Subjective expected utility with imperfect perception

Pivato, Marcus and Vergopoulos, Vassili

THEMA, Université de Cergy-Pontoise, Paris School of Economics, and University Paris 1 Panthéon-Sorbonne

8 April 2018

Subjective expected utility with imperfect perception*

Marcus Pivato[†] and Vassili Vergopoulos[‡]

April 8, 2018

Abstract

In many decisions under uncertainty, there are constraints on both the available information and the feasible actions. The agent can only make certain observations of the state space, and she cannot make them with perfect accuracy —she has *imperfect perception*. Likewise, she can only perform acts that transform states *continuously* into outcomes, and perhaps satisfy other regularity conditions. To incorporate such constraints, we modify the Savage decision model by endowing the state space $\mathcal{S}$ and outcome space $\mathcal{X}$ with topological structures. We axiomatically characterize a Subjective Expected Utility (SEU) representation of conditional preferences, involving a continuous utility function on $\mathcal{X}$ (unique up to positive affine transformations), and a unique probability measure on a Boolean algebra $\mathfrak{B}$ of regular open subsets of $\mathcal{S}$. We also obtain SEU representations involving a Borel measure on the Stone space of $\mathfrak{B}$ — a "subjective" state space encoding the agent's imperfect perception.
Keywords: Subjective expected utility; imperfect perception; technological feasibility; topological space; continuous utility; regular open set; Borel measure.
JEL classification: D81.

1 Introduction

Consider the following decision problems.

(a) Doctor Ali is considering several drug treatment options for a patient. The efficacy of each drug is a continuous function of the patient's blood chemistry, blood pressure, and other physiological variables. But many of these variables are unknown and either they are unmeasurable, or the available instruments are unreliable and imprecise.

(b) Bryant Heavy Industries (BHI) is about to build a new factory. Several different factory designs are available, using different machines and production processes. The

*We thank Jean Baccelli, Jean-Marc Tallon and Peter Wakker for their detailed comments. We also thank Sophie Bade, Denis Bouyssou, Alain Chateauneuf, Eric Danan, Franz Dietrich, Adam Dominiak, Jürgen Eichberger, Spyros Galanis, Ani Guerdjikova, Simon Grant, Andrew Mackenzie, Michael Mandler, Efe Ok, Emre Ozdenoren, Michael Richter, Klaus Ritzberger and Jingyi Xue for many helpful suggestions.
[†]THEMA, Université de Cergy-Pontoise, supported by Labex MME-DII (ANR11-LBX-0023-01).
[‡]Paris School of Economics, and University Paris 1 Panthéon-Sorbonne.

future profitability of each design is a continuous function of the unpredictable market prices of several raw materials and of BHI's final products.

(c) The Climate Emergency Panel (CEP) is considering policies to deal with climate change, including decarbonisation, mitigation, and geoengineering. The effectiveness of each policy is a continuous function of atmospheric, ecological, and economic parameters. But many parameter values are unknown, and existing measurements are unreliable and imprecise.

In all three examples, an agent must make a choice under uncertainty. The Subjective Expected Utility (SEU) model is the standard paradigm for these kinds of decisions. In the classic axiomatic foundations of Savage (1954) (and most subsequent treatments), uncertainty is described by a state space, and "acts" are functions from this state space into a space of outcomes. But the Savage theory assumes that *all* possible functions from states into outcomes are feasible, so that an agent can meaningfully form preferences over them. This makes sense in decision problems where the state space has a *discrete* topology (e.g. bets on coins, dice, or urns; Arrow-Debreu economies). But in the three examples above, only *continuous* functions are feasible, because of the underlying technological constraints. There is no drug that will transform the patient's physiological state discontinuously into a health outcome. Likewise, there is no production process where profit is a discontinuous function of market prices. So it is ill-conceived and potentially misleading to suppose that an agent can form preferences over such infeasible acts.

Another feature of the three examples is the nature of the *information* available to the agent. In the standard Savage model, an agent can acquire information by observing an "event" —that is, a subset of the state space —and can form preferences conditional on this event. In the Savage model, *any* subset of the state space is a potentially observable event. But in the three examples above, agents can only acquire information using unreliable and imprecise measurement devices, and many events remain unobservable.

For example, Dr. Ali might have instruments to measure the patient's high-density lipoprotein cholesterol (HDLC) level, in milligrams per decilitre (mg/dL). Suppose her instrument can only measure concentration to the nearest mg/dL. If the instrument reports the cholesterol level as "8 mg/dL", this means that the measured value is somewhere between 7.5 and 8.5 mg/dL. Furthermore, the machine is susceptible to a random error of 0.3 mg/dL, so Dr. Ali can only be sure that the true level is between 7.2 and 8.8 mg/dL. If she carefully repeats the measurement several times, she can reduce this error down to 0.05 mg/dL, in which case she will know that the true value is between 7.45 and 8.55 mg/dL.

Dr. Ali can also measure the patient's low-density lipoprotein cholesterol (LDLC) level. Again, her instrument is imperfect, so it can only report an integer value, and is susceptible to error; a measurement of "11 mg/dL" means the measured value is between 10.5 and 11.5 mg/dL, which means that the true value is between 10.2 and 11.8 mg/dL. Unfortunately, to prescribe the correct medecine, she needs to know whether the *total* cholesterol level —the sum of HDLC and LDLC levels —is below 20 mg/dL, and she has no instrument which can directly measure this. According to her measurements, the ordered pair (HDLC,LDLC) is confined to the box $[7.45, 8.55] \times [10.2, 11.8]$. But the line described by the equation "HDLC + LDLC = 20 mg/dL" cuts through this box.

For another example, suppose that BHI has the option of investing in a production process that requires 0.1 kg of tellurium (Te) and 1 kg of ferro tungsten (FeW) per unit of output, and which will only be profitable if the combined cost of the two materials is below $70 per unit. After considerable market research (e.g. geological surveys of tellurium deposits in Kyrgyzstan), BHI believes the world price of Te will remain between $200/kg and $300/kg for the foreseeable future, while the price of FeW will be between $20/kg and $60/kg. Thus, the price vector (Te,FeW) must be in the box $[200, 300] \times [20, 60]$. But the line described by the equation "$0.1\,\mathrm{Te} + \mathrm{FeW} = \$70$" cuts diagonally through this box.

Furthermore, some events are simply impossible to observe. For example, it might be diagnostically useful for Dr. Ali to measure the patient's triglyceride levels, but perhaps she has no device which can do this. Likewise, the effects of different climate policies may depend partly on the structure of the methane clathrate deposits at the bottom of the Barents Sea. But it may be impossible to make any precise measurements of these deposits. So the CEP cannot condition its plans on such measurements.

One *could* apply the Savage approach directly to these kinds of decision problems. But this would require the agent to form preferences over infeasible acts and to condition these preferences on unobservable events. This undermines the plausibility of the preference relation, the axioms, and the resulting SEU representation —whether interpreted normatively or descriptively. For these reasons, imperfect perception and technological constraints require a substantial departure from the Savage framework; this is the topic of this paper. First, we introduce a new model of imperfect perception; then, we use this model to analyse decisions under uncertainty with imperfect perception and technological constraints. We do this by enriching Savage's state space and outcome space with topological structure.

Our model of imperfect perception has three aspects. First, we assume that the agent is only able to acquire information about the state by observing it through a *finite partition* of the state space. For example, Dr. Ali is only able to measure the patient's HDLC levels to the nearest mg/dL. It might be diagnostically useful to measure it to the nearest *microgram* per decilitre, but her instruments cannot do this. Second, only *some* finite partitions are observable. For example, Dr. Ali is only able to measure *certain* physiological variables; she has no way to directly measure total cholesterol or trigliceride levels. Likewise, BHI cannot directly estimate the price of the Te-FeW mixture, and the CEP cannot precisely measure the methane clathrate deposits. Finally, even for those measurements the agent *can* make, there may be a small amount of random error. The agent might be able to make this error arbitrarily small, but she cannot reduce it to zero. So Dr. Ali can only really determine that HDLC levels are between $7.5 - \epsilon$ and $8.5 + \epsilon$ mg/dL, for some small ϵ.

To capture these three aspects, we suppose that the agent's information about the state arrives through a *regular partition* of the state space $\mathcal{S}$. For example, there might be an open, continuous function ϕ from $\mathcal{S}$ to some closed interval $[a, b]$ of real numbers (representing some numerical "measurement", e.g. the HDLC level in mg/dL), along with a finite set of threshold values $a = h_0 < h_1 < \cdots < h_N = z$ such that the agent can only observe events of the form "$h_{n-1} - \epsilon < \phi(s) < h_n + \epsilon$" (for some $n \in [1 \ldots N]$), where s is the true state of the world and $\epsilon > 0$ is some measurement error. If she can make ϵ arbitrarily small, the agent can, in the limit, observe events of the form "$h_{n-1} \leq \phi(s) \leq h_n$". In other words, she can observe the events $\phi^{-1}[h_{n-1}, h_n]$, for all $n \in [1 \ldots N]$. These are closed subsets of $\mathcal{S}$, which overlap on their boundaries. For technical reasons, it is simpler (and for our purposes, equivalent) to represent these sets by their *interiors*, which are the events $\mathcal{R}_1, \ldots, \mathcal{R}_N$, where $\mathcal{R}_n := \phi^{-1}(h_{n-1}, h_n)$ (because ϕ is continuous and open). These are *regular subsets* of $\mathcal{S}$, and their union $\mathcal{R}_1 \sqcup \cdots \sqcup \mathcal{R}_N$ is *dense* in $\mathcal{S}$ —we call this a *regular partition* of $\mathcal{S}$. This is the basic unit of information available to the agent in our model.

Furthermore, only *certain* regular partitions might be available —for example, because only *certain* real-valued functions ϕ can be used in the above construction, because only certain "measurements" are technologically feasible. The collection of all feasible regular partitions generates a Boolean algebra $\mathfrak{B}$. This is the algebra of all events which are observable, either directly or indirectly, by the agent. Any conditional preferences she forms must be conditional on events in $\mathfrak{B}$. Meanwhile, our model represents technological constraints on the feasible actions by defining them as *continuous* functions from the state space onto the outcome space. The domain $\mathcal{A}$ of feasible acts need not contain *all* continuous functions; thus, our framework can incorporate further technological restrictions. The agent's conditional preferences rank acts in $\mathcal{A}$ conditional upon events in $\mathfrak{B}$.

Despite these limitations, our main results provide axiomatic characterizations of SEU representations for conditional preferences. In these representations, utility is a *continuous* function; thus, similar outcomes yield similar utility levels. This makes our representations particularly relevant to applications in economics and finance, which usually take continuity for granted (Gollier, 2001). Moreover, beliefs are described by what we call a *credence*, a structure like a finitely additive probability measure on the Boolean algebra $\mathfrak{B}$ (Theorem 1). Finally, under an additional assumption on $\mathfrak{B}$, beliefs can also be represented by a classical Borel probability measure on the *Stone space* of $\mathfrak{B}$ —a "subjective" state space extending the initial state space (Theorem 2).

Technological constraints introduce some obstacles into the axiomatization of SEU. For example, Savage's axioms (e.g. the *Sure Thing Principle*) and his construction of conditional preferences depend on the ability to splice any two acts on any binary partition of the state space. Furthermore, Savage obtains the subjective probability measure and utility function by restricting preferences to two-valued acts and finitely-valued acts respectively. But both spliced acts and finitely-valued acts are typically discontinuous, and hence inadmissible in our framework. Furthermore, Savage's axiom P6 (*Small event continuity*) relies on a rich collection of classical partitions of the state space. But in our model of imperfect perception, only certain *regular* partitions of the state space are available.

The rest of this paper is organized as follows: Section 2 presents our model of imperfect perception. Section 3 introduces the notation and terminology of our model of decisions under uncertainty. Section 4 introduces the six axioms used in our results. Section 5 presents the SEU representation, in terms of a credence on a Boolean subalgebra $\mathfrak{B}$ of regular sets, while Section 6 presents the Stonean SEU representation, which relies on a Borel probability measure on the Stone space of $\mathfrak{B}$. Section 7 contains variants and extensions of our SEU representations. Section 8 reviews prior literature. All the proofs are in the Appendices. Appendix A contains the proof of Theorem 1, which is the lynchpin result of the paper. Appendix B proves the variants and extensions of Theorem 1.

2 Imperfect perception

Let $\mathcal{S}$ be a topological space, which we interpret as the set of possible states of nature. An open subset $\mathcal{R} \subseteq \mathcal{S}$ is ***regular*** if $\mathcal{R}$ is the interior of its own closure. For example, any open interval is a regular subset of $\mathbb{R}$. The union of any two non-touching open intervals is also regular. However, the union $(0,1) \sqcup (1,2)$ is *not* regular, because the interior of its closure is the interval $(0,2)$. The interior of any closed set is regular.

The intersection of two regular subsets is another regular subset. Given any two regular subsets $\mathcal{D}, \mathcal{E} \subseteq \mathcal{S}$, we define their ***join*** to be $\mathcal{D} \vee \mathcal{E} := \text{int}[\text{clos}(\mathcal{D} \cup \mathcal{E})]$ (the interior of the closure of $\mathcal{D} \cup \mathcal{E}$). This is the smallest regular set containing both $\mathcal{D}$ and $\mathcal{E}$. Meanwhile, given a regular subset $\mathcal{D}$, we define $\neg\mathcal{D}$ to be the interior of $\mathcal{S} \setminus \mathcal{D}$ —another regular subset. The set $\mathfrak{R}(\mathcal{S})$ of all regular subsets of $\mathcal{S}$ forms a Boolean algebra under the operations $\vee$, $\cap$, and $\neg$ (Fremlin, 2004, Theorem 314P). For example, we noted that $(0,1) \sqcup (1,2)$ is not a regular subset of $\mathbb{R}$. But $(0,1) \vee (1,2) = (0,2)$ is indeed regular. Likewise, the set-theoretic complement of $(0,1)$ is not regular, but its interior, $\neg(0,1) = (-\infty, 0) \cup (1, \infty)$, is.[1]

A ***regular partition*** of $\mathcal{S}$ is a collection $\mathcal{R}_1, \ldots, \mathcal{R}_N$ of disjoint regular subsets such that $\mathcal{R}_1 \vee \cdots \vee \mathcal{R}_N = \mathcal{S}$ —equivalently, such that $\mathcal{R}_1 \sqcup \cdots \sqcup \mathcal{R}_N$ is dense in $\mathcal{S}$. As we already hinted in the introduction, we will represent the agent's imperfect perception of the state space by a collection such regular partitions —or more precisely, by the Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$ which is generated by them. We will illustrate this with two models.

2.1 Imperfect measurement technology

Let $\overline{\mathbb{R}} := [-\infty, \infty]$ be the extended real line, with the natural topology.[2] We will represent a "measurement" as a function $\phi : \mathcal{S} \longrightarrow \overline{\mathbb{R}}$ with two properties:[3]

(i) (*Stability*) Small changes in the state only cause small changes in the measured value.

(ii) (*Sensitivity*) For any state s in $\mathcal{S}$, any small change in the value measured at s can be achieved by some small perturbation of s.

Formally, property (i) means that ϕ is *continuous* everwhere on $\mathcal{S}$. Meanwhile, property (ii) means that ϕ is an *open* function. Thus (i) and (ii) together imply that ϕ is an open, continuous, $\overline{\mathbb{R}}$-valued function on $\mathcal{S}$. As noted in the introduction, the agent cannot perceive the *precise* value of ϕ. Real-life instruments do not display their measurements with infinitely many digits of precision. (Even if they did, humans could not absorb this information.) Thus, the agent's perception of the measurement value is filtered through some finite partition of $\overline{\mathbb{R}}$ into intervals. Formally, we define a ***measurement instrument***

[1] What we call "regular" sets are often called ***regular open*** sets. Symmetrically, a subset $\mathcal{Q} \subseteq \mathcal{S}$ is ***regular closed*** if $\mathcal{Q} = \text{clos}[\text{int}(\mathcal{Q})]$ —or equivalently, if $\mathcal{Q}^{\complement}$ is regular open. The regular closed sets form a Boolean algebra which is dual to the regular open sets. Thus, we could have developed the entire theory of this paper using regular *closed* sets instead of regular *open* sets.

[2] That is: $\mathbb{R}$ has the usual topology, while neighbourhoods of ∞ and $-\infty$ are of the form $(r, \infty]$ and $[-\infty, r)$, respectively, for any $r \in \mathbb{R}$. So $[-\infty, \infty]$ is homeomorphic to $[-1, 1]$ in the obvious way.

[3] We could equivalently regard ϕ as an open continuous function into a finite interval like $[0, 1]$, if desired.

to be an ordered pair $(\phi, \mathcal{H})$, where $\phi : \mathcal{S} \longrightarrow \overline{\mathbb{R}}$ is an open, continuous function, and $\mathcal{H} = \{h_0 < h_1 < h_2 < \cdots < h_N\} \subset \overline{\mathbb{R}}$ is a finite set of "threshold" values, where by convention we fix $h_0 := -\infty$ and $h_N := \infty$. We assume the agent is able to observe events of the form "the measured value is between h_{n-1} and h_n" for each $n \in [1 \ldots N]$.

Formally, this corresponds to the subset $\phi^{-1}[h_{n-1}, h_n]$ in $\mathcal{S}$. If the agent could make one or both of these inequalities sharp, then she could even perceive events like $\phi^{-1}[h_{n-1}, h_n)$ or $\phi^{-1}(h_{n-1}, h_n)$. But the measurement device inevitably has some small amount of error —call it ϵ. Thus, in practice, the agent is only able to observe events of the form $\phi^{-1}(h_{n-1} - \epsilon, h_n + \epsilon)$.[4] By carefully repeating the measurement many times, or by otherwise expending resources to increase precision, the agent can make ϵ arbitrarily small, but she cannot reduce it to zero. In the limit, she can observe the event $\mathcal{F}_n := \bigcap_{\epsilon > 0} \phi^{-1}(h_{n-1} - \epsilon, h_n + \epsilon)$. It is easily verified that in fact $\mathcal{F}_n = \phi^{-1}[h_{n-1}, h_n]$, a closed subset of $\mathcal{S}$.

The family $\{\mathcal{F}_1, \ldots, \mathcal{F}_N\}$ covers $\mathcal{S}$. But for all $n \in [1 \ldots N]$, the sets $\mathcal{F}_{n-1}$ and $\mathcal{F}_n$ overlap on their common boundary $\phi^{-1}\{h_n\}$. So $\{\mathcal{F}_1, \ldots, \mathcal{F}_N\}$ is *not* a partition of $\mathcal{S}$. For all $n \in [1 \ldots N]$, let $\mathcal{G}_n$ be the event that the state is *not* in $\mathcal{F}_m$ for any $m \neq n$. Formally,

$$\mathcal{G}_n \quad := \quad \mathcal{S} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \mathcal{F}_{n-1} \cup \mathcal{F}_{n+1} \cup \cdots \cup \mathcal{F}_N). \tag{1}$$

To understand this construction, note that $\mathcal{G}_n$ is precisely the set of states where the agent can be sure that $h_{n-1} \leq \phi(s) \leq h_n$, after sufficiently precise measurements. The following facts are easily verified: (i) $\mathcal{G}_1 = \phi^{-1}[-\infty, h_1)$, $\mathcal{G}_N = \phi^{-1}(h_{N-1}, \infty]$, and $\mathcal{G}_n = \phi^{-1}(h_{n-1}, h_n)$ for all $n \in [2 \ldots N-1]$; (ii) for all $n \in [1 \ldots N]$, $\mathcal{G}_n = \operatorname{int}[\mathcal{F}_n]$;[5] (iii) thus, $\mathcal{G}_n$ is a regular subset of $\mathcal{S}$; and (iv) the collection $\mathfrak{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_N\}$ is a regular partition of $\mathcal{S}$. We will use $\mathfrak{G}$ to represent the information the agent can obtain from the measurement instrument $(\phi, \mathcal{H})$. When we say "The agent observes $\mathcal{G}_n$", this means that the instrument has returned a reading which tells her that the measured value is in the interval $[h_{n-1}, h_n]$, which means that the *true* value is in the interval $(h_{n-1} - \epsilon, h_n + \epsilon)$ (for some arbitrarily small $\epsilon > 0$). Importantly, when the agent "observes" $\mathcal{G}_n$, the only thing she knows for *sure* is that the true state is in $\mathcal{F}_n$ —in particular, the state may lie on the *boundary* $\partial \mathcal{G}_n = \phi^{-1}\{h_{n-1}, h_n\}$.

A ***measurement technology*** is a collection $\mathcal{M}$ of measurement instruments. Let $\mathfrak{B}_{\mathcal{M}}$ be the Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$ generated by all sets of the form $\phi^{-1}(h_n, h_m)$, for any instrument $(\phi, \mathcal{H}) \in \mathcal{M}$ and any $h_n, h_m \in \mathcal{H}$, under application of the operations $\vee$, $\neg$, and $\cap$. This is the Boolean algebra of all possible events in the state space which the agent can observe through any combination of measurements with her technology.

To see why it is appropriate to close the observable events under $\vee$ and $\neg$, fix $n, m \in [1 \ldots N]$ with $n \leq m$ and let $\mathcal{G}_{n,m}$ be the event that the state is *not* in $\mathcal{F}_p$ for any $p \neq n, m$. Formally,

$$\mathcal{G}_{n,m} \quad := \quad \mathcal{S} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \mathcal{F}_{n-1} \cup \mathcal{F}_{n+1} \cup \cdots \mathcal{F}_{m-1} \cup \mathcal{F}_{m+1} \cup \ldots \cup \mathcal{F}_N).$$

Thus, $\mathcal{G}_{n,m}$ is the set of states where, after sufficiently precise measurements, the agent can be sure that the true state is either in $\mathcal{F}_n$ or in $\mathcal{F}_m$. It is easily verified that $\mathcal{G}_{n,m} = \mathcal{G}_n \vee \mathcal{G}_m$.

[4]Here we define $-\infty - \epsilon = -\infty$ and $\infty + \epsilon = \infty$.

[5]It is important here that ϕ is both continuous and *open*.

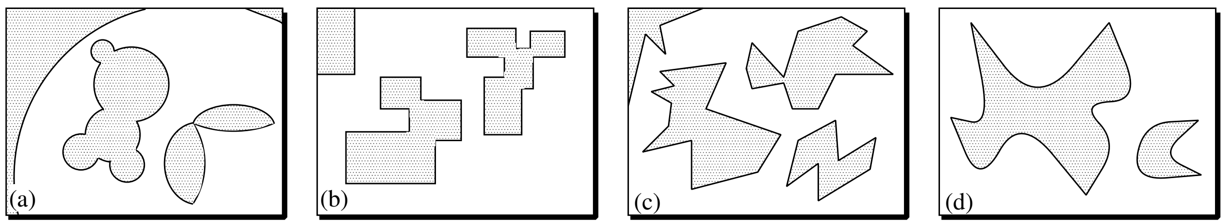

Figure 1: The Boolean subalgebras from Examples 2 to 5. (a) A typical element of $\mathfrak{B}_{\rm prx}(\mathbb{R}^2)$. (b) A typical element of $\mathfrak{B}_{\rm box}(\mathbb{R}^2)$. (c) A typical element of $\mathfrak{B}_{\rm poly}(\mathbb{R}^2)$. (d) A typical element of $\mathfrak{B}_{\rm smth}(\mathbb{R}^2)$. (Note that in each case, the *negation* of the shaded set is also an element of the algebra in question.)

Likewise, for any $n \in [1 \ldots N]$, let $\overline{\mathcal{G}}_n$ be the event that the state is *not* in $\mathcal{F}_n$. Formally, $\overline{\mathcal{G}}_n := \mathcal{S} \setminus \mathcal{F}_n$. It is then easy to see that $\overline{\mathcal{G}}_n = \neg\mathcal{G}_n$. Hence, the regular operations $\vee$ and $\neg$ represent the appropriate logical connectives for the set of observable events.

Typically, a measurement technology $\mathcal{M}$ has the form $\mathcal{M} = \Phi \times \mathfrak{H}$, where Φ is a set of all open, continous functions from $\mathcal{S}$ into $\overline{\mathbb{R}}$ satisfying certain "regularity" conditions, and where $\mathfrak{H}$ is the set of *all* finite subsets $\mathcal{H} = \{-\infty = h_0 < h_1 < h_2 < \cdots < h_N = \infty\}$. We define $\mathfrak{B}_\Phi := \mathfrak{B}_{\Phi\times\mathfrak{H}}$.

Example 1. Let $\mathcal{S} = \mathbb{R}$, and let Φ be the set of all open, continuous, $\mathbb{R}$-valued functions on $\mathcal{S}$. Let $\mathfrak{Bas}(\mathbb{R})$ be the Boolean algebra consisting of all *basic* sets; that is, finite disjoint unions of the form $(a_1, b_1) \sqcup (a_2, b_2) \sqcup \cdots \sqcup (a_N, b_N)$, where $-\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N \leq \infty$. Then $\mathfrak{Bas}(\mathbb{R})$ is the Boolean algebra of Φ-observable events. To see this, note that any open, continuous function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is either strictly increasing or strictly decreasing. Thus, for any $h, h' \in \overline{\mathbb{R}}$, the preimage $\phi^{-1}(h, h')$ is an open interval. Any element of $\mathfrak{B}_\Phi$ is a finite join of intersections of such intervals. ♢

Example 2. (*Proximity measurements*) Let $\mathcal{S} = \mathbb{R}^N$. For any $s \in \mathcal{S}$, define $\phi_s : \mathcal{S} \longrightarrow \mathbb{R}_+$ by setting $\phi_s(t) := d(s, t)$ for all $t \in \mathcal{S}$. Let $\Phi_{\rm prx} = \{\phi_s;\ \ s \in \mathcal{S}\}$. It is easily verified that these functions are all open and continuous.

Let $\mathfrak{B}_{\rm prx}(\mathbb{R}^N)$ be the collection of all regular subsets of $\mathbb{R}^N$ constructed by taking joins and/or intersections of finite collections of open balls and/or the complements of their closures. Then $\mathfrak{B}_{\rm prx}(\mathbb{R}^N)$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$. A typical element is shown in Figure 1(a). It is easily verified that $\mathfrak{B}_{\rm prx}(\mathbb{R}^N) = \mathfrak{B}_{\Phi_{\rm prx}}$. This algebra describes the information available to an agent whose measurement technology allows her to check whether the true state within a specified proximity of some target state. (For example, she can check the statement, "The true state is within distance 1.6 of the point $(0, 0, 0)$".) ♢

Example 3. (*Coordinate projections and boxes*) Again, let $\mathcal{S} = \mathbb{R}^N$, but now let $\Phi_{\rm box} := \{\pi_1, \pi_2, \ldots, \pi_N\}$, where $\pi_n : \mathbb{R}^N \longrightarrow \mathbb{R}$ is the projection onto the nth coordinate. Clearly these projections are open and continuous.

A subset of $\mathbb{R}^N$ is an *open box* if it is a Cartesian product of open intervals. Any open box is regular. The intersection of two open boxes is also an open box (if it is nonempty). Let $\mathfrak{B}_{\rm box}(\mathbb{R}^N)$ be the Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$ generated by open boxes. A typical element is shown in Figure 1(b). It is easily verified that $\mathfrak{B}_{\rm box}(\mathbb{R}^N) = \mathfrak{B}_{\Phi_{\rm box}}$. This algebra describes the information available to an agent whose measurement technology allows her to check whether any particular *coordinate* of the true state satisfies some strict inequality. (For example, she can check the statement, "The horizontal coordinate of the state is strictly between 1.16 and 3.24."). ♢

Example 4. (*Affine measurements and polyhedra*) Again, let $\mathcal{S} = \mathbb{R}^N$, but now let $\Phi_{\rm poly}$ be the set of all nonconstant affine functions from $\mathcal{S}$ to $\mathbb{R}$. (A function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is *affine* if $\phi = \phi_0 + r$ for some linear function $\phi_0 : \mathbb{R}^N \longrightarrow \mathbb{R}$ and some constant $r \in \mathbb{R}$.)

A subset $\mathcal{H} \subseteq \mathbb{R}^N$ is a *hyperplane* if there is a (nontrivial) linear function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that $\mathcal{H} := \phi^{-1}\{r\}$ for some $r \in \mathbb{R}$. A regular subset $\mathcal{R} \subseteq \mathbb{R}^N$ is a *polyhedron* if there is a finite collection $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ of hyperplanes such that $\partial\mathcal{R} = (\mathcal{H}_1 \cap \partial\mathcal{R}) \cup \cdots \cup (\mathcal{H}_N \cap \partial\mathcal{R})$. (Heuristically, each of the sets $\mathcal{H}_n \cap \partial\mathcal{R}$ is one of the "faces" of the polyhedron. Note that we do not require these polyhedra to be convex, or even connected.) Let $\mathfrak{B}_{\rm poly}(\mathbb{R}^N)$ be the set of regular polyhedra; then $\mathfrak{B}_{\rm poly}(\mathbb{R}^N)$ is a Boolean subalgebra of $\mathfrak{R}(\mathbb{R}^N)$. A typical element is shown in Figure 1(c). It is easily verified that $\mathfrak{B}_{\rm poly}(\mathbb{R}^N) = \mathfrak{B}_{\Phi_{\rm poly}}$. This algebra describes the information available to an agent whose measurement technology allows her to check whether the state satisfies any finite collection of strict linear inequalities.[6] ♢

Example 5. (*Differentiable measurements*) Let $\mathcal{S}$ be an open subset of $\mathbb{R}^N$, and let $\Phi_{\rm smth} := \{\phi : \mathcal{S} \longrightarrow \mathbb{R};\ \phi$ is differentiable and $\mathrm{d}\phi$ is everywhere nonzero$\}$. Then $\Phi_{\rm smth}$ is a collection of continuous, open functions (by the Open Mapping Theorem).

