representations using residual charges

A topological space $\mathcal{S}$ is ***Hausdorff*** if any pair of points in $\mathcal{S}$ can be placed in two disjoint open neighbourhoods. For example, any metrizable space (e.g. any subset of $\mathbb{R}^N$) is Hausdorff. The space $\mathcal{S}$ is ***compact*** if, for any collection $\mathfrak{O}$ of open sets whose union is $\mathcal{S}$, there is a finite subcollection $\{\mathcal{O}_1, \ldots \mathcal{O}_N\} \subseteq \mathfrak{O}$ such that $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_N = \mathcal{S}$. For example, any closed, bounded subset of $\mathbb{R}^N$ is compact. Our first result will give an SEU representation for conditional preference structures on any compact Hausdorff state space.

Let $\mathfrak{Bor}(\mathcal{S})$ be the Borel sigma-algebra of $\mathcal{S}$ —that is, the smallest sigma-algebra containing all open sets. Observe that $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{Bor}(\mathcal{S})$ as sets, but the Boolean algebra operations are different. A subset $\mathcal{N} \subseteq \mathcal{S}$ is ***nowhere dense*** if $\text{int}[\text{clos}(\mathcal{N})] = \emptyset$. For example, for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, the boundary $\partial\mathcal{R}$ is nowhere dense in $\mathcal{S}$. A subset $\mathcal{M} \subseteq \mathcal{S}$ is ***meager*** if it is a countable union of nowhere dense sets. Heuristically, meager sets are "small". For example, the set $\mathbb{Q}$ of rational numbers is meager in the space $\mathbb{R}$.

Let $\mathfrak{B}$ be a Boolean algebra of subsets of $\mathcal{S}$ (for example, $\mathfrak{B} = \mathfrak{Bor}(\mathcal{S})$). A ***probability charge*** on $\mathfrak{B}$ is a function $\nu : \mathfrak{B} \longrightarrow [0,1]$ such that (1) $\nu[\mathcal{S}] = 1$ and (2) $\nu[\mathcal{A} \sqcup \mathcal{B}] = \nu[\mathcal{A}] + \nu[\mathcal{B}]$ for any disjoint $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$. We say that ν is a ***residual charge*** if, furthermore, $\nu[\mathcal{M}] = 0$ for any meager $\mathcal{M} \in \mathfrak{B}$. We say that ν has ***full support*** if $\nu(\mathcal{R}) > 0$ for any nonempty regular subset $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. In particular, if every nonempty open subset of $\mathcal{S}$ contains a nonempty regular subset (i.e. $\mathcal{S}$ is *quasiregular*), then ν has full support if and only if $\nu(\mathcal{O}) > 0$ for any nonempty open subset $\mathcal{O}$. Here is our first representation theorem.

Theorem 1 *Let $\mathcal{S}$ be a nonsingleton, compact Hausdorff space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ satisfying* (Rch). *Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies* (CEq), (C), (Dom), (Sep), (CCP) *and* (TC) *if and only if there is a residual probability charge ν on $\mathfrak{Bor}(\mathcal{S})$ with full support, and a continuous function $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$, we have*

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \left(\int_{\mathcal{R}} u \circ \alpha \ {\rm d}\nu \geq \int_{\mathcal{R}} u \circ \beta \ {\rm d}\nu\right). \tag{2}$$

Furthermore, ν is unique, and u is unique up to positive affine transformation.

Theorem 1 is a special case of a more general result. A subset $\mathcal{B} \subseteq \mathcal{S}$ has the *Baire property* if $\mathcal{B} = \mathcal{O}\triangle\mathcal{M}$ for some open $\mathcal{O} \subseteq \mathcal{S}$ and some meager $\mathcal{M} \subset \mathcal{S}$. Heuristically, this means that $\mathcal{B}$ is "almost open" in $\mathcal{S}$. Let $\mathfrak{Bai}(\mathcal{S})$ be the collection of all subsets of $\mathcal{S}$ with the Baire property; then $\mathfrak{Bai}(\mathcal{S})$ is a Boolean algebra under the standard set operations. Again, $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{Bai}(\mathcal{S})$ as sets, but the Boolean algebra operations are different.

A topological space $\mathcal{S}$ is a *Baire space* if the intersection of any countable family of open dense sets is dense. For example, any open subset of the Euclidean space $\mathbb{R}^N$ is a Baire space. More generally, any completely metrizable space is Baire (Willard, 2004, Corollary 25.4). In particular, every topological manifold is Baire. Finally, any locally compact Hausdorff space is Baire. Intuitively, *non*-Baire spaces are extremely "sparse" or "porous"; they are unlikely to arise naturally in economic models. (For example, a countable Hausdorff space is not Baire. Also, the product topology on $\mathbb{Q} \times \mathbb{R}$ is not Baire.) Finally, $\mathcal{S}$ is *nondegenerate* if it contains a nonempty open subset which is *not* dense —or equivalently, a proper closed subset with nonempty interior. This means that $\mathfrak{R}(\mathcal{S})$ is not trivial. Nondegeneracy is a very mild condition; for example, any nonsingleton Hausdorff space is nondegenerate (Lemma A2(a)). Our second result gives an SEU representation for conditional preference structures on any nondegenerate Baire state space.

Theorem 2 *Let $\mathcal{S}$ be a nondegenerate Baire space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ satisfying* (Rch). *Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies* (CEq), (C), (Dom), (Sep), (CCP) *and* (TC) *if and only if there is a residual probability charge ν on $\mathfrak{Bai}(\mathcal{S})$ with full support, and a continuous function $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that statement* (2) *holds for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$. Furthermore, ν is unique, and u is unique up to positive affine transformation.*

Imperfect perception via observational equivalence. Section 2 presented a model of imperfect perception based on regular partitions of the state space (or equivalently, maximally decisive UHC multifunctions). But Theorems 1 and 2 suggest *another* model of imperfect perception. Let $\mathfrak{B}$ be a Boolean algebra of subsets of $\mathcal{S}$ (e.g. $\mathfrak{Bor}(\mathcal{S})$ or $\mathfrak{Bai}(\mathcal{S})$), and let μ be a probability charge on $\mathcal{S}$. Given $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, recall that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *equal μ-almost everywhere* (or *μ-a.e.*) if $\mu[\mathcal{B}_1\triangle\mathcal{B}_2] = 0$ (where $\mathcal{B}_1\triangle\mathcal{B}_2 = (\mathcal{B}_1 \setminus \mathcal{B}_2) \sqcup (\mathcal{B}_2 \setminus \mathcal{B}_1)$). We will indicate this by writing "$\mathcal{B}_1 \underset{\mu}{\approx} \mathcal{B}_2$". This is an equivalence relation on $\mathfrak{B}$. Likewise, if $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ and $\mathfrak{P}' = \{\mathcal{P}'_1, \ldots, \mathcal{P}'_N\}$ are two $\mathfrak{B}$-measurable partitions of $\mathcal{S}$, then we say $\mathfrak{P}$ and $\mathfrak{P}'$ are *equal μ-a.e.* if $\mathcal{P}_n \underset{\mu}{\approx} \mathcal{P}'_n$ for all $n \in [1 \ldots N]$. This is an equivalence relation on the set of $\mathfrak{B}$-measurable partitions. In probability theory, if two partitions are equal μ-a.e., then they are regarded as *observationally equivalent*; an agent will never be able to tell whether she is observing the world via the partition $\mathfrak{P}$ or via the partition $\mathfrak{P}'$. The event that these two partitions generate different observations (i.e. the set $(\mathcal{P}_1 \triangle \mathcal{P}'_1) \cup \cdots \cup (\mathcal{P}_N \triangle \mathcal{P}'_N)$) has probability zero. Thus, probabilistically speaking, it *never happens*; even if the agent performed a million experiments, she would never observe a situation where $\mathfrak{P}$ and $\mathfrak{P}'$ disagree. For all intents and purposes, $\mathfrak{P}$ and $\mathfrak{P}'$ are "the same partition". Indeed, in probability theory and real analysis, this informal observation is often formalized by working only with a.e.-equivalence classes of sets and functions.

For residual charges, these observations have a particular significance. For any $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, write "$\mathcal{B}_1 \approx \mathcal{B}_2$" if $\mathcal{B}_1 \triangle \mathcal{B}_2$ is meager; this is an equivalence relation on $\mathfrak{B}$. If $\mathcal{B}_1 \approx \mathcal{B}_2$, then *a fortiori* $\mathcal{B}_1 \underset{\mu}{\approx} \mathcal{B}_2$ for any residual charge μ, because every meager set has μ-measure zero. Likewise, if $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ and $\mathfrak{P}' = \{\mathcal{P}'_1, \ldots, \mathcal{P}'_N\}$ are two $\mathfrak{B}$-measurable partitions of $\mathcal{S}$, then we will write $\mathfrak{P} \approx \mathfrak{P}'$ if $\mathcal{P}_n \approx \mathcal{P}'_n$ for all $n \in [1 \ldots N]$; this is an equivalence relation on the set of $\mathfrak{B}$-measurable partitions.

If $\mathfrak{P} \approx \mathfrak{P}'$, then $\mathfrak{P} \underset{\mu}{\approx} \mathfrak{P}'$ for any residual charge μ. Thus, if an agent's beliefs about the state space are represented by a residual charge (as in Theorems 1 and 2), then to her, $\mathfrak{P}$ and $\mathfrak{P}'$ are *observationally equivalent*, exactly as described above. Thus, as in probability theory and analysis, it makes sense to work with $\approx$-equivalence classes of events.

Proposition 2 *Suppose that either* (i) *$\mathcal{S}$ is a compact space and $\mathfrak{B} = \mathfrak{Bor}(\mathcal{S})$, or* (ii) *$\mathcal{S}$ is a Baire space, and $\mathfrak{B} = \mathfrak{Bai}(\mathcal{S})$. For any $\mathcal{B} \in \mathfrak{B}$, there is a unique $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ such that $\mathcal{B} \approx \mathcal{R}$; define $\phi(\mathcal{B}) := \mathcal{R}$. This yields a surjective Boolean algebra homomorphism $\phi : \mathfrak{B} \longrightarrow \mathfrak{R}(\mathcal{S})$.*

In other words, every $\approx$-equivalence class of events in $\mathfrak{B}$ can be identified with a unique regular set, and every $\approx$-equivalence class of $\mathfrak{B}$-measurable partitions can be identified with a unique regular partition. Section 2 introduced a "non-classical" model of information, in which the agent could only obtain information about the state of the world through regular partitions. But Proposition 2 renders this equivalent to a totally *classical* model, where the agent obtains information about the state through (ordinary) $\mathfrak{B}$-measurable partitions. But she assigns zero probability to all meager sets, and as in standard probability theory, she cannot distinguish between two events that are equal almost everywhere.

