Voting in the Goods and Service Tax Council of India

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Abstract

In 2017, India enacted a new taxation law called the Goods and Services Tax (GST). This law created a GST Council with representatives of the Union government and the Provincial governments. The decision making procedure in the GST Council is specified to be by weighted voting. This work performs a detailed study of such a mechanism using the framework of formal voting games. On a theoretical note, we introduce several new notions regarding blocking dynamics of voting games. These are then applied to the study of voting in the GST context. We identify a set of basic desiderata and propose some modifications to the voting rule in the GST Act.

1 Introduction

Presently, India accounts for about 18% of the world population[1], is the sixth largest economy in terms of nominal GDP[2] and is one the fastest growing major economy. Taxation policy is a crucial factor in determining the overall health of the economy of a nation and the well being of its citizens. Determining taxation policy is a political process. India is a multi-party democracy with a federal structure consisting of a Union government (called the Centre) and provincial governments (called the States). Achieving a stable taxation policy requires the participation of the relevant political parties governing the Centre and the States.

On 1st July 2017, India adopted a new law on taxation called the Goods and Services Tax (GST). The GST is a major overhaul of the taxation system which existed in India. The law also created a new constitutional body called the GST Council which has been empowered with making recommendations to the Centre and the States on almost all aspects of taxation policy in India. The GST Council has two members representing the Centre and one member representative for each of the States. This makes the GST Council one of the most powerful constitutional bodies in the country. Consequently, the decision making procedure in the GST Council is of major interest.

Presently, the law specifies that all decisions in the GST Council will be taken by a weighted voting procedure where the Centre will have one-third weightage of the total votes cast and the States together will have two-thirds weightage of the total votes cast. Passing of a resolution requires the support of at least three-fourth weightage of the total votes cast.

[1]http://www.un.org/en/sections/issues-depth/population/
[2]https://en.wikipedia.org/wiki/Economy_of_India

The purpose of the present work is to carry out a detailed study of the voting procedure in the GST Council. This is done in the formal framework of voting games. Our contributions include the following.

Mechanism to determine voting weights. The GST Act specifies that the States should have two-thirds weightage of the total votes cast. It does not specify how this two-third weightage is to be distributed among the States. One option is that all States which participate in the voting will have the same weights. While this is a possibility, whether the non-mention of the weight distribution procedure necessarily implies equal weights is open to interpretation. We consider the more general problem of assigning weights to the States in proportion to some socio-economic parameter. More formally, suppose the Centre gets λ fraction of the total votes cast so that all the States together get a fraction $1 - \lambda$ of the total votes cast. Further suppose that there is some a priori weight vector $\mathbf{v}$ which assigns to each of the States a value according to some socio-economic parameter. If n States participate in the voting, we show how to assign voting weights to the Centre and the States such that the Centre's fraction of the total weight is λ and each of the participating States get a weight which is proportional to its value in $\mathbf{v}$. If λ is a rational number and $\mathbf{v}$ is a vector of positive integers, then our mechanism derives voting weights which are also positive integers. We show the following.

1. The weights are minimal in the sense that their greatest common divisor is one.
2. They are unique in the sense that any other set of voting weights which assign a fraction λ of the total voting weight to the Centre and distribute the other $1-\lambda$ fraction of the total voting weight to the States in proportion to their value in $\mathbf{v}$ is necessarily a scaled version of the weights that we obtain.

Formal analysis. The voting weight assigned to the Centre by the GST Act makes it a blocker, i.e., no resolution can be passed without the Centre. As a result, no coalition of the States can pass a resolution without the Centre. A coalition of States, on the other hand, can form a blocking coalition. In a political set-up, it would be important for a coalition of States to understand the dynamics of how they can prevent the passing of a resolution. This leads us into a detailed analysis of voting games from the point of view of a player's (or, a group of players) ability to block a resolution. While the notions of blocker and blocking coalitions have already appeared in the literature, we introduce the notions of *blocking swing* and *minimal blocking coalitions*. Using these concepts, we introduce the notions of efficiency of a game and the influence of a non-blocker with respect to a designated blocker. Further, we use the notion of blocking swings to introduce a measure of blocking power of a player.

The game which arises in the GST Council is not pre-determined. This is because the actual weights of the Centre and the States depend on the number of States which actually take part in the voting. More generally, we provide a formal definition of GST Voting Games which is parameterised by the maximum number of states, their a priori weights, the number of States which participate in the voting, the fraction of the total voting weight which has to be assigned to the Centre and the winning threshold. In this formal set-up, we prove characterisations of some basic requirements including when the Centre is a blocker; when the weights of the States are less than that of the Centre and when no State is a blocker.

The GST Act specifies a meeting quorum whereby the proceedings of a meeting is considered to be valid only if more than half the members of the Council are present. We argue that the notion of meeting quorum is not sufficient to rule out the theoretical possibilities of a State having more weight than the Centre or, a State becoming a blocker. Such possibilities can be ruled out by imposing a requirement of voting quorum in addition to certain conditions on the a priori weight vector. The voting quorum would require a voting to be valid only if a certain number of States participate in the voting.

A detailed formal study is made of the GST Voting Game that arises when all the a priori voting weights are taken to be equal. We call such a game to be a simple GST game. For such games, it is possible to perform a threadbare analysis yielding results on the number of swings, the number of minimal winning coalitions, the number of blocking swings, the number of minimal blocking coalitions and also a collection of results which capture the scenario when a group of players work as a bloc in the game. A GST game is called (α, β)-efficient with respect to the Centre if the Centre requires at least α States to pass a resolution and if a coalition of at least β States are required to block a resolution. The values of α and β depend on the a priori weight vector. We prove that for simple GST voting game, $\alpha + \beta$ is one more than the number of States which participate in the voting. We prove the interesting result that the values of α and β are maximised when the a priori weights are all equal.