A subset $\mathcal{H} \subseteq \mathbb{R}^N$ is a *smooth hypersurface* if there is a differentiable function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that $\mathcal{H} := \phi^{-1}\{r\}$ for some $r \in \mathbb{R}$, and such that $\mathrm{d}\phi(h) \neq 0$ for all $h \in \mathcal{H}$. We will say that a regular subset $\mathcal{R} \subseteq \mathcal{S}$ has a *piecewise smooth boundary* if there is a finite collection $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ of smooth hypersurfaces such that $\partial\mathcal{R} = (\mathcal{H}_1 \cap \partial\mathcal{R}) \cup \cdots \cup (\mathcal{H}_N \cap \partial\mathcal{R})$. Let $\mathfrak{B}_{\rm smth}(\mathbb{R}^N)$ be the set of regular subsets of $\mathbb{R}^N$ with piecewise smooth boundaries. A typical element is shown in Figure 1(d). It is easily verified that $\mathfrak{B}_{\rm smth}(\mathbb{R}^N) = \mathfrak{B}_{\Phi_{\rm smth}}$. This algebra describes the information available to an agent whose measurement technology allows her to check whether the state satisfies any finite collection of strict inequalities based on differentiable functions. We normally assume that the output of any scientific instrument is a differentiable function of the true state of the world; thus, $\mathfrak{B}_{\rm smth}(\mathbb{R}^N)$ describes the information available through such scientific instruments.[7] ♢

In Examples 1 to 5, all measurements ranged over $\mathbb{R}$. But if $\mathcal{S}$ is a *compact* space, then we must allow $\overline{\mathbb{R}}$-valued measurements, because there are no open continuous functions from a compact space into $\mathbb{R}$. For instance, Example 5 can be generalized to any smooth manifold $\mathcal{S}$, by defining $\Phi_{\rm smth}$ to be the set of *Morse functions* on $\mathcal{S}$ (Pivato and Vergopoulos, 2018c, Example 4.2(b)). But if $\mathcal{S}$ is a compact manifold, then these functions must range over a closed interval (which we can take to be $\overline{\mathbb{R}}$ without loss of generality).

[6] This construction works if $\mathcal{S}$ is any topological vector space. But we must then stipulate that ϕ is *continuous* as well as linear.

[7] We can also construct subalgebras of $\mathfrak{B}_{\rm smth}(\mathbb{R}^N)$ using subsets of $\Phi_{\rm smth}$ (e.g. polynomials).

2.2 Imperfect observations of a metric space

The construction in Section 2.1 is fairly general, but it does not cover all cases of interest. First, it does not seem possible to obtain the entire Boolean algebra $\mathfrak{R}(\mathbb{R}^N)$ from a measurement technology. Second, if $\mathcal{S}$ is a totally disconnected space (e.g. a Cantor set), then there are *no* open continuous functions from $\mathcal{S}$ into $\overline{\mathbb{R}}$; thus, we cannot represent *any* regular subset of $\mathcal{S}$ in terms of such "measurement instruments". So we will now introduce a second and more general model of imperfect perception. In this case, we will suppose that the state space $\mathcal{S}$ is a metric space.

Let $\psi : \mathcal{S} \longrightarrow [1 \ldots N]$ be an arbitrary function, representing an "observation device". For all $n \in [1 \ldots N]$, let $\mathcal{P}_n := \psi^{-1}\{n\}$; then $\{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ be is a partition of $\mathcal{S}$ —that is, $\mathcal{P}_1, \ldots, \mathcal{P}_N$ are disjoint, and $\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_N = \mathcal{S}$. As in Section 2.1, these observations are subject to some error of size $\epsilon > 0$, but now this error is measured in terms of the metric of $\mathcal{S}$. If the device reports the reading "n", then the agent does not know for sure that the true state lies in $\mathcal{P}_n$; she only knows that it lies in the *ϵ-neighbourhood* $\mathcal{P}_n^\epsilon$, defined

$$\mathcal{P}_n^\epsilon \quad := \quad \{s \in \mathcal{S} \; ; \; d(s, p) < \epsilon \text{ for some } p \in \mathcal{P}_n\}. \tag{2}$$

Again, by making repeated, careful observations, the agent can make ϵ very small, but she cannot reduce it to zero. In the limit, she can observe the set

$$\mathcal{F}_n \quad := \quad \bigcap_{\epsilon > 0} \mathcal{P}_n^\epsilon. \tag{3}$$

It is easily verified that $\mathcal{F}_n := \text{clos}(\mathcal{P}_n)$.[8] Once again, the collection $\{\mathcal{F}_1, \ldots, \mathcal{F}_N\}$ covers $\mathcal{S}$, but it is is *not* a partition of $\mathcal{S}$, because these sets may overlap on the topological boundaries of the sets $\mathcal{P}_1, \ldots, \mathcal{P}_N$. For any $n \in [1 \ldots N]$, we could define $\mathcal{G}_n$ as in formula (1). But assertions (ii)-(iv) beneath formula (1) are not necessarily true. The reason is simple: if $\mathcal{P}_1, \ldots, \mathcal{P}_N$ are *arbitrary* subsets of $\mathcal{S}$, then their boundaries may cover large regions in $\mathcal{S}$. (Indeed, if $\mathcal{P}_n$ is *dense* in $\mathcal{S}$, then $\partial \mathcal{P}_n = \mathcal{S}$). Thus, the aforementioned ϵ-imprecision in observation, even in the limit when ϵ is reduced to zero, may almost completely destroy whatever information was carried in the original partition $\mathfrak{P}$.

To avoid this problem, the agent must make observations using "neat" partitions of $\mathcal{S}$. A subset $\mathcal{P} \subseteq \mathcal{S}$ is ***neat*** if (i) $\text{int}[\text{clos}(\mathcal{P})] \subseteq \mathcal{P}$ and (ii) $\mathcal{P} \subseteq \text{clos}[\text{int}(\mathcal{P})]$. Inclusion (ii) means that every element of $\mathcal{P}$ is a cluster point of its interior. Inclusion (i) is equivalent to saying that $\text{int}(\mathcal{P}) = \text{int}[\text{clos}(\mathcal{P})]$; in other words, the interior of $\mathcal{P}$ is as large as it can be, inside $\text{clos}(\mathcal{P})$. In particular, $\text{int}(\mathcal{P})$ is regular. It is easily verified that $\mathcal{P}$ is neat if and only if $\mathcal{P}^\complement$ is neat. (However, the collection of neat sets does not form a Boolean algebra.) Also, if $\mathcal{P}$ is neat, then its boundary $\partial \mathcal{P}$ is nowhere dense. A partition $\{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ is ***neat*** if $\mathcal{P}_1, \ldots, \mathcal{P}_N$ are neat. For example, if $(\phi, \mathcal{H})$ is a measurement technology (as defined in Section 2.1), then for all $h, h' \in \mathcal{H}$, the set $\phi^{-1}[h, h')$ is neat; thus, the partition $\{\phi^{-1}[-\infty, h_1), \ \phi^{-1}[h_1, h_2), \ \ldots, \ \phi^{-1}[h_{n-2}, h_{n-1}), \phi^{-1}[h_{n-1}, \infty]\}$ is neat.

[8] The argument of this section does *not* crucially rely on the existence of a metric on $\mathcal{S}$. Assuming only a topology, we would get the same results by directly defining $\mathcal{F}_n := \text{clos}(\mathcal{P}_n)$.

Let $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ be a neat partition. For all $n \in [1 \ldots N]$, define $\mathcal{F}_n$ using formulae (2) and (3), and then define $\mathcal{G}_n$ via formula (1). It is now easy to verify assertions (ii)-(iv) beneath formula (1). In particular, $\mathfrak{G} := \{\mathcal{G}_1, \ldots, \mathcal{G}_N\}$ is a regular partition of $\mathcal{S}$. We will use $\mathfrak{G}$ to represent the information the agent learns from the observation represented by $\mathfrak{P}$. When we say "The agent observes $\mathcal{G}_n$", this means her device reports "n", which only means that the *true* value is in the neighbourhood $\mathcal{P}_n^\epsilon$ (for some arbitrarily small $\epsilon > 0$). Importantly, when she "observes" $\mathcal{G}_n$, the only thing she knows for *sure* is that the true state is in $\mathcal{F}_n$ —in particular, the state may lie on the *boundary* $\partial\mathcal{G}_n$.

We define an *observation technology* to be a collection $\mathfrak{O}$ of neat partitions of $\mathcal{S}$; this can be seen as a generalization of the measurement technologies introduced in Section 2.1. Given an observation technology $\mathfrak{O}$, let $\mathfrak{B}_{\mathfrak{O}}$ be the subalgebra of $\mathfrak{R}(\mathcal{S})$ generated by all elements $\mathcal{G}_n$ defined using formula (1) as in the previous paragraph, under application of $\vee$, $\cap$ and $\neg$. This is the Boolean algebra of all possible events which the agent can learn through any combination of observations via her technology. In particular, if we allow $\mathfrak{O}$ to be the set of *all* neat partitions of $\mathcal{S}$, then $\mathfrak{B}_{\mathfrak{O}} = \mathfrak{R}(\mathcal{S})$.

There are several other models of "imperfect perception" which lead to regular partitions. In one of these, observations are represented by upper hemicontinuous multifunctions from $\mathcal{S}$ into a finite set. In another, observations are represented by *random* neat partitions of $\mathcal{S}$ (with the randomness concentrated on the boundaries). Finally, if we suppose that meager subsets of $\mathcal{S}$ have "measure zero", then we only need to define partitions "almost everywhere", as is typically done in classical probability theory. In this case, regular partitions emerge as canonical representatives of these a.e. equivalence classes. However, the technical details of these alternative interpretations are beyond the scope of this paper; we refer the reader to Pivato and Vergopoulos (2018a) for details.

3 Acts and preferences

Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces. As in Section 2, elements of $\mathcal{S}$ represent states of nature. Elements of $\mathcal{X}$ are called *outcomes*; they represent the possible consequences of actions. We will assume $\mathcal{X}$ is connected.

Information. We will represent the agent's imperfect perception by means of two Boolean subalgebras $\mathfrak{B} \subseteq \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} \subseteq \mathfrak{R}(\mathcal{X})$, as explained in Section 2. If $\mathcal{B} \in \mathfrak{B}$, then a $\mathfrak{B}$-*partition* of $\mathcal{B}$ is a collection $\{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ (for some $N \in \mathbb{N}$) of disjoint elements of $\mathfrak{B}$ such that $\mathcal{B} = \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_N$. We define $\mathfrak{D}$-partitions similarly. We suppose that the agent can only observe states and outcomes through $\mathfrak{B}$-partitions and $\mathfrak{D}$-partitions.

Acts. Like Savage, we will suppose that the agent can choose from a menu of *acts*, where each act is a function from the state space onto the outcome space. This function describes the outcome that would result from the choice of this act at each possible state of the world. Unlike Savage, we will assume that only *continuous* acts are feasible.

Recall that a subset $\mathcal{Y} \subseteq \mathcal{X}$ is ***relatively compact*** if its closure $\mathrm{clos}(\mathcal{Y})$ is compact. (It follows that any continuous, real-valued function on $\mathcal{X}$ is bounded when restricted to $\mathcal{Y}$.) For example, if $\mathcal{X}$ is a metric space, then $\mathcal{Y}$ is relatively compact if and only if $\mathcal{Y}$ is a bounded subset of $\mathcal{X}$. A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is ***bounded*** if its image $\alpha(\mathcal{S})$ is relatively compact in $\mathcal{X}$. If $\mathcal{X}$ is a metric space, then this agrees with the usual definition of "bounded". But this definition makes sense even if $\mathcal{X}$ is nonmetrizable. Let $\mathcal{C}(\mathcal{S}, \mathcal{X})$ be the set of all continuous functions from $\mathcal{S}$ into $\mathcal{X}$, and let $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$ be the set of all *bounded* continuous functions from $\mathcal{S}$ into $\mathcal{X}$. We will assume that all feasible acts lie in $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathcal{X})$.

An act can indirectly yield information about the state. To see this, let $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ be an act, and let $s \in \mathcal{S}$ be the state. The agent can acquire information about s by first applying α to s and then obtaining $\mathfrak{D}$-observable information about $\alpha(s)$. In a model with perfect perception, we would formalize this by saying that, for any $\mathcal{D} \in \mathfrak{D}$, the agent can check whether $\alpha(s)$ is in $\mathcal{D}$. But we are assuming *imperfect* perception. So the agent can only learn whether $\alpha(s)$ is in $\mathrm{clos}(\mathcal{D})$. Thus, if she is obtaining information about the state of the world via α, then she can only learn whether s is $\alpha^{-1}[\mathrm{clos}(\mathcal{D})]$. As in Section 2, we represent this observation with the *regular* set $\mathrm{int}\,(\alpha^{-1}[\mathrm{clos}(\mathcal{D})])$. Roughly speaking, we say that α is "comeasurable" if this observation conveys no new information about the state, beyond the information already contained in the algebra $\mathfrak{B}$. Formally, a continuous function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is ***comeasurable*** with respect to $\mathfrak{B}$ and $\mathfrak{D}$ (or $(\mathfrak{B}, \mathfrak{D})$-***comeasurable***) if $\mathrm{int}\,(\alpha^{-1}[\mathrm{clos}(\mathcal{D})]) \in \mathfrak{B}$ for all $\mathcal{D} \in \mathfrak{D}$.

Example 6. (a) If $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is any continuous function, then α is comeasurable with respect to $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{R}(\mathcal{X})$ (because the interior of any closed set is regular).

(b) Let $\mathcal{S} = \mathcal{X} = \mathbb{R}$, and let $\mathfrak{B} = \mathfrak{D} = \mathfrak{Bas}(\mathbb{R})$ be the Boolean algebra of basic sets from Example 1. A continuous function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is $(\mathfrak{B}, \mathfrak{D})$-comeasurable if there is a finite sequence of points $-\infty = r_0 < r_1 < r_2 < r_3 < \cdots < r_N = \infty$ such that for each $n \in [1 \ldots N]$, either ϕ is non-increasing on (r_{n-1}, r_n), or ϕ is non-decreasing on (r_{n-1}, r_n). In particular, any polynomial is $(\mathfrak{B}, \mathfrak{D})$-comeasurable, as is any non-decreasing or non-increasing continuous function. But $\phi(x) = \sin(x)$ is not $(\mathfrak{B}, \mathfrak{D})$-comeasurable.

(c) Let $\mathcal{S} = \mathbb{R}^N$ and $\mathcal{X} = \mathbb{R}^M$ and let $\mathfrak{B} = \mathfrak{B}_{\mathrm{poly}}(\mathbb{R}^N)$ and $\mathfrak{D} = \mathfrak{B}_{\mathrm{poly}}(\mathbb{R}^M)$ be the algebras of regular polyhedra, from Example 4. A function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is ***affine*** if $\phi = f_0 + r$ for some linear function $f_0 : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ and some constant $r \in \mathbb{R}^M$. We say ϕ is ***piecewise affine*** if there is a partition $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ of $\mathbb{R}^N$ into regular polyhedra, and a set $\phi^1, \ldots, \phi^N : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ of affine functions, such that $f_{\upharpoonright \mathcal{R}_n} = \phi^n_{\upharpoonright \mathcal{R}_n}$ for all $n \in [1 \ldots N]$. Any continuous piecewise affine function from $\mathbb{R}^N$ to $\mathbb{R}^M$ is $(\mathfrak{B}, \mathfrak{D})$-comeasurable.

(d) Again, let $\mathcal{S} = \mathbb{R}^N$ and $\mathcal{X} = \mathbb{R}^M$ and let $\mathfrak{B} = \mathfrak{B}_{\mathrm{smth}}(\mathbb{R}^N)$ and $\mathfrak{D} = \mathfrak{B}_{\mathrm{smth}}(\mathbb{R}^M)$ be the Boolean algebras of regular sets with piecewise smooth boundaries, from Example 5. If $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is any differentiable function such that the Jacobian matrix $\mathsf{D}\phi(s)$ is nonsingular for all $s \in \mathbb{R}^N$, then ϕ is $(\mathfrak{B}, \mathfrak{D})$-comeasurable. ♢

Let $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$ denote the set of bounded, $(\mathfrak{B}, \mathfrak{D})$-comeasurable and continuous functions from $\mathcal{S}$ to $\mathcal{X}$. We will assume that all feasible acts lie in $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$. There may be additional feasibility restrictions on acts, beyond boundedness, comeasurability and continuity. Thus, we introduce an exogenously given subset $\mathcal{A} \subseteq \mathcal{C}_b(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$; this is the set of *feasible acts*. In general, $\mathcal{A}$ could be much smaller than $\mathcal{C}_b(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$. For instance, in the case where $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$, if feasible production plans must be infinitely differentiable, then we could define $\mathcal{A}$ to be the set of all bounded and infinitely differentiable functions from $\mathcal{S}$ to $\mathcal{X}$. However, the collection $\mathcal{A}$ cannot be *too* small; it must be large enough to satisfy structural condition (Rch) below, and must contain all constant acts; these represents *riskless* alternatives. The inclusion of such acts in $\mathcal{A}$ means that we can risklessly obtain any outcome by a feasible act.

Conditional preference structures. Savage (1954) started from a preference order on the set of unconditional acts. He then obtained conditional preferences via axiom P2 (the *Sure Thing Principle*). Axiom P2 assumes that, for any two feasible acts α and β, and any event $\mathcal{B}$, the "spliced" act $\alpha_{\mathcal{B}}\beta$ (which is equal to α on $\mathcal{B}$ and to β on the complement $\mathcal{B}^{\complement}$) is also feasible. But such "spliced" acts are often discontinuous, hence, inadmissible in our framework. So instead of defining conditional preferences *implicitly* via P2, we must assume they exist explicitly. But we will only assume that these preferences can rank *feasible acts*, and we only assume preferences conditional on *observable* events. Thus, in terms of its primitive behavioral data, our model is not directly comparable to the Savage (1954) theory: while Savage assumed a *single* preference order on the universal domain of acts, our approach relies on a *collection* of preference orders on a more restrictive domain. But compared to other conditional versions of SEU (e.g. Ghirardato, 2002), our approach requires less data, both in terms of the number of preference orders and their domain.

For any $\mathcal{B} \in \mathfrak{B}$, and any $\alpha \in \mathcal{A}$, let $\alpha_{\upharpoonright\mathcal{B}}$ denote the restriction of α to a function on $\mathcal{B}$. Let $\mathcal{A}(\mathcal{B}) := \{\alpha_{\upharpoonright\mathcal{B}};\ \alpha \in \mathcal{A}\}$ be the set of acts conditional upon $\mathcal{B}$. Let $\succeq_{\mathcal{B}}$ be a preference order on $\mathcal{A}(\mathcal{B})$. We interpret $\succeq_{\mathcal{B}}$ as the *conditional preferences* over $\mathcal{A}(\mathcal{B})$ of an agent who, after sufficiently precise measurements, learns that the true state lies in the closure of $\mathcal{B}$. We will therefore refer to the system $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ as a *conditional preference structure*; this will be the primitive data of the model. Our goal is to axiomatically characterize an SEU representation for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$.

The richness condition. As already noted, the restriction to continuous acts means that we cannot rely on "spliced" acts the way that Savage did. Instead, we will require the set $\mathcal{A}$ of feasible acts to satisfy a "richness" condition with respect to the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$. Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ be disjoint regular subsets of $\mathcal{S}$. For any $\alpha_1 \in \mathcal{A}(\mathcal{B}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{B}_2)$, say that α_1 and α_2 are *compatible* if there is some $\alpha \in \mathcal{A}$ with $\alpha_{\upharpoonright\mathcal{B}_1} = \alpha_1$ and $\alpha_{\upharpoonright\mathcal{B}_2} = \alpha_2$. We need $\mathcal{A}$ to satisfy the following condition:

(Rch) For any disjoint regular subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, and any $\alpha_1 \in \mathcal{A}(\mathcal{B}_1)$ and $\alpha_2 \in \mathcal{A}(\mathcal{B}_2)$, there is an act $\beta_2 \in \mathcal{A}(\mathcal{B}_2)$ which is compatible with α_1, such that $\alpha_2 \approx_{\mathcal{B}_2} \beta_2$.

In other words, the values of an act on a regular subset $\mathcal{B}_1$ do not restrict the indifference class of that act conditional upon the disjoint regular subset $\mathcal{B}_2$, in spite of the continuity requirement on feasible acts. If there is a "gap" between $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathcal{S}$, then (Rch) is

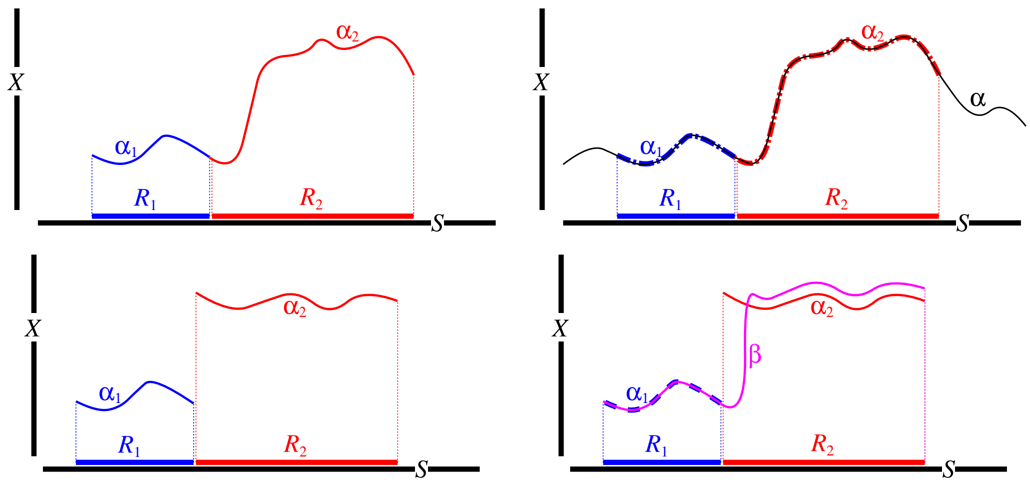

Figure 2: *Top row.* α_1 is compatible with α_2. *Bottom row.* The richness condition.

not very restrictive; often, *every* element of $\mathcal{A}(\mathcal{B}_2)$ is compatible with α_1. The nontrivial case of (Rch) is when $\mathcal{B}_1$ and $\mathcal{B}_2$ are "touching" –e.g. when $\mathcal{B}_1 = \neg\mathcal{B}_2$. In this case, (Rch) provides a weak version of Savage's act splicing: For any $\mathcal{B} \in \mathfrak{B}$, and any $\alpha, \beta \in \mathcal{A}$, there is some $\gamma \in \mathcal{A}$ that is equal to α on $\mathcal{B}$ and *indifferent* to $\beta_{\upharpoonright\neg\mathcal{B}}$ conditional on $\neg\mathcal{B}$. (Rch) is also similar to *solvability*, a condition often used in axiomatizations of additive utility.

$\mathcal{A}$ need not contain *all* bounded continuous functions from $\mathcal{S}$ to $\mathcal{X}$, as long as it satisfies (Rch) and contains all constant acts. For example, suppose $\mathcal{S}$ and $\mathcal{X}$ are differentiable manifolds (e.g. open subsets of Euclidean spaces $\mathbb{R}^N$ and $\mathbb{R}^M$, for some $N, M \geq 1$), and let $\mathcal{A}$ be the set of all *differentiable* functions from $\mathcal{S}$ to $\mathcal{C}$; then a conditional preference structure on $\mathcal{A}$ can easily satisfy (Rch) along with our other axioms.[9] Alternatively, let $\mathcal{S}$ and $\mathcal{X}$ be metric spaces, let $c \in (0, 1]$, and let $\mathcal{A}$ be the set of all *c-Hölder-continuous* functions from $\mathcal{S}$ to $\mathcal{X}$; then (Rch) is easily satisfied.[10] Or, let $\mathcal{S}$ be a bounded interval in $\mathbb{R}$, let $\mathcal{X}$ be a path-connected metric space, and let $\mathcal{A}$ be the set of all continuous functions from $\mathcal{S}$ into $\mathcal{X}$ having *bounded variation*; then again (Rch) is easily satisfied.[11] But if $\mathcal{S}$ and $\mathcal{X}$ are open subsets of Euclidean spaces, and $\mathcal{A}$ is a set of *analytic* functions from $\mathcal{S}$ to $\mathcal{X}$ (e.g. polynomials), then a conditional preference structure on $\mathcal{A}$ *cannot* satisfy (Rch).[12]

[9]The same is true if $\mathcal{A}$ is the set of N-times differentiable functions, for any $N \in [2 \ldots \infty]$.

[10]A function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *c-Hölder-continuous* if there is some constant $K > 0$ such that $d[\phi(s_1), \phi(s_2)] \leq K \cdot d(s_1, s_2)^c$ for all $s_1, s_2 \in \mathcal{S}$. In the special case when $c = 1$, these are called *Lipschitz-continuous* functions. Any continuously differentiable function is Lipschitz.

[11]A function $\alpha : [0, S] \longrightarrow \mathcal{X}$ has *bounded variation* if its "total variation" $\sup\{\sum_{n=1}^{N} d[\alpha(s_n), \alpha(s_{n-1})];\ N \in \mathbb{N}$ and $0 \leq s_0 < s_1 < \cdots < s_N \leq S\}$ is finite. Heuristically, this means that α does not oscillate too violently; it describes a path through $\mathcal{X}$ of finite total length.

[12]An infinitely differentiable function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is *analytic* if it is the limit of its own Taylor series in a neighbourhood around each point in $\mathcal{S}$. An analytic function can be completely reconstructed from its behaviour in a tiny neighbourhood around any point in its domain. This means that an analytic function defined on an open subset $\mathcal{R} \subseteq \mathcal{S}$ has at most *one* extension to an analytic function on all of $\mathcal{S}$.

4 Axioms

Throughout the paper, we assume that each order $\succeq_{\mathcal{B}}$ in the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ is *complete* (for any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, at least one of $\alpha \succeq_{\mathcal{B}} \beta$ or $\beta \succeq_{\mathcal{B}} \alpha$ holds), *transitive* (for any $\alpha, \beta, \gamma \in \mathcal{A}(\mathcal{B})$, if $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$, then $\alpha \succeq_{\mathcal{B}} \gamma$), and *nontrivial* (there exist $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ such that $\alpha \succ_{\mathcal{B}} \beta$). These assumptions are more natural in our framework than in Savage's: they only require a transitive ordering on *feasible* acts, not on all logically possible acts. To understand the implications of this distinction, consider a case where an agent observes event $\mathcal{B} \in \mathfrak{B}$, and must choose between two feasible acts α and γ in $\mathcal{A}(\mathcal{B})$. Suppose that she has preferences over unfeasible acts, and there is an unfeasible act β such that $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$. A blind application of transitivity would yield $\alpha \succeq_{\mathcal{B}} \gamma$. But the unfeasibility of β undermines the meaningfulness of both rankings $\alpha \succeq_{\mathcal{B}} \beta$ and $\beta \succeq_{\mathcal{B}} \gamma$. Why should these two rankings influence the choice between α and γ? By restricting preferences to feasible acts, we eliminate such spurious influences.

The separability axioms. Additive separability over disjoint events is a characteristic feature of SEU theories. In a Savage framework, it is captured by P2. In Ghirardato's (2002) axiomatization, where an agent is endowed with conditional preferences, separability is captured by the axiom of Dynamic Consistency. Dynamic Consistency also plays a central role in Hammond's (1988) derivation of SEU maximization on decision trees; see also Hammond (1998, §6-§7). Our next axiom captures separability through a version of Dynamic Consistency that only applies to regular partitions of a regular event.

(Sep) For any event $\mathcal{B} \in \mathfrak{B}$, any disjoint events $\mathcal{D}, \mathcal{E} \in \mathfrak{B}$ such that $\mathcal{D} \vee \mathcal{E} = \mathcal{B}$, and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ with $\alpha_{\upharpoonright\mathcal{D}} \approx_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$, we have $\alpha \succeq_{\mathcal{B}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \succeq_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$.

In (Sep), the "forward implication" (from $\alpha \succeq_{\mathcal{B}} \beta$ to $\alpha_{\upharpoonright\mathcal{E}} \succeq_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$) says that a feasible act that was deemed optimal conditional on $\mathcal{B}$ will be still be optimal conditional on $\mathcal{E}$. The "backward implication" says that a more-informed decision is more reliable than a less-informed decision; thus, decisions based on inferior information should be guided by the hypothetical decisions that *would* have been made with *superior* information. In this case, the agent should choose α over β given inferior information ($\mathcal{B}$), because she recognizes that she *would* be willing to choose α given superior information (either $\mathcal{D}$ or $\mathcal{E}$).

Just as the restriction to feasible acts strengthens the appeal of the ordering axiom, the restriction to events in $\mathfrak{B}$ strengthens the appeal of (Sep) —more specifically, its "backward implication". To see this, suppose the agent must choose between two feasible acts α and β, conditional on some $\mathcal{B} \in \mathfrak{B}$. Say that she has preferences conditional upon events $\mathcal{D}$ and $\mathcal{E}$, with $\mathcal{D} \vee \mathcal{E} = \mathcal{B}$, such that $\alpha_{\upharpoonright\mathcal{D}} \approx_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}} \succeq_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$. A naïve application of separability would then yield $\alpha \succeq_{\mathcal{B}} \beta$. But if $\mathcal{D}$ and $\mathcal{E}$ are not elements of $\mathfrak{B}$, then they are *unobservable* events, so it is not clear that the preferences conditional on $\mathcal{D}$ and $\mathcal{E}$ are even meaningful, much less that they should determine the choice between α and β. The restriction to $\mathfrak{B}$ eliminates this problem.

It is easy to see that the logical equivalence in Axiom (Sep) also holds for indifference and for strict preference: for any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ with $\alpha_{\upharpoonright\mathcal{D}} \approx_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$, we have:

(i) $\alpha \succ_{\mathcal{B}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \succ_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$; and

(ii) $\alpha \approx_{\mathcal{B}} \beta$ if and only if $\alpha_{\upharpoonright\mathcal{E}} \approx_{\mathcal{E}} \beta_{\upharpoonright\mathcal{E}}$.

Statement (i) means that *no event in* $\mathfrak{B}$ *is null*. Thus, any SEU representation must give nonzero probability to all events in $\mathfrak{B}$. Conversely, statement (ii) says that *the boundary of any event in* $\mathfrak{B}$ *is null*: the behaviour of α and β on that small part of $\mathcal{B}$ that is not covered by $\mathcal{D} \cup \mathcal{E}$ is irrelevant for decisions conditional on $\mathcal{B}$. This seems to suggest that the SEU representation must give *zero probability* to the boundary of any regular set. But $\mathcal{A}$ is a set of *continuous* functions; thus, the behaviour of α and β on the open sets $\mathcal{D}$ and $\mathcal{E}$ entirely determines their behaviour on the common boundary $\partial\mathcal{D} \cap \partial\mathcal{E}$. Thus, statement (ii) does not mean that we *ignore* the behaviour of α and β on $\partial\mathcal{D} \cap \partial\mathcal{E}$, as if $\partial\mathcal{D} \cap \partial\mathcal{E}$ had zero probability; it just means that we have *already* implicitly accounted for this behaviour in our rankings of $\alpha_{\upharpoonright\mathcal{D}}$ versus $\beta_{\upharpoonright\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}}$ versus $\beta_{\upharpoonright\mathcal{E}}$.