Thus, in this alternative model of imperfect perception, the Boolean algebra $\mathfrak{R}(\mathcal{S})$ is merely a convenient mathematical shorthand for dealing with $\approx$-equivalence classes of events in $\mathfrak{Bor}(\mathcal{S})$ or $\mathfrak{Bai}(\mathcal{S})$. But this model has two limitations. First, it only works for compact spaces and Baire spaces, whereas we want a theory which applies to a wider class of state spaces. Second, this approach *assumes* the agent already has probabilistic beliefs, and that furthermore these beliefs are a residual charge. But following the tradition of Savage, our goal in this paper is to *derive* the agent's beliefs from her preferences over acts, via a subjective expected utility representation. Thus, while this alternative model provides *another* justification for using $\mathfrak{R}(\mathcal{S})$ to represent the information available to the agent, it is not our primary justification. Our main justification is the model in Section 2.

6 SEU representations with liminal structures

Theorems 1 and 2 yield "classical" SEU representations, quite similar to Savage. But they use residual charges, which assign probability zero to the boundaries of regular sets, as if the agent thinks that such boundary events "never" occur. This means that she does not perceive her measurement devices as truly ambiguous: although a state on the knife-edge between two measurement outcomes could occur in principal, such a situation never arises in practice. We will now introduce an SEU representation where the agent recognizes that boundary events *can* occur; this is encoded in what we call a *liminal* structure, which encodes the probabilities of different measurement outcomes at each knife-edge state. Also, the residual charges from Theorems 1 and 2 are only finitely additive, and not normal. In contrast, the SEU representations in this section use normal Borel probability measures.

Let $\mathfrak{Bor}(\mathcal{S})$ be the Borel sigma-algebra of $\mathcal{S}$. Let ν be a *Borel probability measure* on $\mathcal{S}$ — that is, a (countably additive) probability measure on $\mathfrak{Bor}(\mathcal{S})$. Recall that ν is *normal* if, for every $\mathcal{B} \in \mathfrak{Bor}(\mathcal{S})$, we have $\nu[\mathcal{B}] = \sup\{\nu[\mathcal{C}];\ \mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{C}$ closed in $\mathcal{S}\}$ and $\nu[\mathcal{B}] = \inf\{\nu[\mathcal{O}];\ \mathcal{B} \subseteq \mathcal{O} \subseteq \mathcal{S}$ and $\mathcal{O}$ open in $\mathcal{S}\}$. For any $\mathcal{B} \in \mathfrak{Bor}(\mathcal{S})$, let $\nu_{\mathcal{B}}$ be the restriction of ν to a Borel measure on $\mathcal{B}$, and let $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}})$ be the Banach space of real-valued, $\nu_{\mathcal{B}}$-integrable functions on $\mathcal{B}$, modulo equality $\nu_{\mathcal{B}}$-almost everywhere. Finally, let $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}}; [0,1])$ be the set of $[0,1]$-valued functions in $\mathbb{L}^1(\mathcal{B}, \nu_{\mathcal{B}})$. A *liminal density structure subordinate to* ν is a collection $\{\phi_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$, where, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{R})$, $\phi_{\mathcal{R}} \in \mathbb{L}^1(\partial\mathcal{R}, \nu_{\partial\mathcal{R}}; [0,1])$ is a function such that, for any regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, we have

$$\phi_{\mathcal{R}_1} + \cdots + \phi_{\mathcal{R}_N} \quad = \quad 1, \quad \nu\text{-almost everywhere on } \partial\mathcal{R}_1 \cup \cdots \cup \partial\mathcal{R}_N. \tag{3}$$

In the model we develop in this section, the agent's "beliefs" will be represented by a Borel probability measure ν, along with a liminal density structure $\{\phi_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ subordinate to ν. Heuristically, ν describes the agent's *ex ante* beliefs, while $\{\phi_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ describes how she copes with her informational constraints. To be more precise, suppose the agent obtains information through an upper hemicontinuous multifunction $f : \mathcal{S} \rightrightarrows [1 \ldots N]$, which we represent using a regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$, as in Proposition 1. For all $n \in [1 \ldots N]$, her *ex ante* probability of receiving the signal "n" is

$$\mu[\mathcal{R}_n] \quad := \quad \nu[\mathcal{R}_n] + \int_{\partial\mathcal{R}_n} \phi_{\mathcal{R}_n} \, \mathrm{d}\nu.$$

Now suppose the agent receives the signal "n". Then she knows that the state of the world lies in $\mathrm{clos}(\mathcal{R}_n)$. But there is a chance that the state is not in $\mathcal{R}_n$ itself, but instead on its boundary $\partial\mathcal{R}_n$. Indeed, the state is in $\mathcal{R}_n$ with probability $\nu[\mathcal{R}_n]/\mu[\mathcal{R}_n]$, whereas it is on the boundary of $\mathcal{R}_n$ with probability $1 - (\nu[\mathcal{R}_n]/\mu[\mathcal{R}_n])$. Furthermore, the density $\phi_{\mathcal{R}_n}$ tells the agent *where* the state is likely to be on $\partial\mathcal{R}_n$, given that this latter case occurs. To be precise, she assigns the following conditional probability to any event $\mathcal{B} \in \mathfrak{Bor}(\mathcal{S})$:

$$\frac{\nu(\mathcal{B} \cap \mathcal{R}_n) + \displaystyle\int_{\mathcal{B} \cap \partial\mathcal{R}_n} \phi_{\mathcal{R}_n} \, \mathrm{d}\nu}{\nu(\mathcal{R}_n) + \displaystyle\int_{\partial\mathcal{R}_n} \phi_{\mathcal{R}_n} \, \mathrm{d}\nu}. \tag{4}$$

Thus, receiving the signal "n" increases the probability of $\mathcal{R}_n$ to $\nu[\mathcal{R}_n]/\mu[\mathcal{R}_n]$, but not necessarily to certainty. However, this "spillover" probability is confined to the *closure* of $\mathcal{R}_n$; formula (4) implies that the probability of $\mathrm{clos}(\mathcal{R}_n)$ given $\mathcal{R}_n$ always equals one.

For example, suppose $\mathcal{S} = [-1, 1]$, and consider the multifunction $f : \mathcal{S} \rightrightarrows \{L, R\}$ such that $f(s) = \{L\}$ for all $s < 0$, $f(s) = \{R\}$ for all $s > 0$, and $f(0) = \{L, R\}$. This corresponds to the regular partition $\{\mathcal{L}, \mathcal{R}\}$, where $\mathcal{L} := [-1, 0)$ and $\mathcal{R} := (0, 1]$. Note that $\partial\mathcal{L} = \partial\mathcal{R} = \{0\}$, so that $\phi_{\mathcal{L}}$ and $\phi_{\mathcal{R}}$ are entirely determined by their values at 0. Suppose that $\nu\{0\} > 0$. Formula (8) says that $\phi_{\mathcal{L}}(0) + \phi_{\mathcal{R}}(0) = 1$. Let $\mu[\mathcal{L}] := \nu(\mathcal{L}) + \phi_{\mathcal{L}}(0)\,\nu\{0\}$, while $\mu[\mathcal{R}] := \nu(\mathcal{R}) + \phi_{\mathcal{R}}(0)\,\nu\{0\}$; these are the agent's subjective probabilities of receiving the signals "L" and "R", respectively. The agent believes that, if the true state were $s = 0$, then she would receive the signal "L" with probability $\phi_{\mathcal{L}}(0)$, whereas she would receive the signal "R" with probability $\phi_{\mathcal{R}}(0)$. Once she has received "L" she thinks that actually $s = 0$ with probability $p := \phi_{\mathcal{L}}(0)\,\nu\{0\}/\mu(\mathcal{L})$, whereas $s < 0$ with probability $1 - p$. On the other hand, if she receives "R" , then she thinks that actually $s = 0$ with probability $q := \phi_{\mathcal{R}}(0)\,\nu\{0\}/\mu(\mathcal{R})$, whereas $s > 0$ with probability $1 - q$.

We will explain this interpretation more formally below, by representing the agent's perception of her environment with *stochastic functions*. But first we will state our next representation theorem using these liminal structures.

Liminal density SEU representation. Let $\mathcal{X}$ be a topological space, let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$ be a set of feasible acts , and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A *liminal density SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is given by a Borel probability measure ν on $\mathfrak{Bor}(\mathcal{S})$ and a liminal density structure $\{\phi_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ subordinate to ν, along with a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \tag{5}$$
$$\left(\int_{\mathcal{R}} u \circ \alpha \;\mathrm{d}\nu + \int_{\partial\mathcal{R}} (u \circ \alpha)\,\phi_{\mathcal{R}} \;\mathrm{d}\nu \;\geq\; \int_{\mathcal{R}} u \circ \beta \;\mathrm{d}\nu + \int_{\partial\mathcal{R}} (u \circ \beta)\,\phi_{\mathcal{R}} \;\mathrm{d}\nu\right).$$

(Here we use the fact $u \circ \alpha$ and $u \circ \beta$ have unique extensions to $\partial\mathcal{R}$, by continuity.) Note that the "boundary" terms in (5) do *not* violate consequentialism or dynamic consistency. To see this, let $\alpha, \beta \in \mathcal{A}$. Because α, β and u are all continuous, the values of $u \circ \alpha$ and $u \circ \beta$ on $\partial\mathcal{R}$ are completely determined by their values on $\mathcal{R}$. Thus, (5) satisfies *consequentialism*: if $\alpha_{\upharpoonright\mathcal{R}} = \beta_{\upharpoonright\mathcal{R}}$, then α and β must have the same expected utility, conditional on $\mathcal{R}$. Meanwhile, equation (8) ensures that the expected utility of α on $\mathcal{S}$ is a weighted average of the conditional-expected utilities of α on $\mathcal{R}$ and on $\neg\mathcal{R}$ (as expressed on the right side of (5)), and likewise for β. Thus, if the conditional-expected utility of α is greater than that of β on both $\mathcal{R}$ and $\neg\mathcal{R}$, then it must be greater on all of $\mathcal{S}$, in accord with the principle of dynamical consistency. If $\mathcal{S} = \mathcal{R}$, then (5) simplifies to

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \left(\int_{\mathcal{S}} u \circ \alpha \;\mathrm{d}\nu \geq \int_{\mathcal{S}} u \circ \beta \;\mathrm{d}\nu\right). \tag{6}$$

In other words, the unconditional preference order $\succeq_{\mathcal{S}}$ always has a "classical" SEU representation. The liminal density structure only emerges in the representation of the *con-*

ditional preference orders: it influences the way the agent updates her preferences and beliefs as new information is acquired. Our third result provides a characterization of liminal density SEU representations in terms of the axioms of Section 4.

