Random Matching and Aggregate Uncertainty

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RANDOM MATCHING AND AGGREGATE UNCERTAINTY

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Abstract. Random matching is often used in economic models as a means of introducing uncertainty in sequential decision problems. We show that random matching schemes that satisfy standard conditions on proportionality are not unique. Hence economic models that use random matching as a shock may not be well defined unless additional conditions are imposed on the matching scheme. Two examples show that in a simple growth model, radically different optimal behavior can result from distinct matching schemes satisfying identical proportionality conditions. We give conditions on the reward and transition structures of sequential decision models so the models are well defined in spite of non-uniqueness of the matching scheme. Finally, information entropy is introduced as a method for selecting unique matching structures for these models.

Keywords and Phrases: random matching, aggregate uncertainty, information entropy
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1. Introduction

Random matching of agents plays a fundamental role in models of economic, social and biological systems. These models often attempt to explain how interaction among individuals will impact the system as a whole. Since the manner in which the individuals interact depends upon how they are matched in the first place, a clear understanding of the matching process is essential. Important examples of the role random matching plays in models can be found in [18], [17], and [15].[1]

One frequent objective in random matching models is a deterministic model that captures the aggregate behavior of the agents. The early paper by Hardy and Weinberg [15] approximated a stochastic system with random matching in population genetics by a deterministic system so that the problem of computing limiting population types became quite straightforward. Indeed, one of the main objectives of the extensive literature on random matching has been to show that it is reasonable to approximate a complicated discrete dynamical system in which agents are randomly matched by a deterministic system that is useful for computation. The attempt to get a tractable deterministic system has led many authors to consider a continuum of interacting agents. This is the implicit assumption in the paper by Hardy and Weinberg.

In order to obtain a deterministic model, an independence condition with respect to types is often implicitly assumed. If the agents are of distinct types, then the event that agent g is matched to an agent of a particular type should, in some sense, be independent of the event that agent $\widetilde{g}$ is matched to an agent of a particular type. This assumption implies that the random bilateral matching generates a process of i.i.d. random variables (indexed by the set of agents taking values in the set of types). The non-existence of measurable continuous time stochastic processes of nontrivial i.i.d. random variables is contained in the work of Doob [9]. Judd [16] elaborated on these issues for economists by showing that a random matching process for a continuum of agents that satisfies an independence of types condition is not measurable with respect to the natural product $\sigma - algebra$ placed on the product of matches and agents.

R. Boylan [7] used another approach to obtain deterministic systems through matching models. He constructed special matching schemes for both finite and countably infinite sets of agents. His random matching schemes satisfy properties that make it possible to approximate stochastic dynamical systems by deterministic systems. Deterministic systems are therefore obtained as the limit of the stochastic systems as the number of agents tends to infinity.

C. Alós-Ferrer [4] showed that it is possible to construct random matching models for a continuum of agents such that the probability that a given agent is matched to an agent of a given type is proportional to the frequency with which the type is present in the population. The stochastic process he constructs is measurable with respect to the standard product measure on the set of agents and the set of matches. This process is observable with the standard

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[1] In the economics literature, random matching appears in many subfields, including game theory, monetary theory, labor economics and experimental economics. Since many of the papers we cite in this work give excellent reviews of the use of random matching in economics, we do not repeat those surveys here.

tools of analysis. On the other hand, his process does not have the property that distinct agents are independently matched with a particular type. D. Duffie and Y. Sun [10] were able to construct a matching process that satisfies an independence in types condition. Their process is measurable with respect to a product $\sigma - algebra$ that is larger than the standard $\sigma - algebra$ on the product of agents and matches, and their existence result uses methods from nonstandard analysis.

Since a significant amount of work has been done that attempts to settle the foundational problems of random matching, it is fair to ask if the issue, at this point, is simply a technical one. Economists have a good sense of how random matching should work in their models, and are understandably reluctant to drop a model or theory because the foundations might have minor technical difficulties. We shall show how the mechanics of the random matching process can have significant implications for economic models. In this sense, our contribution complements that of Aliprantis, Camera, Puzzello [2, 3] where the anonymity properties of random matching schemes are emphasized.

In this paper we discuss the role nonuniqueness plays in random matching models. In section 5 we give simple examples of growth models that use random matching as shock. The matching schemes used in the examples are identical from the point of view of the standard properties imposed on matching in the economics literature. However, in our examples, the behavior of the systems these growth models describe are radically different from one another. These examples describe reasonable situations where the mechanics of matching have substantive economic implications, and they can be extended in several directions.

The fundamental problem that our examples illustrate stems from the non-uniqueness of random matching schemes that satisfy standard assumptions about the matching. Since the existence problem for random matching is nontrivial, [7, 4, 13, 10, 11], it is perhaps not surprising that the uniqueness question has not been examined up to this point. We think that it is important to study nonuniqueness since it improves our understanding of the workings of random matching systems.

Before one attempts to address the problems posed by non-uniqueness of random matching schemes that satisfy proportionality conditions, it is necessary to have a clear understanding of the aggregate uncertainty concept for random matching models. An extensive discussion of aggregate uncertainty for the case of shocks that do not come from matching can be found in Alós-Ferrer [6]. We make the concept of *no aggregate uncertainty* for dynamic random matching models precise in Definition 3.5.

This paper presents two methods for dealing with the uniqueness issue. The first method examines the relationship between the random matching structure and the reward and transition structures of the models. This approach provides sufficient conditions for the formulation of random matching models with no aggregate uncertainty where nonuniqueness is no longer relevant. In some cases the model of interest may exhibit aggregate uncertainty, and nonuniqueness may be relevant. Our second method introduces the concept of *information entropy* to obtain uniqueness of a random matching scheme.

2. Outline of Paper

We present a brief outline of the paper so that the reader can quickly see the main ideas without having to read through the technical details of each section. If one is willing to accept some of the intermediate results, one can start with Section 3, and then go on to read the Sections 5, 6, and 8. One can then read the sections that discuss specific models of different cardinalities for the populations, 4.1, 4.2, and 4.3. Finally, the section on the information entropy of random matching, Section 7, can be read.

In Section 3, Sequential Decision Problems with Random Matching, we describe typical classes of economic models that use random matching as shock. The set of agents in these models can be finite, countably infinite or a continuum. In this section, we do not assume any special cardinality for the set of agents, but describe the three essential structures of these models: reward, transition, and matching. We describe how the matching structure interacts with the reward structure and the transition structure, and we clarify why one must be concerned with the "joint measurablity" of the agent and matching measures. We also describe three standard assumptions that relate the "type" of an agent to the random matching, and we shall refer to them as "Proportionality" assumptions in this overview. We also give a definition of *aggregate uncertainty* for these models.

In Section 4, Matching Structures and Proportionality, we describe in more detail the methods of constructing random matching schemes that satisfy the proportionality conditions. We deal with each of the three possibilities for the cardinality of the set of agents in separate subsections. For each of the three cardinalities, we show that the set of matching schemes that satisfy proportionality is a nonempty convex set in some large dimensional space (large can mean infinite). In particular, the set of random matching schemes that satisfy a particular set of proportionality conditions is far from unique. In fact, the problem of non-uniqueness becomes more difficult as the cardinality of the set of agents increases.

The difficulties caused by non-uniqueness of random matching schemes is described in Section 5, A Growth Model with Random Matching. Here we present a very simple growth model, derived from a standard well known example, that demonstrates two points.

(1) Two matching schemes that satisfy identical proportionality conditions can give radically different economic results.
(2) For our model, the structure of the probability distribution on the set of matches is important to the determination of optimal policies.

Although we use a model with a finite set of agents to demonstrate the problem, a simple modification shows that the probem persists if the set of agents is infinite (countable or a continuum). The problem is the result of the non-uniqueness of the matching structure and its relationship with the reward and transition structures.

We next describe, in Section 6, No Aggregate Uncertainty, how one can mitigate the problem caused by non-uniqueness. We show that if the reward and transition structures of a model depend only on the types of the agents, then two random matching schemes that satisfy the same proportionality conditions will give the same well defined deterministic model. Thus uncertainty will disappear in the aggregate. The results are again independent of the cardinality of the set of agents and level of sophistication of the matching scheme. Although one must make fairly strong assumptions on the reward and transition structures, it will always be possible to use this technique if one is willing to assume that the system can be described by a finite set of states.

Section 7, The Entropy of a Matching Structure, introduces a new idea to economic modeling with random matching. In some cases it will not be desirable to limit the state space of individual agents to finite sets. For example, one might want to allow an individual to possess any real number amount of capital up to some bound. This will make the state space for the individual agents uncountable, and the method of dealing with non-uniqueness described in Section 6 will not work. To make such a model well defined one needs a way to single out a unique random matching scheme that satisfies the proportionality conditions. In some cases one can do this by using the information entropy of the matching scheme.

We finally summarize our results and indicate some future directions for research in Section 8. Although random matching schemes are typically assumed to be exogenous, one might consider random matching as a control variable. Indeed this is already being done for complicated biological and economic systems. For example, control of epidemics is often linked to control over a random matching structure.

3. Sequential Decision Problems with Random Matching

An economic model with random matching of agents frequently has the structure of a sequential decision problem. We consider discrete time models defined over an infinite time horizon. The state of the system is determined as the aggregate of the states of the individuals that make up the population. In this section, we take a general approach to the problem of matching and so the set of agents can be finite, countably infinite, or a continuum. We denote the state of an individual agent at time t by $s(t, g)$. We write $s(g)$ to indicate the state of the agent g when the time is understood by the context. Generally $s(g)$ will be a vector consisting of various attributes of the agent such as type of good produced, capital holdings, buyer or seller, and so on. The state of the population as a whole is then described by the function $s(*)$ that gives the state of each individual. The sequential decision problem can be roughly described as follows.

(1) Agents are randomly matched and interact. The interaction results in a reward (or cost) for the individual agents. The individual rewards are determined by the state of the agent, the state of the agent's partner, and a control variable. The state of the economy is the aggregate of the states of individuals.
(2) After the interaction or matching of agents, the individual states of agents change. The new states are determined as a result of the interaction and hence depend upon the matching scheme.

To make these models precise, it is necessary to describe the reward process, the matching process, and the transition process. There are generally two ways of selecting values for the control parameter. Perhaps the simplest approach is to stipulate that the control parameter is selected by a *central planner*. Thus the interaction of individual agents and the subsequent transition to a new state is independent of decisions made by agents. The central planner might select values of the control to optimize total expected reward for the entire population subject to some set of constraints. Alternatively, the control parameter might be selected by a rule linked to the information available to individual agents. In this case individual agents select values of the control to optimize expected individual reward.