If $\mathcal{D}, \mathcal{E} \in \mathfrak{B}$ are disjoint and $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$, then Axiom (Sep) says that the $\succeq_{\mathcal{B}}$-ranking of two acts $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ is partly determined by the $\succeq_{\mathcal{D}}$-ranking of $\alpha_{\upharpoonright\mathcal{D}}$ versus $\beta_{\upharpoonright\mathcal{D}}$ and the $\succeq_{\mathcal{E}}$-ranking of $\alpha_{\upharpoonright\mathcal{E}}$ versus $\beta_{\upharpoonright\mathcal{E}}$. The next axiom says that this dependency is continuous.

(CCP) (*Continuity of conditional preferences*) Let $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$ as in axiom (Sep). Let $\underline{\beta}, \alpha, \overline{\beta} \in \mathcal{A}(\mathcal{B})$ be three acts with $\underline{\beta} \prec_{\mathcal{B}} \alpha \prec_{\mathcal{B}} \overline{\beta}$. Then there exist $\underline{\delta}, \overline{\delta} \in \mathcal{A}(\mathcal{D})$ and $\underline{\epsilon}, \overline{\epsilon} \in \mathcal{A}(\mathcal{E})$, with $\underline{\delta} \prec_{\mathcal{D}} \alpha_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{E}} \alpha_{\upharpoonright\mathcal{E}} \prec_{\mathcal{E}} \overline{\epsilon}$ such that, for any $\alpha' \in \mathcal{A}(\mathcal{B})$, if $\underline{\delta} \prec_{\mathcal{D}} \alpha'_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \overline{\delta}$ and $\underline{\epsilon} \prec_{\mathcal{E}} \alpha'_{\upharpoonright\mathcal{E}} \prec_{\mathcal{E}} \overline{\epsilon}$ then $\underline{\beta} \prec_{\mathcal{B}} \alpha' \prec_{\mathcal{B}} \overline{\beta}$.

The intuition here is that a small variation in $\alpha_{\upharpoonright\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}}$ (relative to the order topologies on $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\mathcal{E})$) should not affect the $\succeq_{\mathcal{B}}$- ranking of α versus $\underline{\beta}$ and $\overline{\beta}$.

Measurability of *ex post* preferences. For any $x \in \mathcal{X}$, let κ^x be the constant x-valued act on $\mathcal{S}$. Let $\mathcal{K} := \{\kappa^x;\ x \in \mathcal{X}\}$. We have assumed $\mathcal{K} \subseteq \mathcal{A}$, so the preference order $\succeq_{\mathcal{S}}$, restricted to $\mathcal{K}$, induces a preference order $\succeq_{\text{xp}}$ on $\mathcal{X}$ as follows: for any $x, y \in \mathcal{X}$,

$$\left(x \succeq_{\text{xp}} y\right) \iff \left(\kappa^x \succeq_{\mathcal{S}} \kappa^y\right). \tag{4}$$

$\succeq_{\text{xp}}$ describes the *ex post* preferences of the agent on $\mathcal{X}$ when there is no uncertainty. The next axiom says that these preferences are compatible with the Boolean subalgebra $\mathfrak{D}$.

(M) The *ex post* order $\succeq_{\text{xp}}$ is $\mathfrak{D}$-measurable. That is: for all $x \in \mathcal{X}$, the contour sets $\{y \in \mathcal{X};\ y \succ_{\text{xp}} x\}$ and $\{y \in \mathcal{X};\ y \prec_{\text{xp}} x\}$ are elements of $\mathfrak{D}$.

The intuition here is straightforward. The agent is aware of her own *ex post* preferences. So for any $x, y \in \mathcal{X}$, she can discern whether y satisfies the properties "$y \succ_{\text{xp}} x$" or "$y \prec_{\text{xp}} x$" —i.e. whether y belongs to the open upper or lower contour set of x. In other words, these open contour sets must be "observable" subsets of $\mathcal{X}$. But then they must belong to $\mathfrak{D}$. Moreover, note that axiom (M) implies the continuity of *ex post* preferences with respect to the topology on $\mathcal{X}$ (because every element of $\mathfrak{D}$ is open in $\mathcal{X}$).

Certainty equivalents. For any $\mathcal{B} \in \mathfrak{B}$ and $x \in \mathcal{X}$, let $\kappa^x_{\mathcal{B}} := (\kappa^x)_{\upharpoonright\mathcal{B}}$; this is the constant x-valued act, conditional on $\mathcal{B}$. Given an act $\alpha \in \mathcal{A}(\mathcal{B})$, we say x is a *certainty equivalent* for α on $\mathcal{B}$ if $\kappa^x_{\mathcal{B}} \approx_{\mathcal{B}} \alpha$. The next axiom is a mild richness condition on $\mathcal{X}$.

(CEq) For any event $\mathcal{B} \in \mathfrak{B}$, any act $\alpha \in \mathcal{A}(\mathcal{B})$ has a certainty equivalent on $\mathcal{B}$.

Axiom (CEq) may appear somewhat implausible or technical. But it is a logical consequence of the following axiom of "constant measurability" which may seem more natural and has an interpretation similar to that of axiom (M).

(CM) For any event $\mathcal{B} \in \mathfrak{B}$ and any act $\alpha \in \mathcal{A}(\mathcal{B})$, the sets $\{x \in \mathcal{X};\ \kappa^x_{\mathcal{B}} \succ_{\mathcal{B}} \alpha\}$ and $\{x \in \mathcal{X};\ \alpha \succ_{\mathcal{B}} \kappa^x_{\mathcal{B}}\}$ are elements of $\mathfrak{D}$.

If $\mathcal{X}$ is connected and $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$, then (CM) is equivalent to the conjunction of (M) and (CEq). So we could state our results with (CM) in place of (M) and (CEq).

The statewise dominance axiom. Our next axiom imposes some consistency between the agent's conditional preference structure and her *ex post* preferences. It says that the agent always prefers a statewise dominating act.

(Dom) For any $\mathcal{B} \in \mathfrak{B}$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, if $\alpha(b) \succeq_{\mathrm{xp}} \beta(b)$ for all $b \in \mathcal{B}$, then $\alpha \succeq_{\mathcal{B}} \beta$. Furthermore, for any $\mathcal{B} \in \mathfrak{B}$ and any $x, y \in \mathcal{X}$, if $x \succ_{\mathrm{xp}} y$, then $\kappa^x_{\mathcal{B}} \succ_{\mathcal{B}} \kappa^y_{\mathcal{B}}$.

Recall that, although the agent has observed the event $\mathcal{B}$, the state of the world might actually be in $\partial\mathcal{B}$. (Dom) says that the values of acts on $\partial\mathcal{B}$ do not matter for statewise dominance. To see why this is reasonable, recall that α and β are continuous functions on $\mathcal{B}$, so they have unique extensions to $\mathrm{clos}(\mathcal{B})$, and these extensions preserve weak statewise dominance. Thus, weak statewise dominance over $\mathcal{B}$ implies weak statewise dominance over $\mathrm{clos}(\mathcal{B})$; that is, over all the states that remain possible given the observation of $\mathcal{B}$.

(Dom) appears similar to (Sep), and thus to Savage's axiom P2. The difference is that (Sep) applies to regular partitions, while (Dom) applies to partitions into singleton sets, which, in general, are *not* regular. Thus, (Dom) cannot be obtained as a special case of (Sep). Axiom (Dom) is also related to Savage's axioms P3 and P7. Axiom P3 requires the ranking of outcomes to be independent of the events that yield the outcomes. (Dom) entails a similar form of state independence: it implies that $\succeq_{\mathcal{S}}$ can be replaced by $\succeq_{\mathcal{B}}$ for any $\mathcal{B} \in \mathfrak{B}$, in formula (4). Thus, the *ex post* preference orders obtained from different conditional preference orders must agree with one other. To see how (Dom) and P7 overlap, consider the special case of (Dom) where one of α or β is a constant act. In fact, under (CM) and as long as $\mathcal{A}$ only contains continuous and bounded acts, (Dom) *implies* P7.

Tradeoff consistency. To state the last axiom, we need some preliminary definitions. Let $\mathcal{B} \in \mathfrak{B}$, and let $\mathcal{Q} := \neg\mathcal{B}$. Consider an outcome $x \in \mathcal{X}$ and an act $\alpha \in \mathcal{A}(\mathcal{Q})$. Structural condition (Rch) yields an act $(x_{\mathcal{B}}\alpha) \in \mathcal{A}$ with two properties:

(B1) $(x_{\mathcal{B}}\alpha)_{\upharpoonright\mathcal{B}} \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$, and **(B2)** $(x_{\mathcal{B}}\alpha)_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \alpha$.

We will call $(x_{\mathcal{B}}\alpha)$ an (x,α)-*bet* for $\mathcal{B}$; if $\mathcal{B}$ obtains, this bet is indifferent to the outcome x, while it is indifferent to α conditional on the complement of $\mathcal{B}$. Note that $(x_{\mathcal{B}}\alpha)$ is not uniquely defined by (B1) and (B2). But if $(x_{\mathcal{B}}\alpha)$ and $(x_{\mathcal{B}}\alpha)'$ are two acts satisfying (B1) and (B2), then axiom (Sep) implies that $(x_{\mathcal{B}}\alpha) \approx_{\mathcal{S}} (x_{\mathcal{B}}\alpha)'$.

Fix now four outcomes $x, y, v, w \in \mathcal{X}$, and a regular subset $\mathcal{B} \in \mathfrak{B}$. Let $\mathcal{Q} := \neg\mathcal{B}$. We write $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ if there exist $\alpha, \beta \in \mathcal{A}(\mathcal{Q})$, an (x,α)-bet $(x_{\mathcal{B}}\alpha) \in \mathcal{A}$, a (y,β)-bet $(y_{\mathcal{B}}\beta) \in \mathcal{A}$, a (v,α)-bet $(v_{\mathcal{B}}\alpha) \in \mathcal{A}$ and a (w,β)-bet $(w_{\mathcal{B}}\beta) \in \mathcal{A}$ such that $(x_{\mathcal{B}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}}\beta)$ while $(v_{\mathcal{B}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}}\beta)$. By the remark in the previous paragraph, this implies that for *any* such bets $(x_{\mathcal{B}}\alpha), (y_{\mathcal{B}}\beta), (v_{\mathcal{B}}\alpha), (w_{\mathcal{B}}\beta) \in \mathcal{A}$, we have $(x_{\mathcal{B}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}}\beta)$ and $(v_{\mathcal{B}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}}\beta)$.

If $(x_{\mathcal{B}}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}}\beta)$, then the "gain" obtained by changing x to y on $\mathcal{B}$ is at least enough to compensate for the "loss" incurred by changing α to β on $\mathcal{Q}$. In contrast, if $(v_{\mathcal{B}}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}}\beta)$, then the gain obtained by changing v to w on $\mathcal{B}$ is at *most* enough to compensate for the loss incurred by changing α to β on $\mathcal{Q}$. Together, these two observations imply that the gain obtained from changing x to y on $\mathcal{B}$ is at least as large as the gain from changing v to w on $\mathcal{B}$; hence the notation $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$. If $\succeq_{\mathcal{S}}$ has an SEU representation with utility function u, then $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ means that $u(y) - u(x) \geq u(w) - u(v)$.

Conversely, we write $(x \overset{\mathcal{B}}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}}{\rightsquigarrow} w)$ if there exist $\gamma, \delta \in \mathcal{A}(\mathcal{Q})$, an (x,γ)-bet $(x_{\mathcal{B}}\gamma) \in \mathcal{A}$, a (y,δ)-bet $(y_{\mathcal{B}}\delta) \in \mathcal{A}$, a (v,γ)-bet $(v_{\mathcal{B}}\gamma) \in \mathcal{A}$ and a (w,δ)-bet $(w_{\mathcal{B}}\delta) \in \mathcal{A}$ such that $(x_{\mathcal{B}}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{B}}\delta)$ while $(v_{\mathcal{B}}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{B}}\delta)$. Again, this implies that $(x_{\mathcal{B}}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{B}}\delta)$ and $(v_{\mathcal{B}}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{B}}\delta)$ for *any* such bets $(x_{\mathcal{B}}\gamma), (y_{\mathcal{B}}\delta), (v_{\mathcal{B}}\gamma), (w_{\mathcal{B}}\delta) \in \mathcal{A}$. If $\succeq_{\mathcal{S}}$ had an SEU representation, then this means that $u(y) - u(x) < u(w) - u(v)$. Here is our final axiom:

(TC) For any two regular subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, there are no $x, y, v, w \in \mathcal{X}$ such that $(x \overset{\mathcal{B}_1}{\rightsquigarrow} y) \succeq (v \overset{\mathcal{B}_1}{\rightsquigarrow} w)$ while $(x \overset{\mathcal{B}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}_2}{\rightsquigarrow} w)$.

In the case $\mathcal{B}_1 = \mathcal{B}_2$, (TC) requires "tradeoff attitudes" over outcomes to be well-defined, independently of the acts that are used to reveal them. In the case $\mathcal{B}_1 \neq \mathcal{B}_2$, (TC) requires tradeoff attitudes at different regular subsets to be consistent with each other: they must be independent of the event over which outcomes are traded. Thus, tradeoff attitudes can be evaluated independently from the choice situation used to reveal them.

Previous axiomatizations of SEU using a tradeoff consistency axiom (e.g. Wakker's (1988) *Cardinal Coordinate Independence*) did not also require a separability axiom, because it was implied by tradeoff consistency. But our axiom (Sep) is not superseded by (TC); to the contrary, (Sep) is necessary to even *state* (TC). Axiom (TC) needs "bets"' satisfying conditions (B1) and (B2). To construct these bets, we use (Rch). But the resulting construction is non-unique. To show that this non-uniqueness doesn't matter in our formulation of (TC), we must invoke (Sep).

5 SEU representations

Beliefs. We will represent the probabilistic "beliefs" of the agent by a *credence* —a structure like a finitely additive probability measure on the Boolean algebra $\mathfrak{B}$. To be precise, a *credence* on $\mathfrak{B}$ is a function $\mu : \mathfrak{B} \longrightarrow [0,1]$ such that $\mu[\mathcal{S}] = 1$ and such that, for any finite collection $\{\mathcal{B}_n\}_{n=1}^N$ of disjoint elements of $\mathfrak{B}$, we have

$$\mu\left[\bigvee_{n=1}^N \mathcal{B}_n\right] \quad = \quad \sum_{n=1}^N \mu[\mathcal{B}_n]. \tag{5}$$

A credence μ behaves like an ordinary probability measure. For example, for any $\mathcal{B} \in \mathfrak{B}$ with $\mu[\mathcal{B}] > 0$, we can define the *conditional credence* $\mu_{\mathcal{B}}$ by setting $\mu_{\mathcal{B}}[\mathcal{R}] := \mu[\mathcal{B} \cap \mathcal{R}]/\mu[\mathcal{B}]$ for all $\mathcal{R} \in \mathfrak{B}$. Then $\mu_{\mathcal{B}}$ also satisfies equation (5). We say that a credence μ has *full support* if $\mu[\mathcal{B}] > 0$ for all nonempty $\mathcal{B} \in \mathfrak{B}$.

An important difference from the usual definition of a measure is that additivity is defined with respect to the operation $\vee$, rather than ordinary union. To explain this, it helps to reformulate the additivity property of probability in logical terms, as follows:

> If B and B' are two mutually exclusive statements, then the probability of the logical *disjunction* of B and B' is the sum of their probabilities.

Now, suppose that statements B and B' corresponds to some disjoint events $\mathcal{B}, \mathcal{B}' \in \mathfrak{B}$. As explained in Section 2.1, in our model of imperfect perception, the event corresponding to the disjunction of B and B' is not $\mathcal{B} \sqcup \mathcal{B}'$, but rather, $\mathcal{B} \vee \mathcal{B}'$. So the above principle requires that $\mu[\mathcal{B} \vee \mathcal{B}'] = \mu[\mathcal{B}] + \mu[\mathcal{B}']$.

Example 7. (a) Let $\mathcal{S} := (0,1)$, and let $\mathfrak{Bas}(0,1)$ be the Boolean algebra of basic open subsets of $(0,1)$, as defined in Example 1. For any $\mathcal{B} \in \mathfrak{Bas}(0,1)$, if $\mathcal{B} = (a,b)$ for some $a < b$, then let $\mu[\mathcal{B}] := b - a$. Next, if $\mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_N$ for some disjoint open intervals $\mathcal{B}_1, \ldots, \mathcal{B}_N$, then define $\mu[\mathcal{B}] := \mu[\mathcal{B}_1] + \cdots + \mu[\mathcal{B}_N]$. Then μ is a credence on $\mathfrak{Bas}(0,1)$.

(b) More generally, let $\mathcal{S} = (0,1)^N$, and let λ be the Lebesgue measure on $\mathcal{S}$. Let $\mathfrak{B}$ be the set of all regular subsets $\mathcal{B} \subseteq \mathcal{S}$ such that $\lambda[\partial\mathcal{B}] = 0$. It is easily verified that $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$. Define $\mu[\mathcal{B}] = \lambda[\mathcal{B}]$ for all $\mathcal{B} \in \mathfrak{B}$; then μ is a credence on $\mathfrak{B}$. (Pivato and Vergopoulos, 2018c, Proposition 3.4). The same construction works for any subalgebra of $\mathfrak{B}$; in particular, example (a) is a special case when $\mathcal{S} = (0,1)$. ♢

Conditional expectation structures. From any credence, we can construct a system of *expectation functionals*, which assign "expected values" to bounded, comeasurable and continuous real-valued measurements and, in particular, to the utility profiles of feasible acts. First, we need a bit of background. A continuous function $h : \mathcal{S} \longrightarrow \mathbb{R}$ is *$\mathfrak{B}$-comeasurable* if, for any $r \in \mathbb{R}$, we have $\mathrm{int}\,(h^{-1}(-\infty, r]) \in \mathfrak{B}$ and $\mathrm{int}\,[h^{-1}[r, \infty)] \in \mathfrak{B}$. Let $\mathcal{C}(\mathcal{S}, \mathbb{R})$ denote the vector space of all continuous, real-valued functions on $\mathcal{S}$. Let $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$ be the Banach space of *bounded*, continuous, real-valued functions, with the uniform norm $\|\cdot\|_\infty$. Let $\mathcal{C}_{\mathfrak{B}}(\mathcal{S})$ be the set of all $\mathfrak{B}$-comeasurable functions in $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$. This set is not necessarily closed under addition. So, let $\mathcal{G}_{\mathfrak{B}}(\mathcal{S})$ be the closed linear subspace of $\mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$ spanned by $\mathcal{C}_{\mathfrak{B}}(\mathcal{S})$. (If $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$, then $\mathcal{G}_{\mathfrak{B}}(\mathcal{S}) = \mathcal{C}_{\mathfrak{B}}(\mathcal{S}) = \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathbb{R})$.) For any subset $\mathcal{B} \subseteq \mathcal{S}$, let $\mathcal{G}_{\mathfrak{B}}(\mathcal{B}) := \{g_{\restriction\mathcal{B}};\ g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{S})\}$. This is a linear subspace of $\mathcal{C}_{\mathrm{b}}(\mathcal{B}, \mathbb{R})$.

An *expectation functional* on $\mathcal{B}$ is a linear functional $\mathbb{E} : \mathcal{G}_{\mathfrak{B}}(\mathcal{B}) \longrightarrow \mathbb{R}$ such that $\|\mathbb{E}\|_\infty = 1$, and such that, for any $f, g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, if $f(b) \leq g(b)$ for all $b \in \mathcal{B}$, then $\mathbb{E}[f] \leq \mathbb{E}[g]$. If $\mathbf{1}$ is the constant function with value 1, then it follows that $\mathbb{E}[\mathbf{1}] = 1$.

Now let μ be a credence on $\mathfrak{B}$. A *conditional expectation structure* for $\mathcal{S}$ that is *compatible with* μ is a collection $\mathbf{E} := \{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$, where, for all $\mathcal{B} \in \mathfrak{B}$, $\mathbb{E}_{\mathcal{B}}$ is an expectation functional on $\mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, and furthermore, if $\mu[\mathcal{B}] > 0$, then for any $\mathfrak{B}$-partition $\{\mathcal{B}_n\}_{n=1}^N$ of $\mathcal{B}$, and any $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, we require

$$\mathbb{E}_{\mathcal{B}}[g] \quad = \quad \frac{1}{\mu[\mathcal{B}]} \sum_{n=1}^{N} \mu[\mathcal{B}_n] \, \mathbb{E}_{\mathcal{B}_n} \left[g_{\restriction \mathcal{B}_n} \right]. \tag{6}$$

In particular, $\mathbb{E} = \mathbb{E}_{\mathcal{S}}$ is an expectation functional on $\mathcal{G}_{\mathfrak{B}}(\mathcal{S})$, and a version of equation (6) holds for every $\mathfrak{B}$-partition of $\mathcal{S}$. Equation (6) captures a key feature of Bayesianism: conditional expectations are additively separable over the disjoint events of a $\mathfrak{B}$-partition. Indeed, for any regular event $\mathcal{B} \in \mathfrak{B}$ with $\mu[\mathcal{B}] > 0$, the subcollection $\{\mathbb{E}_{\mathcal{R}}\}_{\mathcal{R}\in\mathfrak{B}'}$, where $\mathfrak{B}'$ is the collection of sets in $\mathfrak{B}$ that are contained in $\mathcal{B}$, is itself a conditional expectation structure on $\mathcal{B}$, compatible with the conditional credence $\mu_{\mathcal{B}}$.

If $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{S})$ and $\mathcal{B} \in \mathfrak{B}$, we will abuse notation and write "$\mathbb{E}_{\mathcal{B}}[g]$" to mean $\mathbb{E}_{\mathcal{B}}[g_{\restriction \mathcal{B}}]$. We say $\mathbf{E}$ is *strictly monotonic* if, for all $\mathcal{B} \in \mathfrak{B}$ and $g \in \mathcal{C}_{\mathfrak{B}}(\mathcal{B})$, if $g(b) > 0$ for all $b \in \mathcal{B}$, then $\mathbb{E}_{\mathcal{B}}[g] > 0$. For every credence μ, there is a unique compatible conditional expectation structure $\mathbf{E}$; furthermore, if μ has full support, then $\mathbf{E}$ is strictly monotonic (see Theorem 4.3 from Pivato and Vergopoulos (2018c)). For example, let $\mathcal{S} = (0,1)^N$, $\mathfrak{B} \subset \mathfrak{R}(\mathcal{S})$ and μ be as in Example 7(b). Then the unique μ-compatible conditional expectation structure is defined as follows: for any $\mathcal{B} \in \mathfrak{B}$ and any $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, $\mathbb{E}_{\mathcal{B}}[g] = \int_{\mathcal{B}} g \, \mathrm{d}\lambda$, where λ is the Lebesgue measure on $\mathcal{S}$. (Pivato and Vergopoulos, 2018c, Example 4.1). For other examples, see Pivato and Vergopoulos (2018c).

SEU representations. A continous function $u : \mathcal{X} \longrightarrow \mathbb{R}$ is $\mathfrak{D}$-*measurable* if, for all $r \in \mathbb{R}$, the preimage sets $h^{-1}(-\infty, r)$ and $h^{-1}(r, \infty)$ are elements of $\mathfrak{D}$. Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$, and let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be a $\mathfrak{B}$-indexed conditional preference structure on $\mathcal{A}$. Let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a continuous $\mathfrak{D}$-measurable "utility" function. Let μ be a credence on $\mathfrak{B}$. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the (unique) conditional expectation structure that is compatible with μ. The pair (u, μ) is a *subjective expected utility* (SEU) representation for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ if, for any $\mathcal{B} \in \mathfrak{B}$ and any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, we have

$$\Big(\alpha \succeq_{\mathcal{B}} \beta\Big) \iff \Big(\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \geq \mathbb{E}_{\mathcal{B}}[u \circ \beta]\Big). \tag{7}$$

Since we suppose that the utility function u is $\mathfrak{D}$-measurable, and that all acts $\alpha \in \mathcal{A}$ are $(\mathfrak{B}, \mathfrak{D})$-comeasurable, it follows that all utility profiles $u \circ \alpha : \mathcal{S} \longrightarrow \mathbb{R}$ are $\mathfrak{B}$-comeasurable (Pivato and Vergopoulos, 2018c, Proposition 5.4(a)). Thus, all utility profiles of feasible acts can be evaluated by any conditional expectation structure compatible with a credence on $\mathfrak{B}$, and the SEU representation (7) is well-defined. In this SEU representation, the utility function is continuous, but it need not be bounded, unlike Savage's utility. This is because we assume only *bounded* acts are feasible. We now come to our first main result.

Theorem 1 *Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces, with $\mathcal{X}$ connected. Let $\mathfrak{B}$ and $\mathfrak{D}$ be non-trivial Boolean subalgebras of $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{R}(\mathcal{X})$ respectively. Let $\mathcal{A}$ be a collection of bounded continuous and $(\mathfrak{B}, \mathfrak{D})$-comeasurable functions from $\mathcal{S}$ into $\mathcal{X}$. Let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be a conditional preference structure on $\mathcal{A}$ which satisfies condition* (Rch)*. Then, it satisfies* (CEq)*,* (M)*,* (Dom)*,* (Sep)*,* (CCP)*, and* (TC) *if and only if it has an SEU representation* (7) *with full support. Finally, μ is unique, and u is unique up to positive affine transformation.*

The SEU representation (7) axiomatically characterized in Theorem 1 portrays an agent facing three sorts of technological constraints. First, she only has limited information about states and outcomes —perhaps in the form of a small collection of real-valued functions representing "feasible measurements". Even worse, due to limitations in her instruments or her own perception, she cannot even make these measurements precisely; instead of a continuum of measurement values, she can only discriminate a finite set of "measurement intervals", which determine regular partitions of $\mathcal{S}$ and $\mathcal{X}$. These two informational limitations mean that she can only perceive events arising from some Boolean subalgebras $\mathfrak{B}$ and $\mathfrak{D}$ of regular subsets of $\mathcal{S}$ and $\mathcal{X}$. Finally, her actions are restricted to a collection $\mathcal{A}$ of continuous acts, which furthermore must be comeasurable relative to $\mathfrak{B}$ and $\mathfrak{D}$. However, if she defines probabilistic beliefs in the form of a "credence" on $\mathfrak{B}$, then she can compute the expected utility of every element of $\mathcal{A}$, conditional on any event in $\mathfrak{B}$, and in this way, she can define a conditional preference structure on $\mathcal{A}$. Theorem 1 says that, given certain rationality axioms, every conditional preference structure arises in this way.

Sketch of proof. We will sketch the main steps in the construction of the SEU representation from the axioms. In the first step, we fix a $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ of $\mathcal{S}$ and use (Rch) to construct a mapping $\Phi_{\mathfrak{P}} : \mathcal{X}^N \longrightarrow \mathcal{A}$ such that $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright\mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa_{\mathcal{E}_n}^{x_n}$ for any $n \in [1 \ldots N]$ and any $\mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$. This mapping induces a preference order $\succeq_{\mathfrak{P}}$ on $\mathcal{X}^N$ in the following way: For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \succeq_{\mathcal{S}} \Phi_{\mathfrak{P}}(\mathbf{y})\Big).$$

We then invoke (CCP), (M), (Dom) and (CEq) to show that this new preference order $\succeq_{\mathfrak{P}}$ is continuous. By (TC), it also satisfies Cardinal Coordinate Independence. Since $\mathcal{X}$ is connected, Theorem 6.2 from Wakker (1988) provides a continuous function $u_{\mathfrak{P}} : \mathcal{X} \longrightarrow \mathbb{R}$ and a probability vector $(\mu_{\mathfrak{P}}(\mathcal{E}_1), \ldots, \mu_{\mathfrak{P}}(\mathcal{E}_N)) \in \Delta([1 \ldots N])$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{E}_n) \cdot u_{\mathfrak{P}}(x_n) \geq \sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{E}_n) \cdot u_{\mathfrak{P}}(y_n)\right).$$

In the next step, we show that the functions $u_{\mathfrak{P}}$ can be taken to be independent of $\mathfrak{P}$ and that the numbers $\mu_{\mathfrak{P}}(\mathcal{E}_n)$ are independent from $\mathfrak{P}$, provided $\mathfrak{P}$ contains $\mathcal{E}_n$ as a cell. This follows from (Sep) and the uniqueness of Wakker's theorem. Thus, we obtain a continuous function u from $\mathcal{X}$ to $\mathbb{R}$, which can be shown to be $\mathfrak{D}$-measurable by axiom (M). We also obtain a function μ from $\mathfrak{B}$ to $[0, 1]$, which is a credence with full support.

Finally, we show that the expected utility of any feasible act conditional on any observable event is equal to the utility of any of its certainty equivalents. The argument uses (Sep) and (Dom), and relies on the boundedness and comeasurability of feasible acts. From there, since (CEq) provides certainty equivalents, the SEU representation easily follows.

6 Stonean SEU representations

The Stone Representation Theorem shows that any Boolean algebra $\mathfrak{B}$ can be represented as a Boolean algebra of subsets of some set —the *Stone space* of $\mathfrak{B}$. This can be used to obtain an alternative SEU representation where the agent's beliefs are represented by the more familiar notion of a Borel probability measure.

Borel probability measures. Let $\mathcal{S}$ be a topological space. Let $\mathfrak{Bor}(\mathcal{S})$ be the Borel sigma-algebra of $\mathcal{S}$ —that is, the smallest sigma-algebra containing all open sets. A *Borel probability measure* on $\mathcal{S}$ is a (countably additive) probability measure on $\mathfrak{Bor}(\mathcal{S})$. A Borel probability measure μ is *normal* if, for every $\mathcal{B} \in \mathfrak{Bor}(\mathcal{S})$, we have $\mu[\mathcal{B}] = \sup\{\mu[\mathcal{C}];\ \mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{C}$ closed in $\mathcal{S}\}$ and $\mu[\mathcal{B}] = \inf\{\mu[\mathcal{O}];\ \mathcal{B} \subseteq \mathcal{O} \subseteq \mathcal{S}$ and $\mathcal{O}$ open in $\mathcal{S}\}$. Finally, it has *full support* if $\mu(\mathcal{O}) > 0$ for any open set $\mathcal{O}$ in $\mathcal{S}$.