Theorem 3 *Let $\mathcal{S}$ be a (nonsingleton) compact Hausdorff space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ satisfying* (Rch). *Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies* (CEq), (C), (Dom), (Sep), (CCP) *and* (TC) *if and only if it admits a liminal density SEU representation* (5), *where ν is a normal Borel probability measure with full support. Furthermore, ν is unique, the elements of $\{\phi_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ are unique* (ν-almost everywhere)*, and u is unique up to positive affine transformation.*

At first glance, there appears to be a direct contradiction between the "uniqueness" claims in Theorems 1 and 3, since both theorems could be applied to the *same* conditional preference structure. But there is no contradiction: Theorem 3 is formulated in terms of a *countably additive* probability measure, while Theorem 1 was formulated in terms of a *residual charge*. They are not the same type of representation. (Likewise, none of our later SEU representations will contradict each other's "uniqueness" claims.)

Imperfect perception via stochastic functions. In Sections 2 and 5, we introduced two models of imperfect perception in terms of regular partitions —one in terms of upper hemicontinuous multifunctions, and one in which partitions are regarded as observationally equivalent if they differ on a meager set. But Theorem 3 suggests a *third* model of imperfect perception, in terms of measurement devices with a random or erratic output.

For any $N \in \mathbb{N}$, let $\Delta[1 \ldots N]$ be the set of probability measures on $[1 \ldots N]$. A ***stochastic function*** from $\mathcal{S}$ to $[1 \ldots N]$ is a function $\zeta : \mathcal{S} \longrightarrow \Delta[1 \ldots N]$. This can be interpreted as an observation of the state made by an "erratic" measurement device which can emit N possible signals, but which has a somewhat random behaviour: for any $s \in \mathcal{S}$, $\zeta(s)$ is the probability distribution of the signal the device will emit if the true state is s. We will indicate this by writing "$\zeta : \mathcal{S} \rightsquigarrow [1 \ldots N]$". Any ordinary function from $\mathcal{S}$ into $[1 \ldots N]$ can be treated as a stochastic function which maps each state in $\mathcal{S}$ to a point mass on $[1 \ldots N]$. Conversely, if ζ is a stochastic function which maps each state in $\mathcal{S}$ to a point mass, then we can regard it as an ordinary function in the obvious way.

A stochastic function $\zeta : \mathcal{S} \rightsquigarrow [1 \ldots N]$ is ***harmonic*** if, for all $s \in \mathcal{S}$, there is an open neighbourhood $\mathcal{Q}_s \subseteq \mathcal{S}$ around s, and a probability measure $\mu \in \Delta(\mathcal{Q}_s)$ such that

$$\zeta(s) \quad = \quad \int_{\mathcal{Q}_s} \zeta(q) \; \mathrm{d}\mu[q], \tag{7}$$

and furthermore, for any $q \in \mathcal{Q}_s$, μ can be chosen such that $\mu\{q\} > 0$. In other words, $\zeta(s)$ is a weighted average of ζ-values at nearby points.[4] Let $\mathcal{M}_s$ be the set of all μ in $\Delta(\mathcal{Q}_s)$ that satisfy (7). Then $\mathcal{M}_s$ has "full support" in a strong sense: *any* point in $\mathcal{Q}_s$ gets positive probability from some element of $\mathcal{M}_s$. If ζ represents a measurement device, then (7) says that the output is subject to a random "jitter" in the state which is observed.

[4]The name *harmonic* is by analogy to a similar property of *harmonic functions* in classical analysis.

If the true state is s, then the device will actually return a reading at some nearby point $q \in \mathcal{Q}_s$. But q must be a *nearby* point, which limits the error. In particular, if δ_n is the point mass at some $n \in [1 \ldots N]$, then $\zeta(s) = \delta_n$ if and only if $\zeta(q) = \delta_n$ for all $q \in \mathcal{Q}_s \setminus \{s\}$.

Let $\zeta : \mathcal{S} \rightsquigarrow [1 \ldots N]$ be a stochastic function. For all $s \in \mathcal{S}$, define $\text{supp}\zeta(s) := \text{supp}[\zeta(s)]$; this yields a multifunction $\text{supp}\zeta : \mathcal{S} \rightrightarrows [1 \ldots N]$. If ζ is harmonic, then $\text{supp}\zeta$ is upper hemicontinuous —indeed,

$$\text{supp}\zeta(s) \quad = \quad \bigcup_{q \in \mathcal{Q}_s \setminus \{s\}} \text{supp}\zeta(q).$$

Conversely, given any upper hemicontinuous multifunction $f : \mathcal{S} \rightrightarrows [1 \ldots N]$, there is some harmonic $\zeta : \mathcal{S} \rightsquigarrow [1 \ldots N]$ such that $\text{supp}\zeta = f$.

Given two stochastic functions $\zeta, \xi : \mathcal{S} \rightsquigarrow [1 \ldots N]$ and some $s \in \mathcal{S}$, we say that ξ is ***more random*** than ζ at s if there is another probability measure $\gamma_s \in \Delta[1 \ldots N]$ and some $r_s \in (0, 1]$ such that $\xi(s) = r_s\,\zeta(s) + (1 - r_s)\,\gamma_s$. In effect, the signal generated by the ξ-device in state s is the outcome of a two-stage random process: *first*, flip an $(r_s, 1 - r_s)$-biased coin, and then, depending on the outcome, either take a reading using the ζ-device, or generate a random signal drawn from γ_s. We say that ξ is ***more random*** than ζ if it is more random than ζ at every point in $\mathcal{S}$, we indicate this by writing "$\xi \trianglerighteq \zeta$". It is easily verified that relation $\trianglerighteq$ is a preorder (that is, a transitive and reflexive relation). Furthermore, if $\zeta \trianglelefteq \xi$, then $\text{supp}\zeta \subseteq \text{supp}\xi$. (This is because $r_s > 0$ for all $s \in \mathcal{S}$.)

Let $\Xi(\mathcal{S}, N)$ be the set of harmonic stochastic functions from $\mathcal{S}$ into $[1 \ldots N]$. Let $\zeta \in \Xi(\mathcal{S}, N)$; we say that ζ is ***minimally random*** in $\Xi(\mathcal{S}, N)$ if there is no $\xi \in \Xi(\mathcal{S}, N)$ with $\xi \triangleleft \zeta$. From the observations in the previous two paragraph, this implies that $\text{supp}\zeta$ is maximally decisive in $\text{UHC}(\mathcal{S}, N)$. Thus, ζ defines a regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ via Proposition 1. For all $n \in [1 \ldots N]$, $\mathcal{R}_n$ is the set of all states in $\mathcal{S}$ which ζ maps to the point mass δ_n. Thus, if ζ represents an erratic measurement device, then $\mathcal{R}_n$ is the set of states where this device is *guaranteed* to emit the signal n. In contrast, $\partial\mathcal{R}_n$ is the set of states where the device *might* emit n, but might also randomly emit some other signal.

Stochastic functions also arise in Bayesian games, in the form of type-dependent mixed strategies. Recall the notation at the end of Section 2. Given a strategy profile $\boldsymbol{\delta} \in \Delta(\mathcal{A}_2) \times \cdots \times \Delta(\mathcal{A}_M)$, let $f_{\boldsymbol{\delta}} : \mathcal{T}_1 \rightrightarrows \mathcal{A}_1$ be the best response correspondence for Player 1 (where $\mathcal{T}_1$ is Player 1's type space and $\mathcal{A}_1$ is her strategy space). A (type-dependent) ***mixed strategy*** for Player 1 is a stochastic function $\zeta : \mathcal{T}_1 \rightsquigarrow \mathcal{A}_1$; it is a ***best response*** to $\boldsymbol{\delta}$ if $\text{supp}\zeta(t) \subseteq f_{\boldsymbol{\delta}}(t)$ for all $t \in \mathcal{T}_1$. As noted in Section 2, other players can learn about Player 1's type by observing her pattern of play. Player 1 can hinder this by randomizing over all elements of $f_{\boldsymbol{\delta}}(t)$, so that $\text{supp}\zeta(t) = f_{\boldsymbol{\delta}}(t)$ for all $t \in \mathcal{T}_1$. This implies that ζ is harmonic. If $f_{\boldsymbol{\delta}}$ is maximally decisive, then ζ is minimally random.

These considerations yield a third model of imperfect perception, in addition to those suggested in Sections 2 and 5. In this model, the agent has access to a collection of erratic measurement devices, each with a finite set of possible outputs. Due to technological constraints, the associated stochastic functions are harmonic. But the devices are still *as reliable as possible* given this constraint. In other words, the associated stochastic functions are minimally random in $\Xi(\mathcal{S}, N)$. As explained above, each of these erratic measurement devices then defines a regular partition of $\mathcal{S}$. Suppose the set of measurement devices is large enough that every regular subset of $\mathcal{S}$ can be realized in this way; then the Boolean algebra $\mathfrak{R}(\mathcal{S})$ describes the set of all events which are observable by the agent.

This suggests an interesting interpretation of Theorem 3. Let ν be a Borel probability measure on $\mathcal{S}$, and let $\{\phi_\mathcal{R}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a liminal density structure subordinate to ν. Let $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ be a regular partition of $\mathcal{S}$. By modifying the functions $\phi_{\mathcal{P}_1}, \ldots, \phi_{\mathcal{P}_N}$ on a set of ν-measure zero if necessary, we can assume without loss of generality that[5]

$$\phi_{\mathcal{P}_1}(s) + \cdots + \phi_{\mathcal{P}_N}(s) \quad = \quad 1, \quad \text{for all } s \in \partial\mathcal{P}_1 \cup \cdots \cup \partial\mathcal{P}_N. \tag{8}$$

We can now define a stochastic function $\zeta_\mathfrak{P} : \mathcal{S} \rightsquigarrow [1 \ldots N]$ as follows: For all $s \in \mathcal{S}$, if $s \in \mathcal{P}_n$ for some $n \in [1 \ldots N]$, then define $\zeta_\mathfrak{P}(s) := \delta_n$. Meanwhile, if $s \in \partial\mathcal{P}_1 \cup \cdots \cup \partial\mathcal{P}_N$, then define $\zeta_\mathfrak{P}(s) := (\phi_{\mathcal{P}_1}(s), \ldots, \phi_{\mathcal{P}_N}(s))$. Suppose that for all $n \in [1 \ldots N]$, we have $\phi_{\mathcal{P}_n}(s) > 0$ for all $s \in \partial\mathcal{P}_n$. Then $\zeta_\mathfrak{P}$ is harmonic, and is minimally random in $\Xi(\mathcal{S}, N)$.

Let us say that the liminal density structure $\{\phi_\mathcal{R}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ has *full support* if $\phi_\mathcal{R}$ is nonzero ν-almost everywhere, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. In this case, via the construction in the previous paragraph, we can associate to each regular partition $\mathfrak{P}$ a minimally random, harmonic stochastic function $\zeta_\mathfrak{P}$ (unique ν-almost everywhere). Thus, the liminal density SEU representation (5) describes an agent whose prior beliefs about the world are given by the Borel probability measure ν, and whose information about the world is described by the collection of stochastic functions $\{\zeta_\mathfrak{P}\}$, where $\mathfrak{P}$ ranges over all regular partitions of $\mathcal{S}$.