Since both methods of selecting control values result in very complicated stochastic dynamical systems, one tries to simplify the model through assumptions on all three parts of the sequential model. One goal of the simplifying assumptions is to develop a model of the aggregate economy that is deterministic. Hence, the simplifying assumptions should make it possible to obtain relationships (generally functional equations) for quantitative measures of the aggregate economy that depend only on the proportionality of types. The resulting model is often said to contain *no aggregate uncertainty.*

We introduce some notation to make the ideas more precise. Let A denote the set of agents, with a typical agent in A denoted by g. Assume we have a way of measuring the size of subsets of A so that we have a measure space, $(A, \mathcal{L}, \mu)$. In case μ is not a finite measure on A (e.g. A is the set of natural numbers and μ is counting measure) one might encounter convergence problems. To handle this problem we assume we have an exhaustion of A by sets, A_n of finite μ measure, such that $A = \cup A_n$ with $A_n \subset A_{n+1}$. Let Φ denote the set of bilateral matches of agents, that is the set of involutions of A with no fixed point. Let φ denote a typical element of Φ. In subsequent sections, we discuss the set Φ in detail, but for now the only additional structure we place on Φ is a probability distribution, Q, with a $\sigma - algebra$ of subsets that we denote by $\mathcal{F}$, so that $(\Phi, \mathcal{F}, Q)$ is a probability space.

The reward structure for the model is determined by reward functions for the agents which we denote by

$$R_g\left(s(g), s(\varphi(g)), u\right).$$

The variable u is the control. This notation is consistent with our assumption that rewards for individual agents depend only on the state of the agent, the state of the agent's partner, and the control. The expected aggregate reward for the population for a single time period is

$$E_Q\left[R\left(s,u\right)\right].$$

A computation of this expected reward will involve integration (or summation) over the product measure space $A \times \Phi$. We use integration with the understanding that the integral is traditionally written as a sum in the case of discrete distributions on discrete probability spaces. In case the measure μ is not finite the aggregate reward over all agents would not generally be finite, so we define expected reward as a limit with respect to the exhaustion of A. In case μ is finite the limit is not needed. The aggregate expected reward is computed as

$$E_Q\left[R\left(s,u\right)\right] = \lim_{n\to\infty} \frac{1}{\mu(A_n)} \int\int_{A_n\times\Phi} R_g\left(s(g), s(\varphi(g)), u\right)\left(d\mu(g) \times dQ(\varphi)\right). \tag{3.1}$$

Remark 3.1. One could allow more complicated reward structures that include stochastic shocks independent of the shock introduced by random matching. We want to focus on the stochastic shock introduced by the matching process, and so we do not include this complicating feature in our discussion.

We next describe the transition structure of the general model. We assume that the state of a typical agent g at time $t+1$ is a deterministic function of the state of g at time t, the state the agent with whom g is matched at time t, and the control variable u. In other words

$$s(t+1, g) = f\left(s(t,g), s(t,\varphi(g)), u\right).$$

Therefore, the transition probabilities for agents are given by

$$\mathrm{Prob}_Q\left(s(t+1,g)\left|s(t,g), u\right.\right) = Q\left\{\varphi\left|s(t+1,g) = f\left(s(t,g), s(t,\varphi(g)), u\right)\right.\right\}.$$

The transition probabilities depend explicitly on the probability distribution Q defined on Φ. The transition probabilities for the entire system are the aggregates of the transition probabilities for individual agents. In terms of the state function this can be written

$$\mathrm{Prob}_Q\left(s(t+1,*)\left|s(t,*), u\right.\right) = Q\left\{\varphi\left|s(t+1,*) = f\left(s(t,*), s(t,\varphi(*)), u\right)\right.\right\}$$

where the equality in the expression

$$Q\left\{\varphi\left|s(t+1,*) = f\left(s(t,*), s(t,\varphi(*)), u\right)\right.\right\}$$

must be regarded as equality of functions defined on the set of agents, A.

At this point, we have described the general sequential decision model that uses random matching as shock. We have mentioned two approaches to solving a model of this type. Either a central planner selects values of the control to optimize total expected (discounted) rewards, or the control is selected by some rule based on states of individual agents and information available to individual agents. (Perhaps the history of the entire system up to the present time.) We now turn our attention to the matching structure and the assumptions that are made in an attempt to simplify the model.

Perhaps the most common assumption made about the random matching structure involves the *type* of an agent. Suppose there exists a set of types $S = \{\tau_1, \tau_2, \tau_3, \cdots, \tau_L\}$, and a measurable function

$$\tau : A \Rightarrow S$$

that assigns a type to each agent. In general, the set of types need not be finite; however, this additional assumption is often made, so we follow that convention here.

Remark 3.2. One might want to allow stochastic type functions. For example, if agents become buyers or sellers in a random manner after matching, it might be more appropriate to define τ as a random variable. An examples of models of this type can be found in Lagos and Wright [19]. Here we shall restrict the discussion to the case of a deterministic type assignment because we wish to concentrate on the random matching as shock.

The *type* of an agent becomes a component of the state variable for the agent. Hence, one can write the state of the agent at time t as

$$s(t,g) = (\tau(t,g), k(t,g))$$

where we incorporate state information other than type in the state variable $k(t,g)$. For example, in a number of matching models in monetary theory, agents are assigned a type based on a good produced by the agent. The agents are also assigned an amount of money via an endowment or through a trade, production, and consumption process. See for example the paper of Kiyotaki and Wright [18]. As before, we shall drop the t from the notation in case the time period is understood by context.

We need a way to measure the relative size of the set of agents of various types. The relative size of the set of agents of a given type can be computed by exhausting the set of agents A by an increasing sequence of subsets, A_n, of finite measure, and then computing the proportion of agents in the sets A_n of a given type. We say A_n exhausts A if $A = \cup A_n$ with $A_n \subset A_{n+1}$. Denote the proportion of agents in the population of type τ_k by P_k. Hence, we define P_k by

$$P_k = \lim_{n\to\infty} \left[\frac{\mu\left\{g \in A_n \,|\tau(g) = \tau_k\right\}}{\mu\left\{g \in A_n\right\}} \right]. \tag{3.2}$$

whenever the limit exists.

Remark 3.3. If the measure μ on the set of agents is bounded, there is no need for the limit. This is the case for finite sets of agents or if the set of agents is $[0,1)$ and μ is Lebesgue measure. In the case $A = \mathbb{N}$, there is a natural choice for the sequence of exhausting sets; let $A_n = \{1, 2, 3, \cdots, n\}$, with μ the counting measure.

The desire for a deterministic model with matching leads to a set of assumptions on the matching process. Each match should preserve the measure of sets of agents. Next, the probability that a given agent is matched to an agent of some type should equal the proportion of agents of that type in the population. Finally, for each match, the proportion of agents of a given type that are matched to a second type should equal the product of the proportions of the two types in the population. We formalize this with the following set of three random matching properties related to types. We follow the labels used by Alós-Ferrer in [5] here.

Definition 3.4. Suppose the proportions P_k are defined by equation 3.2 for an exhaustion of A by A_n. The random matching scheme defined by $(A, \mathcal{L}, \mu)$ and $(\Phi, \mathcal{F}, Q)$, satisfies the *Proportionality Laws* if the following conditions are satisfied.

M1. Measure Preserving	$\mu(E) = \mu(\varphi(E))$ for all measurable E for φ
M2. Proportional Law	$Q\left\{\varphi \,|\tau(\varphi(g)) = \tau_l\right\} = P_l$ for a.e. $g \in A$
M3. Quasi-independence	$\lim_{n\to\infty} \mu\left\{g \in A_n \,|\tau(g) = \tau_i \text{ and } \tau(\varphi(g)) = \tau_j\right\}/\mu\left(A_n\right)$ $= P_i P_j$ for a.e. $\varphi \in \Phi$

Note that in the case that $\mu(A)$ is finite, there is no need for the limit in property M3.[2]

We can now make the concept of *aggregate uncertainty* for the dynamic random matching model precise. This concept has a long history in the economics literature, and several methods have been proposed to construct models that exhibit the no aggregate uncertainty property. See for example [16].

Definition 3.5. The reward structure of the model contains *no aggregate uncertainty* if

$$E_Q\left[R\left(s,u\right)\right] = E_{\widetilde{Q}}\left[R\left(s,u\right)\right]$$

whenever Q and $\widetilde{Q}$ satisfy the same Proportionality Laws. Similarly, the transition structure of the model contains *no aggregate uncertainty* if

$$\mathrm{Prob}_Q\left(s(t+1,*)\,|s(t,*),u\right) = \mathrm{Prob}_{\widetilde{Q}}\left(s(t+1,*)\,|s(t,*),u\right)$$

whenever Q and $\widetilde{Q}$ satisfy the same Proportionality Laws. If both of these conditions are satisfied, we say the model contains no aggregate uncertainty.

[2] Alós-Ferrer refers to property M3 as the "Mixing Property". We prefer a different term since mixing could be confused with a uniform dispersal of agents among types. The product nature of the property seems closer to the idea of independent events to us.

The above definition of no aggregate uncertainty implies that for models that satisfy this condition, the aggregate expected reward can be computed in terms of the proportionality constants P_i. Hence our definition is consistent with the concept of no aggregate uncertainty in case random shocks are applied to individual agents (without matching). We next discuss the matching structure of these models in more detail. We consider each of the three possibilities for the cardinality of the set of agents, and show why non-uniqueness of the matching scheme occurs in each case.

4. Matching Structures and Proportionality

Recall that the set of matches, Φ, is the set of involutions defined on the set of agents such that no agent is matched with herself. So if $\varphi \in \Phi$ then φ is a bijection, $\varphi(\varphi(g)) = g$, and $\varphi(g) \neq g$. A random matching structure is a probability space with Φ as the set of elements of the space, and we denote this space by $(\Phi, \mathcal{F}, Q)$.

In this section, we discuss the existence and non-uniqueness of probability distributions on Φ that satisfy the three conditions M1, M2, and M3 of Definition 3.4. Most of the work that has been done in trying to establish rigorous foundations for matching models have focused on the existence question. It does not seem to be widely known that one can construct random matching structures for finite sets of agents that will satisfy the Proportionality Laws. In particular, this makes it possible to construct matching models with finitely many agents that exhibit the no aggregate uncertainty condition. We expect that the construction below will also be useful in experimental economics since one must work with finite sets of agents. Some examples may be found in [8, 12].

4.1. **Matching for Finite Sets of Agents.** Take as the set of agents, $A = \{1, 2, 3, \cdots, N\}$, with $N = 2n$. We can represent any bilateral match, φ, as follows. Write

$$\varphi \sim (a_{11}, a_{12})(a_{21}, a_{22}) \cdots (a_{n1}, a_{n2})$$

to mean that $\varphi(a_{i1}) = a_{i2}$ and $\varphi(a_{i2}) = a_{i1}$ for $1 \leq i \leq n$. In other words, a_{i1} and a_{i2} are matched.