Stone spaces. Let $\mathfrak{B}$ be any Boolean algebra. A *truth valuation* on $\mathfrak{B}$ is a Boolean algebra homomorphism $v : \mathfrak{B} \longrightarrow \mathfrak{T}$, where $\mathfrak{T} := \{T, F\}$ is the two-element Boolean algebra, with the usual operations $\vee$, $\wedge$ and $\neg$. If $\mathfrak{B}$ is a set of propositions about the world, each of which may be true or false, then v is a complete, logically consistent assignment of truth values to these propositions. Let $\sigma(\mathfrak{B})$ be the set of all truth valuations of $\mathfrak{B}$. For any $\mathcal{B} \in \mathfrak{B}$, let $\mathcal{B}^* := \{v \in \sigma(\mathcal{B});\ v(\mathcal{B}) = T\}$. The collection $\{\mathcal{B}^*;\ \mathcal{B} \in \mathfrak{B}\}$ is a base of clopen sets for a topology on $\sigma(\mathfrak{B})$, making it into a compact, totally disconnected Hausdorff space, called the *Stone space* of $\mathfrak{B}$.[13]

In particular, let $\mathcal{S}$ be a topological state space, and let $\mathfrak{B}$ be a Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$ representing all events which are observable to the agent. Let $\mathcal{S}^* := \sigma(\mathfrak{B})$ be the Stone space of $\mathfrak{B}$. Then, each truth valuation $s^* \in \mathcal{S}^*$ is a *complete subjective description* of the world, as perceived by this agent. In turn, $\mathcal{S}^*$ can be interpreted as the "subjective state space" of the agent.

Extensions of the feasible acts. Our Stonean SEU representation will use Borel probability measures on the Stone space $\mathcal{S}^*$ of $\mathfrak{B}$, and therefore needs the feasible acts to be extended into functions on $\mathcal{S}^*$. This requires additional structural assumptions.

A topological space $\mathcal{S}$ is *Hausdorff* if any pair of points in $\mathcal{S}$ can be placed in two disjoint open neighbourhoods. A Hausdorff space $\mathcal{S}$ is *locally compact* if every point in $\mathcal{S}$ has a

[13]The Boolean algebra structure of $\mathfrak{B}$ is completely encoded in the topology of $\sigma(\mathfrak{B})$. To be precise, the Stone Representation Theorem says that the map $\mathcal{B} \mapsto \mathcal{B}^*$ is an isomorphism from $\mathfrak{B}$ to the Boolean algebra of clopen subsets of $\sigma(\mathfrak{B})$. Meanwhile, the Stone Duality Theorem says that σ is a functorial isomorphism between the category of Boolean algebras and the category of compact, totally disconnected Hausdorff spaces; see e.g. Johnstone (1986, §4.1) or Fremlin (2004, §311-§312).

compact neighbourhood. For example, every compact Hausdorff space is locally compact. Other examples include topological manifolds, open or closed subset of $\mathbb{R}^N$, and totally bounded, locally complete metric spaces. In short: most topological spaces which would arise naturally in economic applications are locally compact.

Let $\mathcal{S}$ be a locally compact Hausdorff space. There is a unique compact Hausdorff space $\check{\mathcal{S}}$, called the *Stone-Čech compactification* of $\mathcal{S}$, with the following properties.

(SČ1) $\mathcal{S}$ is an open, dense subset of $\check{\mathcal{S}}$, and the native topology of $\mathcal{S}$ is the same as the subspace topology it inherits from $\check{\mathcal{S}}$.

(SČ2) For any compact Hausdorff space $\mathcal{K}$, and any continuous function $f : \mathcal{S} \longrightarrow \mathcal{K}$, there is a unique continuous function $\check{f} : \check{\mathcal{S}} \longrightarrow \mathcal{K}$ such that $\check{f}_{\upharpoonright \mathcal{S}} = f$.

(SČ3) For any $\mathcal{R} \in \mathfrak{R}(\mathcal{S})$, there is a unique $\check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})$ such that $\check{\mathcal{R}} \cap \mathcal{S} = \mathcal{R}$. Furthermore, the mapping $\mathfrak{R}(\mathcal{S}) \ni \mathcal{R} \mapsto \check{\mathcal{R}} \in \mathfrak{R}(\check{\mathcal{S}})$ is a Boolean algebra isomorphism.[14]

Let $\mathcal{S}$ be any locally compact Hausdorff space, and let $\check{\mathcal{S}}$ be its Stone-Čech compactification. Let $\mathfrak{B}$ be a Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$, and define $\check{\mathfrak{B}} := \{\check{\mathcal{B}};\ \mathcal{B} \in \mathfrak{B}\}$, where we define $\check{\mathcal{B}}$ as in statement (SČ3). We say that $\mathfrak{B}$ is *generative* if $\check{\mathfrak{B}}$ is a base for the topology of $\check{\mathcal{S}}$. For example, the full Boolean algebra $\mathfrak{R}(\mathcal{S})$ is generative (Pivato and Vergopoulos, 2018c, Lemma 8.2). If $\mathfrak{B}$ is generative, then there is a continuous surjection $p : \mathcal{S}^* \longrightarrow \check{\mathcal{S}}$ (Pivato and Vergopoulos, 2018c, Proposition 8.3).[15] Let $\mathcal{X}$ be a Hausdorff space. By statement (SČ2), any feasible act $\alpha \in \mathcal{C}_{\rm b}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$ has a unique continuous extension $\check{\alpha} \in \mathcal{C}(\check{\mathcal{S}}, \mathcal{X})$. If we define $\alpha^* := \check{\alpha} \circ p$, then $\alpha^* \in \mathcal{C}(\mathcal{S}^*, \mathbb{R})$. If the Stone space $\mathcal{S}^*$ of $\mathfrak{B}$ is the subjective state space of the agent representing the uncertainty as she imperfectly perceives it, each function α^* is a subjective representation of the feasible act α incorporating the agent's imperfect perception of the outcomes of α.

Stonean SEU representations. Let $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$, and let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ be a $\mathfrak{B}$-indexed conditional preference structure on $\mathcal{A}$. A *Stonean SEU representation* for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ is given by a normal Borel probability measure μ^* on $\mathfrak{Bor}(\mathcal{S}^*)$ and a $\mathfrak{D}$-measurable utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that, for all $\mathcal{B} \in \mathfrak{B}$ and all $\alpha, \beta \in \mathcal{A}(\mathcal{B})$,

$$\left(\alpha \succeq_{\mathcal{B}} \beta\right) \quad \iff \quad \left(\int_{\mathcal{B}^*} u \circ \alpha^* \, \mathrm{d}\mu^* \quad \geq \int_{\mathcal{B}^*} u \circ \beta^* \, \mathrm{d}\mu^*\right). \tag{8}$$

Theorem 2 *Let $\mathcal{S}$ be a locally compact Hausdorff space, and let $\mathcal{X}$ be a connected Hausdorff space. Let $\mathfrak{B}$ be a generative Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$, let $\mathfrak{D}$ be a nontrivial Boolean subalgebra of $\mathfrak{R}(\mathcal{X})$, and let $\mathcal{A}$ be a collection of bounded, continuous, $(\mathfrak{B}, \mathfrak{D})$-comeasurable functions from $\mathcal{S}$ into $\mathcal{X}$. Let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ be a conditional preference structure on $\mathcal{A}$ which*

[14] Property (SČ2) holds if $\mathcal{S}$ is any *Tychonoff* space; see e.g. Theorem 19.5 of Willard (2004) or Theorem 2.79 of Aliprantis and Border (2006). But property (SČ3) only holds for the somewhat smaller class of locally compact Hausdorff spaces (Pivato and Vergopoulos, 2018c, Lemma 6.4(a)).

[15] If $\mathcal{S}$ is compact, then $\check{\mathcal{S}} = \mathcal{S}$. If also $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$, then $p : \mathcal{S}^* \longrightarrow \mathcal{S}$ is called the *Gleason cover* of $\mathcal{S}$; it plays an important role in categorical topology (Johnstone, 1986, §3.10).

satisfies condition (Rch). *Then, it satisfies* (CEq), (M), (Dom), (Sep), (CCP), *and* (TC) *if and only if it admits a Stonean SEU representation* (8)*, where* μ^* *has full support on* $\mathcal{S}^*$. *Finally,* μ^* *is unique, and* u *is unique up to positive affine transformation.*

Theorems 1 and 2 use exactly the same axioms to characterize the SEU and Stonean SEU representations. The choice between these two representations is then purely a matter of mathematical convenience. An advantage of Stonean SEU representations lies in their use of the classical notion of a normal Borel probability measure to represent the agent's beliefs. But this measure is defined on an extended and abstract version of the original state space —namely, the Stone space of $\mathfrak{B}$. Moreover, Stonean SEU representations rely on more stringent structural assumptions on $\mathcal{S}$, $\mathcal{X}$ and, more importantly, on $\mathfrak{B}$.

Theorem 2 has a natural and appealing interpretation. The agent is only able to observe the events in the algebra $\mathfrak{B}$. So for her, a complete subjective description of the world is given by a (logically consistent) assignment of truth-values to the events in $\mathfrak{B}$ —that is, an element of $\mathcal{S}^*$. The measure μ^* assigns probabilities to such complete subjective descriptions. Given any act $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$, it is possible to represent α as a function converting each complete subjective description into an outcome —that is, a function $\alpha^* : \mathcal{S}^* \longrightarrow \mathcal{X}$. The agent then ranks each act α according to the μ^*-expected utility of α^*. This may seem peculiar, but in fact it is quite psychologically natural. Perhaps $\mathcal{S}$ describes the world "as it really is". But for the agent, $\mathcal{S}^*$ describes the world *as she experiences it*. Thus, for her, an SEU representation on $\mathcal{S}^*$ might be more natural than one on $\mathcal{S}$ itself.

7 Extensions

SEU representations on $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{R}(\mathcal{X})$. If $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$, then we can simplify Theorem 1. The following continuity axiom weakens axiom (M) by only requiring *ex post* preferences to be compatible with the underlying topology on $\mathcal{X}$.

(C) The *ex post* order $\succeq_{\text{xp}}$ is continuous in the topology on $\mathcal{X}$. That is: for all $x \in \mathcal{X}$, the contour sets $\{y \in \mathcal{X};\ y \succeq_{\text{xp}} x\}$ and $\{y \in \mathcal{X};\ y \preceq_{\text{xp}} x\}$ are closed subsets of $\mathcal{X}$.

We will say that a topological space $\mathcal{S}$ is ***nondegenerate*** if it contains a nonempty open subset which is *not* dense —or equivalently, a proper closed subset with nonempty interior. This means that $\mathfrak{R}(\mathcal{S})$ is not trivial. Nondegeneracy is a very mild condition; for example, any nonsingleton Hausdorff space is nondegenerate.

Theorem 3 *Let* $\mathcal{S}$ *be a nondegenerate topological space, let* $\mathcal{X}$ *be a connected topological space, and let* $\mathcal{A} \subseteq \mathcal{C}_{\text{b}}(\mathcal{S}, \mathcal{X})$. *Let* $\{\succeq_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})}$ *be a conditional preference structure on* $\mathcal{A}$ *which satisfies condition* (Rch). *Then, it further satisfies Axioms* (CEq), (C), (Dom), (Sep), (CCP), *and* (TC) *if and only if it has an SEU representation* (u, μ)*, where* u *is a continuous function and* μ *is a credence on* $\mathfrak{R}(\mathcal{S})$ *with full support. Finally,* μ *is unique, and* u *is unique up to positive affine transformation.*

Theorem 3 essentially shows that it is sufficient to weaken axiom (M) into axiom (C) and, correspondingly, weaken the $\mathfrak{D}$-measurability of the utility function into continuity to obtain an SEU representation, in the case where all regular subsets of $\mathcal{S}$ and $\mathcal{X}$ are observable. Importantly, note that the $\mathfrak{D}$-measurability of u is *not* needed here for the SEU representation to be well-defined. Indeed, since u and α are continuous, $u \circ \alpha$ is also continuous, and therefore automatically $\mathfrak{R}(\mathcal{S})$-comeasurable for any $\alpha \in \mathcal{A}$.

SEU representations on unbounded acts. Theorems 1 to 3 assume that all acts in $\mathcal{A}$ are bounded. However, many economic applications of SEU require unbounded acts — e.g. acts that generate Poisson or Gaussian distributions over a Euclidean outcome space. We now present an extension of Theorem 1 to such unbounded acts. We will need three structural assumptions involving $\mathfrak{B}$ and $\mathcal{A}$.

Let $\mathcal{A}$ be a collection of continuous and $(\mathfrak{B}, \mathfrak{D})$-comeasurable functions from $\mathcal{S}$ to $\mathcal{X}$ and let $\mathfrak{B}$ be a Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$. For any $\mathcal{B} \in \mathfrak{B}$, let $\mathcal{A}_{\mathrm{b}}(\mathcal{B})$ be the set of all bounded functions in $\mathcal{A}(\mathcal{B})$. Let $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be a conditional preference structure on $\mathcal{A}$.

(Exh) There exists a countable $\mathfrak{B}$-partition $\{\mathcal{P}_n,\ n \in \mathbb{N}\}$ of $\mathcal{S}$ such that, for any $s \in \mathcal{S}$, $s \in \mathcal{P}_1 \vee \ldots \vee \mathcal{P}_n$ for some $n \in \mathbb{N}$.

Condition (Exh) requires the existence of a countable $\mathfrak{B}$-partition "exhausting" $\mathcal{S}$, as any state must be included in the join of finitely many events in the partition. It is comparable to Savage's axiom (P6). Indeed, (P6) implies the nonatomicity of the subjective probability and, therefore, the existence of countable partitions. For example, $\mathfrak{R}(\mathbb{R}^N)$ satisfies (Exh). Likewise, all the Boolean subalgebras from Examples 1 to 5 also satisfy (Exh).

(Loc) If $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ is continuous, and for any $s \in \mathcal{S}$, there exists a neighborhood $\mathcal{N}_s$ of s and an act $\alpha^s \in \mathcal{A}$ with $\alpha_{\upharpoonright\mathcal{N}_s} = \alpha^s_{\upharpoonright\mathcal{N}_s}$, then $\alpha \in \mathcal{A}$.

Condition (Loc) says that *feasibility is a local property*: if a continuous act is equal to some feasible act locally around each point in $\mathcal{S}$, then it must be feasible itself. Since Savage uses the universal domain of acts, this condition is automatically satisfied in his framework.

(Rch*) For any disjoint $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, and any $\alpha_1 \in \mathcal{A}_{\mathrm{b}}(\mathcal{B}_1)$ and $\alpha_2 \in \mathcal{A}_{\mathrm{b}}(\mathcal{B}_2)$, there is an act $\beta_2 \in \mathcal{A}_{\mathrm{b}}(\mathcal{B}_2)$ which is compatible with α_1, such that $\alpha_2 \approx_{\mathcal{B}_2} \beta_2$. Moreover, for any $x \in \mathcal{X}$, if $\alpha_1(s) \succeq_{\mathrm{xp}} x$ for all $s \in \mathcal{B}_1$ and $\alpha_2 \succeq_{\mathcal{B}_2} \kappa^x_{\mathcal{B}_2}$, then β_2 can be chosen such that $\beta_2(s) \succeq_{\mathrm{xp}} x$ for all $s \in \mathcal{B}_2$. Likewise, for any $x \in \mathcal{X}$, if $\alpha_1(s) \preceq_{\mathrm{xp}} x$ for all $s \in \mathcal{B}_1$ and $\alpha_2 \preceq_{\mathcal{B}_2} \kappa^x_{\mathcal{B}_2}$, then β_2 can be chosen such that $\beta_2(s) \preceq_{\mathrm{xp}} x$ for all $s \in \mathcal{B}_2$.

Condition (Rch*) is a variant of (Rch) and complements it by providing finer control over the values of spliced acts. Roughly speaking, the spliced act will not have "too low", or "too high", outcomes if it does not need to. Thus, (Rch*) is a rather mild strengthening of (Rch). We now come to the extension of Theorem 1 to unbounded acts.

Theorem 4 *Let $\mathcal{S}$ and $\mathcal{X}$ be topological spaces, with $\mathcal{X}$ connected. Let $\mathfrak{B}$ and $\mathfrak{D}$ be non-trivial Boolean subalgebras of $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{R}(\mathcal{X})$ respectively, with $\mathfrak{B}$ satisfying* (Exh). *Let $\mathcal{A}$*

be a collection of continuous and $(\mathfrak{B}, \mathfrak{D})$*-comeasurable functions from* $\mathcal{S}$ *into* $\mathcal{X}$ *satisfying* (Loc). *Let* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ *be a conditional preference structure on* $\mathcal{A}$ *which satisfies condition* (Rch*). *Then, it satisfies Axioms* (CEq), (M), (Dom), (Sep), (CCP), *and* (TC) *if and only if it has an SEU representation* (u, μ), *with* u *bounded,* $\mathfrak{D}$*-measurable and* μ *of full support. Finally,* μ *is unique, and* u *is unique up to positive affine transformation.*

The formal agument deriving Theorem 4 from Theorem 1 is very similar to Savage's extension of his SEU representation from finitely-valued acts to infinitely-valued ones. But his argument heavily used the universal domain of acts and events. In contrast, we need the conditions (Exh), (Loc) and (Rch*) to compensate for our more restricted domains. On the other hand, Savage's argument also relies on an additional axiom, namely (P7). We can dispense with (P7) because we exploit the topological properties of our framework. By comparison with Theorem 1, Theorem 4 seems to be more general because it allows for unbounded acts. Yet it is less general in that the utility function must be bounded. This is unavoidable because we need the utility profiles of each feasible act to be bounded, in order for it to have a well-defined expectation with respect to the credence.

8 Related literature

In mainstream probability theory and its applications in decision theory and game theory, an agent is endowed with a state space and a prior probability. Her information structure is represented by a partition. At any state, she "perfectly" perceives the event in the partition containing that state. Then the agent's posterior beliefs at that state are given by the Bayesian update of her prior on this event. Thus, perfectly perceptive agents can be described in terms of a function mapping each state onto the corresponding posterior.

Caplin and Martin (2015) are among the few authors investigating decision-making under imperfect perception. In their approach, the imperfect perception of events results in a multiplicity of possible posteriors. More precisely, they describe the agent by a *perception function* mapping each state onto a *probability* on possible posteriors. Thus, their approach does not really model imperfect perception *per se*, but only its effect on the agent's posterior beliefs. In contrast, we explicitly model imperfect perception in terms of a family of unreliable, imprecise measurement devices.[16]

Several previous papers have restricted the Savage universal domain of events. In fact, Savage (1954) explicitly notes that his axiomatic construction works equally well over a σ-algebra, but only produces finitely additive probabilities. Arrow (1970) enriches the Savage axioms so as to further derive the σ-additivity of probability measures. See also Kopylov (2010). Epstein and Zhang (2001) construct a theory of "probabilistic sophistication" on "λ-systems". Finally, Kopylov (2007) provides both SEU maximization and probabilistic sophistication on the weaker structure of so-called *mosaics*, which include Boolean algebras.

The SEU representation in Theorem 2 has similarities with previous SEU representations using "subjective state spaces". For example, Stinchcombe (1997, §7) also constructs

[16] Aliev and Huseynov (2014) provide another model of "vague" or "imprecise" information by representing events in terms of fuzzy subsets of the state space.

an SEU representation based on Stone spaces. However, he works in a very different framework, and his results are unrelated to ours. Jaffray and Wakker (1993) and Mukerji (1997) introduce "two-tiered" state spaces; in the model of Jaffray and Wakker, the agent has probabilistic beliefs about one tier and total ignorance about the other, whereas in Mukerji's model, one tier represents the agent's internal epistemic state and the other tier represents objectively payoff-relevant information. Finally, there is an interesting contrast between Theorem 2 and the model of Lipman (1999). In Lipman's model, as in ours, the agent is equipped with a mental vocabulary of propositions, each of which can be either true or false. In Lipman's model, the "true" state space is the set of all *logically consistent* assignments of truth-values to these propositions. But the agent's *subjective* state space also includes some logically *inconsistent* truth-value assignments; these so-called "impossible possible worlds" reflect her lack of logical omniscience. In our model, by contrast, $\mathcal{S}$ is a set of consistent but *logically incomplete* truth-value assignments, whereas $\mathcal{S}^*$ is the set of all consistent and *complete* assignments.

This paper is the first of three on similar themes. In Pivato and Vergopoulos (2018a), we specialize the model of the present paper to the case when $\mathfrak{B} = \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} = \mathfrak{R}(\mathcal{X})$, and obtain SEU representations in terms of residual charges, Borel measures and compactifications of the state space. These results yield deeper insights into the way the agent deals with her imperfect perception, using a probabilistic device we call a *liminal structure*. Finally, in Pivato and Vergopoulos (2018b), we allow for *piecewise continuous* acts (i.e. functions which are continuous on each cell of some regular partition of the state space) and extend the SEU representations accordingly.

Appendices

Appendix A contains the proof of Theorem 1. Appendix B contains the proofs of all other results in the paper, including Theorems 2, 3 and 4. The proofs in the appendices draw heavily on results from a companion paper, which studies credences and their representations by classical probability measures (Pivato and Vergopoulos, 2018c). We will refer to results in the companion paper with the prefix "PV". Thus, "Theorem PV-4.3" should be read as, "Theorem 4.3 from Pivato and Vergopoulos (2018c)."

A Proof of Theorem 1

The proof of Theorem 1 has two preliminary stages. First, Proposition A3 uses (Rch) and axioms (Sep), (TC), (CCP), (Dom), (CEq) and (M) to construct a credence μ on $\mathfrak{B}$ and a $\mathfrak{D}$-measurable utility function u using a theorem of Wakker (1988) for continuous additive representations. Second, Proposition A6 shows that the expected utility of any act, with respect to μ and u, equals the utility of any certainty equivalent of this act. The rest of the appendix uses these findings to construct the SEU representations and establishes the necessity of the axioms, as well as the uniqueness of the representation. Throughout this appendix, we maintain the following standing assumptions:

$\mathcal{S}$ and $\mathcal{X}$ are topological spaces, with $\mathcal{X}$ connected. $\mathfrak{B} \subseteq \mathfrak{R}(\mathcal{S})$ and $\mathfrak{D} \subseteq \mathfrak{R}(\mathcal{X})$ are nontrivial Boolean subalgebras, $\mathcal{A} \subseteq \mathcal{C}_{\rm b}(\mathcal{S}, \mathfrak{B}; \mathcal{X}, \mathfrak{D})$, and $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ is a conditional preference structure on $\mathcal{A}$ that satisfies (Rch).

Lemma A1 *Suppose* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ *satisfies axioms* (M), (Dom) *and* (CEq). *Let* $\mathcal{B} \in \mathfrak{B}$. *Define the function* $\mathsf{K}_{\mathcal{B}} : \mathcal{X} \longrightarrow \mathcal{A}(\mathcal{B})$ *by* $\mathsf{K}_{\mathcal{B}}(x) := \kappa^x_{\mathcal{B}}$ *for all* $x \in \mathcal{X}$. *Then* $\mathsf{K}_{\mathcal{B}}$ *is continuous relative to the* $\succeq_{\mathcal{B}}$*-order topology on* $\mathcal{A}(\mathcal{B})$.

Proof. For any $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$, let $(\alpha, \gamma)_{\succeq_{\mathcal{B}}} := \{\beta \in \mathcal{A}(\mathcal{B});\ \alpha \prec_{\mathcal{B}} \beta \prec_{\mathcal{B}} \gamma\}$. This collection of sets (for all $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$) forms a base for the $\succeq_{\mathcal{B}}$-order topology on $\mathcal{A}(\mathcal{B})$. So it suffices to show that $\mathsf{K}^{-1}_{\mathcal{B}}\left[(\alpha, \gamma)_{\succeq_{\mathcal{B}}}\right]$ is open in $\mathcal{X}$, for all $\alpha, \beta \in \mathcal{A}(\mathcal{B})$.

Axiom (CEq) says there exist $x, z \in \mathcal{X}$ such that $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$ and $\gamma \approx_{\mathcal{B}} \kappa^z_{\mathcal{B}}$. Define $(x, z)_{\succeq_{\rm xp}} := \{y \in \mathcal{X};\ x \prec_{\rm xp} y \prec_{\rm xp} z\}$. Since axiom (M) implies axiom (C), the latter is an open subset of $\mathcal{X}$. Now, for any $y \in \mathcal{X}$, we have

$$\begin{aligned}
\left(y \in \mathsf{K}^{-1}_{\mathcal{B}}\left[(\alpha,\gamma)_{\succeq_{\mathcal{B}}}\right]\right) &\iff \left(\mathsf{K}_{\mathcal{B}}(y) \in (\alpha,\gamma)_{\succeq_{\mathcal{B}}}\right) \iff \left(\alpha \prec_{\mathcal{B}} \kappa^y_{\mathcal{B}} \prec_{\mathcal{B}} \gamma\right)\\
&\underset{(*)}{\iff} \left(\kappa^x_{\mathcal{B}} \prec_{\mathcal{B}} \kappa^y_{\mathcal{B}} \prec_{\mathcal{B}} \kappa^z_{\mathcal{B}}\right) \underset{(\dagger)}{\iff} \left(x \prec_{\rm xp} y \prec_{\rm xp} z\right)\\
&\iff \left(y \in (x,z)_{\succeq_{\rm xp}}\right).
\end{aligned}$$

Here, $(*)$ is because $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$ and $\gamma \approx_{\mathcal{B}} \kappa^z_{\mathcal{B}}$, while $(\dagger)$ is by axiom (Dom) and its contrapositive. Thus, we see that $\mathsf{K}^{-1}_{\mathcal{B}}\left[(\alpha,\gamma)_{\succeq_{\mathcal{B}}}\right] = (x,z)_{\succeq_{\rm xp}}$, which is an open subset of $\mathcal{X}$. Since this holds for all $\alpha, \gamma \in \mathcal{A}(\mathcal{B})$, we conclude that $\mathsf{K}_{\mathcal{B}}$ is continuous. □

It will be convenient to use the following equivalent formulation of axiom (CCP).

(PC′) Let $\mathcal{B} = \mathcal{D} \vee \mathcal{E}$ as in axiom (Sep). Let $\mathcal{O} \subseteq \mathcal{A}(\mathcal{B})$ be open in the $\succeq_{\mathcal{B}}$-order topology, and let $\alpha \in \mathcal{O}$. Then there exist sets $\mathcal{O}_{\mathcal{D}} \subseteq \mathcal{A}(\mathcal{D})$ and $\mathcal{O}_{\mathcal{E}} \subseteq \mathcal{A}(\mathcal{E})$ which are open in the $\succeq_{\mathcal{D}}$-order topology and $\succeq_{\mathcal{E}}$-order topology, with $\alpha_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$ and $\alpha_{\upharpoonright\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$, such that, for any $\beta \in \mathcal{A}(\mathcal{B})$, if $\beta_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$ and $\beta_{\upharpoonright\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$, then $\beta \in \mathcal{O}$.

Lemma A2 *Suppose* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ *satisfies axioms* (CCP), (M), (Dom) *and* (CEq). *Consider a* $\mathfrak{B}$*-partition* $\mathfrak{P} = \{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ *of* $\mathcal{S}$ *with* $N \geq 2$. *There exists a mapping* $\Phi_{\mathfrak{P}} : \mathcal{X}^N \longrightarrow \mathcal{A}$ *that is continuous with respect to the product topology on* $\mathcal{X}^N$ *and the* $\succeq_{\mathcal{S}}$*-order topology on* $\mathcal{A}$ *and satisfies* $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright\mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ *for any* $n \in [1 \ldots N]$ *and any* $\mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$.

Proof. Let $\mathbf{x} := (x_1, \ldots, x_N) \in \mathcal{X}^N$. Define $\alpha^1 := \kappa^{x_1}_{\mathcal{E}_1}$, an element of $\mathcal{A}(\mathcal{E}_1)$. Condition (Rch) yields $\alpha^2 \in \mathcal{A}(\mathcal{E}_1 \vee \mathcal{E}_2)$ such that $\alpha^2_{\upharpoonright\mathcal{E}_1} = \kappa^{x_1}_{\mathcal{E}_1}$ and $\alpha^2_{\upharpoonright\mathcal{E}_2} \approx_{\mathcal{E}_2} \kappa^{x_2}_{\mathcal{E}_2}$. Next, (Rch) yields $\alpha^3 \in \mathcal{A}(\mathcal{E}_1 \vee \mathcal{E}_2 \vee \mathcal{E}_3)$ such that $\alpha^3_{\upharpoonright\mathcal{E}_1\vee\mathcal{E}_2} = \alpha^2$ and $\alpha^3_{\upharpoonright\mathcal{E}_3} \approx_{\mathcal{E}_3} \kappa^{x_3}_{\mathcal{E}_3}$. In particular, this means that $\alpha^3_{\upharpoonright\mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ for all $n \in \{1, 2, 3\}$.

Inductively, let $M \in [4 \ldots N]$, and suppose we have some $\alpha^{M-1} \in \mathcal{A}(\mathcal{E}_1 \vee \cdots \vee \mathcal{E}_{M-1})$ such that $\alpha^{M-1}_{\upharpoonright\mathcal{E}_m} \approx_{\mathcal{E}_m} \kappa^{x_m}_{\mathcal{E}_m}$ for all $m \in [1 \ldots M)$. (Rch) yields $\alpha^M \in \mathcal{A}(\mathcal{E}_1 \vee \cdots \vee \mathcal{E}_M)$

such that $\alpha^M_{\upharpoonright\mathcal{E}_1\vee\cdots\vee\mathcal{E}_{M-1}} = \alpha^{M-1}$ and $\alpha^M_{\upharpoonright\mathcal{E}_M} \approx_{\mathcal{E}_M} \kappa^{x_M}_{\mathcal{E}_M}$. In particular, this means that $\alpha^M_{\upharpoonright\mathcal{E}_m} \approx_{\mathcal{E}_m} \kappa^{x_m}_{\mathcal{E}_m}$ for all $m \in [1\ldots M]$.

Setting $M = N$ in this construction, we obtain some α^N such that $\alpha^N_{\upharpoonright\mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$ for all $n \in [1\ldots N]$. Now define $\Phi_{\mathfrak{P}}(\mathbf{x}) := \alpha^N$. To prove the continuity of $\Phi_{\mathfrak{P}}$, we need a preliminary result, which extends axiom (CCP).