The problem of assigning voting weight to the Centre. Let λ be the fraction of the Centre's voting weight and q be the fraction determining the winning threshold. A resolution passes if it receives *at least* q fraction of the total votes cast. The GST Act specifies $\lambda = 1/3$ and $q = 3/4$. This effectively makes the Centre a blocker in the game since no resolution can be passed without the Centre's approval. We suggest that a possible justification for making the Centre a blocker is that it enjoys the confidence of the Lok Sabha which is the highest directly elected legislative body in the country. On the other hand, we do not find any socio-political rationale for assigning the Centre a weight which is more than that required to make it a blocker. Having identified this issue, we turn to the problem of choosing q and λ such that λ is the minimum possible value required to make the Centre a blocker. We argue that this problem is not solvable in the set-up of weighted majority voting games. A simple modification of the definition of such games, however, provides a natural and intuitive solution to the problem. The modification consists of requiring a resolution of having been passed if it receives *more than* q fraction of the total votes. In other words, the passing condition is changed from "$\geq$" to "$>$." We call such games to be modified weighted majority voting games. In the modified framework, letting $\lambda = 1 - q$ ensures that it is a blocker and this is the smallest value of λ for which the Centre is a blocker.

Based on the notion of modified weighted majority voting games, we introduce the notion of modified GST voting games. Results for the modified GST voting games which are counterparts of the GST voting games are obtained. These include detailed results on simple modified GST voting games which correspond to the a priori weights of the States being equal.

Concrete analysis. We complement the theoretical development by applying the methodological framework to analyse the concrete voting games that can arise in the GST Council. This is done under the following broad frameworks.

Setting A: The a priori weights of the States are all equal. The analysis is done under GST voting game, i.e., with "$\geq$" passing condition. As specified in the GST Act, the values of λ and q are taken to be $1/3$ and $3/4$ respectively.

Setting B: The a priori weights of the States are all equal. The analysis is done under modified GST voting game, i.e., with "$>$" passing condition. The value of q is taken to be $3/4$ and so $\lambda = 1/4$.

Setting C: The a priori weights of the States are taken to be the respective number of seats of the States in the Rajya Sabha. The analysis is done under GST voting game, i.e., with "$\geq$" passing condition. As specified in the GST Act, the values of λ and q are taken to be $1/3$ and $3/4$ respectively.

Setting D: The a priori weights of the States are taken to be the respective number of seats of the States in the Rajya Sabha. The analysis is done under modified GST voting game, i.e., with "$>$" passing condition. The value of q is taken to be $3/4$ and so $\lambda = 1/4$.

Detailed consequences to the political position as on 31 March 2018 have been worked out.

Suggestions for modifying the GST voting rules. Based on our analysis, we identify a list of some basic desiderata. Consequently, we suggest modifications to the decision making procedure specified in the GST Act. These modifications will remove some theoretical gaps in the GST Act as well as provide improved democratic decision making in the context of the federal nature of India.

Relevant Background on Voting Games

Voting is an intrinsic aspect of democratic decision making in a committee. In many cases of practical interest, the members of a committee have voting weights and a decision is made only when it receives a certain amount of support. An example of weighted voting procedure is company boardrooms. In the domain of public policy, perhaps the two most famous examples of weighted voting are the International Monetary Fund (IMF) and the European Union (EU).

Banzhaf (1965) was one of the pioneers to systematically study weighted voting. An earlier work by Penrose (1946) had identified a notion of voting power of a player. Shapley (1953); Shapley and Shubik (1954) studied voting in the framework where the order in which votes are cast is important. Coleman (1971) provided alternative methods of quantifying voting powers of the players. There has been a great deal of work on formal axiomatic treatment of voting powers. We refer the reader to Felsenthal and Machover (1998); Laruelle and Valenciano (2011) for details. A textbook level introduction to the area can be found in Chakravarty et al. (2015).

Application of the formal framework of voting games to the decision making process in the IMF has been carried out by Leech (2002b) and to the EU by Leech (2002a). Recently, an application of voting games to the study of social dynamics of Cryptocurrency communities has been done by Bhattacherjee and Sarkar (2017). The present work can also be viewed as an application of the theory of voting games to the study of the decision making process in the GST Council of India.

Organisation of the Paper

In Section 2 we provide a brief background on the federal structure of India and the taxation system arising from the GST Act. This is followed in Section 3 by some informal observations on the voting procedure suggested in the Act. A background on formal voting games is provided in Section 4. The study of voting game arising in the GST context necessitates introducing a few new notions regarding blocking in voting games. These are covered in Section 5. In Section 6, we provide a mechanism for deriving positive integral voting weights from positive integral a priori weights of the States. The formal definition of GST voting game is introduced in Section 7. Basic properties of such games are characterised and detailed results regarding simple GST voting games are stated. The modified GST voting game whereby the Centre is assigned the minimum possible voting weight to ensure it is a blocker is introduced and studied in Section 8. The theory developed in the earlier sections is applied to the present political context to perform a concrete analysis in Section 9. A set of basic desiderata is identified in Section 10 and we put forward suggestions to modify the voting rule in the GST Act. Finally, Section 11 concludes the work. The proofs are provided in the Appendix.

2 Federal Structure and Taxation System of India

In this section, we provide a brief background of the political structure of India and the taxation policy.

2.1 Federal Structure

The Republic of India[3] has a federal structure consisting of a Union Government and 29 States and 7 Union Territories. The Union Government is also called the Central Government - in short the Union or the Centre. The supreme legislative body is the Parliament. The Parliament consists of the President of India and two houses, namely, the Lok Sabha or, the House of the People and the Rajya Sabha or, the House of the States. The President is the Head of the State. The Head of the Government is the Prime Minister who is appointed by the President along with a Council of Ministers. The members of the Lok Sabha are directly elected by the people of the country based on adult universal suffrage [4]. The Council of Ministers is collectively responsible to the Lok Sabha.

The maximum strength of the Lok Sabha as envisaged by Article 81 of the Constitution is 552. There can be not more than 530 members chosen by direct election from territorial constituencies in the States, and not more than 20 to represent the Union Territories, chosen in such manner as the Parliament may by law provide. Each State is divided into territorial constituencies in such manner that the ratio between the population of each constituency and the number of seats allotted to it is, so far as practicable, the same throughout the State. As of now, there are only 13 representatives from Union Territories. Two representatives may be nominated by the President from the Anglo-Indian community if they are not adequately represented.

Article 80 of the Constitution of India lays down the maximum strength of the Rajya Sabha to be 250[5], out of which 12 members are to be nominated by the President and the other 238 members are representatives of the States and two of the Union Territories namely, Delhi and Puducherry. The allocation of the number of seats in the Rajya Sabha to the States is made on the basis of the population of each State in a manner that smaller states have marginal advantage over more populous states.