SEU representations on normal state spaces. A shortcoming of Theorem 3 is that $\mathcal{S}$ must be compact, which makes it inapplicable in certain situations. In Section 7, we will extend Theorem 3 to *locally compact* Hausdorff spaces. But many spaces are not even locally compact (e.g. infinite-dimensional normed vector spaces), and these could easily arise as state spaces in decision problems. For this reason, we will now obtain an SEU representation analogous to Theorem 3, but for *normal* state spaces. The price we pay for this generality is that ν is no longer a countably additive Borel measure, but only a finitely additive charge. This, in turn, will force us to modify the definition of liminal structure.

A Hausdorff space $\mathcal{S}$ is *normal* (or "T_4") if, for any disjoint closed subsets $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{S}$, there exist disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{S}$ with $\mathcal{C}_1 \subseteq \mathcal{O}_1$ and $\mathcal{C}_2 \subseteq \mathcal{O}_2$. For example, every metrizable space is normal (e.g. any subset of $\mathbb{R}^N$, any topological manifold). Also, every compact Hausdorff space is normal. Finally, the order topology on any strictly ordered set is both Hausdorff and normal. Thus, almost all topological spaces which would arise naturally in economic applications are normal.

Charges and liminal charge structures. Let $\mathfrak{A}(\mathcal{S})$ be the Boolean algebra generated by the open subsets of $\mathcal{S}$. Thus, $\mathfrak{A}(\mathcal{S})$ contains all open subsets, all closed subsets, and all finite unions and intersections of such sets. A function $\nu : \mathfrak{A}(\mathcal{S}) \longrightarrow [0, 1]$ is a *charge* if it is finitely additive —i.e. $\nu[\mathcal{A} \sqcup \mathcal{B}] = \nu[\mathcal{A}] + \nu[\mathcal{B}]$ for any disjoint $\mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{S})$. We say ν

[5] This measure-zero modification of $\phi_{\mathcal{P}_1}, \ldots, \phi_{\mathcal{P}_N}$ must be made *after* we fix the partition $\mathfrak{P}$. Different partitions may require different measure-zero modifications.

is a *probability* charge if, furthermore, $\nu(\mathcal{S}) = 1$. Another charge ρ is *absolutely continuous* relative to ν if $\rho[\mathcal{B}] = 0$ whenever $\nu[\mathcal{B}] = 0$.

Let ν be a probability charge on $\mathfrak{A}(\mathcal{S})$. For any $\mathcal{B} \in \mathfrak{A}(\mathcal{S})$, let $\nu_{\mathcal{B}}$ be the restriction of ν to a charge $\mathfrak{A}(\mathcal{B})$. A *liminal charge structure subordinate to* ν is a collection $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$, where, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{R})$, $\rho_{\mathcal{R}}$ is a charge on $\mathfrak{A}(\partial\mathcal{R})$ which is absolutely continuous with respect to ν, such that, for any regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, we have

$$\rho_{\mathcal{R}_1} + \cdots + \rho_{\mathcal{R}_N} \quad = \quad \nu_{\partial\mathcal{R}_1\cup\cdots\cup\partial\mathcal{R}_N}. \tag{9}$$

As with the liminal density structures introduced earlier in Section 6, the liminal charge structure describes how the agent copes with her informational limitations; once she has received the signal "n", the charge $\rho_{\mathcal{R}_n}$ describes how much probability she conditionally assigns to $\partial\mathcal{R}_n$, and how this probability is distributed. To be precise, for any $\mathcal{U} \in \mathfrak{A}(\mathcal{S})$, the conditional probability she assigns to $\mathcal{U}$, given the signal "n", is the following ratio:

$$\frac{\nu(\mathcal{U} \cap \mathcal{R}_n) + \rho_{\mathcal{R}_n}(\mathcal{U} \cap \partial\mathcal{R}_n)}{\nu(\mathcal{R}_n) + \rho_{\mathcal{R}_n}(\partial\mathcal{R}_n)}. \tag{10}$$

A charge ν is *normal* if, for every $\mathcal{B} \in \mathfrak{A}(\mathcal{S})$, we have $\nu[\mathcal{B}] = \sup\{\nu[\mathcal{C}];\ \mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{C}$ closed in $\mathcal{S}\}$ and $\nu[\mathcal{B}] = \inf\{\nu[\mathcal{O}];\ \mathcal{B} \subseteq \mathcal{O} \subseteq \mathcal{S}$ and $\mathcal{O}$ open in $\mathcal{S}\}$. A liminal charge structure $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is *normal* if $\rho_{\mathcal{R}}$ is a normal charge on $\partial\mathcal{R}$ for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$. Finally, a charge ν is said to have *full support* if $\nu(\mathcal{O}) > 0$ for any open set $\mathcal{O}$ in $\mathcal{S}$.

Liminal charge SEU representation. Let $\mathcal{X}$ be another topological space, let $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A *liminal charge SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is given by a charge ν on $\mathfrak{A}(\mathcal{S})$, a liminal charge structure $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ subordinate to ν, and a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\left(\alpha \succeq_{\mathcal{R}} \beta\right) \iff \left(\int_{\mathcal{R}} u \circ \alpha \ {\rm d}\nu + \int_{\partial\mathcal{R}} u \circ \alpha \ {\rm d}\rho_{\mathcal{R}} \geq \int_{\mathcal{R}} u \circ \beta \ {\rm d}\nu + \int_{\partial\mathcal{R}} u \circ \beta \ {\rm d}\rho_{\mathcal{R}}\right). \tag{11}$$

In this representation, the value of an act conditional on a regular event $\mathcal{R}$ has two components, which correspond to the two ways in which $\mathcal{R}$ could be the outcome of an observation, as explained above. Here is a version of Theorem 3 for normal state spaces.

Theorem 4 *Let $\mathcal{S}$ be a (nonsingleton) normal Hausdorff space, let $\mathcal{X}$ be a connected space, and let $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch). *Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq), (C), (Dom), (Sep), (CCP) *and* (TC) *if and only if it admits a liminal charge SEU representation* (11), *where ν is a normal probability charge on $\mathfrak{A}(\mathcal{S})$ with full support, and $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is a normal liminal charge structure. Furthermore, ν and $\{\rho_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ are unique, and u is unique up to positive affine transformation.*

7 SEU representations on compactifications

In many economic applications, the relevant state space is not compact —for example, it could be an unbounded subset of $\mathbb{R}^N$. Theorem 3 cannot accommodate such state spaces. Theorem 4 can, but only by sacrificing countable additivity. We will now extend Theorem 3 to non-compact spaces by constructing *compactifications* of these spaces.

A Hausdorff space $\mathcal{S}$ is *locally compact* if every point in $\mathcal{S}$ has a compact neighbourhood. For example, every compact Hausdorff space is locally compact. Every topological manifold is locally compact. In particular, any open or closed subset of $\mathbb{R}^N$ is locally compact. (However, the set of rational numbers is *not* locally compact.) Every totally bounded, locally complete metric space is locally compact. In short: most topological spaces which would arise naturally in economic applications are locally compact.

For any other Hausdorff space $\mathcal{X}$, let $\mathcal{C}_{\mathrm{L}}(\mathcal{S},\mathcal{X})$ be the set of all continuous functions $\alpha : \mathcal{S}\longrightarrow\mathcal{X}$ which converge to some limit "at infinity" in the following sense: there exists $x \in \mathcal{X}$ such that, for any open neighbourhood $\mathcal{O} \subseteq \mathcal{X}$ around x, there is a compact subset $\mathcal{K} \subseteq \mathcal{S}$ such that $\alpha(\mathcal{S} \setminus \mathcal{K}) \subseteq \mathcal{O}$. When it exists, this limit x is unique and denoted $\lim_{\infty} f$.

Let $\mathcal{S}$ be a locally compact Hausdorff space. The *Alexandroff* (or *one-point*) *compactification* $\dot{\mathcal{S}}$ is the set $\mathcal{S} \sqcup \{\infty\}$ (where ∞ represents a "point at infinity") equipped with the smallest topology such that every open subset of $\mathcal{S}$ remains open in $\dot{\mathcal{S}}$, while the open neighbourhoods of ∞ are the sets $\dot{\mathcal{S}}\setminus\mathcal{K}$, where $\mathcal{K}$ is any compact subset of $\mathcal{S}$. For example, the Alexandroff compactification of $[0,\infty)$ is $[0,\infty]$. The Alexandroff compactification of $\mathbb{R}$ is homeomorphic to a circle. The Alexandroff compactification of $\mathbb{R}^2$ is homeomorphic to a sphere. In general, Alexandroff compactifications have the following properties:

(A1) $\dot{\mathcal{S}}$ is compact and Hausdorff. If $\mathcal{S}$ is noncompact, then $\mathcal{S}$ is a dense open subset of $\dot{\mathcal{S}}$. (Otherwise, if $\mathcal{S}$ is already compact, then ∞ is an isolated point of $\dot{\mathcal{S}}$.)

(A2) For any Hausdorff space $\mathcal{X}$, any function $\alpha \in \mathcal{C}_{\mathrm{L}}(\mathcal{S},\mathcal{X})$ has a unique extension $\dot{\alpha} \in \mathcal{C}(\dot{\mathcal{S}},\mathcal{X})$ defined by $\dot{\alpha}_{\upharpoonright\mathcal{S}} = \alpha$ and $\dot{\alpha}(\infty) = \lim_{\infty} \alpha$.

(A3) For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a unique $\dot{\mathcal{R}} \in \mathfrak{R}(\dot{\mathcal{S}})$ such that $\dot{\mathcal{R}} \cap \mathcal{S} = \mathcal{R}$.[6]

Let $\mathcal{X}$ be a Hausdorff space. Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{L}}(\mathcal{S},\mathcal{X})$ be a set of feasible acts , and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. An *Alexandroff SEU representation* for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is given by a normal Borel probability measure $\dot{\nu}$ on $\mathfrak{Bor}(\dot{\mathcal{S}})$ and a liminal density structure $\{\dot{\phi}_{\dot{\mathcal{R}}}\}_{\dot{\mathcal{R}}\in\mathfrak{R}(\dot{\mathcal{S}})}$ subordinate to $\dot{\nu}$, along with a continuous utility function $u : \mathcal{X}\longrightarrow\mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\left(\alpha \succeq_{\mathcal{R}} \beta\right) \iff \left(\int_{\dot{\mathcal{R}}} u \circ \dot{\alpha} \,\mathrm{d}\dot{\nu} + \int_{\partial\dot{\mathcal{R}}} (u \circ \dot{\alpha}) \cdot \dot{\phi}_{\dot{\mathcal{R}}} \,\mathrm{d}\dot{\nu} \;\geq\; \int_{\dot{\mathcal{R}}} u \circ \dot{\beta} \,\mathrm{d}\dot{\nu} + \int_{\partial\dot{\mathcal{R}}} (u \circ \dot{\beta}) \cdot \dot{\phi}_{\dot{\mathcal{R}}} \,\mathrm{d}\dot{\nu}\right). \tag{12}$$

[6] See Lemma 7.4(a) in Pivato and Vergopoulos (2018c).