Claim 1: *Let $\mathcal{O} \subseteq \mathcal{A}$ be open in the $\succeq_{\mathcal{S}}$-order topology, and let $\alpha \in \mathcal{O}$. Then for all $n \in [1\ldots N]$, there is a set $\mathcal{O}_n \subseteq \mathcal{A}(\mathcal{E}_n)$ which is open in the $\succeq_{\mathcal{E}_n}$-order topology, with $\alpha_{\upharpoonright\mathcal{E}_n} \in \mathcal{O}_n$, such that, for any $\beta \in \mathcal{A}$, if $\beta_{\upharpoonright\mathcal{E}_n} \in \mathcal{O}_n$ for all $n \in [1\ldots N]$, then $\beta \in \mathcal{O}$.*

Proof. Let $\mathcal{D}_1 := \mathcal{E}_2 \vee \cdots \vee \mathcal{E}_N$. Thus, $\mathcal{B} := \mathcal{E}_1 \vee \mathcal{D}_1$. Setting $\mathcal{D} := \mathcal{D}_1$ and $\mathcal{E} := \mathcal{E}_1$ in axiom (PC′), we obtain some $\mathcal{O}_1 \subseteq \mathcal{A}(\mathcal{E}_1)$ and $\mathcal{Q}_1 \subseteq \mathcal{A}(\mathcal{D}_1)$ with $\alpha_{\upharpoonright\mathcal{E}_1} \in \mathcal{O}_1$ and $\alpha_{\upharpoonright\mathcal{D}_1} \in \mathcal{Q}_1$, such that, for any $\beta \in \mathcal{A}(\mathcal{B})$, if $\beta_{\upharpoonright\mathcal{E}_1} \in \mathcal{O}_1$ and $\beta_{\upharpoonright\mathcal{D}_1} \in \mathcal{Q}_1$ then $\beta \in \mathcal{O}$.

Now let $\mathcal{D}_2 := \mathcal{E}_3 \vee \cdots \vee \mathcal{E}_N$. Thus, $\mathcal{D}_1 := \mathcal{E}_2 \vee \mathcal{D}_2$. Setting $\mathcal{D} := \mathcal{D}_2$ and $\mathcal{E} := \mathcal{E}_2$ in axiom (PC′), we obtain some $\mathcal{O}_2 \subseteq \mathcal{A}(\mathcal{E}_2)$ and $\mathcal{Q}_2 \subseteq \mathcal{A}(\mathcal{D}_2)$ with $\alpha_{\upharpoonright\mathcal{E}_2} \in \mathcal{O}_2$ and $\alpha_{\upharpoonright\mathcal{D}_2} \in \mathcal{Q}_2$, such that, for any $\beta \in \mathcal{A}(\mathcal{D}_1)$, if $\beta_{\upharpoonright\mathcal{E}_2} \in \mathcal{O}_2$ and $\beta_{\upharpoonright\mathcal{D}_2} \in \mathcal{Q}_2$ then $\beta \in \mathcal{Q}_1$. In particular, this means that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\upharpoonright\mathcal{E}_1} \in \mathcal{O}_1$, $\beta_{\upharpoonright\mathcal{E}_2} \in \mathcal{O}_2$ and $\beta_{\upharpoonright\mathcal{D}_2} \in \mathcal{Q}_2$, then $\beta \in \mathcal{O}$.

Inductively, let $M \in [3\ldots N)$, let $\mathcal{D}_{M-1} := \mathcal{E}_M \vee \cdots \vee \mathcal{E}_N$, and suppose that, for all $m \in [1\ldots M)$, we have constructed $\mathcal{O}_m \subseteq \mathcal{A}(\mathcal{E}_m)$ (open in the $\succeq_{\mathcal{E}_m}$-topology) with $\alpha_{\upharpoonright\mathcal{E}_m} \in \mathcal{O}_m$, along with some $\mathcal{Q}_{M-1} \subseteq \mathcal{A}(\mathcal{D}_{M-1})$ (open in the $\succeq_{\mathcal{D}_{M-1}}$-topology) with $\alpha_{\upharpoonright\mathcal{D}_{M-1}} \in \mathcal{Q}_{M-1}$, such that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\upharpoonright\mathcal{E}_m} \in \mathcal{O}_m$ for all $m \in [1\ldots M)$ and $\beta_{\upharpoonright\mathcal{D}_{M-1}} \in \mathcal{Q}_{M-1}$, then $\beta \in \mathcal{O}$. Now let $\mathcal{D}_M := \mathcal{E}_{M+1}\vee\cdots\vee\mathcal{E}_N$. Thus, $\mathcal{D}_{M-1} := \mathcal{E}_M\vee\mathcal{D}_M$. Setting $\mathcal{D} := \mathcal{D}_M$ and $\mathcal{E} := \mathcal{E}_M$ in axiom (PC′), we obtain some $\mathcal{O}_M \subseteq \mathcal{A}(\mathcal{E}_M)$ and $\mathcal{Q}_M \subseteq \mathcal{A}(\mathcal{D}_M)$ with $\alpha_{\upharpoonright\mathcal{E}_M} \in \mathcal{O}_M$ and $\alpha_{\upharpoonright\mathcal{D}_M} \in \mathcal{Q}_M$, such that, for any $\beta \in \mathcal{A}(\mathcal{D}_{M-1})$, if $\beta_{\upharpoonright\mathcal{E}_M} \in \mathcal{O}_M$ and $\beta_{\upharpoonright\mathcal{D}_M} \in \mathcal{Q}_M$ then $\beta \in \mathcal{Q}_{M-1}$. In particular, this means that, for any $\beta \in \mathcal{A}(\mathcal{G})$, if $\beta_{\upharpoonright\mathcal{E}_m} \in \mathcal{O}_m$ for all $m \in [1\ldots M]$ and $\beta_{\upharpoonright\mathcal{D}_M} \in \mathcal{Q}_M$, then $\beta \in \mathcal{O}$.

Suppose $M = N-1$ in the previous paragraph. Then $\mathcal{D}_M = \mathcal{E}_N$. Thus, if we define $\mathcal{O}_N := \mathcal{Q}_{N-1}$, then we have obtained sets $\mathcal{O}_1, \ldots, \mathcal{O}_N$ satisfying the claim. ◇ Claim 1

It remains to show that $\Phi_{\mathfrak{P}}$ is continuous with respect to the product topology on $\mathcal{X}^N$ and the $\succeq_{\mathcal{S}}$-order topology on $\mathcal{A}$. To see this, let $\mathcal{O} \subseteq \mathcal{A}$ be open in the $\succeq_{\mathcal{S}}$-order topology. It is sufficient to show that $\mathcal{U} := \Phi_{\mathfrak{P}}^{-1}(\mathcal{O})$ is open in the product topology on $\mathcal{X}^N$. To do this, let $\mathbf{x} \in \mathcal{U}$; we will construct an open neighbourhood around $\mathbf{x}$ inside $\mathcal{U}$.

Let $\alpha := \Phi_{\mathfrak{P}}(\mathbf{x}) \in \mathcal{A}$. Then, $\alpha \in \mathcal{O}$. For any $n \in [1\ldots N]$, let $\mathcal{O}_n \subseteq \mathcal{A}(\mathcal{E}_n)$ be the open subset in the $\succeq_{\mathcal{E}_n}$-order topology obtained in Claim 1, and define $\mathcal{V}_n := \mathsf{K}^{-1}_{\mathcal{E}_n}(\mathcal{O}_n)$. By Lemma A1, each $\mathsf{K}_{\mathcal{E}_n}$ is a continuous function from $\mathcal{X}$ to $\mathcal{A}(\mathcal{E}_n)$. So $\mathcal{V}_n$ is an open subset of $\mathcal{X}$ for any $n \in [1\ldots N]$. Define $\mathcal{V} := \mathcal{V}_1 \times \ldots \times \mathcal{V}_N$; then $\mathcal{V}$ is an open subset of $\mathcal{X}^N$ in the product topology.

Claim 2: $\mathbf{x} \in \mathcal{V}$.

Proof. Any open set in an order topology is a union of order intervals, and any order interval is a union of indifference classes (because if an order interval contains some element γ, then it also contains all other elements which are indifferent to γ). Thus, any open set is a union of indiffererence classes.

Now fix $n \in [1 \ldots N]$. By Claim 1, we have $\alpha_{\upharpoonright \mathcal{E}_n} \in \mathcal{O}_n$. Moreover, by the definition of α and the construction of $\Phi_{\mathfrak{P}}$, we have $\alpha_{\upharpoonright \mathcal{E}_n} = \Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{x_n}_{\mathcal{E}_n}$. By the remark in the previous paragraph, and since $\mathcal{O}_n$ is open in the $\succeq_{\mathcal{E}_n}$-order topology, we obtain $\kappa^{x_n}_{\mathcal{E}_n} \in \mathcal{O}_n$. Then, $x_n \in \mathcal{V}_n$. Since this holds for any $n \in [1 \ldots N]$, we obtain $\mathbf{x} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_N = \mathcal{V}$. $\diamondsuit$ Claim 2

Claim 3: $\mathcal{V} \subseteq \mathcal{U}$.

Proof. Let $\mathbf{y} = (y_1, \ldots, y_N) \in \mathcal{V}$ and define $\beta = \Phi_{\mathfrak{P}}(\mathbf{y}) \in \mathcal{A}$. Fix $n \in [1 \ldots N]$. Then $y_n \in \mathcal{V}_n$, so by definition of $\mathcal{V}_n$, we have $\kappa^{y_n}_{\mathcal{E}_n} \in \mathcal{O}_n$. By the construction of $\Phi_{\mathfrak{P}}$, we have $\beta_{\upharpoonright \mathcal{E}_n} = \Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright \mathcal{E}_n} \approx_{\mathcal{E}_n} \kappa^{y_n}_{\mathcal{E}_n}$. Since $\mathcal{O}_n$ is open in the $\succeq_{\mathcal{E}_n}$-order topology and $\kappa^{y_n}_{\mathcal{E}_n} \in \mathcal{O}_n$, $\mathcal{O}_n$ must also contain any act that is indifferent to $\kappa^{y_n}_{\mathcal{E}_n}$. Thus, $\beta_{\upharpoonright \mathcal{E}_n} \in \mathcal{O}_n$. This holds for any $n \in [1 \ldots N]$. Then, by Claim 1, $\beta \in \mathcal{O}$. Finally, $y \in \Phi_{\mathfrak{P}}^{-1}(\mathcal{O}) = \mathcal{U}$. $\diamondsuit$ Claim 3

Thus, $\mathcal{V}$ is an open neighbourhood around $\mathbf{x}$ (in the product topology), which is contained in $\mathcal{U}$. We can construct such a neighbourhood around any $\mathbf{x} \in \mathcal{U}$. Thus, $\mathcal{U}$ is open in the product topology. Hence $\Phi_{\mathfrak{P}}$ is continuous, as claimed. $\Box$

Consider any regular partition $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ of $\mathcal{S}$ with $N \geq 2$. Let $\Phi_{\mathfrak{P}}$ be the mapping from Lemma A2. We then define a preference order $\succeq_{\mathfrak{P}}$ on $\mathcal{X}^N$ in the following way: For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \succeq_{\mathcal{S}} \Phi_{\mathfrak{P}}(\mathbf{y})\Big). \tag{A1}$$

Proposition A3 *Suppose* $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ *satisfies axioms* (Sep), (M), (TC), (CCP), (Dom) *and* (CEq). *Then, there exists a credence* μ *on* $\mathfrak{B}$ *with full support and a continuous function* $u : \mathcal{X} \longrightarrow \mathbb{R}$ *such that, for any* $\mathfrak{B}$*-partition* $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ *of* $\mathcal{S}$ *with* $N \geq 2$*, we have*

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(x_n) \geq \sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(y_n)\right), \tag{A2}$$

where $\succeq_{\mathfrak{P}}$ *is defined by formula* (A1). *Moreover,* μ *is unique, and* u *is unique up to positive affine transformation. Finally,* u *is an ordinal utility function for* $\succeq_{\text{xp}}$ *and is* $\mathfrak{D}$*-measurable.*

Proof. Fix a $\mathfrak{B}$-partition $\mathfrak{P} = (\mathcal{B}_1, \ldots, \mathcal{B}_N)$ of $\mathcal{S}$ with $N \geq 2$ (such a partition exists because $\mathfrak{B}$ is nontrivial). Define $\succeq_{\mathfrak{P}}$ on $\mathcal{X}^N$ according to formula (A1).

Claim 1: $\succeq_{\mathfrak{P}}$ *is continuous with respect to the product topology on* $\mathcal{X}^N$.

Proof. Fix $\mathbf{y} \in \mathcal{X}^N$ and define $\beta := \Phi_{\mathfrak{P}}(\mathbf{y}) \in \mathcal{A}$. Let $\mathcal{O} = \{\alpha \in \mathcal{A}, \ \alpha \succ_{\mathcal{S}} \beta\}$; this is an open set in the $\succeq_{\mathcal{S}}$-order topology on $\mathcal{A}$. Then, by Lemma A2, $\mathcal{U} := \Phi_{\mathfrak{P}}^{-1}(\mathcal{O})$ is an open subset of $\mathcal{X}^N$ in the product topology. Moreover, for any $\mathbf{x} \in \mathcal{X}^N$, we have

$$\Big(\mathbf{x} \succ_{\mathfrak{P}} \mathbf{y}\Big) \quad \underset{(*)}{\Longleftrightarrow} \quad \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \succ_{\mathcal{S}} \beta\Big) \quad \underset{(\dagger)}{\Longleftrightarrow} \quad \Big(\mathbf{x} \in \mathcal{U}\Big),$$

where $(*)$ is by formula (A1) and $(\dagger)$ is by the definition of $\mathcal{U}$. Thus, the strict upper contour set of $\succeq_{\mathfrak{P}}$ at $\mathbf{y}$ is equal to $\mathcal{U}$ and, therefore, an open set in the product topology on $\mathcal{X}^N$. A similar proof works for strict lower contour sets. $\diamondsuit$ `Claim 1`

Claim 2: $\succeq_{\mathfrak{P}}$ *satisfies* Cardinal Coordinate Independence*: For all* $n, m \in [1 \ldots N]$, *all* $\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w} \in \mathcal{X}^N$ *and all* $a, b, c, d \in \mathcal{X}$, *if* $a_n\mathbf{x}_{-n} \preceq_{\mathfrak{P}} b_n\mathbf{y}_{-n}$, $c_n\mathbf{x}_{-n} \succeq_{\mathfrak{P}} d_n\mathbf{y}_{-n}$ *and* $a_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} b_m\mathbf{w}_{-m}$, *then* $c_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m\mathbf{w}_{-m}$.

Proof. Define $(a_{\mathcal{B}_n}\alpha), (b_{\mathcal{B}_n}\beta), (c_{\mathcal{B}_n}\alpha), (d_{\mathcal{B}_n}\beta) \in \mathcal{A}$ by $(a_{\mathcal{B}_n}\alpha) := \Phi_{\mathfrak{P}}(a_n\mathbf{x}_{-n})$, $(b_{\mathcal{B}_n}\beta) := \Phi_{\mathfrak{P}}(b_n\mathbf{y}_{-n})$, $(c_{\mathcal{B}_n}\alpha) := \Phi_{\mathfrak{P}}(c_n\mathbf{x}_{-n})$ and $(d_{\mathcal{B}_n}\beta) := \Phi_{\mathfrak{P}}(d_n\mathbf{y}_{-n})$. Then, by the definition (A1) of $\succeq_{\mathfrak{P}}$, we have $(a_{\mathcal{B}_n}\alpha) \preceq_{\mathcal{S}} (b_{\mathcal{B}_n}\beta)$ and $(c_{\mathcal{B}_n}\alpha) \succeq_{\mathcal{S}} (d_{\mathcal{B}_n}\beta)$. Moreover, by the definition of $\Phi_{\mathfrak{P}}$, we have $(a_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^a_{\mathcal{B}_n}$, $(b_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^b_{\mathcal{B}_n}$, $(c_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^c_{\mathcal{B}_n}$ and $(d_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^d_{\mathcal{B}_n}$. Meanwhile, $(a_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{B}_l} \approx_{\mathcal{B}_l} (c_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{B}_l}$ and $(b_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{B}_l} \approx_{\mathcal{B}_l} (d_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{B}_l}$ for all $l \in [1 \ldots N]$ with $l \neq n$. So if we set $\mathcal{Q} := \neg\mathcal{B}_n$, then by (Sep) we obtain $(a_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} (c_{\mathcal{B}_n}\alpha)_{\upharpoonright\mathcal{Q}}$ and $(b_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} (d_{\mathcal{B}_n}\beta)_{\upharpoonright\mathcal{Q}}$. This shows that $(a \overset{\mathcal{B}_n}{\rightsquigarrow} b) \succeq (c \overset{\mathcal{B}_n}{\rightsquigarrow} d)$.

Meanwhile, define $(a_{\mathcal{B}_m}\gamma), (b_{\mathcal{B}_m}\delta), (c_{\mathcal{B}_m}\gamma), (d_{\mathcal{B}_m}\delta) \in \mathcal{A}$ by $(a_{\mathcal{B}_m}\gamma) := \Phi_{\mathfrak{P}}(a_m\mathbf{v}_{-m})$, $(b_{\mathcal{B}_m}\delta) := \Phi_{\mathfrak{P}}(b_m\mathbf{w}_{-m})$, $(c_{\mathcal{B}_m}\gamma) := \Phi_{\mathfrak{P}}(c_m\mathbf{v}_{-m})$ and $(d_{\mathcal{B}_m}\delta) := \Phi_{\mathfrak{P}}(d_m\mathbf{w}_{-m})$. Proceeding as above, we obtain $(a_{\mathcal{B}_m}\gamma) \succeq_{\mathcal{S}} (b_{\mathcal{B}_m}\delta)$ with $(a_{\mathcal{B}_m}\gamma)_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^a_{\mathcal{B}_m}$, $(b_{\mathcal{B}_m}\delta)_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^b_{\mathcal{B}_m}$, $(c_{\mathcal{B}_m}\gamma)_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^c_{\mathcal{B}_m}$ and $(d_{\mathcal{B}_m}\delta)_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^d_{\mathcal{B}_m}$. Moreover, set $\mathcal{Q}' := \neg\mathcal{B}_m$. Then we also have $(a_{\mathcal{B}_m}\gamma)_{\upharpoonright\mathcal{Q}'} \approx_{\mathcal{Q}'} (c_{\mathcal{B}_m}\gamma)_{\upharpoonright\mathcal{Q}'}$ and $(b_{\mathcal{B}_m}\delta)_{\upharpoonright\mathcal{Q}'} \approx_{\mathcal{Q}'} (d_{\mathcal{B}_m}\delta)_{\upharpoonright\mathcal{Q}'}$. Now, if it is *not* the case that $c_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m\mathbf{w}_{-m}$, then (A1) implies that it is also not the case that $(c_{\mathcal{B}_m}\gamma) \succeq_{\mathcal{S}} (d_{\mathcal{B}_m}\delta)$. Thus, $(c_{\mathcal{B}_m}\gamma) \prec_{\mathcal{S}} (d_{\mathcal{B}_m}\delta)$ (because $\succeq_{\mathcal{S}}$ is a complete order). But this means that $(a \overset{\mathcal{B}_m}{\rightsquigarrow} b) \prec (c \overset{\mathcal{B}_m}{\rightsquigarrow} d)$, which contradicts (TC). Thus, we must have $c_m\mathbf{v}_{-m} \succeq_{\mathfrak{P}} d_m\mathbf{w}_{-m}$, as claimed. $\diamondsuit$ `Claim 2`

By Claims 1 and 2, and the connectedness of $\mathcal{X}$, the assumptions of Wakker's (1988) Theorem 6.2 are satisfied. So there exist a continuous function $u_{\mathfrak{P}} : \mathcal{X} \longrightarrow \mathbb{R}$ and a probability vector $(\mu_{\mathfrak{P}}(\mathcal{B}_1), \ldots, \mu_{\mathfrak{P}}(\mathcal{B}_N)) \in \Delta([1 \ldots N])$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \quad \Longleftrightarrow \quad \left(\sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot u_{\mathfrak{P}}(x_n) \geq \sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot u_{\mathfrak{P}}(y_n)\right). \tag{A3}$$

Moreover, the probability vector is unique, and the function is unique up to positive affine transformation.

By the nontriviality of $\succeq_{\mathcal{S}}$ and axiom (Dom), there exist $l, o \in \mathcal{X}$ with $l \succ_{\text{xp}} o$. Then, still by (Dom), $\kappa^l_{\mathcal{B}} \succ_{\mathcal{B}} \kappa^o_{\mathcal{B}}$ for any $\mathcal{B} \in \mathfrak{B}$. Fix $n \in [1 \ldots N]$, and let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$ be such that $x_n = l$, $y_n = o$, and $x_m = y_m = o$ for any $m \in [1 \ldots N] \setminus \{n\}$. Then, $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright\mathcal{B}_m} \approx_{\mathcal{B}_m} \kappa^o_{\mathcal{B}_m} \approx_{\mathcal{B}_m} \Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright\mathcal{B}_m}$ for any $m \in [1 \ldots N] \setminus \{n\}$. Let $\mathcal{Q} = \neg\mathcal{B}_n$. By iterative applications of (Sep), we have $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright\mathcal{Q}}$. Moreover, we have $\Phi_{\mathfrak{P}}(\mathbf{x})_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^l_{\mathcal{B}_n} \succ_{\mathcal{B}_n} \kappa^o_{\mathcal{B}_n} \approx_{\mathcal{B}_n} \Phi_{\mathfrak{P}}(\mathbf{y})_{\upharpoonright\mathcal{B}_n}$ Another application of (Sep) yields $\Phi_{\mathfrak{P}}(\mathbf{x}) \succ_{\mathcal{B}_n} \Phi_{\mathfrak{P}}(\mathbf{y})$ and, by formula (A1), $\mathbf{x} \succ_{\mathfrak{P}} \mathbf{y}$. Then, from (A2) and the definition of $\mathbf{x}$ and $\mathbf{y}$, we get

$$\mu_{\mathfrak{P}}(\mathcal{B}_n) \cdot [u_{\mathfrak{P}}(l) - u_{\mathfrak{P}}(o)] \quad > \quad 0.$$

This inequality first shows that $\mu_{\mathfrak{P}}(\mathcal{B}_n) > 0$, and this holds for any $n \in [1 \ldots N]$. It also shows $u_{\mathfrak{P}}(l) > u_{\mathfrak{P}}(o)$. Thus, we obtain a unique function $u_{\mathfrak{P}}$ providing a representation as in formula (A3) and satisfying $u_{\mathfrak{P}}(l) = 1$ and $u_{\mathfrak{P}}(o) = 0$. From now on, we assume that the functions $u_{\mathfrak{P}}$ are normalized in this way.

Claim 3: *For any two $\mathfrak{B}$-partitions $\mathfrak{P}$ and $\mathfrak{Q}$ of $\mathcal{S}$, each with at least two cells, $u_{\mathfrak{P}} = u_{\mathfrak{Q}}$. Moreover, if $\mathcal{B} \in \mathfrak{B}$ is a cell in each of $\mathfrak{P}$ and $\mathfrak{Q}$, then $\mu_{\mathfrak{P}}(\mathcal{B}) = \mu_{\mathfrak{Q}}(\mathcal{B})$.*

Proof. Let $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ with $\mathcal{P}_n \in \mathfrak{B}$ for all $n \in [1 \ldots N]$, and let $\mathfrak{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_M)$ with $\mathcal{Q}_m \in \mathfrak{B}$ for all $m \in [1 \ldots M]$. Consider first the case where $\mathfrak{Q}$ ***refines*** $\mathfrak{P}$ —that is, for all $m \in [1 \ldots M]$, there is some $n \in [1 \ldots N]$ such that $\mathcal{Q}_m \subseteq \mathcal{P}_n$. For all $n \in [1 \ldots N]$, let $\mathcal{M}_n \subseteq [1 \ldots M]$ be the set of $m \in [1 \ldots M]$ such that $\mathcal{Q}_m \subseteq \mathcal{P}_n$. Then, for all $n \in [1 \ldots N]$, the subcollection $\{\mathcal{Q}_m,\ m \in M_n\}$ is a $\mathfrak{B}$-partition of $\mathcal{P}_n$. For all $n \in [1 \ldots N]$, we define

$$p_n \;:=\; \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m). \tag{A4}$$

Then, the collection $(p_1, \ldots, p_N)$ is a probability vector on $[1 \ldots N]$. Moreover, for any $\mathbf{x} \in \mathcal{X}^N$, define $\mathbf{x}' \in \mathcal{X}^M$ by setting

$$x'_m \;=\; x_n, \qquad \text{for all } m \in \mathcal{M}_n \text{ and } n \in [1 \ldots N]. \tag{A5}$$

Note that by (Sep) we have the following indifference for any $\mathbf{x} \in \mathcal{X}^N$:

$$\Phi_{\mathfrak{P}}(\mathbf{x}) \;\approx_{\mathcal{S}}\; \Phi_{\mathfrak{Q}}(\mathbf{x}') \tag{A6}$$

Thus, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$,

$$\begin{aligned}
\Big(\mathbf{x} \succeq_{\mathfrak{P}} \mathbf{y}\Big) \;\underset{\text{(a)}}{\iff}\; & \Big(\Phi_{\mathfrak{P}}(\mathbf{x}) \;\succeq_{\mathcal{S}}\; \Phi_{\mathfrak{P}}(\mathbf{y})\Big) \;\underset{\text{(b)}}{\iff}\; \Big(\Phi_{\mathfrak{Q}}(\mathbf{x}') \;\succeq_{\mathcal{S}}\; \Phi_{\mathfrak{Q}}(\mathbf{y}')\Big) \\
\underset{\text{(a)}}{\iff}\; & \Big(\mathbf{x}' \succeq_{\mathfrak{Q}} \mathbf{y}'\Big) \;\underset{\text{(c)}}{\iff}\; \left(\sum_{m=1}^{M} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(x'_m) \geq \sum_{m=1}^{M} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(y'_m)\right) \\
\underset{\text{(d)}}{\iff}\; & \left(\sum_{n=1}^{N} \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(x_n) \geq \sum_{n=1}^{N} \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m) \cdot u_{\mathfrak{Q}}(y_n)\right) \\
\underset{\text{(e)}}{\iff}\; & \left(\sum_{n=1}^{N} p_n \cdot u_{\mathfrak{Q}}(x_n) \geq \sum_{n=1}^{N} p_n \cdot u_{\mathfrak{Q}}(y_n)\right).
\end{aligned}$$

Here, both (a) are by equation (A1), (b) is by equation (A6), (c) is by equation (A3), (d) by equation (A5), and (e) is by equation (A4). Thus, $u_{\mathfrak{Q}}$ and $(p_1, \ldots, p_N)$ provide a representation of $\succeq_{\mathfrak{P}}$ as in equation (A3). By uniqueness, we obtain $u_{\mathfrak{P}} = u_{\mathfrak{Q}}$. Moreover, for all $n \in [1 \ldots N]$,

$$\mu_{\mathfrak{P}}(\mathcal{P}_n) \;=\; \sum_{m \in \mathcal{M}_n} \mu_{\mathfrak{Q}}(\mathcal{Q}_m). \tag{A7}$$

Now, if $\mathfrak{P}$ and $\mathfrak{Q}$ have a common cell $\mathcal{B} \in \mathfrak{B}$, then $\mathcal{B} = \mathcal{P}_n = \mathcal{Q}_m$ for some $n \in [1 \ldots N]$ and $m \in [1 \ldots M]$ such that $M_n = \{m\}$. Then, equation (A7) yields $\mu_{\mathfrak{P}}(\mathcal{B}) = \mu_{\mathfrak{Q}}(\mathcal{B})$.

Now consider the general case, where neither $\mathfrak{P}$ nor $\mathfrak{Q}$ refines the other. Let $\mathfrak{P} \otimes \mathfrak{Q} := \{\mathcal{P} \cap \mathcal{Q};\ \mathcal{P} \in \mathfrak{P} \text{ and } \mathcal{Q} \in \mathfrak{Q}\}$. Then $\mathfrak{P} \otimes \mathfrak{Q}$ is a $\mathfrak{B}$-partition which refines both $\mathfrak{P}$ and $\mathfrak{Q}$. Now apply to the previous argument to $\mathfrak{P}$ and $\mathfrak{P} \otimes \mathfrak{Q}$ on the one hand, and to $\mathfrak{Q}$ and $\mathfrak{P} \otimes \mathfrak{Q}$ on the other hand to conclude. ◇ `Claim 3`

Now, we define a set function $\mu : \mathfrak{B} \longrightarrow [0,1]$ by setting $\mu(\mathcal{S}) = 1$, $\mu(\varnothing) = 0$ and, for any $\mathcal{B} \in \mathfrak{B}$, $\mu(\mathcal{B}) = \mu_{\mathfrak{P}}(\mathcal{B})$ where $\mathfrak{P} = \{\mathcal{B}, \neg\mathcal{B}\}$. Note that, for any nonempty $\mathcal{B} \in \mathfrak{B}$, $\mu(\mathcal{B}) > 0$ since we have already proved that $\mu_{\mathfrak{P}}(\mathcal{B}) > 0$.