Each of the States and the two Union Territories of Delhi and Puducherry also have a State Government. The State Legislature may be unicameral in which case it has only one house - the Legislative Assembly or the Vidhan Sabha[6]. Members of the Vidhan Sabha are directly elected by the people of the State. Seven States have bicameral legislature where in addition to the Legislative Assembly, there is an upper house called the Legislative Council or the Vidhan Parishad. The remaining States and the two Union Territories of Delhi and Puducherry have unicameral legislatures. The State Government consists of a Council of Ministers which is headed by the Chief Minister. The Council of Ministers of the State Government has the confidence of the respective Vidhan Sabha.

The members of the Rajya Sabha for the seats allocated to a State or Union Territory are indirectly elected by the respective State or Territorial Legislatures through single transferable votes[7]. To be elected to the Rajya Sabha it is not necessary for a person to be a member of the concerned Vidhan Sabha.

India has a multi-party democracy. There are many political parties in the country who can contest elections to the Lok Sabha and the Vidhan Sabha. The party which enjoys the confidence of the Lok Sabha forms the Union Government. For any State and the two Union Territories, the Party which enjoys the confidence of the respective Vidhan Sabha forms the concerned State Government.

The Constitution of India[8] provides the legislative framework for the country. The powers of the Union and the State Governments are delineated in Seventh Schedule (Article 246) of the Constitution. There are issues[9] which are within the legal and administrative authority of the Union (called the Union

[3] https://www.india.gov.in/india-glance/profile
[4] https://en.wikipedia.org/wiki/Universal_suffrage
[5] http://rajyasabha.nic.in/rsnew/council_state/council_state.asp
[6] https://en.wikipedia.org/wiki/State_governments_of_India
[7] https://en.wikipedia.org/wiki/Single_transferable_vote
[8] https://www.india.gov.in/my-government/constitution-india
[9] http://lawmin.nic.in/olwing/coi/coi-english/Const.Pock%202Pg.Rom8Fsss(35).pdf

List) while there are issues which are within the legal and administrative authority of the States (called the State List). On some issues, both the Union and the States can legislate and such issues are put in the Concurrent List.

The Constitution of India can be amended to make changes to the laws of the country. The Constitution itself lays out the procedure for making such an amendment in Part XX (Article 368)[10] of the Constitution. An amendment of the Constitution can be initiated only by the introduction of a Bill in either House of Parliament. Constitution Amendment Bills can be of three types. The Bill type relevant in the context of this work has to be passed in each House by a majority of the total membership of that House and by a majority of not less than two-thirds of the members of that House present and voting. Additionally, if the amendment requires change in the proviso to Article 368 as mentioned in its Clause (2), it must be "ratified" by not less than half of the State Legislatures. Clause (2) of Article 368 particularly mentions that a bill that intends to shift items between the Union, State or Concurrent Lists of Article 246 would require ratification of more than half of the State Legislatures.

The accounts of the Union are kept in three parts:[11] the Consolidated Fund of India, the Contingency Fund of India and Public Account. The taxes collected by the Union are deposited in the Consolidated fund of India. The taxes collected by the States are deposited in the Consolidated fund of the respective State[12].

Article 110 of the Constitution defines a special kind of Bills called Money Bills and specifies that such Bills can only be introduced in the Lok Sabha. The Rajya Sabha can not stall a Money Bill[13]. It gets only fourteen days to suggest amendments which may or may not be considered by the Lok Sabha.

2.2 Taxation System in India

Tax collection is done by both the Union and the State Governments. The authority to levy taxes is derived from the Constitution. Article 265 of the Constitution states that "No tax shall be levied or collected except by the authority of law". So, any tax has to be backed by an appropriate law which has been passed either in the Parliament or in a respective Vidhan Sabha.

The Goods and Services Tax (GST) is applicable throughout India as a comprehensive indirect tax levied on manufacture, sale and consumption of most goods and services. The tax came into effect from 1st July, 2017 as an implementation of the Constitution (One Hundred and First Amendment) Act, 2016[14]. We shall refer to the Constitution (One Hundred and First Amendment) Act, 2016 as the GST Act and its preceding Bill as the GST Bill.. The GST Bill was passed in accordance with the provisions of Article 368 of the Constitution, with the ratification of more than half of the State Legislatures (24 States and two Union Territories - Delhi and Puducherry), as required under Clause (2) of Article 368[15]. Five states did not ratify.

Before the introduction of GST, the Constitution had delineated separate powers for the Union and the States to impose various taxes. The Union levied excise duty on all goods produced or manufactured in India. The Union also exclusively taxed all services rendered within the country along with applicable cesses, if any. For goods imported into India, the Union levied basic customs duty and additional duties of customs together with applicable cesses, if any. On completion of manufacture as the goods enter the stream of trade, the States levied Value Added Tax (VAT). Additionally, there were State-specific levies

[10] http://lawmin.nic.in/olwing/coi/coi-english/Const.Pock%202Pg.Rom8Fsss(26).pdf

[11] http://ccaind.nic.in/govt_accounts.asp

[12] http://www.cag.gov.in/sites/default/files/cfra_account_files/CFRA_2015_16_overview.pdf

[13] http://rajyasabha.nic.in/rsnew/legislation/introduction.asp

[14] http://lawmin.nic.in/ld/The%20Constitution%20(One%20Hundred%20and%20First%20Amendment)%20Act,%202016.pdf

[15] https://en.wikipedia.org/wiki/One_Hundred_and_First_Amendment_of_the_Constitution_of_India#cite_note-16

like entry tax, luxury tax, entertainment tax, lottery and betting tax, local taxes levied by Panchayats etc.

The apportioning of the net proceeds of taxes between the Union and the States was done through recommendations of various bodies - primarily the Finance Commission and often superseded by bodies like the Planning Commission and the NITI Aayog (National Institute of Transforming India). Article 280 of the Constitution defines the scope of the Finance Commission and Article 270[16] of the Constitution lists provisions of the Constitution bearing on the work of the Finance Commission. Articles 268, 269, 275, 282 and 293 are some of the others that specify ways and means of sharing resources between the Union and the States.

The objective behind the introduction of GST was to replace multiple cascading taxes levied by the Union and the State governments. The GST has subsumed most of the indirect taxes of the Centre and the States including excise duty, service tax, VAT, entertainment tax, luxury tax and octroi. The GST has been projected to bring about a landmark change in the way business is done in the country.