We further order the pairs so that the following inequalities hold.

$$a_{11} < a_{21} < \cdots < a_{n1},$$
$$a_{j1} < a_{j2}.$$

We shall refer to this ordering on the pairs as the lexicographic ordering, and it makes the representation unique. It also gives us an easy way to count the number of elements in the set, Φ, of involutions. We denote this number by $m(2n)$ and we have

$$m(2n) = \mathrm{card}\Phi = (2n-1)(2n-3)(2n-5)\cdots(3)(1).$$

This follows by noting that we must have $a_{11} = 1$ and that there are $2n-1$ choices for a_{12}. Recursively apply this observation until the set A is exhausted to get the result.

It will be convenient to have an alternate expression for the number of involutions when we discuss probability distributions on the set Φ. If we select the elements for matching two at a time and note that the order in which we select pairs is unimportant, then we immediately have

$$m(2n) = \mathrm{card}\Phi = \frac{\begin{pmatrix} 2n \\ 2 \end{pmatrix}\begin{pmatrix} 2n-2 \\ 2 \end{pmatrix}\cdots\begin{pmatrix} 2 \\ 2 \end{pmatrix}}{n!}.$$

Here we use the notation, $\begin{pmatrix} p \\ q \end{pmatrix} = p!/(q!(p-q)!)$ for "p choose q".

Additional properties of involutions, bilateral matching, and equivalent counting arguments are discussed in [1, 2, 17].

We now describe probability distributions on Φ that satisfy the conditions M1, M2 and M3.

Theorem 4.1. *Let $S = \{\tau_1, \tau_2, ..., \tau_L\}$ be a finite set of types and let $P_1, P_2, ..., P_L$ be positive rational numbers with $\sum P_i = 1$. Then there exists a finite set of agents, A, a type assignment, $\tau : A \to S$, and a probability distribution Q on the set of bilateral matches, Φ, of A such that the following conditions hold with μ the counting measure on A.*

(i) The proportion of agents of type τ_l equals P_l, so $\mu\{g \,|\tau(g) = \tau_l\}/\mu(A) = P_l$;

(ii) $Q\{\varphi \in \Phi \,|\tau(\varphi(g)) = \tau_l\} = P_l$ for all $g \in A$;

(iii) $\mu\{g \,|\tau(g) = \tau_i$ and $\tau(\varphi(g)) = \tau_j\}/\mu(A) = P_i P_j$ for a.e. $\varphi \in \Phi$.

Proof. Write $P_l = p_l/q_l$ where p_l and q_l are integers, and the quotients P_l are in reduced form. The set A will consist of a total of $N = 2q_1q_2\cdots q_L$ agents. We shall define a subset $\Psi \subset \Phi$ of bilateral matches having a special form. The number of agents of types τ_l will be $n_l = 2p_lq_1q_2\cdots\check{q}_l\cdots q_L$ where the $\check{q}_l$ indicates that q_l is not present in the product. Since $\sum P_l = 1$ we have

$$2p_1q_2q_3\cdots q_L + \cdots + 2p_Lq_1q_2\cdots q_{L-1} = 2q_1q_2\cdots q_L \tag{4.1}$$

and the total number of agents equals the sum of the number of agents of each type.

To make it easier to keep track of types, we label our agents with a subscript and a superscript with the superscript indicating type. Denote an agent of type τ_l by g_*^l where $*$ runs from 1 through n_l. Now each match, $\varphi \in \Psi$, will match the agents as follows. Each φ will match u_l agents of type τ_l to agents of type τ_l and will match v_{ij} agents of type τ_i to agents of type τ_j where $i \neq j$. So for example the number of agents of type τ_1 matched to agents of type τ_1 will equal u_1, and the number of type τ_1 agents matched to type τ_2 agents is v_{12}. For instance, if there are two types, a typical match can be denoted as

$$\begin{array}{c} g_*^1 g_*^1 \cdots g_*^1 g_*^2 \cdots g_*^2 g_*^1 g_*^1 \cdots g_*^1 \\ g_*^1 g_*^1 \cdots g_*^1 g_*^2 \cdots g_*^2 g_*^2 g_*^2 \cdots g_*^2 \end{array}$$

where the agents in the first row are matched to the corresponding agent in the second row.

We assign the following values to the numbers u_l and v_{ij} for $i \neq j$. Let

$$u_l = \frac{p_l}{q_l}(n_l) = \frac{p_l}{q_l}(2p_l q_1 q_2 \cdots \check{q}_l \cdots q_L)$$

$$\begin{aligned} v_{ij} &= \frac{p_j}{q_j}(2p_i q_1 q_2 \cdots \check{q}_i \cdots q_L) = \\ &= 2p_i p_j q_1 q_2 \cdots \check{q}_i \cdots \check{q}_j \cdots q_L \\ &= \frac{p_i}{q_i}\frac{p_j}{q_j} 2q_1 q_2 q_3 \cdots q_L \\ &= \frac{p_i}{q_i}\frac{p_j}{q_j} N \end{aligned}$$

The numbers v_{ij} are clearly integers and from equation 4.1 it follows that u_l is an even integer. It is also clear from the expression for v_{ij} that v_{ij} is symmetric in i and j as required. From the last expression for v_{ij} we also see that the proportion of agents of type τ_i that are matched with agents of type τ_j equals $P_i P_j$. Hence we obtain property (iii) of the theorem with these choices.

It is easy to compute the total number of matches in the set Ψ described above. First let $m(k)$ denote the total number of bilateral matches of k agents as before. Then the total number of distinct matches of the above form, denoted $M(P_1, P_2, ..., P_L)$, is given by

$$M(P_1, P_2, ..., P_L) = \text{card } \Psi =$$

$$\binom{n_1}{u_1}\binom{n_1 - u_1}{v_{12}}\binom{n_1 - u_1 - v_{12}}{v_{13}} \cdots \binom{n_1 - u_1 - ... - v_{1,L-1}}{v_{1L}} \cdots$$

$$\cdots \binom{n_L}{u_L}\binom{n_L - u_L}{v_{L1}} \cdots \binom{n_L - ... - v_{L,L-2}}{v_{L,L-1}} \prod_l m(u_l) \prod_{i<j} v_{ij}!$$

We assign equal probability to each match in Ψ. In other words, we put the uniform discrete distribution, Q, on the set Ψ of bilateral matches we have described above. The distribution Q has the property given in statement (ii) of the theorem. To see this let g_*^* denote any agent. For example, consider agent g_1^1. The probability that this agent is matched to an agent of type τ_1 equals the number of matches in which g_1^1 is matched to a type τ_1 agent divided by the total number of matches. But this is exactly equal to the proportion of the agents selected to be matched to type τ_1 agents, or in other words, P_1 as desired. Similarly, the probability that agent g_1^1 is matched to a type τ_i agent equals the proportion of type τ_1 agents selected to be matched with type τ_i agents or P_i. □

Example 4.2. An example will help explain the above construction and counting argument. Suppose we want to construct a matching model with two types and the property that the probability that any given agent be matched with an agent of either type equal 1/2. Then $P_1 = P_2 = 1/2$. The number of agents we need is

$$\text{card } A = 2q_1 q_2 = 2(2)(2) = 8$$

with 4 agents of each type. We denote the types as a and b to make the notation a bit cleaner in this example. We match the agents as follows.

$$\begin{array}{cccc} a_* & a_* & a_* & b_* \\ a_* & b_* & b_* & b_* \end{array}$$

where the $*$ subscripts range from 1 through 4. Agents in the top row are matched with the corresponding agent in the second row. There are $\binom{4}{2}$ (4 choose 2) ways to select the a agents to be matched to a agents. Once these two a agents are selected, the remaining 2 type a agents must be matched to b agents. Similarly, there are $\binom{4}{2}$ ways to select the b agents to be matched with b agents. Once the pairs of agents of each type are selected, there are 2! ways to match the type a agents with the type b agents. (In this simple example we do not have the additional complication of counting the ways to match the a agents with the a agents since there are only two of them and hence only one way to match them.) Thus there are

$$\binom{4}{2}\binom{2}{2}\binom{4}{2}\binom{2}{2}2! = 72$$

matches of this form. We assign probability 1/72 to each of the matches. The probability that any given agent, for example agent a_1, will be matched to a type a agent is 1/2 and the probability that a_1 will be matched to a type b agent is 1/2. For each match of the 72 matches, exactly two of the type a agents are matched to type a agents since two of the type a agents are matched to one another. Also, exactly two of the type a agents are matched to type b agents. Hence, conditions M2 and M3 are satisfied. In this case M1 is trivial, so the Proportionality Laws hold.

The above probability distribution is not unique in the sense that there are other probability distributions on the set of all matches of the 8 agents such that the Proportionality Laws are satisfied. In fact, we need only two matches, each equally likely, to accomplish this. Let

$$\varphi_1 = (a_1, a_2)(a_3, b_3)(a_4, b_4)(b_1, b_2)$$

$$\varphi_2 = (a_1, b_1)(a_2, b_2)(a_3, a_4)(b_3, b_4)$$

and assign probability 1/2 to each of these two matches and probability 0 to all other bilateral matches of the 8 agents. It is easy to see that the probability that any given agent be matched to an agent of either type equals 1/2 just as before. The quasi-independence property is also obvious.

If Q and $\widetilde{Q}$ satisfy conditions M2 and M3, then any convex combination of Q and $\widetilde{Q}$ will satisfy M2 and M3. In the case of finitely many agents with the counting measure on the set of agents, M1 holds trivially. Hence, the set of probability distributions, Q, such that the Proportionality Laws hold (with counting measure on the set of agents) is a convex polyhedra with nonempty relative interior. So the distributions that satisfy the Proportionality Laws are far from unique.

4.2. **Countably Infinite Sets of Agents.** Let Φ denote the set of all involutions without fixed point of $\mathbb{N}$, the natural numbers. As in the finite case, every element φ of Φ has a unique representation as an infinite sequence of pairs of natural numbers.

$$(a_{11}, a_{12}), (a_{21}, a_{22}), (a_{31}, a_{32}), \cdots, (a_{j1}, a_{j2}), (a_{j+1,1}, a_{j+1,2}), \cdots$$

where the pairs of natural numbers satisfy the conditions

$$a_{11} < a_{21} < a_{31} < \cdots \tag{4.2}$$

$$a_{j1} < a_{j2} \text{ for all } j \tag{4.3}$$

and the set of all a_{ji} exhausts $\mathbb{N}$. The relationship between the sequence and the involution φ is

$$\varphi(a_{k1}) = a_{k2}$$

for all k as in the finite case. It will be convenient to have a formal name for this representation.

Definition 4.3. Let φ be a bilateral match of either a finite set $\{1, 2, ..., 2n\}$ or of the natural numbers $\mathbb{N}$. The representation of φ by the sequence (finite or infinite) of pairs

$$(a_{11}, a_{12})(a_{21}, a_{22})(a_{31}, a_{32}) \cdots (a_{j1}, a_{j2})(a_{j+1,1}, a_{j+1,2}) \cdots$$

satisfying conditions (4.2) and (4.3) is the lexicographic representation of φ.