Claim 4: *μ is a credence on $\mathfrak{B}$ with full support.*

Proof. Consider a collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ of pairwise disjoint regular subsets in $\mathfrak{B}$, and let $\mathcal{B}$ be its join. Consider first the case where $\mathcal{B} = \mathcal{S}$, and set $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$. Then, $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{S}$. We have

$$\sum_{n=1}^{N} \mu(\mathcal{P}_n) \underset{(*)}{=} \sum_{n=1}^{N} \mu_{\mathfrak{P}}(\mathcal{P}_n) \underset{(\dagger)}{=} 1. \tag{A8}$$

Here $(*)$ is by Claim 3, and $(\dagger)$ is because $\mu_{\mathfrak{P}}$ is a probability distribution. Now, if $\mathcal{B} \neq \mathcal{S}$, set $\mathcal{P}_{N+1} := \neg\mathcal{B}$. Consider $\mathfrak{Q} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N, \mathcal{P}_{N+1}\}$ and $\mathfrak{Q}' = \{\mathcal{B}, \mathcal{P}_{N+1}\}$, two $\mathfrak{B}$-partitions of $\mathcal{S}$. We have

$$\sum_{n=1}^{N} \mu(\mathcal{P}_n) \underset{\text{(a)}}{=} 1 - \mu_{\mathfrak{Q}}(\mathcal{P}_{N+1}) \underset{\text{(b)}}{=} 1 - \mu_{\mathfrak{Q}'}(\mathcal{P}_{N+1}) \underset{\text{(c)}}{=} \mu_{\mathfrak{Q}'}(\mathcal{B}) \underset{\text{(d)}}{=} \mu(\mathcal{B}). \tag{A9}$$

Here, both (a) and (c) are by suitable versions of equation (A8) while both (b) and (d) are by Claim 3. Thus, μ is a credence. Finally, μ is fully supported since it has positive values at any nonempty regular subset in $\mathfrak{B}$. ◇ `Claim 4`

Set $u := u_{\mathfrak{Q}}$ for some $\mathfrak{B}$-partition $\mathfrak{Q}$ of $\mathcal{S}$ with at least two cells. For any $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N\}$ of $\mathcal{S}$ with $N \geq 2$, Claim 3 yields $u = u_{\mathfrak{P}}$ and $\mu_{\mathfrak{P}}(\mathcal{P}_n) = \mu(\mathcal{P}_n)$ for all $n \in [1 \ldots N]$. This, together with equation (A3), completes the proof of formula (A2).

Claim 5: *u is an ordinal utility function for $\succeq_{\text{xp}}$.*

Proof. Fix $x, y \in \mathcal{X}$. Since $\mathfrak{B}$ is nondegenerate, there exists a $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{S}$ with $N \geq 2$. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$ be defined by $x_n := x$ and $y_n := y$ for any $n \in [1 \ldots N]$. Then, we have

$$\begin{aligned}
\left(x \succeq_{\text{xp}} y\right) &\underset{\text{(a)}}{\iff} \left(\kappa^x \succeq_{\mathcal{S}} \kappa^y\right) \underset{\text{(b)}}{\iff} \left(\Phi_{\mathfrak{P}}(\mathbf{x}) \succeq_{\mathcal{S}} \Phi_{\mathfrak{P}}(\mathbf{y})\right) \\
&\underset{\text{(c)}}{\iff} \left(\sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(x) \ \geq\ \sum_{n=1}^{N} \mu(\mathcal{B}_n) \cdot u(y)\right) \\
&\iff \left(u(x) \geq u(y)\right),
\end{aligned}$$

where (a) is by the definition of $\succeq_{\rm xp}$, (b) is because by inductive applications of (Sep) and because $\Phi_{\mathfrak{p}}(\mathbf{x})_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^x_{\mathcal{B}_n}$ and $\Phi_{\mathfrak{p}}(\mathbf{y})_{\upharpoonright\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^y_{\mathcal{B}_n}$, for all $n \in [1\ldots N]$. Meanwhile, (c) is by formula (A2). ◇ Claim 5

Claim 6: *u is $\mathfrak{D}$-measurable.*

Proof. Fix an open interval $\mathcal{O} \subseteq \mathbb{R}$. We must show that $u^{-1}(\mathcal{O}) \in \mathfrak{D}$.

First, suppose $\mathcal{O} = (q, \infty)$ for some $q \in \mathbb{R}$. If $u(x) < q$ for all $x \in \mathcal{X}$, then $u^{-1}(q,\infty) = \emptyset \in \mathfrak{D}$. On the other hand, if $u(x) > q$ for all $x \in \mathcal{X}$, then $u^{-1}(q,\infty) = \mathcal{X} \in \mathfrak{D}$. If neither of these cases holds, then there must exist $x, z \in \mathcal{X}$ with $u(x) \leq q \leq u(z)$. Since u is continuous and $\mathcal{X}$ is connected, the Intermediate Value Theorem yields some $y \in \mathcal{X}$ such that $u(y) = q$. But then $u^{-1}(q,\infty) = \{z \in \mathcal{X};\ \ z \succ_{\rm xp} y\}$, because, by Claim 5 , u is an ordinal utility representation for $\succeq_{\rm xp}$. Thus, $u^{-1}(q,\infty)$ is an open upper contour set of $\succeq_{\rm xp}$, so $u^{-1}(q,\infty) \in \mathfrak{D}$, by axiom (M).

The same argument works if $\mathcal{O} = (-\infty, r)$ for some $r \in \mathbb{R}$. Finally, if $\mathcal{O} = (q,r)$, then $\mathcal{O} = (-\infty, r) \cap (q,\infty)$, so $u^{-1}(\mathcal{O}) = u^{-1}(-\infty,r) \cap u^{-1}(q,\infty)$ is an intersection of two elements of $\mathfrak{D}$, and thus, an element of $\mathfrak{D}$.

Finally, let $\mathcal{B}$ be an arbitrary basic subset of $\mathbb{R}$. Then $\mathcal{B} := (a_1,b_1) \sqcup (a_2,b_2) \sqcup \cdots \sqcup (a_N,b_N)$ for some $-\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N \leq \infty$. For all $n \in [1\ldots N]$, let $\mathcal{D}_n := u^{-1}(a_n,b_n)$; then $\mathcal{D}_n \in \mathfrak{D}$ by the previous paragraph, and $u^{-1}(\mathcal{B}) = \bigsqcup_{n=1}^{N} \mathcal{D}_n$. It remains to show that this union is an element of $\mathfrak{D}$.

Claim 6A: $\bigvee_{n=1}^{N} \mathcal{D}_n \subseteq \bigsqcup_{n=1}^{N} \mathcal{D}_n$.

Proof. (by contradiction) Suppose $x \in \left(\bigvee_{n=1}^{N} \mathcal{D}_n\right) \setminus \left(\bigsqcup_{n=1}^{N} \mathcal{D}_n\right)$. Now,

$$\bigvee_{n=1}^{N} \mathcal{D}_n \quad = \quad \operatorname{int}\left[\operatorname{clos}\left(\bigsqcup_{n=1}^{N} \mathcal{D}_n\right)\right] \quad = \quad \operatorname{int}\left[\bigcup_{n=1}^{N} \operatorname{clos}(\mathcal{D}_n)\right].$$

Thus, $x \in \operatorname{int}\left[\bigcup_{n=1}^{N} \operatorname{clos}(\mathcal{D}_n)\right]$, but $x \notin \mathcal{D}_n = \operatorname{int}[\operatorname{clos}(\mathcal{D}_n)]$ for any $n \in [1\ldots N]$. Thus, if $\mathcal{U}$ is any open neighbourhood around x, then $\mathcal{U}$ overlaps $\bigcup_{n=1}^{N} \operatorname{clos}(\mathcal{D}_n)$ but $\mathcal{U} \not\subseteq \operatorname{clos}(\mathcal{D}_n)$ for any $n \in [1\ldots N]$; hence there must be at least two distinct $n, m \in [1\ldots N]$ such that $\mathcal{U} \cap \operatorname{clos}(\mathcal{D}_n) \neq \emptyset$ and $\mathcal{U} \cap \operatorname{clos}(\mathcal{D}_m) \neq \emptyset$. Define

$$\epsilon \quad := \quad \frac{1}{4} \min_{n \in [1\ldots N)} (a_{n+1} - b_n).$$

Then $\epsilon > 0$ because $b_n < a_{n+1}$ for all $n \in [1\ldots N]$, by hypothesis. Let $r := u(x)$, and let $\mathcal{V} := (r-\epsilon, r+\epsilon)$. Then $\mathcal{V}$ is an open neighbourhood around r. Let $\mathcal{U} := u^{-1}(\mathcal{V})$; then $\mathcal{U}$ is an open neighbourhood around x (because u is continuous), so by the previous paragraph there exist distinct $n < m \in [1 \ldots N]$ such that $\mathcal{U} \cap \text{clos}(\mathcal{D}_n) \neq \emptyset$ and $\mathcal{U} \cap \text{clos}(\mathcal{D}_m) \neq \emptyset$. Now, $u(\mathcal{U}) = \mathcal{V}$ (by definition of $\mathcal{U}$), while $u[\text{clos}(\mathcal{D}_n)] \subseteq \text{clos}(a_n, b_n) = [a_n, b_n]$ and $u[\text{clos}(\mathcal{D}_m)] \subseteq \text{clos}(a_m, b_m) = [a_m, b_m]$ (because u is continuous). Thus, we must have $\mathcal{V} \cap [a_n, b_n] \neq \emptyset$ and $\mathcal{V} \cap [a_m, b_m] \neq \emptyset$. But this is impossible, because $\mathcal{V}$ is an interval of length $2\epsilon \leq (a_m - b_n)/2$, by construction.

To avoid the contradiction, we must have $x \in \mathcal{D}_n$ for some $n \in [1 \ldots N]$. This argument applies to all $x \in \bigvee_{n=1}^{N} \mathcal{D}_n$. Thus, $\bigvee_{n=1}^{N} \mathcal{D}_n \subseteq \bigsqcup_{n=1}^{N} \mathcal{D}_n$, as claimed. $\bigtriangledown$ `Claim 6A`

Clearly, $\bigsqcup_{n=1}^{N} \mathcal{D}_n \subseteq \bigvee_{n=1}^{N} \mathcal{D}_n$. Together with Claim 6A, this implies that $\bigsqcup_{n=1}^{N} \mathcal{D}_n = \bigvee_{n=1}^{N} \mathcal{D}_n$. Thus, it is an element of $\mathfrak{D}$, as desired. $\diamond$ `Claim 6`

This completes the proof. □

Our SEU representation requires one more technical preliminary.

Proposition A4 (Pivato and Vergopoulos, 2018c, Theorem 4.3) *Let $\mathcal{S}$ be any topological space, let $\mathfrak{B}$ be any Boolean subalgebra of $\mathfrak{R}(\mathcal{S})$, and let μ be a credence with full support on $\mathfrak{B}$. There exists a unique, strictly monotonic conditional expectation system $\mathbf{E}$ that is compatible with μ.*

Lemma A5 *Suppose $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies axiom* (Sep). *For any $\mathcal{B} \in \mathfrak{B}$, consider a $\mathfrak{B}$-partition $\mathfrak{P} = \{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{B}$. For any $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, if $\alpha_{\upharpoonright\mathcal{B}_n} \succeq_{\mathcal{B}_n} \beta_{\upharpoonright\mathcal{B}_n}$ for any $n \in [1 \ldots N]$, then $\alpha \succeq_{\mathcal{B}} \beta$.*

Proof. We proceed by induction. Consider first a subset $\mathcal{B} \in \mathfrak{B}$ and a two-cell partition $\mathfrak{P} = \{\mathcal{B}_1, \mathcal{B}_2\}$ of $\mathcal{B}$. Let $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ be such that $\alpha_{\upharpoonright\mathcal{B}_1} \succeq_{\mathcal{B}_1} \beta_{\upharpoonright\mathcal{B}_1}$ and $\alpha_{\upharpoonright\mathcal{B}_2} \succeq_{\mathcal{B}_2} \beta_{\upharpoonright\mathcal{B}_2}$. By (Rch), there exists $\gamma \in \mathcal{A}(\mathcal{B})$ such that $\gamma_{\upharpoonright\mathcal{B}_1} \approx_{\mathcal{B}_1} \alpha_{\upharpoonright\mathcal{B}_1}$ and $\gamma_{\upharpoonright\mathcal{B}_2} \approx_{\mathcal{B}_2} \beta_{\upharpoonright\mathcal{B}_2}$. Then, we have $\alpha_{\upharpoonright\mathcal{B}_1} \approx_{\mathcal{B}_1} \gamma_{\upharpoonright\mathcal{B}_1}$ and $\alpha_{\upharpoonright\mathcal{B}_2} \succeq_{\mathcal{B}_2} \gamma_{\upharpoonright\mathcal{B}_2}$. By (Sep), we obtain $\alpha \succeq_{\mathcal{B}} \gamma$. Similarly, we have $\gamma_{\upharpoonright\mathcal{B}_1} \succeq_{\mathcal{B}_1} \beta_{\upharpoonright\mathcal{B}_1}$ and $\gamma_{\upharpoonright\mathcal{B}_2} \approx_{\mathcal{B}_2} \beta_{\upharpoonright\mathcal{B}_2}$. Still by (Sep), we obtain $\gamma \succeq_{\mathcal{B}} \beta$. Since $\alpha \succeq_{\mathcal{B}} \gamma$ and $\gamma \succeq_{\mathcal{B}} \beta$, we finally obtain $\alpha \succeq_{\mathcal{B}} \beta$ as desired.

Consider now a subset $\mathcal{B} \in \mathfrak{B}$ and an N-cell partition $\mathfrak{P} = \{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ of $\mathcal{B}$ with $N \geq 2$. Let $\alpha, \beta \in \mathcal{A}(\mathcal{B})$ be such that $\alpha_{\upharpoonright\mathcal{B}_n} \succeq_{\mathcal{B}_n} \beta_{\upharpoonright\mathcal{B}_n}$ for any $n \in [1 \ldots N]$. Let $\mathcal{Q} = \mathcal{B}_1 \vee \ldots \vee \mathcal{B}_{N_1}$. By induction, we have $\alpha_{\upharpoonright\mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\upharpoonright\mathcal{Q}}$. But since $\{\mathcal{Q}, \mathcal{B}_N\}$ is a two-cell partition of $\mathcal{B}$, and since we have $\alpha_{\upharpoonright\mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\upharpoonright\mathcal{Q}}$ and $\alpha_{\upharpoonright\mathcal{B}_N} \succeq_{\mathcal{B}_N} \beta_{\upharpoonright\mathcal{B}_N}$, the previous paragraph yields $\alpha \succeq_{\mathcal{B}} \beta$ as desired. □

Proposition A6 *Suppose $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies axioms* (Sep), (M), (TC), (CCP), (Dom) *and* (CEq). *Let μ be the credence and u be the utility function from Proposition A3. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the unique μ-compatible conditional expectation system from Proposition A4. Then, for any $\mathcal{B} \in \mathfrak{B}$, $\alpha \in \mathcal{A}(\mathcal{B})$ and $x \in \mathcal{X}$ such that $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$, we have*

$$\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \quad = \quad u(x). \tag{A10}$$

Proof. For any $\mathcal{B} \in \mathfrak{B}$, and any $g \in \mathcal{G}_{\mathfrak{B}}(\mathcal{B})$, we define $\widetilde{\mathbb{E}}_{\mathcal{B}}[g] := \mu[\mathcal{B}]\,\mathbb{E}_{\mathcal{B}}[g]$. Recall that $\mathbb{E}_{\mathcal{B}}[\mathbf{1}] = 1$. Thus, $\widetilde{\mathbb{E}}_{\mathcal{B}}[\mathbf{1}] = \mu[\mathcal{B}]$. Thus, for any $r \in \mathbb{R}$, the linearity of $\mathbb{E}_{\mathcal{B}}$ implies

$$\widetilde{\mathbb{E}}_{\mathcal{B}}[r\mathbf{1}] \quad = \quad r\,\mu[\mathcal{B}]. \tag{A11}$$

Let $\mathcal{B} \in \mathfrak{B}$ and $\alpha \in \mathcal{A}(\mathcal{B})$. Consider first the case where $u \circ \alpha$ is constant over $\mathcal{B}$. Then there exists $y \in \mathcal{X}$ such that $u \circ \alpha(s) = u(y)$ for any $s \in \mathcal{B}$. Then $\mathbb{E}_{\mathcal{B}}[u \circ \alpha] = u(y)$ by the linearity of $\mathbb{E}_{\mathcal{B}}$. On the other hand, by Proposition A3, u is an ordinal utility function for $\succeq_{\text{xp}}$. Therefore, $\alpha(s) \approx_{\text{xp}} y$ for any $s \in \mathcal{B}$. By (Dom), we obtain $\alpha \approx_{\mathcal{B}} \kappa^y_{\mathcal{B}}$ and thus, $\kappa^x_{\mathcal{B}} \approx_{\mathcal{B}} \kappa^y_{\mathcal{B}}$ by transitivity. Still by (Dom) we have $x \approx_{\text{xp}} y$. Thus, $u(x) = u(y)$, because u is an ordinal utility function for $\succeq_{\text{xp}}$. This shows $\mathbb{E}_{\mathcal{B}}[u \circ \alpha] = u(x)$, as desired.

Consider now the case where $u\circ\alpha$ takes at least two different values over $\mathcal{B}$. Let $\beta \in \mathcal{A}$ be such that $\beta_{\upharpoonright\mathcal{B}} = \alpha$. Let $\mathcal{K} := \operatorname{clos}[\beta(\mathcal{S})]$ (i.e. the closure of the image set $\beta(\mathcal{S})$ in $\mathcal{X}$); then $\mathcal{K}$ is compact, because $\beta \in \mathcal{C}_{\mathrm{b}}(\mathcal{S}, \mathfrak{B}; \mathcal{X}; \mathfrak{D})$. Recall that $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous, by Proposition A3. Thus, u is bounded when restricted to $\mathcal{K}$. If we define $\mathcal{U} := u \circ \beta(\mathcal{S})$, then $\mathcal{U}$ is a bounded subset of $\mathbb{R}$ —say, $\mathcal{U} \subseteq [-M, M]$ for some $M \in \mathbb{N}$.

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be large enough that $\frac{1}{N} < \epsilon$. For all $n \in [-MN \ldots MN)$, let $\mathcal{C}_n := (u \circ \beta)^{-1}[\frac{n}{N}, \frac{n+1}{N}]$. Recall that $u : \mathcal{X} \longrightarrow \mathbb{R}$ is $\mathfrak{D}$-measurable by Proposition A3, while β is $(\mathfrak{B}, \mathfrak{D})$-comeasurable, by the definition of $\mathcal{A}$. Thus, $u \circ \beta$ is $\mathfrak{B}$-comeasurable, by Proposition PV-5.4(a). Thus if $\mathcal{B}_n := \operatorname{int}(\mathcal{C}_n) \cap \mathcal{B}$, then $\mathcal{B}_n$ is a (possibly empty) element of $\mathfrak{B}$. Let $\mathcal{P}_{-MN} = \mathcal{B}_{-MN}$ and, for any $m \in (-MN \ldots MN)$, define $\mathcal{P}_m := \mathcal{B}_m \cap (\neg\mathcal{B}_{m-1})$. Then $\mathcal{P}_{-MN}, \ldots, \mathcal{P}_{MN-1}$ are disjoint (possibly empty) elements of $\mathfrak{B}$. Let $\mathcal{N} := \{n \in [-NM \ldots NM);\ \ \mathcal{P}_n \neq \emptyset\}$. Finally, define $\mathfrak{P} := \{\mathcal{P}_n\}_{n \in \mathcal{N}}$; then $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{B}$. Since $u \circ \alpha$ takes at least two different values over $\mathcal{B}$, we can take N to be large enough to make sure that $\mathfrak{P}$ has at least two cells.

Claim 1: *For any $n \in \mathcal{N}$, there exist values $x_n, y_n \in \mathcal{X}$ such that*

$$\frac{n}{N} \quad\leq\quad u(x_n) \quad\leq\quad u \circ \alpha(p) \quad\leq\quad u(y_n) \quad\leq\quad \frac{n+1}{N}, \tag{A12}$$

for all $p \in \mathcal{P}_n$.

Proof. Recall that $\mathcal{K} := \operatorname{clos}[\beta(\mathcal{S})]$ is a compact subset of $\mathcal{X}$. Thus, its image $u(\mathcal{K})$ is a compact subset of $\mathbb{R}$, because u is continuous. Thus, the set $\mathcal{W}_n := u(\mathcal{K}) \cap [\frac{n}{N}, \frac{n+1}{N}]$ is compact. Thus, $\underline{w}_n := \min(\mathcal{W}_n)$ and $\overline{w}_n := \max(\mathcal{W}_n)$ are well-defined. Let $x_n \in u^{-1}\{\underline{w}_n\}$ and let $y_n \in u^{-1}\{\overline{w}_n\}$. Thus, $u(x_n) = \underline{w}_n \geq \frac{n}{N}$, while $u(y_n) = \overline{w}_n \leq \frac{n+1}{N}$,

For any $p \in \mathcal{P}_n$, we have $\alpha(p) = \beta(p)$ and $\beta(p) \in \mathcal{K}$, and also $u \circ \alpha(p) \in [\frac{n}{N}, \frac{n+1}{N}]$ by definition of $\mathcal{P}_n$; thus, $u \circ \alpha(p) \in \mathcal{W}_n$. Thus, $\underline{w}_n \leq u \circ \alpha(p) \leq \overline{w}_n$, i.e. $u(x_n) \leq u \circ \alpha(p) \leq u(y_n)$, as claimed. ◇ Claim 1

Now define $\underline{u}, \overline{u} \in \mathbb{R}$ in the following way:

$$\underline{u} \quad = \quad \sum_{n \in \mathcal{N}} \mu(\mathcal{P}_n) \cdot u(x_n) \quad \text{and} \quad \overline{u} = \sum_{n \in \mathcal{N}} \mu(\mathcal{P}_n) \cdot u(y_n).$$

Claim 2: $\underline{u} \quad\leq\quad \widetilde{\mathbb{E}}_{\mathcal{B}}[u \circ \alpha] \quad\leq\quad \overline{u}$

Proof. Fix $n \in \mathcal{N}$. For all $p \in \mathcal{P}_n$, formula (A12) says $u(x_n) \leq u \circ \alpha(p) \leq u(y_n)$, and thus,

$$\begin{aligned} \mu[\mathcal{P}_n] \cdot u(x_n) &\underset{(*)}{=} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u(x_n)\,\mathbf{1}] \underset{(\dagger)}{\leq} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u \circ \alpha] \\ &\underset{(\dagger)}{\leq} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u(y_n)\,\mathbf{1}] \underset{(*)}{=} \mu[\mathcal{P}_n] \cdot u(y_n). \end{aligned} \tag{A13}$$

Here, both $(*)$ are by equation (A11), and both $(\dagger)$ are by inequality (A12) and the monotonicity of the conditional expectation operator $\mathbb{E}_{\mathcal{B}}$. Summing the versions of inequality (A13) obtained for every $n \in \mathcal{N}$, we obtain

$$\underline{u} \quad = \quad \sum_{n \in \mathcal{N}} \mu(\mathcal{P}_n) \cdot u(x_n) \quad \leq \quad \sum_{n \in \mathcal{N}} \widetilde{\mathbb{E}}_{\mathcal{P}_n}[u \circ \alpha] \quad \leq \quad \sum_{n \in \mathcal{N}} \mu(\mathcal{P}_n) \cdot u(y_n) \quad = \quad \overline{u}.$$

The result then follows by noting that the middle term in this inequality is equal to $\widetilde{\mathbb{E}}_{\mathcal{B}}[u \circ \alpha]$ thanks to equation (6) ◇ `Claim 2`

Claim 3: $\underline{u} \leq \mu(\mathcal{B}) \cdot u(x) \leq \overline{u}$.

Proof. Fix $o \in \mathcal{X}$. Define $\mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathcal{X}^{\mathcal{N}}$ in the following way:

$$\text{For any } n \in \mathcal{N}, \quad a'_n \quad := \quad x_n, \quad b'_n \quad := \quad x \quad \text{and} \quad c'_n \quad := \quad y_n.$$

Consider the $\mathfrak{B}$-partition $\mathfrak{Q} = \{\neg\mathcal{B}\} \cup \{\mathcal{P}_n,\ n \in \mathcal{N}\}$ made of $M := 1 + |\mathcal{N}|$ cells (so $M \geq 2$). Define $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^M$ by setting $\mathbf{a} := (\mathbf{a}', o)$, $\mathbf{b} := (\mathbf{b}', o)$ and $\mathbf{c} := (\mathbf{c}', o)$. Finally, let $\Phi_{\mathfrak{Q}}$ be the mapping constructed in Lemma A2.

Now, u is an ordinal utility function for $\succeq_{\text{xp}}$, by Proposition A3. Thus, for all $n \in \mathcal{N}$ and $p \in \mathcal{P}_n$, formula (A12) implies that $x_n \preceq_{\text{xp}} \alpha_{\upharpoonright \mathcal{P}_n}(p) \preceq_{\text{xp}} y_n$. Thus, axiom (Dom) implies that

$$\kappa_{\mathcal{P}_n}^{x_n} \quad \preceq_{\mathcal{P}_n} \quad \alpha_{\upharpoonright \mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \kappa_{\mathcal{P}_n}^{y_n}. \tag{A14}$$

Given the defining properties of the mapping $\Phi_{\mathfrak{Q}}$, formula (A14) then implies that

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\upharpoonright \mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \alpha_{\upharpoonright \mathcal{P}_n} \quad \preceq_{\mathcal{P}_n} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\upharpoonright \mathcal{P}_n}. \tag{A15}$$

Since formula (A15) holds for every $n \in \mathcal{N}$, Lemma A5 further yields

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\upharpoonright \mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \alpha \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\upharpoonright \mathcal{B}}. \tag{A16}$$

Meanwhile, we have $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright \mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa_{\mathcal{P}_n}^{x}$ for every $n \in \mathcal{N}$. By iterative applications of (Sep), we obtain $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright \mathcal{B}} \approx_{\mathcal{B}} \kappa_{\mathcal{B}}^{x}$. But by assumption $\alpha \approx_{\mathcal{B}} \kappa_{\mathcal{B}}^{x}$. Thus, $\Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright \mathcal{B}} \approx_{\mathcal{B}} \alpha$, by transitivity. Formula (A16) then gives

$$\Phi_{\mathfrak{Q}}(\mathbf{a})_{\upharpoonright \mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright \mathcal{B}} \quad \preceq_{\mathcal{B}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c})_{\upharpoonright \mathcal{B}}. \tag{A17}$$

Moreover, by construction, we have $\Phi_{\mathfrak{Q}}(\mathbf{a})_{\upharpoonright \mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{Q}}(\mathbf{b})_{\upharpoonright \mathcal{Q}} \approx_{\mathcal{Q}} \Phi_{\mathfrak{Q}}(\mathbf{c})_{\upharpoonright \mathcal{Q}} \approx_{\mathcal{Q}} \kappa_{\mathcal{Q}}^{o}$ where $\mathcal{Q} = \neg\mathcal{B}$. Given this fact and formula (A17), axiom (Sep) implies

$$\Phi_{\mathfrak{Q}}(\mathbf{a}) \quad \preceq_{\mathcal{S}} \quad \Phi_{\mathfrak{Q}}(\mathbf{b}) \quad \preceq_{\mathcal{S}} \quad \Phi_{\mathfrak{Q}}(\mathbf{c}). \tag{A18}$$

By the definition of $\succeq_{\mathfrak{Q}}$ in formula (A1) and its representation obtained in Proposition A3, formula (A18) implies

$$\begin{aligned}\sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(a_n) \quad + \quad \mu(\mathcal{Q})\cdot u(o) \quad &\leq \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(b_n) \quad + \quad \mu(\mathcal{Q})\cdot u(o)\\ &\leq \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(c_n) \quad + \quad \mu(\mathcal{Q})\cdot u(o),\end{aligned}$$

which, given the definition of **a**, **b** and **c**, reduces to the following formula

$$\underline{u} \quad = \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(x_n) \quad \leq \quad \mu(\mathcal{B})\cdot u(x) \quad \leq \quad \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot u(x_n) \quad = \quad \overline{u}.$$

This completes the proof of the claim. ◇ `Claim 3`

Finally, we obtain

$$\begin{aligned}|\widetilde{\mathbb{E}}_{\mathcal{B}}\left[u\circ\alpha\right]-\mu(\mathcal{B})\cdot u(x)| \quad &\underset{\text{(a)}}{\leq} \quad |\overline{u}-\underline{u}| \underset{\text{(b)}}{\leq} \sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n)\cdot|u(y_n)-u(x_n)|\\ &\underset{\text{(c)}}{\leq} \quad \epsilon\cdot\sum_{n\in\mathcal{N}}\mu(\mathcal{P}_n) \underset{\text{(d)}}{\leq} \epsilon\cdot\mu(\mathcal{B}).\end{aligned}$$

Here, (a) is by Claims 2 and 3, (b) is by the definition of $\underline{u}$ and $\overline{u}$, (c) is inequality (A12), because $1/N<\epsilon$ by definition, and (d) is because μ is a credence on $\mathfrak{B}$ and $\mathfrak{P}$ is a $\mathfrak{B}$-partition of $\mathcal{B}$. This argument works for all $\epsilon>0$. Letting $\epsilon\to 0$, we conclude that $\widetilde{\mathbb{E}}_{\mathcal{B}}[u\circ\alpha]=\mu(\mathcal{B})\cdot u(x)$. Last, since μ‘ has full support, we obtain $\mathbb{E}_{\mathcal{B}}[u\circ\alpha]=u(x)$. □

Finally, we come to the proof of the main result.

Proof of Theorem 1.

SEU representation. Let $u:\mathcal{X}\longrightarrow\mathbb{R}$ be the normalized *ex post* utility function and let μ be the credence with full support from Proposition A3. Let $\{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the unique μ-compatible conditional expectation system from Proposition A4.

For any $\mathcal{B}\in\mathfrak{B}$ and $\alpha,\beta\in\mathcal{A}(\mathcal{B})$, axiom (CEq) yields $x,y\in\mathcal{X}$ such that $\alpha\approx_{\mathcal{B}}\kappa^x_{\mathcal{B}}$ and $\beta\approx_{\mathcal{B}}\kappa^y_{\mathcal{B}}$. Then,

$$\begin{aligned}\Big(\alpha\succeq_{\mathcal{B}}\beta\Big) \quad &\Longleftrightarrow \quad \Big(\kappa^x_{\mathcal{B}}\succeq_{\mathcal{B}}\kappa^y_{\mathcal{B}}\Big) \underset{(*)}{\Longleftrightarrow} \Big(x\succeq_{\text{xp}}y\Big)\\ &\underset{(\dagger)}{\Longleftrightarrow} \quad \Big(u(x)\geq u(y)\Big) \underset{(\diamond)}{\Longleftrightarrow} \Big(\mathbb{E}_{\mathcal{B}}[u\circ\alpha]\geq\mathbb{E}_{\mathcal{B}}[u\circ\beta].\Big)\end{aligned}$$

Here, $(*)$ is by axiom (Dom), $(\dagger)$ is because u is an ordinal utility function for $\succeq_{\text{xp}}$ by Proposition A3, and $(\diamond)$ is by Proposition A6. This equivalence establishes the SEU representation. It remains to show that the representation is unique and demonstrate the necessity of the axioms.