2.3 The GST Act

The following points from the GST Act are noteworthy in the context of this work.

- Article 246A empowers the Union and every State to make laws with respect to goods and services tax imposed respectively by the Union or by the State itself. The Union holds exclusive power in devising laws with respect to inter-State GST and any item not covered under the State List and the Concurrent List provided by Article 246.

- Article 249(1) provides the Union the power to legislate on matters in the State List of Article 246 provided the bill is passed in the Rajya Sabha (the representative of the States in the Union) with two-third majority. Article 246A and hence the GST laws of the States have been included in the list of matters of the States that the Union can legislate in this manner. In effect, for the GST laws within a State, the Union and the State have "concurrent power" in devising laws. Also, by Article 250(1), during an Emergency the Union has power to legislate on any matter in the State List which now includes the GST laws of the States as well.

- Article 269A empowers the Union to levy and collect GST on all imports into the country and on all inter-State trade or commerce. It also states that the tax thus collected will be apportioned between the Union and the concerned State(s) in a manner determined by laws set by the Union on the recommendation of the GST Council (described below). The apportioning of the revenue collected from non-inter-State GSTs will be done according to Clause (2) of Article 270.

2.4 The GST Council

Article 279A requires the President of India to constitute the Goods and Service Tax (GST) Council. The GST Council will make recommendations to the Union and the States on various matters related to the unified tax system. The GST Council consists of the following members:

- the Union Finance Minister as the Chairperson,

- the Union Minister of State in charge of Revenue or Finance as a member, and

[16] http://fincomindia.nic.in/writereaddata/html_en_files/oldcommission_html/fcreport/Report_of_The_Finance_Commission_1952/provisions%20of%20the%20constitution%20bearing%20on%20the%20work%20of%20the%20fi-93D.pdf

- the Minister in charge of Finance or Taxation or any other Minister nominated by each State Government.

The official GST website[17] provides various information regarding the composition and functioning of the GST Council. For the purpose of the GST Council, apart from the 29 States, the Union Territories of Delhi and Puducherry having independent Vidhan Sabhas are considered to be States. So, the GST Council consists of 2 members representing the Centre and 31 members representing the 31 States[18].

Voting Procedure. The GST Council specifies that every decision of the GST Council will be taken at meetings through weighted voting by its members. The voting mechanism of the GST Council as stated in Article 279A clause (9) is:

> "Every decision of the Goods and Services Tax Council shall be taken at a meeting, by a majority of not less than three-fourths of the weighted votes of the members present and voting, in accordance with the following principles, namely:
>
> (a) the vote of the Central Government shall have a weightage of one-third of the total votes cast, and
>
> (b) the votes of all the State Governments taken together shall have a weightage of two-thirds of the total votes cast, in that meeting."

Note that there are two members representing the Centre and one from each State in the GST Council. The Centre however gets to cast one vote that carries the entire weight assigned to the Centre by the voting mechanism.

Meeting Quorum. All members of the GST Council may not be present in a meeting. As stated in Article 279A clause (7), the "quorum" of a GST Council meeting (minimum number of members required to be present at the meeting for its proceedings to be valid) is at least half of the total number of members of the Council. It is not explicitly mentioned that one or both members of the Centre have to be present in a meeting for the proceedings to be valid.

Role of the GST Council. Article 279A specifies the roles of the GST Council as follows.

> "The Goods and Services Tax Council shall make recommendations to the Union and the States on
>
> 1. the taxes, cesses and surcharges levied by the Union, the States and the local bodies which may be subsumed in the goods and services tax;
> 2. the goods and services that may be subjected to, or exempted from the goods and services tax;
> 3. model Goods and Services Tax Laws, principles of levy, apportionment of Goods and Services Tax levied on supplies in the course of inter-State trade or commerce under article 269A and the principles that govern the place of supply;
> 4. the threshold limit of turnover below which goods and services may be exempted from goods and services tax;
> 5. the rates including floor rates with bands of goods and services tax;

[17] `http://www.gstcouncil.gov.in/gst-council`
[18] http://www.gstcouncil.gov.in/gst-council-members

6. any special rate or rates for a specified period, to raise additional resources during any natural calamity or disaster;
7. special provision with respect to the States of Arunachal Pradesh, Assam, Jammu and Kashmir, Manipur, Meghalaya, Mizoram, Nagaland, Sikkim, Tripura, Himachal Pradesh and Uttarakhand; and
8. any other matter relating to the goods and services tax, as the Council may decide."

As can be seen from the composition of the GST Council, it is a joint forum for the Centre and the States that makes recommendations to the Centre and the States on the aforementioned wide variety of matters of immense financial importance. This underlines the fact that the GST Council is one of India's most powerful federal bodies. Consequently, it is of interest to perform a detailed study of the decision making procedure in the GST Council.

3 Some Observations on Decision Making in the GST Council

Before getting into a detailed formal study of the decision making procedure in the GST Council, we make a few observations.

1. The decision procedure is in essence a weighted voting procedure. The weights of the players, however, are not fixed a priori. The rules specify that the weight of the Centre shall be 1/3 of the total weight of the votes cast. So, among the members who cast their votes, the Centre's vote is assigned a weight of 1/3 while the rest 2/3 is distributed according to some proportion among the other members. An effect of this voting policy is that in no meeting does the Centre have *more than* 1/3 share of the weight. Suppose that instead of the above, the Centre was a priori given a weight of 1/3 and the other 2/3 was also distributed a priori among the States in some proportion. Then, if in any meeting some State is absent or does not vote, then in that meeting the Centre's weight relative to the sum total of the other weights would become more than 1/3.

2. The Centre has 1/3 of the weight in any meeting while the winning threshold is 3/4. So, it is not possible to pass any resolution without the Centre. In the terminology of voting games, the Centre is a *blocker*. The GST Act does not specify any reason for making the Centre a blocker. One can perhaps deduce that this has been done keeping in mind the fact that the Centre enjoys the confidence of the Lok Sabha which is the highest legislative body in the country whose representatives are directly elected by the people.

3. Considering the winning threshold to be 3/4, giving the Centre any weight greater than 1/4 would ensure that the Centre is a blocker. So, the weight of 1/3 to the Centre is not completely explained by only the desire to make the Centre a blocker. Giving the Centre a substantially higher proportion of the weights has the effect that the States get lower weights. This results in a possible increase in the size of any *minimal blocking coalition* that the States may form without the Centre.