It follows immediately from the definition that the lexicographic representation is unique. We use the above representation to show that Φ is uncountable by using a Cantor diagonalization argument. We first give a preliminary result on the lexicographic representation of involutions.

Lemma 4.4. *Let* $\varphi = (a_{11}, a_{12})(a_{21}, a_{22})(a_{31}, a_{32}) \cdots (a_{j1}, a_{j2})(a_{j+1,1}, a_{j+1,2}) \cdots$ *be any involution of* $\mathbb{N}$ *written as the unique pairwise representation. Then* $a_{11} = 1$ *and for* $k > 1$,

$$a_{k1} = \inf\{a_{11}, a_{12}, \cdots, a_{k-1,1}, a_{k-1,2}\}^c.$$

where $\{\}^c$ *denotes the complement in* $\mathbb{N}$.

Proof. The result follows directly from the definition of the lexicographic order. $\square$

Theorem 4.5. *The set of involutions of the natural numbers is uncountable.*

Proof. Let $F : \mathbb{N} \to \Phi$ be any injection from $\mathbb{N}$ into the set of involutions of A. (We use A to denote the set of agents since we use $\mathbb{N}$ to index matches in the counting argument.) We want to show that F cannot be onto, and to do this we use the standard diagonalization argument together with the above unique representation of an involution. Let $\varphi_n = F(n)$ and write φ_n in the unique pairwise representation given above.

$$\varphi_n = (a_{11}^n, a_{12}^n)(a_{21}^n, a_{22}^n)\cdots$$

We shall define an involution in unique pairwise representation form that differs from each of the n involutions in the n^{th} place. This diagonalization then implies that the set of involutions is uncountable.

Let ψ be the match $\psi = (b_{11}, b_{12})(b_{21}, b_{22})\cdots(b_{k1}, b_{k2})\cdots$, with the b_{kl} to be defined. We put $b_{11} = 1$. Put $b_{12} = \inf\{b_{11}, a_{12}^1\}^c$. So for example if $a_{12}^1 \neq 2$ then we must have $b_{12} = 2$. Now define b_{k1} recursively by

$$b_{k1} = \inf\{b_{11}, b_{12}, b_{21}, b_{22}, \cdots, b_{k-1,1}, b_{k-1,2}\}^c \tag{4.4}$$

$$b_{k2} = \inf\{n : n > b_{k1} \text{ and } n \in \{b_{11}, b_{12}, b_{21}, b_{22}, \cdots, b_{k-1,1}, b_{k-1,2}, a_{k2}^k\}^c\}. \tag{4.5}$$

Note that (4.4) must be satisfied if we use the unique pairwise representation for ψ, and the definition of b_{k2} forces the k^{th} pair in the representation of ψ to differ from the k^{th} pair in the representation of φ_k. Since the k^{th} pair of the representation of ψ differs from the k^{th} pair of the representation of φ_k, the involution ψ differs from every φ_k. Therefore the map F cannot be onto and the result follows. □

Remark 4.6. The un-countability of the set of bilateral matches of $\mathbb{N}$ is implicit in the work of Boylan [7] since he constructs unique bilateral matches from sequences of i.i.d. random variables with values in a finite set of more than one element. The cardinality of the set of such sequences is uncountable.

We are interested in describing distributions on the set of involutions. The above result together with the Continuum Hypothesis, tells us that the set of involutions of $\mathbb{N}$ can be identified with the unit interval of real numbers. However, if we wish to use properties of standard continuous distributions on the interval, we need a precise description of the identification. Fortunately, the unique pairwise representation gives an easy way to do this. We first write Φ as a sequence of disjoint unions. Let

$$\Phi_n = \{(1, n+1)(*, *)\cdots\}$$

be the involutions with $(1, n+1)$ as the first pair in the representation. The sets Φ_n are disjoint and the union of the Φ_n gives us Φ. Next, decompose each Φ_n into a disjoint union,

$$\Phi_{i_1} = \cup \Phi_{i_1 n}$$

with

$$\Phi_{i_1 n} = \{(1, i_1)(a_{21}, a_{22})\cdots\}$$

where a_{21} is uniquely defined by the value of i_1 and Lemma 4.4. The smallest possible value of a_{22} corresponds to $n = 1$ and so on. Continue in this manner, and recursively define disjoint sets of involutions $\Phi_{i_1 i_2 i_3 \cdots i_k}$. For instance, $\Phi_1 = \{(1, 2)...\}$ $\Phi_{11} = \{(1, 2)(3, 4)...\}$, $\Phi_{12} = \{(1, 2)(3, 5)...\}$, and so on.

We collect the properties of these sets in the following lemma.

Lemma 4.7. *The sets $\Phi_{i_1 i_2 i_3 \cdots i_k}$ satisfy the following conditions.*

(1) If $\Phi_{i_1 i_2 i_3 \cdots i_k} = \{(a_{11}, a_{12})(a_{21}, a_{22})\cdots(a_{k1}, a_{k2})\cdots\}$ where the a_{ij} appearing in the first k pairs are fixed by the recursive definition, then

$$\Phi_{i_1 i_2 i_3 \cdots i_k n} = \left\{(a_{11}, a_{12})(a_{21}, a_{22})\cdots(a_{k1}, a_{k2})(a_{(k+1)1}, b_n)\cdots\right\}$$

where $a_{(k+1)1}$ is uniquely defined by Lemma 4.4 and b_n denotes the n^{th} smallest possible value for the second element of the $(k+1)^{st}$ pair.

(2) $\Phi_{i_1} \supset \Phi_{i_1 i_2} \supset \Phi_{i_1 i_2 i_3} \supset \cdots$

(3) For fixed k, the sets $\Phi_{i_1 i_2 i_3 \cdots i_k}$ are disjoint.

(4) For fixed k, $\cup \Phi_{i_1 i_2 i_3 \cdots i_k} = \Phi$.

(5) For each $\varphi \in \Phi$ there exists a unique sequence of sets $\Phi_{i_1} \supset \Phi_{i_1 i_2} \supset \Phi_{i_1 i_2 i_3} \supset \cdots$ with

$$\{\varphi\} = \Phi_{i_1} \cap \Phi_{i_1 i_2} \cap \Phi_{i_1 i_2 i_3} \cap \cdots$$

Proof. The first property is just the formal restatement of the recursive definition of the sets. The second property follows directly from the first. The third property follows since two involutions that differ somewhere in the first k pairs are distinct. The fourth property holds since we list all possible sequences of pairs of length k in the definition of $\Phi_{i_1 i_2 i_3 \cdots i_k}$. Finally, suppose $\varphi = (a_{11}, a_{12})(a_{21}, a_{22})\cdots(a_{k1}, a_{k2})\cdots$. By the construction of the sets $\Phi_{i_1 i_2 i_3 \cdots i_k}$, there exists exactly one set Φ_{i_1} containing φ. Next, there exists exactly (disjoint property) one subset, $\Phi_{i_1 i_2} \subset \Phi_{i_1}$ containing φ. Continue in this manner to generate the sequence of nested sets $\Phi_{i_1 i_2 i_3 \cdots i_k}$ each of which contains φ. Suppose there exists an involution $\psi \neq \varphi$ in the intersection. Then the representation of ψ must differ from the representation of φ. If they differ at the k^{th} pair, then φ and ψ cannot be contained in the same set $\Phi_{i_1 i_2 i_3 \cdots i_k}$. $\square$

With the above decomposition of Φ, we are ready to define a one to one and onto map from Φ to the half open unit interval $I = [0, 1)$. We define a decomposition of $[0, 1)$ into disjoint nested subintervals that correspond to the sets $\Phi_{i_1 i_2 i_3 \cdots i_k}$. First for $n = 1, 2, 3, \cdots$ put

$$I_n = \left[\frac{n-1}{n}, \frac{n}{n+1}\right).$$

This gives $[0, 1)$ as a disjoint union of half open subintervals. Now suppose we have recursively defined half open subintervals $I_{i_1 i_2 i_3 \cdots i_k}$ satisfying properties (2), (3), and (4) of Lemma 4.7 with I replacing Φ. Let

$$I_{i_1 i_2 i_3 \cdots i_k} = [a, b)$$

denote one of these subintervals. We decompose $I_{i_1 i_2 i_3 \cdots i_k} = [a, b)$ into a disjoint union of subintervals as follows. Let

$$I_{i_1 i_2 i_3 \cdots i_k n} = \left[a + \frac{n-1}{n}(b-a), a + \frac{n}{n+1}(b-a)\right)$$

These half open intervals are disjoint and

$$\cup_n I_{i_1 i_2 i_3 \cdots i_k n} = I_{i_1 i_2 i_3 \cdots i_k}.$$

Lemma 4.8. *For each $x \in [0, 1)$, there is a unique sequence of half open intervals*

$$I_{i_1} \supset I_{i_1 i_2} \supset \cdots \supset I_{i_1 i_2 i_3 \cdots i_k} \supset \cdots$$

such that

$$\{x\} = I_{i_1} \cap I_{i_1 i_2} \cap \cdots \cap I_{i_1 i_2 i_3 \cdots i_k} \cdots$$

Proof. By the construction of the disjoint sets $I_{i_1 i_2 i_3 \cdots i_k}$ there is a unique I_{i_1} containing x. There is now a unique $I_{i_1 i_2} \subset I_{i_1}$ containing x. Continue in this manner to generate the unique sequence of sets. The intersection of the intervals contains x. If the intersection contains a second point, y, then there exists an $\varepsilon > 0$ with $|x - y| > \varepsilon$. But since the length of the half open subintervals $I_{i_1 i_2 i_3 \cdots i_k}$ tends to 0 this is impossible. $\square$

Finally, with the above results we are able to identify the set Φ of bilateral matchings of the natural numbers with the half open interval $[0, 1)$. This identification will allow us to be precise in our discussion of probability distributions on the set of involutions.

Theorem 4.9. *Let $\varphi \in \Phi$ with representation*

$$(a_{11}, a_{12})(a_{21}, a_{22})(a_{31}, a_{32}) \cdots (a_{j1}, a_{j2})(a_{j+1,1}, a_{j+1,2}) \cdots$$

Let the sequence $(i_1, i_2, i_3, \cdots)$ be the unique sequence of natural numbers associated with φ by Lemma 4.7 so that

$$\{\varphi\} = \Phi_{i_1} \cap \Phi_{i_1 i_2} \cap \Phi_{i_1 i_2 i_3} \cap \cdots$$

Let $x \in [0, 1)$ with

$$\{x\} = I_{i_1} \cap I_{i_1 i_2} \cap \cdots \cap I_{i_1 i_2 i_3 \cdots i_k} \cdots.$$

Then the map $F(\varphi) = x$ is a bijection.