Uniqueness. Let $u, u' : \mathcal{X} \longrightarrow \mathbb{R}$ be two $\mathfrak{D}$-measurable (hence continuous) functions, and let $\mathbf{E} := \{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{R}\in\mathfrak{B}}$ and $\mathbf{E}' := \{\mathbb{E}'_{\mathcal{B}}\}_{\mathcal{R}\in\mathfrak{B}}$ be two conditional expectation systems. Let μ and μ' be two credences on $\mathfrak{B}$ with which $\mathbf{E}$ and $\mathbf{E}'$ are respectively compatible. Suppose that (u, μ) and (u', μ') are both SEU representations for the conditional preference structure $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$. We must show that $\mu = \mu'$ and u is a positive affine transformation of u'.

Let $\mathfrak{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ be a $\mathfrak{B}$-partition of $\mathcal{S}$, with $N \geq 2$ (such a partition exists because $\mathfrak{B}$ is nontrivial). For any $\mathbf{x} \in \mathcal{X}^N$, Lemma A2 yields an act $\alpha^{\mathbf{x}} \in \mathcal{A}$ such that $\alpha^{\mathbf{x}}_{\restriction\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^{x_n}_{\mathcal{E}_n}$ for all $n \in [1 \ldots N]$. Then we have

$$\begin{aligned} \mathbb{E}_{\mathcal{S}}[u \circ \alpha^{\mathbf{x}}] \;&\underset{(*)}{=}\; \sum_{n=1}^{N} \mu(\mathcal{E}_n)\; \mathbb{E}_{\mathcal{E}_n}[u \circ \alpha^{\mathbf{x}}] \\ &\underset{(\dagger)}{=}\; \sum_{n=1}^{N} \mu(\mathcal{E}_n)\, \mathbb{E}_{\mathcal{E}_n}[u \circ \kappa^{x_n}_{\mathcal{E}_n}] \;=\; \sum_{n=1}^{N} \mu(\mathcal{E}_n)\; u(x_n). \end{aligned} \qquad \text{(A19)}$$

Here, $(*)$ is by equation (6), while $(\dagger)$ is by the SEU representation and the fact that $\alpha^{\mathbf{x}}_{\restriction\mathcal{B}_n} \approx_{\mathcal{B}_n} \kappa^{x_n}_{\mathcal{B}_n}$ for all $n \in [1 \ldots N]$. Define $\succeq_{\mathfrak{E}}$ as in equation (A1). By equation (A19), we have for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$

$$\Big(\mathbf{x} \succeq_{\mathfrak{E}} \mathbf{y}\Big) \quad\Longleftrightarrow\quad \left(\sum_{n=1}^{N} \mu(\mathcal{E}_n)\; u(x_n) \geq \sum_{n=1}^{N} \mu(\mathcal{E}_n)\; u(y_n)\right).$$

The SEU representation (u', μ') provides similarly the following representation: for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^N$

$$\Big(\mathbf{x} \succeq_{\mathfrak{E}} \mathbf{y}\Big) \quad\Longleftrightarrow\quad \left(\sum_{n=1}^{N} \mu'(\mathcal{E}_n)\; u'(x_n) \geq \sum_{n=1}^{N} \mu'(\mathcal{E}_n)\; u'(y_n)\right).$$

Now, by the uniqueness part of Proposition A3, we obtain that μ and μ' are equal to each other, and that u and u' are positive affine transformation of each other.

Necessity of the axioms. Assume that $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies condition (Rch) and has an SEU representation in the sense of Theorem 1 with respect to a $\mathfrak{D}$-measurable (hence continuous) utility function u and a credence μ with full support. Let $\mathbf{E} := \{\mathbb{E}_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ be the unique, strictly monotonic conditional expectation system defined by μ via Proposition A4. Axiom (Dom) is a simple consequence of the strict monotonicity of each expectation functional in $\mathbf{E}$. Axiom (Sep) follows from the fact that $\mathbf{E}$ satisfies Equation (6). The proofs of the other axioms are somewhat more involved.

Axiom (TC): Fix two disjoint subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$, and let $\mathcal{Q}_1 = \neg\mathcal{B}_1$ and $\mathcal{Q}_2 = \neg\mathcal{B}_2$. Fix $x, y, v, w \in \mathcal{X}$. By contradiction, assume that $(x \overset{\mathcal{B}_1}{\leadsto} y) \succeq (v \overset{\mathcal{B}_1}{\leadsto} w)$ but $(x \overset{\mathcal{B}_2}{\leadsto} y) \prec (v \overset{\mathcal{B}_2}{\leadsto} w)$. Then, since $(x \overset{\mathcal{B}_1}{\leadsto} y) \succeq (v \overset{\mathcal{B}_1}{\leadsto} w)$, there exist $\alpha, \beta \in \mathcal{A}(\mathcal{Q}_1)$, an (x, α)-bet $(x_{\mathcal{B}_1}\alpha) \in \mathcal{A}$, a (y, β)-bet $(y_{\mathcal{B}_1}\beta) \in \mathcal{A}$, a (v, α)-bet $(v_{\mathcal{B}_1}\alpha) \in \mathcal{A}$ and a (w, β)-bet $(w_{\mathcal{B}_1}\beta) \in \mathcal{A}$ such that

$(x_{\mathcal{B}_1}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}_1}\beta)$ while $(v_{\mathcal{B}_1}\alpha) \succeq_{\mathcal{S}} (w_{\mathcal{B}_1}\beta)$. We now show that $u(x)-u(y) \leq u(v)-u(w)$. Indeed, first we have

$$\begin{aligned} \mu(\mathcal{B}_1)\Big(u(x)-u(y)\Big) & \\ \underset{(*)}{=} \;& \mathbb{E}_{\mathcal{S}}[u\circ(x_{\mathcal{B}_1}\alpha)] - \mu(\mathcal{Q}_1)\mathbb{E}_{\mathcal{Q}_1}[u\circ(x_{\mathcal{B}_1}\alpha)] - \mathbb{E}_{\mathcal{S}}[u\circ(y_{\mathcal{B}_1}\beta)] + \mu(\mathcal{Q}_1)\mathbb{E}_{\mathcal{Q}_1}[u\circ(y_{\mathcal{B}_1}\beta)] \\ \underset{(\dagger)}{\leq} \;& \mu(\mathcal{Q}_1)\Big(\mathbb{E}_{\mathcal{Q}_1}[u\circ(y_{\mathcal{B}_1}\beta)] - \mathbb{E}_{\mathcal{Q}_1}[u\circ(x_{\mathcal{B}_1}\alpha)]\Big). \end{aligned} \tag{A20}$$

Here, $(*)$ is by formula (B1) in the definition of bets, equation (6) and the SEU representation. Meanwhile, $(\dagger)$ is because $\mathbb{E}_{\mathcal{S}}[u\circ(x_{\mathcal{B}_1}\alpha)] \leq \mathbb{E}_{\mathcal{S}}[u\circ(y_{\mathcal{B}_1}\beta)]$ because $(x_{\mathcal{B}_1}\alpha) \preceq_{\mathcal{S}} (y_{\mathcal{B}_1}\beta)$. Proceeding similarly for $(v_{\mathcal{B}_1}\alpha)$ and $(w_{\mathcal{B}_1}\beta)$, we obtain

$$\mu(\mathcal{B}_1)\Big(u(v)-u(w)\Big) \quad \geq \quad \mu(\mathcal{Q}_1)\Big(\mathbb{E}_{\mathcal{Q}_1}[u\circ(w_{\mathcal{B}_1}\beta)] - \mathbb{E}_{\mathcal{Q}_1}[u\circ(v_{\mathcal{B}_1}\alpha)]\Big). \tag{A21}$$

Meanwhile, by formula (B2) in the definition of bets and the SEU representation, we have $\mathbb{E}_{\mathcal{Q}_1}[u\circ(x_{\mathcal{B}_1}\alpha)] = \mathbb{E}_{\mathcal{Q}_1}[u\circ(v_{\mathcal{B}_1}\alpha)]$ and $\mathbb{E}_{\mathcal{Q}_1}[u\circ(y_{\mathcal{B}_1}\beta)] = \mathbb{E}_{\mathcal{Q}_1}[u\circ(w_{\mathcal{B}_1}\beta)]$. Combining inequalities (A20) and (A21) and using the fact that μ has full support, we obtain

$$u(x)-u(y) \quad \leq \quad u(v)-u(w). \tag{A22}$$

Now, since $(x \overset{\mathcal{B}_2}{\rightsquigarrow} y) \prec (v \overset{\mathcal{B}_2}{\rightsquigarrow} w)$, there exist $\gamma, \delta \in \mathcal{A}(\mathcal{Q}_2)$, an (x,γ)-bet $(x_{\mathcal{B}_2}\gamma) \in \mathcal{A}$, a (y,δ)-bet $(y_{\mathcal{B}_2}\delta) \in \mathcal{A}$, a (v,γ)-bet $(v_{\mathcal{B}_2}\gamma) \in \mathcal{A}$ and a (w,δ)-bet $(w_{\mathcal{B}_2}\delta) \in \mathcal{A}$ such that $(x_{\mathcal{B}_2}\gamma) \succeq_{\mathcal{S}} (y_{\mathcal{B}_2}\delta)$ while $(v_{\mathcal{B}_2}\gamma) \prec_{\mathcal{S}} (w_{\mathcal{B}_2}\delta)$. Thus,

$$\begin{aligned} \mu(\mathcal{B}_2)\Big(u(x)-u(y)\Big) \;&\underset{(*)}{\geq}\; \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u\circ(y_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u\circ(x_{\mathcal{B}_2}\gamma)]\Big) \\ &\underset{(\dagger)}{=}\; \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u\circ(w_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u\circ(v_{\mathcal{B}_2}\gamma)]\Big), \end{aligned} \tag{A23}$$

where $(*)$ is obtained like inequality (A20), while $(\dagger)$ is by formula (B2) in the definition of bets and the SEU representation. Combining inequalities (A22) and (A23), we get

$$\mu(\mathcal{B}_2)\Big(u(v)-u(w)\Big) \quad \geq \quad \mu(\mathcal{Q}_2)\Big(\mathbb{E}_{\mathcal{Q}_2}[u\circ(w_{\mathcal{B}_2}\delta)] - \mathbb{E}_{\mathcal{Q}_2}[u\circ(v_{\mathcal{B}_2}\gamma)]\Big). \tag{A24}$$

Finally, applying equation (6) and the SEU representation to inequality (A24), we obtain $(v_{\mathcal{B}_2}\gamma) \succeq_{\mathcal{S}} (w_{\mathcal{B}_2}\delta)$. But this contradicts the fact that $(v_{\mathcal{B}_2}\gamma) \prec_S (w_{\mathcal{B}_2}\delta)$.

Axioms (C) and (M): Let $x \in \mathcal{X}$. Let $(x, \rightarrow)_{\succeq_{\rm xp}} := \{y \in \mathcal{X};\ \ y \succ_{\rm xp} x\}$ and let $(\leftarrow, x)_{\succeq_{\rm xp}} := \{y \in \mathcal{X};\ \ y \prec_{\rm xp} x\}$. To verify axiom (C), we must show that these sets are open in $\mathcal{X}$. To verify axiom (M), we must show that they are elements of $\mathfrak{D}$. To verify both, let $r := u(x)$, and observe that $(x, \rightarrow)_{\succeq_{\rm xp}} = u^{-1}(r, \infty)$ and $(\leftarrow, x)_{\succeq_{\rm xp}} = u^{-1}(-\infty, r)$, because u is an ordinal utility representation for $\succeq_{\rm xp}$. Since u is continuous, these preimage sets are open in $\mathcal{X}$. Since u is $\mathfrak{D}$-measurable, these preimage sets are elements of $\mathfrak{D}$.

Axiom (CEq): Let $\mathcal{B} \in \mathfrak{B}$ and let $\alpha \in \mathcal{A}(\mathcal{B})$. Then $\alpha = \alpha'_{\upharpoonright\mathcal{B}}$ for some $\alpha' \in \mathcal{A}$.

Claim 1: $\text{clos}[\alpha'(\mathcal{S})]$ *has a* $\succeq_{\text{xp}}$*-maximal element and a* $\succeq_{\text{xp}}$*-minimal element.*[17]

Proof. (By contradiction) Suppose $\text{clos}[\alpha'(\mathcal{S})]$ had no $\succeq_{\text{xp}}$-maximal element. Thus, for any $x \in \text{clos}[\alpha'(\mathcal{S})]$, there exists some $y \in \text{clos}[\alpha'(\mathcal{S})]$ with $y \succ_{\text{xp}} x$. In other words, $x \in (\leftarrow, y)_{\succeq_{\text{xp}}}$. Thus, the collection $\{(\leftarrow, y)_{\succeq_{\text{xp}}};\ y \in \text{clos}[\alpha'(\mathcal{S})]\}$ is an open cover for $\text{clos}[\alpha'(\mathcal{S})]$.

However, $\alpha' \in \mathcal{C}_{\text{b}}(\mathcal{S}, \mathcal{X})$, so its image $\alpha'(\mathcal{S})$ is relatively compact; hence $\text{clos}[\alpha'(\mathcal{S})]$ is compact. Thus, this open cover has a finite subcover; in other words, there exists some $y_1, \ldots, y_N \in \text{clos}[\alpha'(\mathcal{S})]$ such that $\text{clos}[\alpha'(\mathcal{S})]$ is covered by the collection $\{(\leftarrow, y_n)_{\succeq_{\text{xp}}}\}_{n=1}^{N}$. Now, let $\overline{y} := \max_{\succeq_{\text{xp}}}\{y_1, \ldots, y_N\}$ (this maximum exists because the set is finite). Then $\overline{y} \in \text{clos}[\alpha'(\mathcal{S})]$, and $(\leftarrow, y_n)_{\succeq_{\text{xp}}} \subseteq (\leftarrow, \overline{y})_{\succeq_{\text{xp}}}$ for all $n \in [1 \ldots N]$. Thus, $\text{clos}[\alpha'(\mathcal{S})] \subseteq (\leftarrow, \overline{y})_{\succeq_{\text{xp}}}$. But clearly, $\overline{y} \notin (\leftarrow, \overline{y})_{\succeq_{\text{xp}}}$, whereas $\overline{y} \in \text{clos}[\alpha'(\mathcal{S})]$. Contradiction.

To avoid the contradiction, $\text{clos}[\alpha'(\mathcal{S})]$ must have a $\succeq_{\text{xp}}$-maximal element. The proof for $\succeq_{\text{xp}}$-minimal elements is analogous. ◇ Claim 1

Let x be a $\succeq_{\text{xp}}$-minimal element of $\text{clos}[\alpha'(\mathcal{S})]$, and let z be a $\succeq_{\text{xp}}$-maximal element of $\text{clos}[\alpha'(\mathcal{S})]$; these exist by Claim 1. Then $x \preceq_{\text{xp}} \alpha(b) \preceq_{\text{xp}} z$ for all $b \in \mathcal{B}$. Thus, axiom (Dom) implies that $\kappa^x_{\mathcal{B}} \preceq_{\mathcal{B}} \alpha \preceq_{\mathcal{B}} \kappa^z_{\mathcal{B}}$. Thus,

$$u(x) \quad = \quad \mathbb{E}_{\mathcal{B}}[u \circ \kappa^x_{\mathcal{B}}] \quad \underset{(*)}{\leq} \quad \mathbb{E}_{\mathcal{B}}[u \circ \alpha] \quad \underset{(*)}{\leq} \quad \mathbb{E}_{\mathcal{B}}[u \circ \kappa^z_{\mathcal{B}}] \quad = \quad u(z),$$

where both $(*)$ are because of the assumed SEU representation. However, $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous, and $\mathcal{X}$ is connected. Thus, the Intermediate Value Theorem yields some $y \in \mathcal{X}$ such that $u(y) = \mathbb{E}_{\mathcal{B}}[u \circ \alpha]$. Thus, $\mathbb{E}_{\mathcal{B}}[u \circ \kappa^y_{\mathcal{B}}] = \mathbb{E}_{\mathcal{B}}[u \circ \alpha]$. But then the assumed SEU representation yields $\kappa^y_{\mathcal{B}} \approx_{\mathcal{B}} \alpha$, as desired.

Axiom (CCP): Let $\mathcal{D} \in \mathfrak{B}$ and $\mathcal{E} \in \mathfrak{B}$ be disjoint, and let $\mathcal{G} := \mathcal{D} \vee \mathcal{E}$. Let $\mathcal{O} \subseteq \mathcal{A}(\mathcal{G})$ be open in the $\succeq_{\mathcal{G}}$-order topology, and let $\beta \in \mathcal{O}$. Thus, there exist some $\alpha, \gamma \in \mathcal{A}(\mathcal{G})$ such that $\alpha \prec_{\mathcal{G}} \beta \prec_{\mathcal{G}} \gamma$, and $\mathcal{O}$ contains the order-interval $(\alpha, \gamma)_{\succeq_{\mathcal{G}}}$. Let $a := \mathbb{E}_{\mathcal{G}}[u \circ \alpha]$, $b := \mathbb{E}_{\mathcal{G}}[u \circ \beta]$, and $c := \mathbb{E}_{\mathcal{G}}[u \circ \gamma]$; then $a < b < c$. Let $\epsilon := \min\{b - a, c - b\}$. Then $\epsilon > 0$.

Claim 2: *There exist a subset* $\mathcal{O}_{\mathcal{D}} \subseteq \mathcal{A}(\mathcal{D})$*, open in the* $\succeq_{\mathcal{D}}$*-order topology, such that* $\beta_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$*, and such that* $|\mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] - \mathbb{E}_{\mathcal{D}}[u \circ \beta_{\upharpoonright\mathcal{D}}]| < \epsilon$ *for all* $\omega_{\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$.

Proof. (*Case 1*) First, suppose that $\beta_{\upharpoonright\mathcal{D}}$ is neither $\succeq_{\mathcal{D}}$-maximal nor $\succeq_{\mathcal{D}}$-minimal in $\mathcal{A}(\mathcal{D})$. Then there exists some $\phi_{\mathcal{D}}, \psi_{\mathcal{D}} \in \mathcal{A}(\mathcal{D})$ such that $\phi_{\mathcal{D}} \prec_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \psi_{\mathcal{D}}$. Now, $\phi_{\mathcal{D}} := \phi'_{\upharpoonright\mathcal{D}}$ and $\psi_{\mathcal{D}} := \psi'_{\upharpoonright\mathcal{D}}$ for some $\phi, \psi \in \mathcal{A}$. Let w be a $\succeq_{\text{xp}}$-minimal element of $\text{clos}[\phi'(\mathcal{S})]$, and let z be a $\succeq_{\text{xp}}$-maximal element of $\text{clos}[\psi'(\mathcal{S})]$; these exist by Claim 1. Then $w \preceq_{\text{xp}} \phi_{\mathcal{D}}(d)$ and $\psi_{\mathcal{D}}(d) \preceq_{\text{xp}} z$ for all $d \in \mathcal{D}$. Thus,

$$\kappa^w_{\mathcal{D}} \quad \preceq_{\mathcal{D}} \quad \phi_{\mathcal{D}} \quad \prec_{\mathcal{D}} \quad \beta_{\mathcal{D}} \quad \prec_{\mathcal{D}} \quad \psi_{\mathcal{D}} \quad \preceq_{\mathcal{D}} \quad \kappa^z_{\mathcal{D}},$$

[17] Actually, we only need to obtain an upper and lower bound for $\text{clos}[\alpha'(\mathcal{S})]$ in $\mathcal{X}$. But constructing a maximum and minimum is no more difficult.

where the "$\preceq_{\mathcal{D}}$" comparisons are by axiom (Dom), and the "$\prec_{\mathcal{D}}$" comparisons are by the definitions of $\phi_{\mathcal{D}}$ and $\psi_{\mathcal{D}}$. Thus,

$$u(w) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{w}] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{z}] \quad = \quad u(z),$$

where both $(*)$ are because of the assumed SEU representation. Thus, $u(w) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] < u(z)$. Now, u is continuous, and $\mathcal{X}$ is connected. Thus, the Intermediate Value Theorem yields $x, y \in \mathcal{X}$ such that $\mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon < u(x) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] < u(y) < \mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon$. (It is even possible that w and z themselves already satisfy these inequalities). Thus,

$$\mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{x}] \quad = \quad u(x) \quad < \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] \quad < \quad u(y) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{y}],$$

so $\kappa_{\mathcal{D}}^{x} \prec_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}} \prec_{\mathcal{D}} \kappa_{\mathcal{D}}^{y}$, by the assumed SEU representation. Thus, if we define $\mathcal{O}_{\mathcal{D}} := (\kappa_{\mathcal{D}}^{x}, \kappa_{\mathcal{D}}^{y})_{\succeq_{\mathcal{D}}}$, then $\mathcal{O}_{\mathcal{D}}$ is open in the $\succeq_{\mathcal{D}}$-order topology, and $\beta_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$. Furthermore, for any $\omega_{\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$, we have $\kappa_{\mathcal{D}}^{x} \prec_{\mathcal{D}} \omega_{\mathcal{D}} \prec_{\mathcal{D}} \kappa_{\mathcal{D}}^{y}$, and thus,

$$\begin{aligned} \mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon \quad & \underset{(\diamond)}{<} \quad u(x) \quad = \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{x}] \quad \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] \\ & \underset{(*)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{y}] \quad = \quad u(y) \quad \underset{(\diamond)}{<} \quad \mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon, \end{aligned}$$

so that $|\mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] - \mathbb{E}_{\mathcal{D}}[u \circ \beta_{\upharpoonright\mathcal{D}}]| < \epsilon$, as desired. Here, the $(*)$ inequalities are by the assumed SEU representation, and $(\diamond)$ inequalities are by the definitions of x and y.

(*Case 2*) Suppose $\beta_{\upharpoonright\mathcal{D}}$ is $\succeq_{\mathcal{D}}$-maximal in $\mathcal{A}(\mathcal{D})$, but not $\succeq_{\mathcal{D}}$-minimal. The logic is similar to *Case 1*, so we will be more cursory. There exists some $\phi_{\mathcal{D}} \in \mathcal{A}(\mathcal{D})$ such that $\phi_{\mathcal{D}} \prec_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$. As in Case 1, use Claim 1 to obtain some $w \in \mathcal{X}$ such that $w \preceq_{\text{xp}} \phi_{\mathcal{D}}(d)$ for all $d \in \mathcal{D}$. Thus, $\kappa_{\mathcal{D}}^{w} \preceq_{\mathcal{D}} \phi_{\mathcal{D}} \prec_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$, and thus, $u(w) = \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{w}] < \mathbb{E}_{\mathcal{D}}[u \circ \beta]$.

Now, u is continuous, and $\mathcal{X}$ is connected, so the Intermediate Value Theorem yields $x \in \mathcal{X}$ such that $\mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon < u(x) < \mathbb{E}_{\mathcal{D}}[u \circ \beta]$. Thus, $\mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{x}] = u(x) < \mathbb{E}_{\mathcal{D}}[u \circ \beta]$, so $\kappa_{\mathcal{D}}^{x} \prec_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$. Thus, if we define $\mathcal{O}_{\mathcal{D}} := (\kappa_{\mathcal{D}}^{x}, \rightarrow)_{\succeq_{\mathcal{D}}}$, then $\mathcal{O}_{\mathcal{D}}$ is open in the $\succeq_{\mathcal{D}}$-order topology, and $\beta_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$; in fact, $\beta_{\upharpoonright\mathcal{D}}$ is a $\succeq_{\mathcal{D}}$-maximal element of $\mathcal{O}_{\mathcal{D}}$. Thus, for any $\omega_{\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$, we have $\kappa_{\mathcal{D}}^{x} \prec_{\mathcal{D}} \omega_{\mathcal{D}} \preceq_{\mathcal{D}} \beta_{\upharpoonright\mathcal{D}}$, and thus,

$$\mathbb{E}_{\mathcal{D}}[u \circ \beta] - \epsilon \underset{(\diamond)}{<} u(x) \;=\; \mathbb{E}_{\mathcal{D}}[u \circ \kappa_{\mathcal{D}}^{x}] \underset{(*)}{<} \mathbb{E}_{\mathcal{D}}[u \circ \omega_{\mathcal{D}}] \underset{(*)}{\leq} \mathbb{E}_{\mathcal{D}}[u \circ \beta] \;<\; \mathbb{E}_{\mathcal{D}}[u \circ \beta] + \epsilon,$$

as desired. Here, the $(*)$ inequalities are by the assumed SEU representation, and the $(\diamond)$ inequality is by the definition of x.

(*Case 3*) Suppose $\beta_{\mathcal{D}}$ is $\succeq_{\mathcal{D}}$-minimal in $\mathcal{A}(\mathcal{D})$, but not $\succeq_{\mathcal{D}}$-maximal. The logic is exactly the same as *Case 2*, but with all the preferences and inequalities reversed.

(*Case 4*) Suppose $\beta_{\mathcal{D}}$ is both $\succeq_{\mathcal{D}}$-minimal and $\succeq_{\mathcal{D}}$-maximal in $\mathcal{A}(\mathcal{D})$. In this case, $\mathbb{E}_{\mathcal{D}}[u \circ \omega] = \mathbb{E}_{\mathcal{D}}[u \circ \beta]$ for all $\omega \in \mathcal{A}(\mathcal{D})$. Thus, if we define $\mathcal{O}_{\mathcal{D}} := \mathcal{A}(\mathcal{D})$, then the claim is trivially satisfied. $\diamond$ `Claim 2`

Claim 3: *There exist a subset $\mathcal{O}_{\mathcal{E}} \subseteq \mathcal{A}(\mathcal{E})$, open in the $\succeq_{\mathcal{E}}$-order topology, such that $\beta_{\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$, and such that $|\mathbb{E}_{\mathcal{E}}[u \circ \omega_{\mathcal{E}}] - \mathbb{E}_{\mathcal{E}}[u \circ \beta_{\mathcal{E}}]| < \epsilon$ for all $\omega_{\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$.*

Proof. The argument is identical to Claim 2. ◇ `Claim 3`

Now let $\omega \in \mathcal{A}(\mathcal{G})$, and suppose $\omega_{\upharpoonright\mathcal{D}} \in \mathcal{O}_{\mathcal{D}}$ and $\omega_{\upharpoonright\mathcal{E}} \in \mathcal{O}_{\mathcal{E}}$. Then

$$\begin{aligned} \mu[\mathcal{G}]\,\mathbb{E}_{\mathcal{G}}[u\circ\omega] &\underset{(*)}{=} \mu[\mathcal{D}]\,\mathbb{E}_{\mathcal{D}}[u\circ\omega] + \mu[\mathcal{E}]\,\mathbb{E}_{\mathcal{E}}[u\circ\omega] \\ &\underset{(\dagger)}{<} \mu[\mathcal{D}]\left(\mathbb{E}_{\mathcal{D}}[u\circ\beta]+\epsilon\right) + \mu[\mathcal{E}]\left(\mathbb{E}_{\mathcal{E}}[u\circ\beta]+\epsilon\right) \\ &\underset{(*)}{=} \mu[\mathcal{G}]\,\left(\mathbb{E}_{\mathcal{G}}[u\circ\beta]+\epsilon\right) \underset{(\diamond)}{\leq} \mu[\mathcal{G}]\,\mathbb{E}_{\mathcal{G}}[u\circ\gamma]. \end{aligned}$$

Here, both $(*)$ are by equation (6), $(\dagger)$ is by the inequalities in Claims 2 and 3, and the full support of μ while $(\diamond)$ is by the definition of ϵ. Thus, $\mathbb{E}_{\mathcal{G}}[u\circ\omega] < \mathbb{E}_{\mathcal{G}}[u\circ\gamma]$. Thus, by the presumed SEU representation $\omega \prec_{\mathcal{G}} \gamma$. By an identical argument, $\mathbb{E}_{\mathcal{G}}[u\circ\omega] > \mathbb{E}_{\mathcal{G}}[u\circ\alpha]$, and thus, $\omega \succ_{\mathcal{G}} \alpha$. Thus, $\omega \in (\alpha,\gamma)_{\succeq_{\mathcal{G}}}$, and thus, $\omega \in \mathcal{O}$, as desired. □

B Proofs of other results

Proof of Theorem 2. Let $\mathcal{S}^*$ be the Stone space of the Boolean algebra $\mathfrak{B}$ —that is, the set of all Boolean algebra homomorphisms from $\mathfrak{B}$ into $\{T,F\}$. Let $\mathfrak{Clp}(\mathcal{S}^*)$ be the set of all clopen subsets of $\mathcal{S}^*$; this is a Boolean algebra under the standard set-theoretic operations of union, intersection, and complementation. The Stone Representation Theorem says there is a Boolean algebra isomorphism $\Phi : \mathfrak{B}\longrightarrow\mathfrak{Clp}(\mathcal{S}^*)$ given by $\Phi(\mathcal{B}) = \mathcal{B}^*$ for all $\mathcal{B}\in\mathfrak{B}$, where $\mathcal{B}^* := \{s^*\in\mathcal{S}^*;\ s^*(\mathcal{B}) = T\}$.

"$\Longleftarrow$" Suppose μ^* is a Borel probability measure on $\mathcal{S}^*$ with full support, and $u : \mathcal{X}\longrightarrow\mathbb{R}$ is a $\mathfrak{D}$-measurable function that together provide a Stonean SEU representation of $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ as in formula (8). For all $\mathcal{B}\in\mathfrak{B}$, define $\mu[\mathcal{B}] := \mu^*[\mathcal{B}^*]$. In other words, $\mu := \mu^*\circ\Phi$. Then μ is a credence on $\mathfrak{B}$, because Φ is a Boolean algebra isomorphism from $\mathfrak{B}$ to $\mathfrak{Clp}(\mathcal{S}^*)$, and μ^* is a finitely additive probability measure when restricted to $\mathfrak{Clp}(\mathcal{S}^*)$. Furthermore, Theorem PV-8.4 says that $\mathbb{E}^{\mu}_{\mathcal{B}}[u\circ\alpha] = \int_{\mathcal{B}^*} u\circ\alpha^*\ \mathrm{d}\mu^*$ for all $\alpha\in\mathcal{A}$; thus, (u,μ) provides an SEU representation as in formula (7). Meanwhile, μ^* has full support, so $\mu^*[\mathcal{B}^*] > 0$ for all $\mathcal{B}^*\in\mathfrak{Clp}(\mathcal{S}^*)$, and hence, $\mu[\mathcal{B}] > 0$ for all $\mathcal{B}\in\mathfrak{B}$; thus μ also has full support. Thus, Theorem 1 says that $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies the axioms (CEq), (Dom), (Sep), (CCP), (M) and (TC).