4. Suppose a State is present in a meeting, but, chooses to abstain from voting. This affects the voting power of the Centre and the States which do participate in the meeting. The Centre's weight still remains 1/3 of the total votes cast. Since the number of players changes, the Centre obtains proportionately more power.

5. The specified quorum requirement is for a meeting. No quorum requirement is specified for the actual voting, i.e., it is not mentioned that a certain minimum number of States must partake in

the voting for the results of the voting to be valid. The actual weights of the States in the voting is computed based on the number of States which do participate in the voting. So, for example, if all 31 States attend a meeting but, only 5 States participate in a vote, then the result of the voting will be considered to be valid.

6. The quorum requirement for the proceedings of a meeting to be considered valid is more than half of the members must be present. Presently, there are 33 members, of which 2 represent the Centre and 31 represent the 31 States. The quorum consists of 17 members. Assuming that both representatives of the Centre attend a meeting, then the representatives of at least 15 States must also attend the meeting to achieve the meeting quorum. So, if representatives of 17 States are not present in a meeting, then the meeting quorum cannot be achieved and the proceedings of such a meeting are not valid. This implicitly gives the States a certain amount of blocking power. *A coalition of 17 States can block a resolution by not attending the meeting.* On the other hand, if they do attend but, choose not to vote, they lose this blocking power, since, in this case the meeting would be considered to be valid and the weights would be computed based on the States who do cast their votes.

7. The GST Act does not specify any explicit procedure for computing the weights of the States who participate in the voting. Whether this indicates that all the States which participate in the voting get equal weights is open to interpretation. Previously, there had been a proposal[19] to have the voting weights of the States to be proportional to the number of seats in the Rajya Sabha for the States.

4 Background on Voting Games

The following notation will be used.

- The cardinality of a finite set S will be denoted by $\#S$.
- The absolute value of a real number x will be denoted by $|x|$.
- For a real number x, $\lfloor x \rfloor$ will denote the greatest integer not greater than x.
- For a real number x, $\lceil x \rceil$ will denote the least integer not lesser than x.
- The greatest common divisors of the integers $a_1, \ldots, a_n$ will be denoted by $\gcd(a_1, \ldots, a_n)$.

We provide some standard definitions arising in the context of voting games. For details the reader may consult (Felsenthal and Machover, 1998; Chakravarty et al., 2015).

Let $N = \{A_1, A_2, \ldots, A_n\}$ be a set of n players. A subset of N is called a *voting coalition*. The set of all voting coalitions is denoted by 2^N. A voting game G is given by its characteristic function $\widehat{G} : 2^N \to \{0, 1\}$ where a winning coalition is assigned the value 1 and a losing coalition is assigned the value 0. For a voting game G, the set of all winning coalitions will be denoted by $W(G)$ and the set of all losing coalitions will be denoted by $L(G)$. Below we recall some basic notions about voting games.

1. For any $S \subseteq N$ and player $A_i \in N$, A_i is said to be a *swing* in S if $A_i \in S$, $\widehat{G}(S) = 1$ but $\widehat{G}(S \setminus \{A_i\}) = 0$.

2. For a voting game G, the number of coalitions in which A_i is a swing will be denoted by $m_G(A_i)$.

[19] `http://www.thehindu.com/news/resources/full-text-of-the-memorandum-tamil-nadu-chief-minister-jayalalithaa-presented-to-prime-minister-narendra-modi-in-new-delhi/article11624165.ece1`

3. A player $A_i \in N$ is called a *dummy player* if A_i is not a swing in any coalition, i.e., if $m_G(A_i) = 0$.

4. A coalition $S \subseteq N$ is called a *minimal winning coalition* if $\widehat{G}(S) = 1$ and there is no $T \subset S$ for which $\widehat{G}(T) = 1$.

5. A coalition $S \subseteq N$ is called a *blocking coalition* if $\widehat{G}(N \setminus S) = 0$. If A_i is a player such that $\{A_i\}$ is a blocking coalition, then A_i is said to be a *blocker*.

6. A player A_i is said to be the dictator in G, if $\{A_i\}$ is the only minimal winning coalition in G.

7. A voting game G is said to be *proper* if for any coalition $S \subseteq N$, $\widehat{G}(S) = 1$ implies that $\widehat{G}(N \backslash S) = 0$. In other words, in a proper game it is not allowed for both S and its complement to be winning.

Definition 1 (Weighted Majority Voting Game) *Consider a triplet $(N, \mathbf{w}, q)$, where $N = \{A_1, \ldots, A_n\}$ is a set of players; $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is a vector of non-negative weights with w_i being the weight of A_i; and q is a real number in $(0, 1)$. Let $\omega = \sum_{i=1}^{n} w_i$. The triplet $(N, \mathbf{w}, q)$ defines a weighted majority voting game G given by its characteristic function $\widehat{G} : 2^N \to \{0, 1\}$ in the following manner. Let $w_S = \sum_{A_i \in S} w_i$ denote the sum of the weights of all the players in the coalition $S \subseteq N$. Then*

$$\widehat{G}(S) = \begin{cases} 1 & \text{if } w_S/\omega \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

We will write $G = (N, \mathbf{w}, q)$ to denote the weighted majority voting game arising from the triplet $(N, \mathbf{w}, q)$.

The quantity q is called the *winning threshold*. A coalition $S \subseteq N$ wins the weighted majority voting game if its weight w_S is *greater than or equal to q* fraction of the total weight ω of all the players.

Bloc. It is possible that a group of players decide to act together, i.e., all of the players in the group either vote for or vote against a motion. For example, in the GST game, the players which are from the same political party are expected to act in such a manner. Such a group is called a *bloc*. Let $G = (N, \mathbf{w}, q)$ be a weighted majority game and suppose that there are pairwise disjoint subsets $S_1, S_2, \ldots, S_t$ which form blocs. Then the game G is modified to a game $G_{S_1,\ldots,S_t} = (N', \mathbf{w}', q)$ where

1. $N' = (N \setminus (S_1 \cup \cdots \cup S_t)) \cup \{A_{S_1}, \ldots, A_{S_t}\}$. Here $A_{S_1}, \ldots, A_{S_t}$ are new players, i.e., these players are not in N;

2. the weight of A_{S_i} is w_{S_i}, $i = 1, \ldots, t$;

3. $\mathbf{w}'$ is formed from $\mathbf{w}$ by dropping the components corresponding to the players in $S_1 \cup \cdots \cup S_t$ and including the weights $w_{S_1}, \ldots, w_{S_t}$ corresponding to the new players $A_{S_1}, \ldots, A_{S_t}$.