Proof. The result follows immediately from the construction of the unique representations of φ and x given in Lemma 4.7 and Lemma 4.8. $\square$

When we discuss the problem of uniqueness for probability distributions on the set of bilateral matches in the countably infinite case, it will be more convenient to have a slightly different representation of the set of bilateral matches of the natural numbers. The following corollary is obtained by mapping the half open interval $[0, 1)$ onto $[0, \infty)$ by the map $x \to x/(1 - x)$.

Corollary 4.10. *There exists a bijection from the set Φ of bilateral matches of the natural numbers $\mathbb{N}$ to the half open interval $[0, \infty)$.*

For a countable infinite set of agents modeled as $\mathbb{N}$ we now have a good description of the set Φ of bilateral matches. The next step is the construction of probability distributions on this set that satisfy the conditions M1, M2, and M3. Boylan [7] proved the existence of a probability distribution on Φ that satisfies these properties. In this case property M1 is trivial if we take μ to be the counting measure on $\mathbb{N}$. Of course this is not a finite measure and therefore we need the limit in property M3. In his existence result, Boylan uses the Cesaro average to define the proportion of agents of a particular type. This is equivalent to exhausting the set $\mathbb{N}$ by the sets $\{1, 2, 3, \cdots, n\}$.

It is easy to see that the probability distribution constructed by Boylan on Φ that satisfies conditions M1, M2, and M3 is not unique. We recall just enough of his construction to explain the non-uniqueness.

Let $S = \{\tau_1, \tau_2, ..., \tau_L\}$ be a finite set of types. Let μ be the counting measure on $\mathbb{N}$. Exhaust $\mathbb{N}$ by sets $A_n = \{1, 2, 3, \cdots, n\}$. Assume the proportion of each type, τ_l, in the population is P_l as defined by equation 3.2. Let τ be the type assignment map. Let X_i be i.i.d. random variables (indexed by the agents) taking values in S with distribution given by the P_i. In other words,

$$\text{Prob}\ (X_* = \tau_l) = P_l.$$

The value of X_l will tell us the type of the agent to be matched with agent l. So, match agent 1 with agent k where

$$k = \inf \{j \,| X_1 = \tau(j) \text{ and } X_j = \tau(1)\,\}\,.$$

Continue to match the remaining unmatched agents in the same manner.

This defines a probability distribution on the set of all bilateral matches of $\mathbb{N}$ that satisfies the Proportionality Laws. Furthermore every set of bilateral matches of the form,

$$\{\varphi \,| \varphi = (a_{11,} a_{12})(a_{21}, a_{22}) \cdots (a_{k1}, a_{k2}) \cdots \}$$

where the first k pairs are specified and the remaining (infinitely many) pairs are unspecified, has positive probability. This probability tends to zero as the number of specified pairs, k, tends to infinity. One can use these properties and the bijection defined in Corollary 4.10, to identify this distribution with a continuous distribution on $[0, \infty)$.

It is easy to see that this distribution is not unique. To construct a second distribution that satisfies the Proportionality Laws, simply interchange two (or more) agents of the same type and reapply the procedure. For example, assume there are two types of equal proportion. Suppose agents $1, 2$ and agent 1000 have the same type. Assume there are many agents of each type from 3 through 999. Switch agent 2 and agent 1000. This will not change the proportion of agents of each type in the population. With the original order, it is much more likely that agent 1 will be matched with agent 2 than with agent 1000. But with the second order, this is reversed. Hence we have a different distribution.

The set of probability distributions on Φ, the set of bilateral matches of $\mathbb{N}$, that satisfy matching properties M1, M2, and M3 form a convex set in the set of all probability distributions on Φ. There does not appear to be any "natural" way to select one over another. We shall discuss the consequences of this fact below and give one possible approach to selecting a distinguished probability distribution.

Finally, we mention that the results of the previous section on random matching distributions for finite sets of agents can be used to construct probability distributions for countably infinite sets of agents. Suppose our (countable infinite) populations consists of finitely many types and the proportion of each type is a rational number. Find a random matching scheme on a finite set of agents that satisfies the properties M1, M2, and M3. Such a scheme exists by the results of the previous section. Now regroup the agents of the infinite countable population into a countable number of subgroups, each one of which is a copy of this finite set. Randomly match the agents in the infinite population by randomly matching the agents in each subgroup. The random matching scheme on the infinite population inherits the properties from the matching scheme on the finite set of agents.

The above construction should make clear the problem one faces when constructing matching schemes satisfying the conditions M1, M2, and M3. The reader might, at this point, anticipate that the same construction is possible in the case of a continuum of agents.

4.3. **Random Matching for a Continuum of Agents.** We now consider the case of a continuum of agents. The set of agents is perhaps most frequently modeled as the unit interval $[0, 1]$ or the half open interval $[0, 1)$. It is also frequently assumed that subsets of agents are measured using Lebesgue measure on the interval. This makes it easy to make sense of the concept of "the proportion of agents of a given type". The unit interval model could of course be replaced with any other model with the cardinality of the reals, $\mathbb{R}$. For example, if one used the infinite half open interval, $[0, \infty)$, with a non-finite measure, then one could make sense of proportionality as a limit. Whichever model one uses, it is important to keep in mind that the real number identifying an agent is simply a name or label for the agent. If one uses a different name or label, the behavior of the model should not change. We will use the $[0, 1)$ model in our discussion here.

A random matching scheme for the set of agents, $[0, 1)$, requires a distribution on the set of involutions of $[0, 1)$. As before, we assume the involutions have no fixed point so that every agent is matched. Let $S = \{\tau_1, \tau_2, \tau_3, \cdots, \tau_L\}$ be a finite set of types and assume that each agent is assigned a type. Suppose the proportion of agents of type τ_k equals P_k as before. For this model, Alós-Ferrer [4] constructs a probability distribution on the bilateral matches that satisfies the properties M1, M2, and M3. Just as in the case of a countably infinite number of agents, this probability distribution is not unique. Although the same basic idea that we used in the countable infinite case can be used to understand the non-uniqueness, we present a second method.

Consider two copies of the unit interval as the set of agents. In other words, we just have two identical copies of a continuum of agents with each agent assigned a type, and the proportion of agents of each type is identical in the two copies. Construct random matching schemes that satisfy M1, M2, and M3 on each copy separately. Now shrink each copy down by some factor. Glue the two copies together to make an interval of length one. The proportions of agents of each type on the two parts equals the original proportions. Apply the random matching scheme on each part. In this way we get a new probability distribution since we have changed the set of involutions. We provide the details below.

Let Q be a probability distribution on some subset Ψ of the set Φ of all bilateral matches of the agent set $[0, 1)$. Let $\varphi \in \Psi$ denote a typical bilateral match. Assume the distribution Q satisfies the conditions M1, M2, and M3. Fix some $\alpha \in (0, 1)$. For each $\varphi \in \Psi$ define a new bilateral match as follows.

$$\widetilde{\varphi}(x) = \begin{cases} \alpha\varphi(\frac{x}{\alpha}) & \text{if } \ 0 \le x < \alpha \\ \alpha + (1-\alpha)\varphi(\frac{x-\alpha}{1-\alpha}) & \text{if } \ \alpha \le x < 1 \end{cases}$$

We have defined a new set of bilateral matches $\widetilde{\Psi}$ and a bijection with the original set of matches Ψ. Define a σ-algebra on $\widetilde{\Psi}$ using the bijection of matches. Define a probability distribution on the sets in the σ- algebra of subsets of $\widetilde{\Psi}$ by $\widetilde{Q}(\widetilde{E}) = Q(E)$ if E and $\widetilde{E}$ correspond to one another under the bijection of matches. Since the shrinking function $x \to \alpha x$ does not change the proportion of agents of a given type, $\widetilde{Q}$ will also satisfy properties M1, M2, and M3. But we now have two distinct probability distributions (in fact a one parameter family) of distributions on the set of all bilateral matches of $[0, 1)$. Note that the matching scheme we constructed by the glueing process has the property that agents in the first interval $[0, \alpha)$ are never matched with agents in the second interval $[\alpha, 1)$.

The above construction can be applied to any random matching scheme for a continuum of agents. Hence, the standard conditions M1, M2, and M3 are not enough to isolate a single matching scheme in the set of bilateral matches of a continuum of agents.

Remark 4.11. The cardinality of the set of mappings $\mathbb{R} \Rightarrow \mathbb{R}$ is known as $\beth_2$ (Beth 2). It is fairly straightforward to show that this is the cardinality of the set of bilateral matches of a continuum of agents as well. The general continuum hypothesis says that $\aleph_2 = \beth_2$. So the set of bilateral matches of a continuum of agents is indeed a very large set. Therefore the nonuniqueness problem for bilateral matches of a continuum of agents seems to be much less tractable than non-uniqueness for bilateral matches of a countable set of agents.

We finally mention that the random matching scheme we constructed on the finite set of agents can be used to construct a matching scheme on the continuum of agents $[0, 1)$. Let g denote an agent in a finite set of agents $\widetilde{A}$ for which we have constructed a matching scheme satisfying M1, M2, and M3. Suppose $\text{card}(\widetilde{A}) = N$. Let G be a half open interval of agents of length $1/N$ with all of the same type as agent g. We now have a continuum of agents with the same proportion of agents of each type (using Lebesgue measure to measure proportion) as the finite set. Now randomly match the agents in the continuum by matching the intervals (which all have the same length) according to the probability law used to match the agents in the finite set. An interval of length $1/N$ is mapped onto a second interval of length $1/N$ in the obvious way. This matching scheme on the continuum of agents satisfies the Proportionality Laws M1, M2, and M3.

4.4. **Summary.** For each of the three possibilities for the cardinality of the set of agents in the population, we have shown that the probability distributions, Q, that satisfy the Proportionality Laws (for a fixed measure, μ, on the set of agents) are not unique. In fact the set of such distributions is a convex set in a large dimensional space (infinite dimensional if A is infinite). In the case A is infinite, the non-uniqueness is more difficult to quantify since the set of bilateral matches is $\aleph_1$ or $\aleph_2$ (assuming the general continuum hypothesis).

If random matching is used in an economic model, and one only knows that the matching scheme satisfies the Proportionality Laws, the non-uniqueness may make it impossible to compute even fundamental quantities such as transition probabilities and expected rewards. The next section shows this problem can be serious.

5. A Growth Model with Random Matching

In the last section we showed that random matching schemes that satisfy proportionality laws are far from unique. In particular, we showed how to construct a multiplicity of random matching structures that satisfy assumptions M1, M2, and M3 for agents sets of finite, countable, or a continuum cardinality. In this section we present an example that shows the non-uniqueness of the matching scheme can present a serious problem for economic models. The example can be easily modified to demonstrate that the same problem persists in the case of infinitely many agents (countable or uncountable). We show that in a modified version of a basic stochastic growth model (e.g., [21]), the mechanics of random matching have substantive implications for the evolution of the growth model as well as optimality and equilibrium.