A key difference between the Alexandroff SEU representation and the SEU representations of Sections 5 and 6 is in the unconditional preference order $\succeq_{\mathcal{S}}$. While both the residual charge and liminal density SEU representations yield a "classical" representation of type (6), formula (12) yields the following representation, for any $\alpha, \beta \in \mathcal{A}$:

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \iff \tag{13}$$
$$\left(\int_{\mathcal{S}} u \circ \alpha \ \mathrm{d}\dot{\nu} + \dot{\nu}\{\infty\} \cdot \lim_{\infty} (u \circ \alpha) \ \geq \ \int_{\mathcal{S}} u \circ \beta \ \mathrm{d}\dot{\nu} + \dot{\nu}\{\infty\} \cdot \lim_{\infty} (u \circ \beta)\right).$$

Thus, *ex ante* beliefs consist of two components: a Borel probability measure on $\mathcal{S}$, and an additional coefficient weighting the asymptotic utility of the acts "at infinity". Here is our first extension of Theorem 3 to locally compact spaces.

Theorem 5 *Let $\mathcal{S}$ be a noncompact, locally compact Hausdorff space, let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{L}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch). *Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies Axioms* (CEq), (C), (Dom), (Sep), (CCP) *and* (TC) *if and only if it admits an Alexandroff SEU representation* (12), *where $\dot{\nu}$ has full support on $\dot{\mathcal{S}}$. Furthermore, $\dot{\nu}$ is unique, the elements of $\{\dot{\phi}_{\dot{\mathcal{R}}}\}_{\dot{\mathcal{R}} \in \mathfrak{R}(\dot{\mathcal{S}})}$ are unique* ($\dot{\nu}$-almost everywhere)*, and u is unique up to positive affine transformation.*

The main advantage of Theorem 5 is that it applies when $\mathcal{S}$ is an *unbounded* space like $\mathbb{R}^N$, whereas Theorem 3 does not. The main *disadvantage* of Theorem 5 is that the conditional preference structure can only compare acts which converge "at infinity". The problem is that a conditional preference structure defined over a larger domain of acts may be sensitive to the asymptotic behaviour of these acts in a way which eludes an Alexandroff SEU representation. For example, if $\mathcal{S} = \mathbb{R}^2$, then the conditional preference structure could be sensitive in different ways to the asymptotic behaviour of acts along different curves, like $y = x^2$ and $y = x^3$. Intuitively, to capture such sensitivity with an SEU representation, we would need to introduce distinct "endpoints" for these two curves, and then assign different probabilities to these endpoints. But *no such distinct endpoints exist* in $\mathbb{R}^2$, or in its (spherical) Alexandroff compactification. To solve this problem, we must add a plethora of new states to $\mathcal{S}$, each acting like a distinct "point at infinity". To be precise, we must extend $\mathcal{S}$ to its *Stone-Čech compactification.*

Let $\mathcal{S}$ is a locally compact Hausdorff space. There is a unique compact Hausdorff space $\check{\mathcal{S}}$, called the *Stone-Čech compactification* of $\mathcal{S}$, with the following properties.

(SČ1) $\mathcal{S}$ is an open, dense subset of $\check{\mathcal{S}}$, and the native topology of $\mathcal{S}$ is the same as the subspace topology it inherits from $\check{\mathcal{S}}$.

(SČ2) For any compact Hausdorff space $\mathcal{K}$, and any continuous function $f : \mathcal{S} \longrightarrow \mathcal{K}$, there is a unique continuous function $\check{f} : \check{\mathcal{S}} \longrightarrow \mathcal{K}$ such that $\check{f}_{\upharpoonright \mathcal{S}} = f$.

(SČ3) For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a unique $\check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})$ such that $\check{\mathcal{R}} \cap \mathcal{S} = \mathcal{R}$. Furthermore, the mapping $\mathfrak{R}(\mathcal{S}) \ni \mathcal{R} \mapsto \check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})$ is a Boolean algebra isomorphism.[7]

Let $\mathcal{X}$ be another Hausdorff space. For any $\alpha \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, assertion (SČ2) says there is a unique function $\check{\alpha} \in \mathcal{C}(\check{\mathcal{S}}, \mathcal{X})$ such that $\check{\alpha}_{\restriction \mathcal{S}} = \alpha$. Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$, and let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$. A ***Stone-Čech SEU representation*** for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ is given by a normal Borel probability measure $\check{\nu}$ on $\mathfrak{Bor}(\check{\mathcal{S}})$ and a liminal density structure $\{\check{\phi}_{\check{\mathcal{R}}}\}_{\check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})}$ subordinate to $\check{\nu}$, along with a continuous utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{R})$,

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \quad \Longleftrightarrow \quad \Big(\int_{\check{\mathcal{R}}} u \circ \check{\alpha}\ \mathrm{d}\check{\nu} + \int_{\partial \check{\mathcal{R}}} (u \circ \check{\alpha})\, \check{\phi}_{\check{\mathcal{R}}}\ \mathrm{d}\check{\nu} \ \geq\ \int_{\check{\mathcal{R}}} u \circ \check{\beta}\ \mathrm{d}\check{\nu} + \int_{\partial \check{\mathcal{R}}} (u \circ \check{\beta})\, \check{\phi}_{\check{\mathcal{R}}}\ \mathrm{d}\check{\nu}\Big). \tag{14}$$

Similarly to formula (13), the Stone-Čech representation of $\succeq_{\mathcal{S}}$ yields *ex ante* beliefs that may assign some weight outside of $\mathcal{S}$. More precisely, for any $\alpha, \beta \in \mathcal{A}$, we have

$$\Big(\alpha \succeq_{\mathcal{S}} \beta\Big) \quad \Longleftrightarrow \quad \Big(\int_{\mathcal{S}} u \circ \alpha\ \mathrm{d}\check{\nu} + \int_{\check{\mathcal{S}} \setminus \mathcal{S}} u \circ \check{\alpha}\ \mathrm{d}\check{\nu} \ \geq\ \int_{\mathcal{S}} u \circ \beta\ \mathrm{d}\check{\nu} + \int_{\check{\mathcal{S}} \setminus \mathcal{S}} u \circ \check{\beta}\ \mathrm{d}\check{\nu}\Big). \tag{15}$$

The set $\check{\mathcal{S}} \setminus \mathcal{S}$ is called the ***corona*** —intuitively, this is the set of "points at infinity". These points play an essential role in the Stone-Čech representation. It is straightforward to construct examples of SEU representations where much of the probability weight lies in the corona (Pivato and Vergopoulos, 2018c, Examples 4.13 and 6.6). Our last theorem generalizes Theorem 5 by providing an SEU representation on locally compact state spaces without requiring feasible acts to have a limit at infinity.

Theorem 6 *Let $\mathcal{S}$ be a (nonsingleton) locally compact Hausdorff space, let $\mathcal{X}$ be a connected Hausdorff space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ satisfying* (Rch). *Then $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ satisfies* (CEq), (C), (Dom), (Sep), (CCP) *and* (TC) *if and only if it admits a Stone-Čech SEU representation* (14)*, where $\check{\nu}$ has full support on $\check{\mathcal{S}}$. Furthermore, $\check{\nu}$ is unique, the elements of $\{\check{\phi}_{\check{\mathcal{R}}}\}_{\check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})}$ are unique* ($\check{\nu}$-almost everywhere)*, and u is unique up to positive affine transformation.*

Theorems 5 and 6 are special cases of a large family of results. Let $\overline{\mathcal{S}}$ be any *compactification* of $\mathcal{S}$ —that is, a compact Hausdorff space which contains $\mathcal{S}$ as a dense subset. Let $\mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S}, \mathcal{X})$ be the set of continuous functions in $\mathcal{C}(\mathcal{S}, \mathcal{X})$ which can be extended to continuous functions in $\mathcal{C}(\overline{\mathcal{S}}, \mathcal{X})$. (For example, let $\mathcal{S} := \mathbb{R}$. Then $\overline{\mathbb{R}} := [-\infty, \infty]$ is a compactification, and $\mathcal{C}_{\overline{\mathbb{R}}}(\mathbb{R}, \mathcal{X})$ is the set of continuous functions from $\mathbb{R}$ to $\mathcal{X}$ which converge to (possibly

[7] Property (SČ2) holds if $\mathcal{S}$ is any *Tychonoff* space; see e.g. Theorem 19.5 of Willard (2004) or Theorem 2.79 of Aliprantis and Border (2006). But property (SČ3) only holds for the somewhat smaller class of locally compact Hausdorff spaces (Pivato and Vergopoulos, 2018c, Lemma 6.4(a)).

different) limits at $\pm\infty$.) If $\mathcal{A} \subseteq \mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S},\mathcal{X})$, and $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ is a conditional preference structure on $\mathcal{A}$ which satisfies (Rch), (CEq), (C), (Dom), (Sep), (CCP) and (TC), then we can obtain a normal Borel probability measure $\overline{\nu}$ on $\mathfrak{Bor}(\overline{\mathcal{S}})$, a liminal density structure $\{\overline{\phi}_{\overline{\mathcal{R}}}\}_{\overline{\mathcal{R}}\in\mathfrak{R}(\overline{\mathcal{S}})}$ subordinate to $\overline{\nu}$, and a continuous utility function $u:\mathcal{X}\longrightarrow\mathbb{R}$, which together yield an SEU representation for $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ analogous to representations (12) and (14).

There are generally *many* compactifications which will yield such a representation for a given conditional preference structure on $\mathcal{A}$. We just need $\overline{\mathcal{S}}$ to be "large enough" that $\mathcal{A}$ is contained in $\mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S},\mathcal{X})$. But if $\mathcal{S}$ is a locally compact Hausdorff space, then the set of all compactifications of $\mathcal{S}$ is a complete lattice (Engelking, 1989, Theorems 3.5.9-11, p.169). Thus, for any $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S},\mathcal{X})$, there exists a unique *minimal* compactification $\overline{\mathcal{S}}$ such that $\mathcal{A} \subseteq \mathcal{C}_{\overline{\mathcal{S}}}(\mathcal{S},\mathcal{X})$; this is the smallest compactification on which we can construct SEU representations for conditional preference structures on $\mathcal{A}$. For example, if $\mathcal{A} \subseteq \mathcal{C}_{\rm L}(\mathcal{S},\mathcal{X})$, then Theorem 5 says we can use the *smallest* compactification of $\mathcal{S}$, namely $\dot{\mathcal{S}}$. At the opposite extreme, if $\mathcal{A} = \mathcal{C}_{\rm b}(\mathcal{S},\mathcal{X})$, then we must use the *largest* compactification, which is $\check{\mathcal{S}}$. Other collections of feasible acts lead to other choices of compactification.