As pointed out above, in the GST voting game it is very likely that all players from the same party will be in one bloc. In such a situation, studying the original game may not be very useful. The effect of bloc formation can be formally studied through the game $G_{S_1,\ldots,S_t}$ instead of the original game G. Whenever we talk about some property of a game G with blocs $S_1, \ldots, S_t$, we will formally mean the corresponding property of the game $G_{S_1,\ldots,S_t}$.

5 Blocking Coalitions

We identify two notions related to blocking coalitions that are not usually studied in the literature. Let G be a voting game on a set $N = \{A_1, \ldots, A_n\}$ of players.

1. A blocking coalition S is said to be a *minimal blocking coalition* if for any $A_i \in S$, $S \setminus \{A_i\}$ is not a blocking coalition. In other words, if any player is dropped from S, then S no longer remains a blocking coalition.

2. Let S be a blocking coalition and $A_i \in S$. Then A_i is said to be a *blocking swing* in S, if $S \setminus \{A_i\}$ is not a blocking coalition. The number of coalitions where A_i is a blocking swing will be denoted by $\eta_G(A_i)$.

The above notions are not restricted to weighted majority voting games and apply to all voting games. Clearly $\eta_G(A_i) \geq 0$. Also, since any coalition S in which A_i is a blocking swing contains A_i, it can contain at most $n-1$ other players. So, the maximum possible value of $\eta_G(A_i)$ is 2^{n-1}. These bounds are formally stated as follows.

Theorem 1 *Let G be a voting game on a set $N = \{A_1, \ldots, A_n\}$ of players. For $i = 1, \ldots, n$, $0 \leq \eta_G(A_i) \leq 2^{n-1}$.*

The upper bound is attained if A_i is a blocker and for all $S \subseteq N \setminus \{A_i\}$, S is not a blocking coalition.

If $\eta_G(A_i) = 0$, we will call A_i a *blocking dummy*. Such a player has no role in blocking a resolution from being passed. If $\eta_G(A_i) = 2^{n-1}$, we will call A_i a *blocking dictator*. If a player A_i is a dictator, then $\{A_i\}$ is both the only minimal winning coalition and the only minimal blocking coalition in the game. Consequently, a dictator is necessarily a blocking dictator though the converse need not be true.

The following result is easy to see.

Proposition 1 *Let G be a voting game.*

1. *Each blocker in G forms a singleton minimal blocking coalition.*

2. *A minimal blocking coalition has cardinality greater than one if and only if all the players in it are non-blockers.*

We next define a modified version of a weighted majority voting game.

Definition 2 (Modified Weighted Majority Voting Game) *Consider a triplet $(N, \mathbf{w}, q)$, where $N = \{A_1, \ldots, A_n\}$ is a set of players; $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is a vector of non-negative weights with w_i being the weight of A_i; and q is a real number in $(0, 1)$. Let $\omega = \sum_{i=1}^{n} w_i$. The triplet $(N, \mathbf{w}, q)$ defines a weighted majority voting game G given by its characteristic function $\widehat{G} : 2^N \to \{0, 1\}$ in the following manner. Let $w_S = \sum_{A_i \in S} w_i$ denote the sum of the weights of all the players in the coalition $S \subseteq N$. Then*

$$\widehat{G}(S) = \begin{cases} 1 & \text{if } w_S/\omega > q, \\ 0 & \text{otherwise.} \end{cases}$$

We will write $G = (N, \mathbf{w}, q)$ to denote the modified weighted majority voting game arising from the triplet $(N, \mathbf{w}, q)$.

A modified weighted majority voting game is a weighted majority voting game where only the winning condition is modified and everything else remains unchanged. The modification consists of the following. In a weighted majority voting game a coalition S is winning if "$w_S/\omega \geq q$"; whereas in a modified weighted majority voting game a coalition S is winning if "$w_S/\omega > q$". So, a more precise terminology would be "weighted majority voting game with modified winning condition". This, however, is too long to be conveniently used and so we have compressed this to *modified weighted majority voting game*.

A minimal blocking coalition in a weighted majority voting game turns out to be a minimal winning coalition in an appropriate modified weighted majority voting game and vice versa. This is formally stated in the following result.

Theorem 2 *Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game and let $S \subseteq N$. Then S is a minimal blocking coalition in G if and only if S is a minimal winning coalition in the modified weighted majority voting game $G' = (N, \mathbf{w}, 1 - q)$.*

Conversely, let $G = (N, \mathbf{w}, q)$ be a modified weighted majority voting game and let $S \subseteq N$. Then S is a minimal blocking coalition in G if and only if S is a minimal winning coalition in the weighted majority voting game $G' = (N, \mathbf{w}, 1 - q)$.

From the proof of Theorem 2 (provided in the Appendix) it can be seen that a coalition S is winning in G' if and only if it is a blocking coalition in G and vice versa. In particular, it is not necessary for the coalition to be "minimal" in winning and blocking in G and G' respectively. The following result can be proved in a manner similar to that of Theorem 2.

Theorem 3 *Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game and let $A_i \in N$. The following are equivalent.*

1. *In G, for any $S \subset N$, A_i is a blocking swing in S.*

2. *In the modified weighted majority voting game $G' = (N, \mathbf{w}, 1 - q)$, A_i is a swing in S.*

Conversely, let $G = (N, \mathbf{w}, q)$ be a modified weighted majority voting game and let $A_i \in N$. The following are equivalent.

1. *In G, for any $S \subset N$, A_i is a blocking swing in S.*

2. *In the weighted majority voting game $G' = (N, \mathbf{w}, 1 - q)$, A_i is a swing in S.*

Based on Theorems 2 and 3, we say that the weighted majority voting game $G = (N, \mathbf{w}, q)$ and the modified weighted majority voting game $G' = (N, \mathbf{w}, 1 - q)$ are *dual games*.

One consequence of Theorem 2 is that finding minimal blocking coalitions in G is equivalent to finding minimal winning coalitions in G'. As a consequence of Theorem 3, finding the number of blocking swings for a player A_i in G can be done by finding the number of swings for the player A_i in G'. As a consequence of these two results, known algorithms (see for example Matsui and Matsui (2000); Chakravarty et al. (2015)) for finding the number of minimal winning coalitions and number of swings can be used to find the number of minimal blocking coalitions and the number of blocking swings.