First recall a very simple growth model that incorporates a stochastic shock. Suppose production of a good occurs during discrete time periods. Let $k(t)$ denote capital available at epoch t, or the start of period t. Let production during period t be denoted by $F(k(t))$ and let $c(t)$ be the fraction, between 0 and 1, of production that is consumed. Let $z(t)$ denote a stochastic shock, so we may think of the $z(t)$ as i.i.d. random variables with values in the unit interval $[0,1]$. The updating rule for capital is given by

$$k(t+1) = k(t) + z(t)F(k(t)) - c(t)z(t)F(k(t)).$$

This model is similar to one described in [21]. We express consumption in relative rather than absolute terms since it will be easier to explain how distinct matching distributions result in distinct outcomes by using a relative expression for consumption. In this model, one may think of the shock, $z(t)$, as a result of natural or man made disasters, technology changes, or some other unpredictable factor.

We now modify the above example to describe an economy with a finite number agents such that the updating rule for the capital involves a shock that is the result of random matching of the agents. Again we consider a discrete time model over an infinite time horizon.

Suppose our economy consists of an even number, $2N$, of agents of two types, *"a"* and *"b"*. Denote the set of agents by

$$A = \{a_1, ..., a_N, b_1, ..., b_N\}$$

Suppose the two types of agents use complementary resources to produce a good. Assume production occurs only when agents of different types are matched. We use common normalized units to measure resources, production, and the good that is consumed. Production and consumption occur during discrete time periods, and a *planner's* optimization problem is to maximize total discounted utility of consumption for the entire set of agents over the infinite time horizon.

The state of the economy at the start of period t is given by the vector of resources available to the agents which we denote by

$$k(t) = (k_{a_1}(t), k_{a_2}(t), ..., k_{a_N}(t), k_{b_1}(t), ..., k_{b_N}(t)).$$

Let $k_g(t)$ denote a typical component of this state vector where the subscript indicates the agent. We assume $k_g(t)$ to be nonnegative. At the beginning of each period, the state vector, $k(t)$, is observed. A *planner* selects a fraction of the potential production to be consumed for the period for each agent. Denote this fraction by the consumption vector, $c(t)$, with a typical component denoted $c_g(t)$ for agent g. Hence the actual consumption of a given agent will be the product of $c_g(t)$ and the production amount. After $k(t)$ is observed and $c(t)$ is selected, agents are randomly bilaterally matched. As before, let Φ denote the set of all possible bilateral matches of these $2N$ agents. The randomness for matching is taken care of by placing a probability distribution on the set of matches. If Q denotes a probability distribution on Φ, then $Q(\varphi)$ denotes the probability of match φ. Let $F_g(k(t), \varphi)$ denote the production amount of agent g. This production amount will depend upon an agent's capital at epoch t and the capital of the agent with whom agent g is matched. Hence the arguments of F_g are $k(t)$ and φ. The updating rule for agent g given the match φ is

$$k_g(t+1) = k_g(t) + F_g(k(t), \varphi) - c_g(t)F_g(k(t), \varphi).$$

We now describe the production function $F_g(k(t), \varphi)$. Since agents possess complementary resources, production occurs only when agents of opposite type meet through the match φ. A simple and reasonable choice for F_g is

$$F_g(k(t), \varphi) = \begin{cases} f\left(\min[k_g(t), k_{\varphi(g)}(t)]\right) & \text{if} \quad g \text{ and } \varphi(g) \text{ have different types} \\ 0 & \text{if} \quad g \text{ and } \varphi(g) \text{ have the same type} \end{cases}.$$

Here f denotes a bounded increasing continuous function with $f(0) = 0$. Since the match φ is chosen randomly from some given distribution, our model incorporates a random shock just as in the simple growth model.

Suppose now that our planner wishes to optimize total discounted utility of consumption over an infinite time horizon with discount factor β. A policy for the planner will be a map from the information sets $\{k(s) : s \leq t\}$ to the set of possible consumption vectors $c(t)$. For sequential decision problems of the type we are considering, a stationary optimal policy exists, so we may assume that $c(t)$ depends only on the current state of the economy, $k(t)$. We first write down an expression for the total discounted utility of consumption.

Let u denote a utility function for consumption of an individual agent. Let π denote a stationary policy. Then the sequence of state vectors, $k(t)$ will be a Markov process. The total discounted expected utility of consumption for the policy π is given by

$$V_\pi = E_\pi \left[\sum_t \beta^t \left(\sum_g u\left(c_g(t) F_g\left(k(t), *\right)\right) \right) \right].$$

The expectation depends on the policy π since the Markov process of state vectors depends on π. Expectation also depends on the random matching distribution.

This type of optimization problem can be solved by using the associated Bellman equation,

$$V(k) = \max_c \left(R(k,c) + \beta \sum P_{k\tilde{k}}(c) V(\tilde{k}) \right).$$

Here $R(k,c)$ is the expected reward for a single period given the initial state vector $k = (k_{a_1}, ..., k_{b_N})$ and consumption vector $c = (c_{a_1}, ..., c_{b_N})$. The $P_{k\tilde{k}}(c)$ are the transition probabilities from state vector k to state vector $\tilde{k}$ under the choice c of consumption vector for the agents during the period. Both expected reward and the transition probabilities depend on the random matching distribution.

Suppose we know the distribution of resources at the start of a period. For example, suppose both type *"a"* agents and type *"b"* agents both have 0 or 1 unit of resource, each with probability .5. Suppose also that the probability that any given agent meets a type "a" agent is .5 and the probability that she meets a type "b" agent is also .5. One might be tempted to compute the expected reward for the first period using the following argument.

> An individual agent will produce only if she has 1 unit of resource, she meets an agent of opposite type, and the agent she meets also has 1 unit of resource. Since each of these events occur with probability .5 and they occur independently, production occurs with probability (.5)(.5)(.5). If these events occur, then the production of the agent will be $f(\min[1,1])$ or $f(1)$ since the agent and partner both have one unit of resource. Hence the expected reward will be
>
> $$\sum_g (.5)(.5)(.5) u\left(c_g f(\min[1,1])\right).$$

This reasoning uses only the proportionality conditions on the matching and the distribution of wealth. It does not use the explicit nature of the matching scheme. However, the argument contains a gap. We give the correct expression for expected per period reward below and show that it depends explicitly on the distribution placed on the set of bilateral matches, and cannot be computed if we only know the proportionality condition satisfied by the probability distribution on matches.

Let Q denote a probability distribution on the set of bilateral matches of the $2N$ agents that satisfies the following proportionality condition. The probability that any given agent meets a type *"a"* agent is .5 and the the probability that she meets a type *"b"* agent is also .5. Let $Q(\varphi)$ denote the probability of match φ. The total expected reward (utility of consumption) for a period is

$$R(k,c) = \sum_g \sum_\varphi Q(\varphi) u\left(c_g(t) F_g(k,\varphi)\right).$$

Here we see the explicit dependence on the distribution Q. We have a corresponding expression for the transition probabilities.

$$P_{k\tilde{k}}(c) = Q(\varphi) \quad \text{if} \quad \tilde{k_g} = k_g + F_g(k,\varphi) - c_g F_g(k,\varphi) \ .$$

Again, we emphasize that these quantities cannot be computed without explicit knowledge of the matching distribution. We give two examples below of distributions that satisfy the same proportionality condition, but give distinct expected rewards, transition probabilities, and consequently distinct optimal policies.

We first describe two distinct probability distributions on the set of bilateral matches of agents in our model economy. We want to consider a specific example so we take $N = 4$ with four type *"a"* agents and four type *"b"* agents. There are $7 * 5 * 3$ or 105 possible bilateral matches of the eight agents. The two distributions are described by tables listing the bilateral match and corresponding probability. Matches that do not appear are assigned probability 0. The lexicographic representation is used to specify matches.

Distribution I

Match	Probability
$(a_1,a_2)(a_3,b_3)(a_4,b_4)(b_1,b_2)$	.5
$(a_1,b_1)(a_2,b_2)(a_3,a_4)(b_3,b_4)$	.5

Distribution II

Match	Probability
$(a_1,a_2)(a_3,b_3)(a_4,b_4)(b_1,b_2)$	.25
$(a_1,a_2)(a_3,b_1)(a_4,b_2)(b_3,b_4)$	.25
$(a_1,b_1)(a_2,b_2)(a_3,a_4)(b_3,b_4)$	.25
$(a_1,b_3)(a_2,b_4)(a_3,a_4)(b_1,b_2)$	.25

Note that both distributions satisfy the proportionality condition, M2, since any individual agent has probability .5 of being matched with a type *"a"* agent and probability .5 of being matched with a type *"b"* agent. Both distributions also satisfy the quasi-independence condition, M3, since for each listed match, exactly one-half of the "a" agents are matched with type "a" agents and one-half are matched with type "b" agents. Condition M1 is trivially satisfied in this case, so both distributions satisfy the Proportionality Laws.

Next we assign initial capital to the agents. We assign the distribution of capital so that each agent has either 0 or 1 unit, and the probability (using the discrete uniform distribution on the set of agents) that a type "a" agent has 1 unit is .5. A table gives the distribution of capital.

Initial Distribution of Capital

Agent	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
Capital	1	1	0	0	0	0	1	1

Now consider the problem of selecting the consumption - investment behavior of the set of agents to maximize total discounted expected utility of consumption over an infinite time horizon.

In the case of Distribution I there is nothing to be done. For both matches, either two agents of the same type are paired or a pair involves one agent with 0 capital. Therefore the distribution of capital does not change from one period to the next. The optimal (maximum) total utility of consumption over the infinite time horizon is 0.

In the case of Distribution II, if one of the first three distributions is realized during the first period, no production takes place because two agents of the same type meet or agents of opposite type meet but one of them has 0 capital. However if the fourth match is realized (and this occurs with probability .25) then agents a_1, a_2, b_3, and b_4 all produce an amount $f\left(\min[1,1]\right) = f(1)$. Therefore, if we take c_g to be positive for all agents g, total expected utility of consumption over the infinite time horizon will be positive in contrast to the zero optimal utility of consumption we see in the case of Distribution I.

Since the expected reward, transition probabilities, total discounted utility, and Bellman equation depend explicitly on the probability distribution on the set of matches, it should be clear that the qualitative properties of this growth model are robust when random matching is used as a shock. Indeed, one can obtain similar results for models with arbitrarily large sets of agents and for models with infinitely many agents (countable or uncountable) and different reward structures. Even if the matching scheme satisfies an independence condition on types, the non-uniqueness of the probability distribution allows one to construct simple and natural examples such that optimal policies depend on the explicit nature of the matching probability distribution. It should also be clear that this behavior does not depend on our choice of production function.