8 Prior literature

Several previous papers have derived continuous utility functions from preferences. For example, Grandmont (1972) obtained continuous utility functions in a von Neuman and Morgenstern (1947) framework. Other papers consider acts from a measurable state space into a topological outcome space, typically assumed to be connected and separable. For example, Wakker (1985, 1988) and Wakker (1989a, Chapter 5) characterized continuous, state-independent SEU representations in this setting. Wakker (1987) characterized continuous and state-*dependent* SEU over a finite state space, while Wakker and Zank (1999) characterized it over any measurable space. Wakker (1989b) characterized continuous Choquet expected utility representations over a finite state space. Finally, Casadesus-Masanell et al. (2000) characterized continuous maximin expected utility representations.

As far as we know, Zhou (1999) is the only previous paper to consider the case where both the state space and the outcome space are topological spaces, and acts are continuous functions. But unlike the present paper, Zhou restricts attention to the case where the outcomes are themselves lotteries over some finite set of consequences, so that acts correspond to "two-stage lotteries" of the kind considered by Anscombe and Aumann (1963). In this framework, Zhou proves versions of Anscombe and Aumann's SEU representation theorem, as well as the Choquet expected utility representation theorem of Schmeidler (1989), in both cases obtaining continuous utility functions. Unlike Zhou, we do not assume any special structure on the outcome space; our framework is more like the framework of Savage, rather than that of Anscombe and Aumann.

Section 7 obtained SEU representations by compactifying the state space $\mathcal{S}$ —in effect, we enlarged $\mathcal{S}$ by adding "ideal points at infinity". Such enlargement via "ideal points" has many precedents in decision theory. For example, Stinchcombe (1997) used such an enlargement to solve certain paradoxes which arise from the failure of countable additivity in the Savage SEU representation. In many models of ambiguity aversion, the agent's beliefs are not even finitely additive. But these failures of additivity in the original state space are sometimes consistent with finite additivity in some extended state space (Gilboa and Schmeidler, 1994). Other authors have used extended state spaces to distinguish between objective reality and the agent's internal representation of that reality. For example, Lipman (1999) augments the original state space with "impossible possible worlds" to model the agent's lack of logical omniscience. Jaffray and Wakker (1993) and Mukerji (1997) introduce "two-tiered" state spaces; in the model of Jaffray and Wakker, the agent has probabilistic beliefs about one tier and total ignorance about the other, whereas in Mukerji's model, one tier represents the agent's internal epistemic state and the other tier represents objectively payoff-relevant information. In a similar way, we could interpret $\mathcal{S}$ as the "true" state space and $\overline{\mathcal{S}}$ as the agent's internal model of this space; in this view, the extra elements of $\overline{\mathcal{S}} \setminus \mathcal{S}$ would be like the "impossible possible worlds" of Lipman (1999).

Chichilnisky (2000, 2009) has proposed a model of "catastrophic risks", where the agent's preferences are represented by a sum $\int_{\mathcal{S}} u \circ \alpha(s) \, \mathrm{d}\nu[s] + \Phi(u \circ \alpha)$. Here, the ν-integral represents subjective expected utility, while Φ is a linear functional that encodes sensitivity to catastrophic risks. One way to represent Φ is as an integral on the Stone-Čech compactification of the state space (Chichilnisky and Heal, 1997), so there is a clear similarity between Chichilnisky's representation and our Stone-Čech SEU representation (15) (although her axioms are very different from ours).

More recently, Alon (2015) has proposed a model of unawareness based on an augmented state space. In her model, the agent *knows* that the initial state space $\mathcal{S}$ is incomplete, but is unable to precisely describe the missing states. So she adds a single state s_0 to $\mathcal{S}$; the fact that s_0 obtains means that none of the states in $\mathcal{S}$ obtains. Each act α on $\mathcal{S}$ is extended into an act $\overline{\alpha}$ on $\overline{\mathcal{S}} = \mathcal{S} \cup \{s_0\}$ by defining $\overline{\alpha}(s_0)$ to be the *worst outcome* of α over $\mathcal{S}$. Thus, s_0 can again be interpreted as a "catastrophe". However, Alon's state space has no topology, so s_0 cannot be described in terms of a compactification.

Conclusion

This paper has presented a series of SEU representations for preferences under uncertainty. However, it is now well-established that the SEU model is often not descriptively accurate. In some cases, it may not even be normatively compelling, e.g. when the agent faces *ambiguity*, where she lacks even sufficient information to form probabilistic beliefs about the state of the world. Thus, there has been much recent interest in "non-SEU" models of decision-making under uncertainty. But there has been little exploration of such non-SEU models in explicitly topological environments like the ones considered in this paper.[8] In that respect, the results in the present paper can be seen as benchmarks, which set the stage for future research into non-SEU representations on topological spaces.

We have assumed that only *continuous* acts are feasible. This may seem unduly restrictive. Of course, Borel-measurable functions can be extremely complex, and it is unlikely

[8] Wakker (1989b), Casadesus-Masanell et al. (2000), and Zhou (1999) are exceptions. But the first two only put a topology on the outcome space.

that all such functions could be technologically feasible acts. But it seems plausible that *piecewise continuous* acts could be feasible (i.e. functions which are continuous on each cell of some regular partition of the state space). By restricting ourselves to *continuous* acts to obtain our SEU representations, we have actually solved a harder problem. It is straightforward to extend these SEU representations to preferences over piecewise continuous acts (Pivato and Vergopoulos, 2018b).

Appendix

Proof of Proposition 1. (a) Clearly, $\mathcal{R}_1, \ldots, \mathcal{R}_N$ are regular. It remains to show that they are disjoint and that $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_N$ is dense $\mathcal{S}$.

Disjoint (by contradiction) Suppose $\mathcal{R}_1, \ldots, \mathcal{R}_N$ are not disjoint. For simplicity, suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are not disjoint. Then $\mathcal{R}_2 \cap \mathcal{K}_1 \neq \emptyset$. Let $\mathcal{K}'_1 := \mathcal{K}_1 \setminus \mathcal{R}_2$; then $\mathcal{K}'_1$ is a closed proper subset of $\mathcal{K}_1$. However, $\mathcal{K}'_1 \cup \mathcal{K}_2 = \mathcal{K}_1 \cup \mathcal{K}_2$. (To see this, note that if $s \in \mathcal{K}_1$, then either $s \in \mathcal{K}'_1$, or $s \in \mathcal{R}_2$ in which case $s \in \mathcal{K}_2$; either way, $s \in \mathcal{K}'_1 \cup \mathcal{K}_2$.) Thus, $\mathcal{K}'_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$; hence $\{\mathcal{K}'_1, \mathcal{K}_2, \ldots, \mathcal{K}_N\}$ is a closed covering of $\mathcal{S}$. Let $g : \mathcal{S} \rightrightarrows [1 \ldots N]$ be the corresponding multifunction; then $g \in \mathrm{UHC}(\mathcal{S}, N)$ and $g \subset f$, contradicting the maximal decisiveness of f.

Density. For all $n \in [1 \ldots N]$, let $\mathcal{Q}_n := \mathrm{clos}(\mathcal{R}_n)$. Thus, $\mathrm{clos}(\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_N) = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_N$. Thus, $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_N$ is dense in $\mathcal{S}$ if and only if $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_N = \mathcal{S}$.

Now, $\mathcal{K}_2 \cup \cdots \cup \mathcal{K}_N$ is closed; thus, $(\mathcal{K}_2 \cup \cdots \cup \mathcal{K}_N)^\complement$ is open, and is a subset of $\mathcal{K}_1$, because $\mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$. Thus, $(\mathcal{K}_2 \cup \cdots \cup \mathcal{K}_N)^\complement \subseteq \mathcal{R}_1$, since $\mathcal{R}_1 = \mathrm{int}(\mathcal{K}_1)$. Thus, $\mathcal{R}_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$. Thus, $\mathcal{Q}_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$.

Next, $\mathcal{Q}_1 \cup \mathcal{K}_3 \cup \cdots \cup \mathcal{K}_N$ is closed; thus, $(\mathcal{Q}_1 \cup \mathcal{K}_3 \cup \cdots \cup \mathcal{K}_N)^\complement$ is open, and is a subset of $\mathcal{K}_2$, because $\mathcal{Q}_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$. Thus, $(\mathcal{Q}_1 \cup \mathcal{K}_3 \cup \cdots \cup \mathcal{K}_N)^\complement \subseteq \mathcal{R}_2$, since $\mathcal{R}_2 = \mathrm{int}(\mathcal{K}_2)$. Thus, $\mathcal{Q}_1 \cup \mathcal{R}_2 \cup \mathcal{K}_3 \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$. Thus, $\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{K}_3 \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$.

Next, let $m \in [2 \ldots N]$, and suppose (by induction) that we have shown that $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{m-1} \cup \mathcal{K}_m \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$. Now, $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{m-1} \cup \mathcal{K}_{m+1} \cup \cdots \cup \mathcal{K}_N$ is closed; thus, $(\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{m-1} \cup \mathcal{K}_{m+1} \cup \cdots \cup \mathcal{K}_N)^\complement$ is open, and is a subset of $\mathcal{K}_m$, because $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{m-1} \cup \mathcal{K}_m \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$ by hypothesis. Thus, $(\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{m-1} \cup \mathcal{K}_{m+1} \cup \cdots \cup \mathcal{K}_N)^\complement \subseteq \mathcal{R}_m$, because $\mathcal{R}_m = \mathrm{int}(\mathcal{K}_m)$. Thus, $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{m-1} \cup \mathcal{R}_m \cup \mathcal{K}_{m+1} \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$ Thus, $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{m-1} \cup \mathcal{Q}_m \cup \mathcal{K}_{m+1} \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$.

Proceeding by induction on m, we establish that $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_m \cup \mathcal{K}_{m+1} \cup \cdots \cup \mathcal{K}_N = \mathcal{S}$ for all $m \in [1 \ldots N]$. Setting $m = N$, we obtain $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_N = \mathcal{S}$, as desired.

(b) To see that $\{\mathcal{K}_1, \ldots, \mathcal{K}_N\}$ is a covering, note that $\mathcal{K}_1 \cup \cdots \cup \mathcal{K}_N = \mathrm{clos}(\mathcal{R}_1) \cup \cdots \cup \mathrm{clos}(\mathcal{R}_N) = \mathrm{clos}(\mathcal{R}_1 \sqcup \cdots \sqcup \mathcal{R}_N) = \mathcal{S}$, because $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_N$ is dense in $\mathcal{S}$ because $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ is a regular partition. Let $f : \mathcal{S} \rightrightarrows [1 \ldots N]$ be the multifunction defined by $\{\mathcal{K}_1, \ldots, \mathcal{K}_N\}$. Then f is upper hemicontinuous because $\mathcal{K}_1, \ldots, \mathcal{K}_N$ are closed.