The next result states that for weighted majority voting games, the number of coalitions where a blocker is a blocking swing is greater than the number of coalitions where a non-blocker is a blocking swing and that the same is true for a modified weighted majority voting game.

Theorem 4 *Let $G = (N, \mathbf{w}, q)$ be a weighted majority game and $A_i, A_j \in N$ be such that A_i is a blocker and A_j is not a blocker. Then $\eta_G(A_i) > \eta_G(A_j)$.*

The same holds for modified weighted majority game.

We introduce a notion of efficiency for voting games.

Definition 3 *Let G be a voting game. Then G is said to be (α, β)-efficient if any minimal winning coalition in G contains at least α players* and *any minimal blocking coalition in G contains at least β players.*

Note that the definition of (α, β)-efficient game does not require G to be a weighted majority voting game.

An (α, β)-efficient game guarantees that passing a resolution will require the support of at least α players while to block a resolution at least β of the players have to come together. If α and β are comparatively high, then it is assured that a small coalition of players ($< \alpha$ players) cannot pass a resolution and at the same time a small coalition of players ($< \beta$ players) cannot play an obstructionist role in the decision making procedure.

It may turn out that a game has one or more blockers. Let us fix a blocker. The blocker can block the passing of any resolution, but, need not be able to ensure by itself that a resolution passes. So, for passing a resolution, a blocker would require support of other players. On the other hand, it is also of interest to know which other coalitions not containing the blocker can prevent a resolution from being passed. These considerations motivate us to define the following notion.

Definition 4 *Suppose G is a voting game on a set of players $N = \{A_0, A_1, \ldots, A_n\}$ and let A_0 be a blocker. The game G is said to be (α, β)*-efficient with respect to A_0 *if the following two conditions hold.*

- *Any minimal winning coalition containing A_0 contains at least α players other than A_0.*
- *Any minimal blocking coalition not containing A_0 contains at least β players.*

In the GST game arising directly out of the GST Act, the Centre is a blocker. Such a game is (α, β)-efficient with respect to the Centre, if the Centre requires the support of at least α States to pass a resolution while at least β States must come together to form a blocking coalition not containing the Centre. More generally, the political grouping ruling the Centre may also be ruling a few of the States. These States and the Centre will operate as a bloc and this bloc is to be considered as a blocker.

Theorem 5 *Suppose G is a voting game on a set of players $N = \{A_0, A_1, \ldots, A_n\}$ where A_0 is a blocker. Further suppose G is (α, β)-efficient with respect to A_0. The following are equivalent.*

1. $\alpha = 0$.
2. *A_0 is a dictator.*
3. *β is undefined.*

Suppose $\alpha = 0$. Then the followings hold.

1. *A_0 is a blocking dictator and so $\eta_G(A_0) = 2^n$.*
2. *For $i = 1, \ldots, n$, A_i is a blocking dummy and so $\eta_G(A_i) = 0$.*

Definition 5 *Suppose G is a voting game on a set of players $N = \{A_0, A_1, \ldots, A_n\}$ where A_0 is a blocker. In the game G, a coalition $\emptyset \neq S \subseteq \{A_1, \ldots, A_n\}$ has $(\gamma_S, \kappa_S, \mu_S, \nu_S)$-influence with respect to A_0 if the following conditions hold.*

- *Any minimal winning coalition containing S contains at least γ_S players other than A_0.*
- *Any minimal winning coalition not containing S contains at least κ_S players other than A_0.*

- *Any minimal blocking coalition containing S contains at least μ_S players.*
- *Any minimal blocking coalition not containing either A_0 or S contains at least ν_S players.*

Since A_0 is a blocker, any winning coalition must contain A_0. So, the influence of a coalition S, in passing a resolution is to some extent captured by considering whether a winning coalition contains S or not. If S has a large influence, then including S will mean a smaller number of other players will be required, while excluding S will result in a larger number of players, i.e., we would expect κ_S to be greater than γ_S. On the other hand, if we wish to consider the influence of S on blocking a resolution, then we also have to consider the two situations of minimal blocking coalitions with and without containing S. With S, a set of size at least μ_S is sufficient to block a resolution while without S (and A_0), a set of size at least ν_S will be required. If S has a significant blocking influence then we would expect ν_S to be greater than μ_S. The coalition S can be a singleton set $S = \{A_i\}$, in which case we talk about the influence of the player A_i. Notationally, the influence of A_i will be written as $(\gamma_i, \kappa_i, \mu_i, \nu_i)$.

Theorem 6 *Let G be a voting game on a set of players $N = \{A_0, A_1, \ldots, A_n\}$ where A_0 is a blocker. Suppose G is (α, β)-efficient with respect to A_0 and $\alpha = 0$. Let $\emptyset \neq S \subseteq N \setminus \{A_0\}$ and suppose that in G, S has $(\gamma_S, \kappa_S, \mu_S, \nu_S)$-influence with respect to A_0. Then the following holds.*

1. *γ_S is undefined.*
2. *$\kappa_S = 0$.*
3. *μ_S and ν_S are undefined.*

The following result relates influence of a coalition to the efficiency of the game with respect to a blocker. It follows directly from the definitions of efficiency and influence.

Theorem 7 *Let G be a voting game on a set of players $N = \{A_0, A_1, \ldots, A_n\}$ where A_0 is a blocker and G is (α, β)-efficient with respect to A_0. Let $\emptyset \neq S \subseteq N \setminus \{A_0\}$ and suppose that in G, S has $(\gamma_S, \kappa_S, \mu_S, \nu_S)$-influence with respect to A_0. Assume that A_0 is not a dictator. Then $\gamma_S, \kappa_S \geq \alpha$ and $\mu_S, \nu_S \geq \beta$.*

Theorem 8 *Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game on a set of players $N = \{A_0, A_1, \ldots, A_n\}$ where $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ and $w_0 \geq w_1 \geq \cdots \geq w_n$. Suppose that A_0 is a blocker and G is (α, β)-efficient with respect to A_0.*

1. *A_0 is a dictator if and only if $\alpha = 0$.*
2. *β is undefined if and only if $\alpha = 0$; $\beta > 0$ if and only if $\alpha > 0$.*

Both the above statements hold if G is a modified weighted majority voting game.