6. No Aggregate Uncertainty

In the previous section we saw that the expected aggregate reward and transition probabilities will depend on the specific nature of a random matching scheme even when the matching scheme satisfies a proportionality condition with respect to types. This naturally leads to the following question. When is it possible to compute expected aggregate rewards and transition probabilities for a matching scheme if we only know that the scheme satisfies the proportionality conditions? In other words, when will a model that uses random matching (that satisfies proportionality) as shock be well defined?

This question can be posed in terms of aggregate uncertainty. We have seen that simply stipulating that the matching scheme satisfy a proportionality condition does not eliminate aggregate uncertainty in an economic model. For the second model of the last section (Distribution II), the aggregate reward is stochastic at each stage. In general, the aggregate expected reward may remain stochastic even if the matching scheme satisfies the Proportionality Laws. We would like to know conditions under which the aggregate uncertainty disappears and expected rewards for the population can be computed strictly in terms of the proportion of each type in the population. It turns out that the answer to this question does not depend on the cardinality of the set of agents, but rather on the reward structure. The same observation holds for the transition structure. Models with finite, countably infinite, or a continuum of agents will have the "no aggregate uncertainty" property if the reward and transition structures for individual agents depends only on types. If the reward or transition structure depends on properties of agents other than types, then aggregate uncertainty may persist regardless of the cardinality of the set of agents of the model.

Suppose that the reward of an individual agent depends only upon the agent's type, the type of the agent's partner under a match, and some control parameter c. Precisely, we assume

$$R_g\left(\tau, c, \varphi\right) = f\left(\tau(g), \tau(\varphi(g)), c\right).$$

Remark 6.1. This can be accomplished in our example from the previous section by redefining types to include the amount of money held by an agent. One would then have four types that could be denoted $(a, 0)$, $(a, 1)$, $(b, 0)$, and $(b, 1)$. Alternatively, if all agents within a type had identical capital endowments, the reward structure would depend only on type. If one allows a continuous distribution of money, it is impossible to describe the possible states of individual agents with a finite set of types.

We show that under this assumption aggregate uncertainty disappears from the reward structure if the matching scheme satisfies the proportionality condition with respect to types. First consider the case of finitely many agents with counting measure on the agent set, so $\mu(A) = N$.

Assume the economy consists of finitely many types as before. Let $S = \{\tau_1, \tau_2, \tau_3, \cdots, \tau_L\}$ denote the set of types. Assume type τ_k makes up the proportion P_k of the population. Assume Q is a distribution on the set of bilateral matches that satisfies the proportionality condition with respect to types, so we are assuming that

$$Q\{\varphi\,|\tau(\varphi(g)) = \tau_k\} = P_k.$$

Alternatively, this can be written as

$$\sum_{\varphi} \delta_{\tau_k \tau(\varphi(g))} Q(\varphi) = P_k$$

where

$$\delta_{pq} = \left\{ \begin{array}{ll} 1 & \text{if} \quad p = q \\ 0 & \text{if} \quad p \neq q \end{array} \right. .$$

With the above notation, we compute the aggregate expected reward in terms of the proportionality constants P_k.

Theorem 6.2. *Let A be a set of N agents. Suppose $R_g(\tau, c, \varphi) = f(\tau(g), \tau(\varphi(g)), c)$ is a reward function that depends only on types, a control parameter, and a match of agents. Assume Q is a probability distribution on the set of bilateral matches that satisfies the proportionality condition, M2, so that for all $g \in A$ we have $Q\{\varphi\,|\tau(\varphi(g)) = \tau_k\} = P_k$. Then the expected reward for the population is deterministic and is given by*

$$E[R(\tau, c)] = \frac{1}{N} \sum_g \sum_k f(\tau(g), \tau_k, c)\, P_k$$

Proof. The proof is a straightforward calculation. From the definition of aggregate reward,

$$E[R(\tau, c)] =$$

$$\frac{1}{N} \sum_g \sum_{\varphi} R_g(\tau, c, \varphi)$$

$$\frac{1}{N} \sum_g \sum_{\varphi} f(\tau(g), \tau(\varphi(g)), c)\, Q(\varphi) =$$

$$\frac{1}{N} \sum_g \sum_{\varphi} \sum_k f(\tau(g), \tau_k, c)\, \delta_{\tau_k \tau(\varphi(g))} Q(\varphi) = \tag{6.1}$$

$$\frac{1}{N} \sum_g \sum_k \sum_{\varphi} f(\tau(g), \tau_k, c)\, \delta_{\tau_k \tau(\varphi(g))} Q(\varphi) =$$

$$\frac{1}{N} \sum_g \sum_k f(\tau(g), \tau_k, c)\, P_k$$

□

The case of a countable infinity of agents follows by the same argument. One replaces summation by integration since we shall assume a continuous (non-atomic) probability distribution on the set Φ of bilateral matches. We assume that the exhaustion of $\mathbb{N}$ is $A_n = \{1, 2, 3, ..., n\}$ so $\mu(A_n) = n$.

Theorem 6.3. *Let $R_g(\tau, c, \varphi) = f(\tau(g), \tau(\varphi(g)), c)$ be a bounded reward function that depends only upon types and the control parameter c. Let Q be a continuous probability distribution on the the set Φ of bilateral matches of the set $\mathbb{N}$ of agents. Assume Q satisfies the proportionality condition M2 so that for all $g \in \mathbb{N}$ we have $Q\{\varphi\,|\tau(\varphi(g)) = \tau_k\} = P_k$. Then the expected reward for the entire population is deterministic and is given by*

$$E[R(\tau, c)] = \lim_{n \to \infty} \frac{1}{n} \sum_{g \in A_n} \sum_k f(\tau(g), \tau_k, c)\, P_k.$$

Proof. The proof follows the same argument as in the case of finitely many agents. We first rewrite the proportionality condition as

$$\int \chi_{\{\varphi|\tau(\varphi(g))=\tau_k\}} dQ(\varphi) = P_k$$

which is assumed to hold for all $g \in \mathbb{N}$, and where χ denotes the characteristic function. Then

$$\begin{gathered}
E[R(\tau, c)] = \\
\lim_{n\to\infty} \frac{1}{n} \sum_{g\in A_n} \int_{\Phi} f\left(\tau(g), \tau(\varphi(g)), c\right) dQ(\varphi) = \\
\lim_{n\to\infty} \frac{1}{n} \sum_{g\in A_n} \int_{\Phi} \sum_{k} f\left(\tau(g), \tau_k, c\right)\right) \chi_{\{\varphi|\tau(\varphi(g))=\tau_k\}} dQ(\varphi) = \\
\lim_{n\to\infty} \frac{1}{n} \sum_{g\in A_n} \sum_{k} \int_{\Phi} f\left(\tau(g), \tau_k, c\right)\right) \chi_{\{\varphi|\tau(\varphi(g))=\tau_k\}} dQ(\varphi) = \\
\lim_{n\to\infty} \frac{1}{n} \sum_{g\in A_n} \sum_{k} f\left(\tau(g), \tau_k, c\right) \int_{\Phi} \chi_{\{\varphi|\tau(\varphi(g))=\tau_k\}} dQ(\varphi) = \\
\lim_{n\to\infty} \frac{1}{n} \sum_{g\in A_n} \sum_{k} f\left(\tau(g), \tau_k, c\right) P_k
\end{gathered}$$

as claimed. □

The above results show that the aggregate uncertainty disappears from the reward structure as a consequence of two features. First, the matching scheme must satisfy a proportionality condition with respect to types, and second, the reward function must depend only upon types. An independence in types condition is not needed. Furthermore, an infinity of agents is not needed. One only needs sufficiently many agents so that it is possible to construct a matching scheme that satisfies the proportionality condition. Clearly, from our earlier construction of the probability distribution satisfying proportionality in the case of finitely many agents, as the number of types increases, the number of agents needed for existence of the matching scheme increases.

In the case of a continuum of agents a completely analogous result holds. In this case one has for expected reward

$$\begin{gathered}
E[R(\tau, c)] = \\
\int_{A\times} \int_{\Phi} f\left(\tau(g), \tau(\varphi(g)), c)\right) d\mu(g) \times dQ(\varphi).
\end{gathered}$$

At this point one must impose some additional conditions on the set of matches and the function f. First the bilateral matches φ must be measurable with respect to the measure space $(A, \mathcal{L}, \mu)$. If the measure μ is not finite, additional conditions will be needed on f to get convergence. For the matching schemes constructed by Alós-Ferrer in [4], this is not a problem since the matches are constructed to be measurable and μ is finite. For some matching schemes, such as those constructed by Duffie and Sun [10], the product measure must be modified in an appropriate way so the integral makes sense.

The important point in this section is that aggregate uncertainty in the reward structure disappears if the rewards depend only on type and the control, and if the random matching structure satisfies the proportionality condition of the Proportionality Laws. Similar conditions will make the transition structure deterministic in the aggregate. Since the arguments follow along the same lines as those for the reward structure, we do not present them here.

7. The Entropy of a Random Matching Scheme

For a typical economic model that uses random matching to introduce uncertainty, very little is typically assumed about the matching scheme. Similarly, many models seek to impose minimal assumptions on how types and other properties are assigned to agents. Therefore one is faced with the problem of defining reasonable probability distributions on the set of matches consistent with some minimal conditions on the distribution. In the last section we showed that if one is willing to restrict the reward and transition structure of the model to depend upon a set of types and the random matching structure satisfies proportionality laws given in terms of the types, then the model is well defined and deterministic in the aggregate.

Now suppose that the reward structure or transition structure depends on state parameters of the individuals in the population beyond the types specified in the matching structure. One is faced with a non-uniqueness problem that can make a model ill posed as our examples showed. In general, there are many distributions compatible with some desired properties, and we need a method that will select one of them.

If we have no reason or information to prefer one random event over another, then the distribution we use to measure the likelihood of the events should reflect this lack of information. The entropy of a distribution measures in some sense the information contained in the distribution - the less information we have the higher the entropy. We first review some needed definitions and results on entropy, and then apply the results to the random matching problem.

Remark 7.1. In the remainder of this section, we do not assume our probability distributions on matches necessarily satisfy the Proportionality Laws. The point here is to develop a method of making matching models well defined in cases where the rewards and transition probabilities depend upon state variables other than type.

7.1. **Information Entropy.** The idea of using entropy as a measure of information encoded in a distribution goes back to Shannon [20]. An excellent review of the topic and basic properties can be found in [14].

Definition 7.2. Let X be a discrete random variable on a probability space such that

$$P\{X = x_k\} = p_k$$

for all x_k in the image of X. The Shannon entropy of the distribution is

$$H(X) = -\sum p_k \log(p_k)$$

where *log* means natural logarithm.

We have a similar definition in case X is a continuous random variable.