"$\Longrightarrow$" If $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ satisfies Axioms (CEq), (Dom), (Sep), (CCP), (M), and (TC), then Theorem 1 says it has an SEU representation (7) given by a credence μ on $\mathfrak{B}$ with full support, and a $\mathfrak{D}$-measurable utility function $u : \mathcal{X}\longrightarrow\mathbb{R}$. Let $\mathbf{E}$ be the μ-compatible conditional expectation structure from Proposition A4. Define the function $\mu^* : \mathfrak{Clp}(\mathcal{S}^*)\longrightarrow[0,1]$ by setting $\mu^*(\mathcal{B}^*) := \mu[\mathcal{B}]$ for all $\mathcal{B}\in\mathfrak{B}$ —in other words, $\mu^* := \mu\circ\Phi^{-1}$. This is a finitely additive probability measure on $\mathfrak{Clp}(\mathcal{S}^*)$ because Φ^{-1} is a Boolean algebra isomorphism from $\mathfrak{Clp}(\mathcal{S}^*)$ to $\mathfrak{B}$. Theorem PV-8.4 says that μ^* extends to a unique Borel probability measure μ^* on $\mathcal{S}^*$ such that, for any $g\in\mathcal{G}_{\mathfrak{B}}(\mathcal{S})$ and $\mathcal{B}\in\mathfrak{B}$, we have $\mathbb{E}^{\mu}_{\mathcal{B}}[g] = \int_{\mathcal{B}^*} g^*\ \mathrm{d}\mu^*$. In particular, for any $\alpha\in\mathcal{A}$, we have

$\mathbb{E}^{\mu}_{\mathcal{B}}[u\circ\alpha] = \int_{\mathcal{B}^*} u\circ\alpha^*\,\mathrm{d}\mu^*$ (because $(u\circ\alpha)^* = u\circ\alpha^*$). Applying this identity to the SEU representation (7), we obtain a Stonean SEU representation as in formula (8).

Full support. $\mu[\mathcal{B}] > 0$ for every nonempty $\mathcal{B}\in\mathfrak{B}$. Thus, $\mu^*[\mathcal{B}^*] > 0$ for every nonempty $\mathcal{B}^*\in\mathfrak{Clp}(\mathcal{S}^*)$. But $\mathcal{S}^*$ is totally disconnected, so $\mathfrak{Clp}(\mathcal{S}^*)$ is a base for the topology of $\mathcal{S}^*$. Thus, we deduce that $\mu^*[\mathcal{O}^*] > 0$ for every nonempty open subset $\mathcal{O}^*\subseteq\mathcal{S}^*$.

Uniqueness. Suppose that both (u_1,μ_1^*) and (u_2,μ_2^*) provide Stonean SEU representation for $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$. Let $\mu_1 := \mu_1^*\circ\Phi$ and $\mu_2 := \mu_2^*\circ\Phi$; these are credences on $\mathfrak{B}$, and by Theorem PV-8.4, they both provide SEU representations as in formula (7). By uniqueness in Theorem 1, u_1 and u_2 are positive affine transformations of each other, while $\mu_1 = \mu_2$. Thus, the "uniqueness" part of Theorem PV-8.4 says that $\mu_1^* = \mu_2^*$. □

Proof of Theorem 3. In Theorem 1, the $\mathfrak{D}$-measurability of u first serves to obtain axiom (M). But this is not needed here. Moreover, $\mathfrak{D}$-measurability is used to show the continuity of u, which we invoke in the proofs of (CEq) and (CCP). But here the continuity of u is part of the representation. To prove the necessity of the axioms, we can therefore proceed here exactly as in Theorem 1. As for the sufficiency of the axioms, note that Lemmas A1 and A2, as well as Proposition A3, all remain valid if one replaces axiom (M) with the weaker axiom (C) — at least, if one removes the conclusion that u be $\mathfrak{D}$-measurable from the latter result. Likewise, Proposition A6 is still true with axiom (C) instead of axiom (M). Indeed, in the proof of this proposition, the $\mathfrak{D}$-measurability of u is only used to make sure that $u\circ\alpha$ is $\mathfrak{B}$-comeasurable for any $\alpha\in\mathcal{A}$. Here, since u and α are continuous, $u\circ\alpha$ is also continuous, and therefore automatically $\mathfrak{R}(\mathcal{S})$-measurable for any $\alpha\in\mathcal{A}$, by Proposition PV-5.4(a). Thus, essentially the same proof as for Theorem 1 provides the SEU representation. Finally, the uniqueness of the representation can be obtained exactly as in Theorem 1 since the argument invoked there uses neither axiom (M) nor $\mathfrak{D}$-measurability, but only axiom (C) and continuity. □

Proof of Theorem 4. The proofs for the necessity of Axioms (M), (Dom), (Sep), (CCP), and (TC) are the same as in Theorem 1 because these proofs do not invoke the boundedness of the acts in $\mathcal{A}$. However, the proof of (CEq) is different. Let $\mathcal{B}\in\mathfrak{B}$ be nonempty and let $\alpha\in\mathcal{A}$. We need to find $x\in\mathcal{S}$ such that $\alpha_{\upharpoonright\mathcal{B}} \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$. Given the SEU representation, it is sufficient to show that $\mathbb{E}_{\mathcal{B}}[u\circ\alpha]\in u(\mathcal{X})$. Let $a := \inf\, u(\mathcal{X})$ and $b := \sup\, u(\mathcal{X})$. Since $\mathcal{S}$ is connected and u is continuous, $u(\mathcal{X})$ is an interval of $\mathbb{R}$. Therefore, we are in one of the following four cases.

Case 1: $u(\mathcal{X}) = [a,b]$. There exist then $x,y\in\mathcal{X}$ such that $u(x) = a$ and $u(y) = b$. Moreover, we necessarily have $u(y)\geq u\circ\alpha(s)\geq u(x)$ for any $s\in\mathcal{B}$. By the monotonicity of the expectation functionals, we have $b = u(y)\geq\mathbb{E}_{\mathcal{B}}[u\circ\alpha]\geq u(x) = a$. This shows that $\mathbb{E}_{\mathcal{B}}[u\circ\alpha]\in u(\mathcal{X})$.

Case 2: $u(\mathcal{X}) = [a, b)$. By proceeding as in Case 1, we obtain $\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \geq a$. The remainder of the proof requires the following claim.

Claim 1: *There exists $r < b$ such that $\mathcal{B} \cap \operatorname{int}((u \circ \alpha)^{-1}(-\infty, r])$ is nonempty.*

Proof. For any $r \in u(\mathcal{X})$, let $\mathcal{B}_r := \operatorname{int}((u \circ \alpha)^{-1}(-\infty, r])$. To conclude, it is sufficient to show the following equality:

$$\mathcal{S} = \bigcup_{r \in u(\mathcal{X})} \mathcal{B}_r \tag{B1}$$

Indeed, then we will have

$$\mathcal{B} = \bigcup_{r \in u(\mathcal{X})} \mathcal{B} \cap \mathcal{B}_r. \tag{B2}$$

Since $\mathcal{B}$ is nonempty by assumption, formula (B2) will imply that $\mathcal{B} \cap \mathcal{B}_r$ is nonempty for some $r \in u(\mathcal{X})$. But $u(\mathcal{X}) = [a, b)$. So we will have $r < b$.

To show formula (B1), fix $s \in \mathcal{S}$. Then, $u \circ \alpha(s) \in u(\mathcal{X}) = [a, b)$. So we can find r such that $u \circ \alpha(s) < r < b$. Therefore, $s \in (u \circ \alpha)^{-1}(-\infty, r)$. But the latter set is open in $\mathcal{S}$ because both u and α are continuous. Moreover, it is contained in $(u \circ \alpha)^{-1}(-\infty, r]$. So $s \in \operatorname{int}((u \circ \alpha)^{-1}(-\infty, r]) := \mathcal{B}_r$. Hence formula (B1). ◇ Claim 1

Now, let $\mathcal{P} := \mathcal{B} \cap \operatorname{int}((u \circ \alpha)^{-1}(-\infty, r])$. Since u is $\mathfrak{D}$-measurable and α is $(\mathfrak{B}, \mathfrak{D})$-comeasurable, $u \circ \alpha$ is $\mathfrak{B}$-comeasurable. See Proposition PV-5.4(a). Therefore, $\mathcal{P} \in \mathfrak{B}$. Set $\mathcal{Q} := \neg\mathcal{P} \cap \mathcal{B}$, another (possibly empty) element of $\mathfrak{B}$. Then, $\{\mathcal{P}, \mathcal{Q}\}$ is a of incompatible subsets in $\mathfrak{B}$ whose join equals $\mathcal{B}$. (If $\mathcal{Q}$ was nonempty, this would be a $\mathfrak{B}$-partition of $\mathcal{B}$.) Moreover, for any $s \in \mathcal{P}$, we have $(u \circ \alpha)(s) \leq r$. Then, by the monotonicity of the expectation functionals, we obtain $\mathbb{E}_{\mathcal{P}}[u \circ \alpha] \leq r$. Moreover, we have $(u \circ \alpha)(s) \leq b$ for any $s \in \mathcal{Q}$ so, still by monotonicity, $\mathbb{E}_{\mathcal{Q}}[u \circ \alpha] \leq b$. Therefore,

$$\begin{aligned}
\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \; &\underset{(a)}{=} \; \mu[\mathcal{P}]\, \mathbb{E}_{\mathcal{P}}[u \circ \alpha] + \mu[\mathcal{Q}]\, \mathbb{E}_{\mathcal{Q}}[u \circ \alpha] \; \underset{(b)}{\leq} \; \mu[\mathcal{P}]\; r + \mu[\mathcal{Q}]\; b \\
&\underset{(c)}{<} \; \mu[\mathcal{P}]\; b + \mu[\mathcal{Q}]\; b \; \underset{(d)}{=} \; b,
\end{aligned}$$

where (a) is by Formula 6, (b) is because $\mathbb{E}_{\mathcal{P}}[u \circ \alpha] \leq r$ and $\mathbb{E}_{\mathcal{Q}}[u \circ \alpha] \leq b$, (c) is by Claim 1, because $r < b$ and $\mu(\mathcal{P}) > 0$ since $\mathcal{P}$ is nonemptu and μ has full support, and finally (d) is because μ is a credence. Summing up, we have $a \leq \mathbb{E}_{\mathcal{B}}[u \circ \alpha] < b$ and, therefore, $\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \in u(\mathcal{X})$.

Case 3: $u(\mathcal{X}) = (a, b]$. The proof is similar to that of Case 2.

Case 4: $u(\mathcal{X}) = (a, b)$. The proof uses the same arguments as in Cases 2 and 3.

In all cases, we have $\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \in u(\mathcal{X})$. Thus, (CEq) holds, and all axioms are necessary for the SEU representation. As for its uniqueness, suppose that (u, μ) provides an SEU representation of $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ on $\mathcal{A}$. Then, (u, μ) also provides an SEU representation to the restriction of $\{\succeq_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$ to the set $\mathcal{A}_{\mathrm{b}}$ of all bounded acts in $\mathcal{A}$. Moreover, thanks to (Rch*), these restricted preferences satisfy (Rch). So, by Theorem 1, u is unique up to positive affine transformation and μ is unique.

We now come to the sufficiency of the axioms. By (Rch*), the restriction of $\{\succeq_{\mathcal{B}}\}_{\mathcal{B}\in\mathfrak{B}}$ to the set $\mathcal{A}_{\mathrm{b}}$ of all bounded acts in $\mathcal{A}$ satisfies (Rch). Moreover, it satisfies all of Axioms (CEq), (M), (Dom), (Sep), (CCP), and (TC). Then, by Theorem 1, these restricted preference have an SEU representation (u, μ), where u is $\mathfrak{D}$-measurable (hence continuous) and μ has full support. That is, for any $\mathcal{B} \in \mathfrak{B}$ and $\alpha, \beta \in \mathcal{A}_{\mathrm{b}}(\mathcal{B})$,

$$\Big(\alpha \succeq_{\mathcal{B}} \beta\Big) \quad \iff \quad \Big(\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \geq \mathbb{E}_{\mathcal{B}}[u \circ \beta].\Big) \tag{B3}$$

Now, suppose that u is not bounded from above. Then, we can easily find a sequence $\{x_n,\, n \in \mathbb{N}\}$ of points in $\mathcal{X}$ such that, for any $n \in \mathbb{N}$,

$$u(x_n) \cdot \mu(\mathcal{P}_n) \;\geq\; 1 \quad \text{and} \quad u(x_{n+1}) \;>\; u(x_n). \tag{B4}$$

Moreover, let $\{\mathcal{P}_n,\, n \in \mathbb{N}\}$ be a countable $\mathfrak{B}$-partition as in condition (Exh).

Claim 2: *There exists $\alpha \in \mathcal{A}$ such that $\alpha_{\restriction \mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa_{\mathcal{P}_n}^{x_n}$ for any $n \in \mathbb{N}$ and $\alpha \succeq_{\mathcal{S}} \kappa^x$ for any $x \in \mathcal{X}$.*

Proof. We first construct a sequence $\{\alpha^n,\, n \in \mathbb{N}\}$ of elements of conditional acts. Let $\alpha^1 := \kappa_{\mathcal{P}_1}^{x_1} \in \mathcal{A}(\mathcal{P}_1)$. Clearly, we have $\alpha^1 \approx_{\mathcal{P}_1} \kappa_{\mathcal{P}_1}^{x_1}$ and $\alpha^1(s) \succeq_{\mathrm{xp}} x_1$ for any $s \in \mathcal{P}_1$.

Now, we have $u(x_2) > u(x_1)$ by formula (B4). So, by formula B3, we obtain $\kappa_{\mathcal{P}_2}^{x_2} \succ_{\mathcal{P}_2} \kappa_{\mathcal{P}_2}^{x_1}$. Since we have additionally $\alpha^1(s) \succeq_{\mathrm{xp}} x_1$ for any $s \in \mathcal{P}_1$, (Rch*) provides an act $\alpha^2 \in \mathcal{A}(\mathcal{P}_1 \vee \mathcal{P}_2)$ such that $\alpha^2_{\restriction \mathcal{P}_1} = \alpha^1$, $\alpha^2 \approx_{\mathcal{P}_2} \kappa_{\mathcal{P}_2}^{x_2}$ and $\alpha^2(s) \succeq_{\mathrm{xp}} x_1$ for any $s \in \mathcal{P}_2$. But, since $\alpha^2_{\restriction \mathcal{P}_1} = \alpha^1$, by the previous paragraph, we also have $\alpha^2(s) \succeq_{\mathrm{xp}} x_1$ for any $s \in \mathcal{P}_1$. In other words, $\mathcal{P}_1 \cup \mathcal{P}_2 \subseteq {\alpha^2}^{-1}[\{x \in \mathcal{X},\, x \succeq_{\mathrm{xp}} x_1\}]$. Since the latter set is closed by axiom (M) and the continuity of α, we further have $\mathrm{clos}(\mathcal{P}_1 \cup \mathcal{P}_2) \subseteq {\alpha^2}^{-1}[\{x \in \mathcal{X},\, x \succeq_{\mathrm{xp}} x_1\}]$. Then, $\mathcal{P}_1 \vee \mathcal{P}_2 = \mathrm{int}(\mathrm{clos}(\mathcal{P}_1 \cup \mathcal{P}_2)) \subseteq {\alpha^2}^{-1}[\{x \in \mathcal{X},\, x \succeq_{\mathrm{xp}} x_1\}]$. In other words, $\alpha^2(s) \succeq_{\mathrm{xp}} x_1$ for any $s \in \mathcal{P}_1 \vee \mathcal{P}_2$.

Iterative applications of this construction provide, for each $n \geq 3$, an act $\alpha^n \in \mathcal{A}(\mathcal{P}_1 \vee \ldots \vee \mathcal{P}_n)$ such that

(a) $\alpha^n_{\restriction \mathcal{P}_1 \vee \ldots \vee \mathcal{P}_{n-1}} = \alpha^{n-1}$,

(b) $\alpha^n_{\restriction \mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa_{\mathcal{P}_n}^{x_n}$,

(c) $\alpha^n(s) \succeq_{\mathrm{xp}} x_1$ for any $s \in \mathcal{P}_1 \vee \ldots \vee \mathcal{P}_n$.

Now, fix $s \in \mathcal{S}$. By condition (Exh), there exists $n \geq 1$ such that $s \in \mathcal{P}_1 \vee \ldots \vee \mathcal{P}_n$. Then, set $\mathcal{N}_s := \mathcal{P}_1 \vee \ldots \vee \mathcal{P}_n$, an open neighborhood of s. For any $m > n$, Property (a) implies $\alpha^m(s) = \alpha^{m-1}(s) = \ldots = \alpha^n(s)$. Thus, the sequence $\{\alpha^m(s),\, m \in \mathbb{N}\}$ is constant from a certain rank and, therefore, converges. Let $\alpha(s)$ be its limit. This defines a function α from $\mathcal{S}$ to $\mathcal{X}$ and we have $\alpha(s) = \alpha^n(s)$. In fact, proceeding the way we just did for s, we can show $\alpha(s') = \alpha^n(s')$ for any $s' \in \mathcal{N}_s$. But α^n is continuous. So α is locally continuous at any point in $\mathcal{S}$ and, therefore, continuous as well. Moreover, we have just constructed a neighborhood $\mathcal{N}_s$ of each $s \in \mathcal{S}$ such that $\alpha_{\restriction \mathcal{N}_s} = \alpha^n_{\restriction \mathcal{N}_s}$ with $\alpha^n \in \mathcal{A}(\mathcal{P}_1 \vee \ldots \vee \mathcal{P}_n)$. Then, there exists $\beta^n \in \mathcal{A}$ such that

$\alpha_{\upharpoonright\mathcal{N}_s} = \beta^n_{\upharpoonright\mathcal{N}_s}$. By (Loc), α must be an element of $\mathcal{A}$. In addition, since we have $\alpha(s) = \alpha^n(s)$, property (c) implies

$$\alpha(s) \ \succeq_{\mathrm{xp}} \ x_1, \tag{B5}$$

and this holds for any $s \in \mathcal{S}$. Moreover, for any $n \in \mathbb{N}$, $\alpha_{\upharpoonright\mathcal{P}_1\vee\ldots\vee\mathcal{P}_n} = \alpha^n_{\upharpoonright\mathcal{P}_1\vee\ldots\vee\mathcal{P}_n}$. Then, $\alpha_{\upharpoonright\mathcal{P}_n} = \alpha^n_{\upharpoonright\mathcal{P}_n}$ So, by property (b),

$$\alpha_{\upharpoonright\mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa^{x_n}_{\mathcal{P}_n} \tag{B6}$$

Now, fix $x \in \mathcal{X}$. We will show $\alpha \succeq_{\mathcal{S}} \kappa^x$. Let $N \geq 1$ be such that

$$\sum_{n=1}^{N} \mu(\mathcal{P}_n) \cdot u(x_n) \ \geq \ u(x). \tag{B7}$$

Such an integer N exists thanks to Formula (B4). Let $\mathcal{Q} := \neg(\mathcal{P}_1 \vee \ldots \vee \mathcal{P}_N)$. By finitely many iterations of (Rch*), we can construct $\beta \in \mathcal{A}_{\mathrm{b}}$ such that $\beta_{\upharpoonright\mathcal{Q}} = \kappa^{x_1}_{\mathcal{Q}}$ and $\beta_{\upharpoonright\mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa^{x_n}_{\mathcal{P}_n}$ for any $n \in [1 \ldots N]$. Since β is bounded, its expected utility $\mathbb{E}_{\mathcal{S}}[u \circ \beta]$ is well defined. We even have

$$\begin{aligned}
\mathbb{E}_{\mathcal{S}}[u \circ \beta] &\underset{\text{(a)}}{=} \mu(\mathcal{P}_1 \vee \ldots \vee \mathcal{P}_n) \cdot \mathbb{E}_{\mathcal{P}_1\vee\ldots\vee\mathcal{P}_n}[u \circ \beta] \ + \ \mu(\mathcal{Q}) \cdot \mathbb{E}_{\mathcal{Q}}[u \circ \beta] \\
&\underset{\text{(b)}}{=} \sum_{n=1}^{N} \mu(\mathcal{P}_n) \cdot \mathbb{E}_{\mathcal{P}_n}[u \circ \beta] \ + \ \mu(\mathcal{Q}) \cdot \mathbb{E}_{\mathcal{Q}}[u \circ \beta] \\
&\underset{\text{(c)}}{=} \sum_{n=1}^{N} \mu(\mathcal{P}_n) \cdot u(x_n) \ + \ \mu(\mathcal{Q}) \cdot u(x_1) \ \underset{\text{(d)}}{\geq} \ \sum_{n=1}^{N} \mu(\mathcal{P}_n) \cdot u(x_n) \ \underset{\text{(e)}}{\geq} \ u(x),
\end{aligned}$$

where (a) and (b) are by formula (6), (c) is because $\beta_{\upharpoonright\mathcal{Q}} = \kappa^{x_1}_{\mathcal{Q}}$ and $\beta_{\upharpoonright\mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa^{x_n}_{\mathcal{P}_n}$ for any $n \in [1 \ldots N]$ and by formula (B3), (d) is because $u(x_1) \geq 0$ by formula (B4), and (e) is by formula (B7). Then, by formula (B3), we obtain $\beta \succeq_{\mathcal{S}} \kappa^x$. On the other hand, we have $\beta_{\upharpoonright\mathcal{P}_n} \approx_{\mathcal{P}_n} \kappa^{x_n}_{\mathcal{P}_n}$ for any $n \in [1 \ldots N]$. By formula (B6), we obtain $\beta_{\upharpoonright\mathcal{P}_n} \approx_{\mathcal{P}_n} \alpha_{\upharpoonright\mathcal{P}_n}$ for any $n \in [1 \ldots N]$. Then, by Lemma A5, $\beta_{\upharpoonright\neg\mathcal{Q}} \approx_{\neg\mathcal{Q}} \kappa^{x_n}_{\neg\mathcal{Q}}$. Moreover, we have $\beta_{\upharpoonright\mathcal{Q}} = \kappa^{x_1}_{\mathcal{Q}}$ and, by formula (B5) and (Dom), $\alpha_{\mathcal{Q}} \succeq_{\mathcal{Q}} \kappa^{x_1}_{\mathcal{Q}}$. So $\alpha_{\mathcal{Q}} \succeq_{\mathcal{Q}} \beta_{\mathcal{Q}}$ and, by (Sep), $\alpha \succeq_{\mathcal{S}} \beta$. Since we have $\beta \succeq_{\mathcal{S}} \kappa^x$, we finally obtain $\alpha \succeq_{\mathcal{S}} \kappa^x$. ◇ `Claim 2`

Claim 3: *There exists $\alpha' \in \mathcal{A}$ such that $\alpha' \succ \alpha$.*

Proof. Let $\{\mathcal{P}_n, n \in \mathbb{N}\}$ be the countable $\mathfrak{B}$-partition of $\mathcal{S}$ from condition (Exh) and set $\mathcal{Q} := \neg\mathcal{P}_1 \in \mathcal{B}$. By (CEq), the act α from Claim 3 has a certainty equivalent for $\succeq_{\mathcal{Q}}$; that is, $\alpha \approx_{\mathcal{Q}} \kappa^x_{\mathcal{Q}}$ for some $x \in \mathcal{X}$. By (Rch*), there exists $\alpha' \in \mathcal{A}$ such that $\alpha'_{\upharpoonright\mathcal{P}_1} = \kappa^{x_2}_{\mathcal{P}_1}$ and $\alpha'_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \kappa^x_{\mathcal{Q}}$. Then, α' satisfies

$$\alpha'_{\upharpoonright\mathcal{P}_1} \approx_{\mathcal{P}_1} \kappa^{x_2}_{\mathcal{P}_1} \quad \text{and} \quad \alpha'_{\upharpoonright\mathcal{Q}} \approx_{\mathcal{Q}} \alpha_{\upharpoonright\mathcal{Q}}. \tag{B8}$$

Now, since we have assumed $u(x_2) > u(x_1)$, formula (B3) yields $\kappa^{x_2}_{\mathcal{P}_1} \succ_{\mathcal{P}_1} \kappa^{x_1}_{\mathcal{P}_1}$. By Claim 2 and the first indifference from formula (B8), we obtain $\alpha'_{\upharpoonright\mathcal{P}_1} \succ_{\mathcal{P}_1} \alpha_{\upharpoonright\mathcal{P}_1}$. Then, the second indifference from formula (B8) and (Sep) finally give $\alpha' \succ \alpha$. ◇ `Claim 3`

By (CEq), there exist $x, x' \in \mathcal{X}$ such that $\alpha \approx_{\mathcal{S}} \kappa^x$ and $\alpha' \approx_{\mathcal{S}} \kappa^{x'}$. Then, by Claim 2, $\alpha \succeq_{\mathcal{S}} \kappa^{x'}$ so $\alpha \succeq_{\mathcal{S}} \alpha'$. But this contradicts the conclusion of Claim 3, thereby showing that u must be bounded from above. A symmetric argument shows that u is bounded from below and, therefore, bounded. Then, the utility profile $u \circ \alpha$ of any $\alpha \in \mathcal{A}(\mathcal{B})$ is bounded on any $\mathcal{B} \in \mathfrak{B}$ (as well as continuous and $\mathfrak{B}$-comeasurable), and the expected utility $\mathbb{E}_{\mathcal{B}}[u \circ \alpha]$ of α on $\mathcal{B}$ is well-defined.

Claim 4: *For any $\mathcal{B} \in \mathfrak{B}$, any $\alpha \in \mathcal{A}(\mathcal{B})$ and any $x \in \mathcal{X}$, if $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$, then $\mathbb{E}_{\mathcal{B}}[u \circ \alpha] = u(x)$.*

Proof. The proof is very similar to that of Proposition A6. There, the boundedness of acts is not really used *per se*, but rather because it implies bounded utility profiles. Since u is here bounded, all utility profiles are bounded and the same argument as in Proposition A6 applies. $\diamond$ `Claim 4`

Finally, for any $\mathcal{B} \in \mathfrak{B}$ and $\alpha, \beta \in \mathcal{A}(\mathcal{B})$, axiom (CEq) yields $x, y \in \mathcal{X}$ such that $\alpha \approx_{\mathcal{B}} \kappa^x_{\mathcal{B}}$ and $\beta \approx_{\mathcal{B}} \kappa^y_{\mathcal{B}}$. Then,

$$\Big(\alpha \succeq_{\mathcal{B}} \beta\Big) \iff \Big(\kappa^x_{\mathcal{B}} \succeq_{\mathcal{B}} \kappa^y_{\mathcal{B}}\Big) \underset{(*)}{\iff} \Big(x \succeq_{\text{xp}} y\Big) \underset{(\dagger)}{\iff} \Big(u(x) \geq u(y)\Big) \underset{(\diamond)}{\iff} \Big(\mathbb{E}_{\mathcal{B}}[u \circ \alpha] \geq \mathbb{E}_{\mathcal{B}}[u \circ \beta]\Big).$$

Here, $(*)$ is by axiom (Dom), $(\dagger)$ is because u is an ordinal utility function for $\succeq_{\text{xp}}$ by formula (B3), and $(\diamond)$ is by Claim 4. Hence the SEU representation on $\mathcal{A}$. $\square$

References

Aliev, R. A., Huseynov, O. H., 2014. Decision Theory with Imperfect Information. World Scientific Publishing Co., Inc., River Edge, NJ.

Aliprantis, C. D., Border, K. C., 2006. Infinite dimensional analysis: A hitchhiker's guide, 3rd Edition. Springer, Berlin.

Arrow, K. J., 1970. Essays in the Theory of Risk-Bearing. North-Holland.

Caplin, A., Martin, D., 2015. A testable theory of imperfect perception. The Economic Journal 125, 184–202.

Epstein, L. G., Zhang, J., 2001. Subjective probabilities on subjectively unambiguous events. Econometrica 69 (2), 265–306.

Fremlin, D. H., 2004. Measure theory. Vol. 3: Measure Algebras. Torres Fremlin, Colchester, U.K.

Ghirardato, P., August 2002. Revisiting savage in a conditional world. Economic Theory 20 (1), 83–92.

Gollier, C., 2001. The Economics of Risk and Time. MIT Press.

Hammond, P. J., 1988. Consequentialist foundations for expected utility. Theory and decision 25 (1), 25–78.

Hammond, P. J., 1998. Objective expected utility: A consequentialist perspective. In: Barberà, S., Hammond, P. J., Seidl, C. (Eds.), Handbook of Utility Theory. Vol. 1. Kluwer, Ch. 5, pp. 143–211.

Jaffray, J.-Y., Wakker, P., 1993. Decision making with belief functions: Compatibility and incompatibility with the sure-thing principle. Journal of Risk and Uncertainty 7 (3), 255–271.

Johnstone, P., 1986. Stone Spaces. Cambridge Studies in Advanced Mathematics. Cambridge University Press.

Kopylov, I., 2007. Subjective probabilities on "small" domains. J. Econom. Theory 133 (1), 236–265.

Kopylov, I., 2010. Simple axioms for countably additive subjective probability. Journal of Mathematical Economics 46, 867–876.

Lipman, B. L., 1999. Decision theory without logical omniscience: toward an axiomatic framework for bounded rationality. Rev. Econom. Stud. 66 (2), 339–361.

Mukerji, S., 1997. Understanding the nonadditive probability decision model. Econom. Theory 9 (1), 23–46.

Pivato, M., Vergopoulos, V., 2018a. Subjective expected utility with topological constraints. (preprint available at https://mpra.ub.uni-muenchen.de/85749/ or https://ssrn.com/abstract=3157647).

Pivato, M., Vergopoulos, V., 2018b. Contingent plans on topological spaces.

Pivato, M., Vergopoulos, V., 2018c. Measure and integration on Boolean algebras of regular open subsets in a topological space. (preprint available at http://arxiv.org/abs/1703.02571).

Savage, L. J., 1954. The foundations of statistics. John Wiley & Sons, Inc., New York; Chapman & Hill, Ltd., London.

Stinchcombe, M. B., 1997. Countably additive subjective probabilities. Rev. Econom. Stud. 64 (1), 125–146.

Wakker, P., 1988. The algebraic versus the topological approach to additive representations. J. Math. Psych. 32 (4), 421–435.

Willard, S., 2004. General topology. Dover Publications, Inc., Mineola, NY.