Claim 1: *For all $n \in [1 \ldots N]$, f is single-valued on $\mathcal{R}_n$.*

Proof. Let $n, m \in [1 \ldots N]$, with $m \neq n$; it suffices to show that $\mathcal{R}_n$ is disjoint from $\mathcal{K}_m$. By hypothesis, $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ is a regular partition of $\mathcal{S}$; thus, $\mathcal{R}_n$ is disjoint from $\mathcal{R}_m$. Thus, $\mathcal{R}_n \subseteq \mathcal{R}_m^{\complement}$, and hence $\mathcal{R}_n \subseteq \mathrm{int}(\mathcal{R}_m^{\complement})$, because $\mathcal{R}_n$ is open. But $\mathrm{int}(\mathcal{R}_m^{\complement}) = \mathrm{clos}(\mathcal{R}_m)^{\complement}$. Thus, $\mathcal{R}_n$ is disjoint from $\mathrm{clos}(\mathcal{R}_m)$. But $\mathrm{clos}(\mathcal{R}_n) = \mathcal{K}_m$, by definition. Thus, $\mathcal{R}_n$ is disjoint from $\mathcal{K}_m$. This holds for all $m \neq n$. Thus, $f(s) = \{n\}$ for all $s \in \mathcal{R}_n$, as claimed. ◇ `Claim 1`

Let $g : \mathcal{S} \rightrightarrows [1 \ldots N]$ be another upper hemicontinuous multifunction such that $g \subseteq f$. To show that f is maximally decisive, it suffices to show that $g = f$. Let $s \in \mathcal{S}$ and let $\mathcal{M} := f(s)$, a subset of $[1 \ldots N]$. Thus, for all $m \in \mathcal{M}$, we have $s \in \mathcal{K}_m$. But $\mathcal{K}_m = \mathrm{clos}(\mathcal{R}_m)$, so there is a net $\{r_i^m\}_{i \in \mathcal{I}}$ in $\mathcal{R}_m$ converging to s (where $\mathcal{I}$ is some directed set). For all $i \in \mathcal{I}$, Claim 1 yields $f(r_i^m) = \{m\}$; thus, $g(r_i^m) = \{m\}$, because $g \subseteq f$ and $g(r_i^m) \neq \emptyset$. But then $m \in g(s)$, because g is upper hemicontinous. This argument holds for all $m \in \mathcal{M}$; thus, $\mathcal{M} \subseteq g(s)$. In other words, $f(s) \subseteq g(s)$. On the other hand, we must have $g(s) \subseteq f(s)$ because $g \subseteq f$. Thus, we conclude that $g(s) = f(s)$. This holds for all $s \in \mathcal{S}$; hence $g = f$.

(c) Let $\Phi : \mathrm{UHC}^*(\mathcal{S}, N) \longrightarrow \mathrm{RgPrt}(\mathcal{S}, N)$ and $\Psi : \mathrm{RgPrt}(\mathcal{S}, N) \longrightarrow \mathrm{UHC}^*(\mathcal{S}, N)$ be the mappings described in parts (a) and (b). We will show they are inverses. It follows that both are bijective.

Let $\mathfrak{P} = \{\mathcal{R}_1, \ldots, \mathcal{R}_N\} \in \mathrm{RgPrt}(\mathcal{S}, N)$, and let $f := \Psi(\mathfrak{P})$. Then $f : \mathcal{S} \rightrightarrows [1 \ldots N]$, and for all $n \in [1 \ldots N]$, if we define $\mathcal{K}_n := \{s \in \mathcal{S};\ n \in f(s)\}$, then $\mathcal{K}_n = \mathrm{clos}(\mathcal{R}_n)$, by the construction in part (b). Thus, $\mathcal{R}_n = \mathrm{int}(\mathcal{K}_n)$, because $\mathcal{R}_n$ is regular. Thus, $\mathfrak{P} = \Phi(f)$, by the construction in part (a).

Conversely, let $f \in \mathrm{UHC}^*(\mathcal{S}, N)$, let $\Phi(f) = \mathfrak{P} = \{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ and then let $g := \Psi(\mathfrak{P})$; we must show that $g = f$. For all $n \in [1 \ldots N]$, let $\mathcal{K}_n := \{s \in \mathcal{S};\ n \in f(s)\}$; then $\mathcal{R}_n = \mathrm{int}(\mathcal{K}_n)$. Meanwhile, let $\mathcal{Q}_n := \{s \in \mathcal{S};\ n \in g(s)\}$; then $\mathcal{Q}_n := \mathrm{clos}(\mathcal{R}_n)$, by the construction in part (b). Now, clearly $\mathcal{Q}_n \subseteq \mathcal{K}_n$, because $\mathcal{K}_n$ is a closed set that contains $\mathcal{R}_n$. Thus, $g \subseteq f$. But f is maximally decisive in $\mathrm{UHC}(\mathcal{S}, N)$; thus, $f = g$. □

A ***credence*** is a function $\mu : \mathfrak{R}(\mathcal{S}) \longrightarrow [0, 1]$ with $\mu[\mathcal{S}] = 1$, and such that for any disjoint $\mathcal{Q}, \mathcal{R} \in \mathfrak{R}(\mathcal{S})$, we have $\mu[\mathcal{Q} \vee \mathcal{R}] = \mu[\mathcal{Q}] + \mu[\mathcal{R}]$. In other words, a credence is like a "finitely additive probability measure" defined on the Boolean algebra $\mathfrak{R}(\mathcal{S})$. Given any bounded continuous function $f : \mathcal{S} \longrightarrow \mathbb{R}$ and any regular subset $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a natural definition of the "conditionally expected value" of f on $\mathcal{R}$ with respect to μ, which we will denote by $\mathbb{E}_{\mathcal{R}}^{\mu}[f]$. This conditional expectation operator satisfies the standard properties; for example, it is linear (i.e. $\mathbb{E}_{\mathcal{R}}^{\mu}[c_1 f_1 + c_2 f_2] = c_1 \mathbb{E}_{\mathcal{R}}^{\mu}[f_1] + c_2 \mathbb{E}_{\mathcal{R}}^{\mu}[f_2]$ for any functions f_1 and f_2 and constants $c_1, c_2 \in \mathbb{R}$), and if $\mathcal{R} = \mathcal{R}_1 \vee \mathcal{R}_2$ for some disjoint $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}(\mathcal{S})$, then $\mu[\mathcal{R}]\, \mathbb{E}_{\mathcal{R}}^{\mu}[f] = \mu[\mathcal{R}_1]\, \mathbb{E}_{\mathcal{R}_1}^{\mu}[f] + \mu[\mathcal{R}_2]\, \mathbb{E}_{\mathcal{R}_2}^{\mu}[f]$ (Pivato and Vergopoulos, 2018c, Theorem 4.4). Thus, if $u : \mathcal{X} \longrightarrow \mathbb{R}$ is a continuous "utility function", then for any bounded continuous "act" $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$, we can define the "conditional expected utility" $\mathbb{E}_{\mathcal{R}}^{\mu}[u \circ \alpha]$. The pair (u, μ) is a ***subjective expected utility*** (SEU) representation for a conditional preference structure

$\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$. if, for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{R})$, we have

$$\Big(\alpha \succeq_{\mathcal{R}} \beta\Big) \iff \Big(\mathbb{E}_{\mathcal{R}}\left[u \circ \alpha\right] \geq \mathbb{E}_{\mathcal{R}}\left[u \circ \beta\right]\Big). \tag{A1}$$

The following axiomatic characterizaton of such subjective expected utility representations is the main result of a companion paper (Pivato and Vergopoulos, 2018a). We will make essential use of this representation theorem in the proofs which follow.

Theorem A1 *Let $\mathcal{S}$ be a nondegenerate topological space, let $\mathcal{X}$ be a connected topological space, and let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$. Let $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch). *Then, it further satisfies Axioms* (CEq), (C), (Dom), (Sep), (CCP), *and* (TC) *if and only if it has an SEU representation (u, μ), where $u : \mathcal{X} \longrightarrow \mathbb{R}$ is a continuous function and μ is a credence on $\mathfrak{R}(\mathcal{S})$ with full support. Finally, μ is unique, and u is unique up to positive affine transformation.*

Proof. See Theorem 13 in (Pivato and Vergopoulos, 2018a). □

What is the relationship between credences and Borel probability measures? What is the relationship between the conditional expectation $\mathbb{E}^{\mu}_{\mathcal{R}}[f]$ and the Lebesgue integral? These questions are addressed in another companion paper (Pivato and Vergopoulos, 2018c). The following proofs proceed by combining the Theorem A1 with the main results of Pivato and Vergopoulos (2018c). We will refer to results in this paper with the prefix "PV". Thus, "Theorem PV-4.4" should be read as, "Theorem 4.4 from Pivato and Vergopoulos (2018c)."

Proof of Theorem 2. The proof is based upon a natural equivalence between credences and residual charges. Let $\nu : \mathfrak{Bai}(\mathcal{S}) \longrightarrow [0, 1]$ be a residual probability charge, as defined in Section 5. Recall that $\mathfrak{R}(\mathcal{S}) \subseteq \mathfrak{Bai}(\mathcal{S})$, but $\mathfrak{R}(\mathcal{S})$ is not a *subalgebra* of $\mathfrak{Bai}(\mathcal{S})$, because the Boolean algebra operations are different (i.e. $\vee$ vs. $\cup$). But if we restrict ν to $\mathfrak{R}(\mathcal{S})$, then we get a credence on $\mathfrak{R}(\mathcal{S})$.[9] Conversely, if $\mu : \mathfrak{R}(\mathcal{S}) \longrightarrow [0, 1]$ is a credence, then we can define a residual probability charge $\nu : \mathfrak{Bai}(\mathcal{S}) \longrightarrow [0, 1]$ in a natural way. For any $\mathcal{B} \in \mathfrak{Bai}(\mathcal{S})$, there is a unique $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ such that $\mathcal{B} \triangle \mathcal{R}$ is meager (Fremlin, 2004, §314Q). Define $\nu(\mathcal{B}) := \mu(\mathcal{R})$. It is easily verified that ν is a residual probability charge (see Proposition PV-3.8). Furthermore, the μ-conditional expectation structure on $\mathfrak{R}(\mathcal{S})$ satisfies

$$\mathbb{E}^{\mu}_{\mathcal{R}}[f] \quad = \quad \frac{1}{\nu[\mathcal{R}]} \int_{\mathcal{R}} f \; \mathrm{d}\nu, \tag{A2}$$

for any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ and $f \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$ (see Proposition PV-6.1).