Definition 7.3. Let X be a continuous real valued random variable with probability density function f. Then the differential entropy of X is

$$H(X) = -\int_{-\infty}^{\infty} f(x)\log[f(x)]dx.$$

The following result demonstrates the link between maximal entropy and minimal information. If X is a random variable taking on a finite set of values, and we have no reason to prefer one value over another, then it is most appropriate to assign equal probability to each possible outcome. This can be stated in terms of entropy as follows.

If Ω is a finite discrete probability space, then the distribution that maximizes the entropy is the uniform discrete distribution.

The discrete uniform distribution maximizes entropy if no constraints are imposed on the probability distribution. Hence, the more or less standard assumption that we should take all possible matches (of finitely many agents) as equally likely in the absence of additional information, is consistent with taking the distribution of maximum entropy.

The Differential Entropy is not as satisfactory a measure of information as the Shannon entropy and it cannot be realized as a limiting case of the Shannon entropy. However, the differential entropy still does give us some measure of information and both the uniform distribution on the unit interval and the usual normal distribution can be characterized in terms of the differential entropy. For example, the normal distribution is the unique maximal (differential) entropy distribution with finite mean and variance.

The Proportionality Laws can be interpreted as information about a random matching scheme, and imply constraints on the probability distribution on the set of matches. Thus, from the set of all probability distributions satisfying a set of constraints, the distribution with maximal entropy appears to be a natural candidate. Indeed, if we select a distribution of lower entropy, we are assuming information we do not have.

We next show that the maximal entropy distribution for distributions on the set of bilateral matches of a finite set of agents that satisfy the proportionality condition is unique.

Theorem 7.4. *Let A be a finite set of agents, $S = \{\tau_1, \tau_2, ..., \tau_L\}$ be a finite set of types, and let $\tau : A \to S$ be an assignment of types. Let Φ be the set of bilateral matches of A. Suppose $P_1, P_2, ..., P_L$ are positive real numbers such that $\sum P_l = 1$. Let $\mathcal{P}$ be the set of probability distributions on Φ that satisfy the proportionality condition M2. If $Q \in \mathcal{P}$ has maximal entropy, then Q is unique.*

Proof. We have already noted that the Proportionality Laws define $\mathcal{P}$ as a bounded convex polyhedra in the real vector space of dimension cardΦ. (In fact this convex polyhedra lies in an affine subspace of much smaller dimension.) Now suppose, by way of contradiction, that Q and $\widetilde{Q}$ are two distinct probability distributions in $\mathcal{P}$ and they have maximal entropy. Then, by the convexity of $\mathcal{P}$, the distribution $tQ + (1-t)\widetilde{Q}$ is in $\mathcal{P}$ for $0 \le t \le 1$. The entropy of the convex combination of distributions, denoted $E(t)$, is

$$E(t) = H(tQ + (1-t)\widetilde{Q}) = -\sum (tq_k + (1-t)\widetilde{q_k}) log(tq_k + (1-t)\widetilde{q_k}).$$

Compute the first and second derivatives with respect to t of the terms in the sum. We get

$$E'(t) = -\sum (q_k - \widetilde{q_k})[1 + \log(t(q_k - \widetilde{q_k}) + \widetilde{q_k})]$$

$$E''(t) = -\sum \frac{(q_k - \widetilde{q_k})^2}{tq_k + (1-t)\widetilde{q_k}}$$

If $E(0) = E(1)$, so the entropy of Q and $\widetilde{Q}$ are equal, then by the strict negativity of the second derivative of $E(t)$ in case Q is not identically equal to $\widetilde{Q}$, we see that we cannot have $H(Q) = H(\widetilde{Q})$ with $H(Q)$ and $H(\widetilde{Q})$ maximal. Hence the maximal entropy distribution is unique. $\square$

Although the following example is quite simple, it illustrates a method that can be used to check if a distribution satisfying a proportionality condition is maximal or not.

Example 7.5. Consider the matching problem for four agents, $\{1, 2, 3, 4\}$ of two types, $\{a, b\}$. Suppose agent 1 and 2 each have type $"a''$ and agents 3 and 4 have type $"b''$. With four agents, there are three distinct bilateral matches. We name the match corresponding to the given representation and corresponding probability in the table below.

Match	Representation	Probability
φ_1	$(1,2)(3,4)$	p_1
φ_2	$(1,3)(2,4)$	p_2
φ_3	$(1,4)(2,3)$	p_3

Let Q_1 be the distribution that assigns values $p_1 = .5$, $p_2 = .5$, and $p_3 = 0$. Let Q_2 be the distribution that assigns values $p_1 = .5$, $p_2 = 0$, and $p_3 = .5$. Both of these distributions have the property that the probability that any given agent is matched to a type $"a''$ agent is .5 and the probability that the agent is matched to a type $"b''$ agent is also .5. Note that interchanging agents 3 and 4 in match φ_2 gives match φ_3. Since agents 3 and 4 both have the same type, type $"b''$, a maximal entropy distribution that satisfies the proportionality condition should assign the same probability to two distinct matches that can be obtained from one another by an interchange of agents of the same type. Therefore, neither distribution Q_1 nor Q_2 can have maximal entropy.

We now formulate the maximal entropy problem for this example. Let Q_{max} denote the maximal entropy distribution. Then Q_{max} is determined by solving the following maximization problem.

$$\text{Maximize} \quad E = -\textstyle\sum_{k=1}^{3} p_k \log(p_k) \quad \text{subject to}$$

$$p_1 = .5 \qquad\qquad p_2 + p_3 = .5 \qquad\qquad 0 \leq p_k \leq 1$$

The condition $p_1 = .5$ occurs because we want the proportionality condition, M2, to be satisfied. So, for example the probability that agent 1 is matched to a type $"a''$ agent must be .5, and the probability that agent 3 is matched to a type $"b''$ agent must be .5. This only occurs for match φ_1. The condition $p_1 = .5$ then implies the second condition, $p_2 + p_3 = .5$ since we want a probability measure on the set of matches.

It is easy to solve this convex optimization problem and the solution is $p_1 = .5$ and $p_2 = p_3 = .25$. Hence, we see that the two matches φ_2 and φ_3 that can be obtained from one another by interchanging agents 3 and 4 (or by interchanging 1 and 2) have the same probability.

Remark 7.6. In the above example condition M3 is not satisfied. To give an example with M2 and M3 satisfied we would need a larger set of agents (eight will do). However the idea is exactly as in the above example. Imposing condition M3 in addition to M2 simply makes the dimension of the convex set of distributions smaller than it would be with condition M2 alone.

7.2. **Differential Entropy.** For random matching schemes for finite sets of agents, a unique distribution of maximal entropy always exists. (If the random matching scheme satisfies the Proportionality Laws, a unique maximal entropy distribution exists as well.) This is not the case for continuous distributions and the Differential Entropy. In fact, we show below that in the case of bilateral matches of a countably infinite set of agents, there is no maximal entropy distribution.

Theorem 7.7. *Let $A = \mathbb{N}$, the set of natural numbers, and let Φ be the set of bilateral matches of A. Identify Φ with the nonnegative real numbers, $[0, \infty)$ through the bijection given by Corollary 4.10. Then there is no continuous probability distribution on Φ of maximal entropy.*

Proof. A straightforward calculation, using the definition of the Differential Entropy, shows that the exponential distribution with density function

$$f(x) = \begin{cases} \frac{1}{\mu}\exp(-\frac{x}{\mu}) & \text{if} \quad x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

has entropy

$$H(f) = \log \mu$$

Since we can take μ arbitrarily large, there is no distribution of maximal entropy. $\square$

The above result shows that in the case of infinitely many agents, one will need to place additional restrictions on the set of probability distributions on Φ in order to get a unique maximal entropy distribution. One such condition is evident from the above result. If we simply specify the mean of the distribution, then there is a unique maximal entropy distribution on $[0, \infty)$ with the given mean. However, specifying the mean is not really satisfactory from the point of view of bilateral matching since there is no obvious natural relationship between the mean and properties of the matching scheme.

We can obtain a positive uniqueness result using the Differential Entropy for bilateral matching of countably infinite sets of agents when the distribution satisfies a proportionality condition and the distribution comes from a class for which the entropy is bounded. More work needs to be done in this direction.

8. Conclusions

In this paper, we have examined some of the issues posed by the non-uniqueness of random matching schemes that satisfy conditions typically found in economic models. We took a general approach since these problems occur regardless of the cardinality of set of agents in the model. We concentrated on sequential models in discrete time since they are commonly used in many areas of economics.

Random matching schemes are frequently assumed to satisfy a set of Proportionality Laws that involves the "type" of an agent. The "type" can be considered to be a component of the state variable for the agent and in some cases it may be the only component of the state of an agent. The motivation for the assumption that the random matching scheme satisfy the Proportionality Laws is so that the model contains no aggregate uncertainty. In other words, the aggregate behavior of the system is deterministic.

We showed with a simple example that if the reward or transition structure of the sequential model depends on state components other than the "type" of an agent, then the Proportionality Laws will not in general eliminate aggregate uncertainty from the model. This is essentially a consequence of the non-uniqueness of random matching schemes that satisfy the Proportionality Laws. We constructed an example to show that if one only postulates a random matching scheme that satisfies the Proportionality Laws without explicitly specifying the probability distribution, the model may not be well defined.

One way to deal with the lack of uniqueness for random matching schemes is to assume that the state of an agent is completely determined by the type of an agent. In our growth model example, one could incorporate the capital holding of an agent into the type. This approach may well lead to additional complications if the number of types becomes infinite.

One should not be surprised by these results since they are entirely consistent with economic reality. It is easy to purchase automotive liability insurance, but difficult to purchase terrorist insurance. The proportion of terrorists in the general population is most likely quite small, and thus our chance encounter with a terrorist correspondingly small. However, our "reward" (cost) function for a match with a terrorist will depend on state variables other than type. For example, the cost will likely depend upon some "state" of the terrorist's mentality at the time of the match.

In some cases it may be undesirable or impossible to define the state variable of an agent in terms of type alone. In such cases, to pin down the model, one must explicitly select a random matching distribution from the many that are available. The selection should be consistent with the information available. We showed that the entropy of a random matching distribution can be used to make the selection in the case of finite sets of agents. In the case of infinite sets of agents this approach is problematic since maximal entropy distributions may not exist. Additional constraints on the random matching distribution will generally be needed to guarantee a maximal differential entropy distribution, and more work is needed in this direction.

Finally, we believe that especially in cases where a unique random matching scheme cannot be obtained in a natural way, it may be more appropriate to regard the matching distribution as a control variable for a sequential decision model. In the case of sequential decision optimality problems, this can be a difficult but very interesting problem. We performed computer simulations on models similar to the ones presented in Section 5, and found very subtle relationships between the matching distribution, the reward structure, and optimal solutions. In some cases, increasing the entropy of the matching scheme resulted in lower expected aggregate reward, and in other cases the situation was reversed. The identification of a more systematic relationship between random matching technologies and efficiency of allocations is left for further research.

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