"$\Longleftarrow$" Let ν be a residual probability charge on $\mathcal{S}$ with full support, and $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a continuous function that together provide an SEU representation of $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ as

[9] *Proof.* For any disjoint $\mathcal{R}, \mathcal{Q} \in \mathfrak{R}(\mathcal{S})$, we have $\mathcal{R} \vee \mathcal{Q} = \mathcal{R} \sqcup \mathcal{Q} \sqcup \mathcal{M}$, where $\mathcal{M}$ is meager. Thus, $\nu[\mathcal{R} \vee \mathcal{Q}] = \nu[\mathcal{R}] + \nu[\mathcal{Q}] + \nu[\mathcal{M}] = \nu[\mathcal{R}] + \nu[\mathcal{Q}]$, because $\nu[\mathcal{M}] = 0$, because ν is a residual charge.

in equation (2). Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained by restricting ν to $\mathfrak{R}(\mathcal{S})$ as explained above. By combining equations (2) and (A2), we obtain an SEU representation (A1). Moreover, ν has full support, so μ also has full support. Thus, $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ must satisfy all of axioms (CEq), (C), (Dom), (Sep), (CCP) and (TC), by Theorem A1.

"$\Longrightarrow$" If $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}(\mathcal{S})}$ satisfies Axioms (CEq), (C), (Dom), (Sep), (CCP) and (TC), then Theorem A1 says it has an SEU representation (A1) given by a credence μ on $\mathfrak{R}(\mathcal{S})$ with full support, and a continuous utility function $u : \mathcal{X}\longrightarrow\mathbb{R}$. Let $\mathbb{E}^{\mu}$ be the μ-compatible conditional expectation structure from Theorem PV-4.4. Since $\mathcal{S}$ is a Baire space, Propositions PV-3.8 and PV-6.1 yields a residual probability charge ν on $\mathcal{S}$ representing μ, as explained above. Combining equations (A1) and (A2), we obtain SEU representation (2). Since μ has full support, ν has also full support.

Finally, suppose that both (u, ν) and (u', ν') provide residual charge SEU representation of $\{\succeq_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{R}}$. Let μ and μ' be the credences obtained by restricting respectively ν and ν' to $\mathfrak{R}(\mathcal{S})$. Then, by combining these SEU representations with equation (A2), we deduce that (u, μ) and (u', μ') both provide SEU representations as in formula (A1). By uniqueness in Theorem A1, u and u' are positive affine transformations of each other. Moreover, μ and μ' are equal to each other. Thus, Proposition PV-3.8 implies that $\nu = \nu'$. □

Proof of Theorem 1. The proof is very similar to the proof of Theorem 2, but uses Proposition PV-3.7 instead of Proposition PV-3.8. □

Proof of Proposition 2. In Case (ii), the statement follows from Proposition 314Q(b) of Fremlin (2004) or Proposition 514I(b,f) of Fremlin (2008). In Case (i), it follows from Case (ii), along with Lemma 2 on p.274 of Givant and Halmos (2009). □

The proofs of Theorems 3, 4, 5 and 6 are very similar to the proof of Theorem 2, and we only briefly sketch them. They require the following lemma.

Lemma A2 **(a)** *Any nonsingleton Hausdorff space is nondegenerate.*

(b) *If $\mathcal{S}$ is a nondegenerate space, then $\mathfrak{R}(\mathcal{S})$ is nontrivial.*

(c) *Suppose $\mathcal{S}$ is either locally compact or normal Hausdorff. For any nonempty open $\mathcal{O} \subseteq \mathcal{S}$, there is a nonempty $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ with* $\text{clos}(\mathcal{R}) \subseteq \mathcal{O}$.

Proof. (a) Since $|\mathcal{S}| \geq 2$, there exist $s_1, s_2 \in \mathcal{S}$ with $s_1 \neq s_2$. Since $\mathcal{S}$ is Hausdorff, there are disjoint open neighbourhoods $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{S}$ around s_1 and s_2. Thus, $\mathcal{O}_1$ and $\mathcal{O}_2$ are both nonempty open subsets which are not dense in $\mathcal{S}$.

(b) Let $\mathcal{O} \subset \mathcal{S}$ be a nonempty, non-dense open subset. Let $\mathcal{C} := \mathcal{S} \setminus \mathcal{O}$. Then $\mathcal{C}$ is a proper closed subset of $\mathcal{S}$ with a nonempty interior (because $\mathcal{O}$ is not dense). Let $\mathcal{R} := \mathrm{int}(\mathcal{C})$. Then $\emptyset \neq \mathcal{R} \neq \mathcal{S}$, and $\mathcal{R}$ is regular, because it is the interior of a closed set. Thus, $\mathfrak{R}(\mathcal{S})$ is nontrivial.

(c) First suppose $\mathcal{S}$ is locally compact. Let $s \in \mathcal{O}$. By local compactness, $\mathcal{O}$ contains a compact subset $\mathcal{K}$ which is also a neighbourhood of s. Let $\mathcal{R} := \mathrm{int}(\mathcal{K})$. Then $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{K} \subseteq \mathcal{O}$, and $\mathcal{R} \neq \emptyset$, because $s \in \mathcal{R}$. Finally, $\mathcal{R}$ is regular, because it is the interior of the closed set $\mathcal{K}$.

Now suppose $\mathcal{S}$ is normal. If $\mathcal{O} = \mathcal{S}$, the statement is trivial. So assume $\mathcal{O} \neq \mathcal{S}$. Let $s \in \mathcal{O}$ and let $\mathcal{C} := \mathcal{S} \setminus \mathcal{O}$. Then $\mathcal{C}$ is closed, and $\{s\}$ is also closed, because $\mathcal{S}$ is Hausdorff. By normality, there exist disjoint open sets $\mathcal{U}_1, \mathcal{U}_2$ containing $\{s\}$ and $\mathcal{C}$. Let $\mathcal{K} = \mathrm{clos}(\mathcal{U}_1)$; the $\mathcal{K}$ is closed and disjoint from $\mathcal{U}_2$. Thus, $\mathcal{K}$ is disjoint from $\mathcal{C}$, so $\mathcal{K} \subseteq \mathcal{O}$. Let $\mathcal{R} := \mathrm{int}(\mathcal{K})$. Then $\mathcal{R}$ is regular (being the interior of a closed set), $\mathcal{R}$ is nonempty (it contains $\mathcal{U}_1$) and $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{K} \subseteq \mathcal{O}$. □

Proof of Theorem 3. By Lemma A2(a,b), $\mathfrak{R}(\mathcal{S})$ is nontrivial. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 2, with equations (PV-6T) and (PV-6U) and Corollary PV-6.7 playing respectively the same role as equation (A2) and Proposition PV-3.8. Finally, there is a minor difference in the proof of the full support property of the probability measure in the sufficiency of the axioms. Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained by applying Theorem A1. Let $\mathbb{E}^\mu$ be the μ-compatible conditional expectation structure from Theorem PV-4.4. Let ν be the Borel probability measure obtained by applying Corollary PV-6.7 to $\mathbb{E}^\mu$. Now, suppose $\nu[\mathcal{O}] = 0$ for some nonempty open subset $\mathcal{O}$ of $\mathcal{S}$. By Lemma A2(c), there exists a nonempty $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$ such that $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{O}$. Thus,

$$\begin{aligned} \mu(\mathcal{R}) \underset{\text{(a)}}{=} \nu(\mathcal{R}) + \int_{\partial\mathcal{R}} \phi_{\mathcal{R}} \, \mathrm{d}\nu \underset{\text{(b)}}{\leq} \nu(\mathcal{R}) + \nu(\partial\mathcal{R}) \\ = \nu(\mathrm{clos}(\mathcal{R})) \underset{\text{(c)}}{\leq} \nu(\mathcal{O}) = 0, \end{aligned}$$

where (a) is by Corollary PV-6.7, (b) is by equation (8) and (c) is because $\mathrm{clos}(\mathcal{R}) \subseteq \mathcal{O}$. But this contradicts the fact that μ has full support. Thus, $\nu[\mathcal{O}] > 0$ for every nonempty open subset $\mathcal{O}$ of $\mathcal{S}$, so ν has full support. □

Proof of Theorem 4. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 3, with equations (PV-6D) and (PV-6E) and Proposition PV-6.4 playing respectively the same role as equations (PV-6T) and (PV-6U) and Corollary PV-6.7. The proof of full support is as in Theorem 3. □

Proof of Theorem 5. The sufficiency and necessity of the axioms, as well as the uniqueness of the representation, are obtained as in Theorem 3, with equations (PV-7A) and (PV-7b), Theorem PV-7.2, and Example PV-7.3(a) playing respectively the same role as equations (PV-6T) and (PV-6U) and Corollary PV-6.7. However, in the sufficiency of the axioms, the proof that the Borel probability measure $\dot{\nu}$ on $\dot{\mathcal{S}}$ has full support is slightly different. Let μ be the credence on $\mathfrak{R}(\mathcal{S})$ obtained from Theorem A1. Let $\mathbb{E}^{\mu}$ be the μ-compatible conditional expectation structure from Theorem PV-4.4. Let $\dot{\nu}$ be the Borel probability measure on $\dot{\mathcal{S}}$ obtained by applying Theorem PV-7.2 to $\mathbb{E}^{\mu}$. Now, suppose $\dot{\nu}[\dot{\mathcal{O}}] = 0$ for some nonempty open subset $\dot{\mathcal{O}}$ of $\dot{\mathcal{S}}$. Since $\dot{\mathcal{S}}$ is compact Hausdorff, Lemma A2(c) gives a nonempty regular subset $\dot{\mathcal{R}} \in \mathfrak{R}(\dot{\mathcal{S}})$ such that $\mathrm{clos}_{\dot{\mathcal{S}}}(\dot{\mathcal{R}}) \subseteq \dot{\mathcal{O}}$. Define $\mathcal{R} := \dot{\mathcal{R}} \cap \mathcal{S}$. By Lemma PV-7.4(a), $\mathcal{R}$ is a regular subset of $\mathcal{S}$. By equation (PV-7A), we have $\mu(\mathcal{R}) = \dot{\nu}(\dot{\mathcal{R}}) + \int_{\partial \dot{\mathcal{R}}} \dot{\phi}_{\dot{\mathcal{R}}} \, \mathrm{d}\dot{\nu}_{\dot{\mathcal{R}}}$. At this point, we can proceed as in Theorem 4 to obtain $\mu(\mathcal{R}) = 0$, which contradicts the fact that μ has full support. Hence the full support of $\dot{\nu}$. □

Proof of Theorem 6. The proof is very similar to Theorem 5, but using Example PV-7.3(c) instead of Example PV-7.3(a). □

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