5.1 Blocking Power

The power of an individual player in a voting game has been studied extensively in the literature. Starting from the seminal works of Banzhaf (1965); Penrose (1946) several indices and measures have been proposed for capturing the notion of power in a voting game. An extensive account of the relevant literature can be obtained in Felsenthal and Machover (1998); Laruelle and Valenciano (2011); Chakravarty et al. (2015). The underlying theme that all of the known power indices and measures try to capture is the influence that an individual player has on the passing of a resolution. The rationale being that the reward a player can expect will be proportional to his/her ability in passing a resolution. For example, the various versions of the Banzhaf and the Coleman indices/measures are all scaled versions of the number of swings of a player in the game.

In the context of multi-party democracy, there is another notion of voting power which is not captured by any of the indices/measures available in the literature. We explain this issue in the context of voting in the GST Council though the central idea is applicable to other policy making contexts consisting of representatives of both the ruling party and the various opposition parties. In the GST Council, the Centre and the States politically aligned with the Centre will typically vote as a bloc. Since the Centre is a blocker, no resolution can be passed without the Centre's consent. This will mean that any resolution which is introduced for discussion will be done so by the Centre, since any resolution introduced by an opposition party and not agreed to by the Centre has no chance of passing.

If the bloc controlled by the Centre has sufficient majority to pass a resolution, then such a bloc is dictatorial and none of the States outside the Centre's bloc has any power. In the constantly changing political dynamics, such a situation even if it occurs cannot be assumed to be a permanent state of affairs. Typically, the Centre's bloc will not have sufficient majority to pass a resolution on its own. It will require support from States outside the bloc. In such a situation, the States outside the bloc have two options.

1. Align with the Centre to pass the resolution.

2. Form a coalition so that the resolution is blocked.

There could be benefits to a State to align with the Centre. For example, it may strike a hard bargain with the Centre to obtain an advantageous economic package for the State. On the other hand, there could also be tangible gains for a State to block a resolution. Such gains could be economic which could occur for example if the tax rate is proposed to be changed on a goods for which the particular State is a large producer. Gains for blocking a resolution could also be political which are perhaps no less important than economic gains.

The power of a State in aligning with the Centre's bloc to pass a resolution can be captured by the various indices/measures available in the literature. As mentioned above, the swing based Banzhaf and the Coleman indices/measures may be used. On the other hand, the question of capturing the power of a State in blocking a resolution is not captured by any of the indices/measures available in the literature.

Moving from the specific context of voting in the GST Council to the setting of general voting games, we pose the question of how to measure the power of an individual player in blocking a resolution. As an answer to this question, we suggest using the notion of blocking swings to measure the blocking power of a player. Recall that for a player A_i in a game G with player set N, the number of blocking swings for A_i is denoted by $\eta_G(A_i)$. The quantity $\eta_G(A_i)$ can be used as a raw measure of blocking power. From Theorem 4 we have that for weighted majority games, the power of a blocker is greater than the power of any non-blocker. For two blockers A_i and A_j, the values of $\eta_G(A_i)$ and $\eta_G(A_j)$ need not be equal, so the measure $\eta_G(\cdot)$ provides a way to distinguish between the powers of two blockers. A scaled version of the raw measure of blocking power can be defined which assigns to A_i the value

$$\mathfrak{M}_G(A_i) = \frac{\eta_G(A_i)}{2^{n-1}}. \tag{1}$$

From Theorem 1, the minimum and maximum values of the above measure are 0 and 1 respectively. The value 0 corresponds to A_i being a blocking dummy and the value 1 corresponds to A_i being a blocking dictator.

Based upon the different variants of the Banzhaf and Coleman measures, it is also possible to consider other variants of measuring blocking power using blocking swings. The formal axiomatic derivation of such indices and measures for blocking power remains to be done and could be an interesting research direction.

We note that the situation described above becomes visible only in a multi-party democratic set-up. If there are only two parties, then one party will be in government while the other party will be in opposition. There is no third political party which depending on the situation can ally with either the government or the major opposition party. So, for example, the notion of blocking swings will not be much relevant in policy decision making processes in the UK or the USA. In some ways, India with its many political parties provide a fertile ground for studying complex decision making processes.

6 Mechanism for Determining Voting Weights

The GST voting rules specify that the Centre has 1/3 weight of the total votes cast while the States which participate in the voting have the remaining 2/3 weight of the total votes cast. This formulation leaves open the following issues.

1. The actual weights of the Centre and the States are not fixed. These weights depend upon the number of States which participate in the voting.

2. The method of apportioning the 2/3 weight among the States which participate in the voting is not specified. A simple method would be to assign equal weights to all the States. Alternatively, one may consider distributing the 2/3 weight among the States in proportion to some socio-economic parameter such as population or gross domestic product. The values of such a parameter for the States would constitute a priori weights of the States.

In more general terms, one can consider the following set-up. Suppose the Centre and n other States participate in the voting with the Centre having a λ fraction of the total weight. Further suppose that the States have some a priori weights $u_1, \ldots, u_n$ and the requirement is to distribute the other $1 - \lambda$ fraction of the total weight in proportion to the a priori weights $u_1, \ldots, u_n$. A mechanism is required to come up with weights $w_0, w_1, \ldots, w_n$ such that w_0 is the weight of the Centre and $w_1, \ldots, w_n$ are the weights of the States satisfying the required conditions. The following result provides such a mechanism.

Theorem 9 *Let $u_1, \ldots, u_n$ be positive integers and $\lambda = a/b$ where a and b are positive integers. Let*

$$w_0 = \frac{ua}{d}, \qquad w_i = \frac{u_i(b-a)}{d} \quad i = 1, \ldots, n; \tag{2}$$

where $u = u_1 + \cdots + u_n$ and $d = \gcd(ua, u_1(b-a), \ldots, u_n(b-a))$. Then $w_0, w_1, \ldots, w_n$ satisfy

$$\frac{w_0}{\omega} = \lambda, \qquad \frac{w_i}{\omega} = (1-\lambda)\frac{u_i}{u}. \tag{3}$$

where $\omega = w_0 + w_1 + \cdots + w_n$.

Further, if $w'_0, w'_1, \ldots, w'_n$ are integer values such that for $\omega' = w'_0 + w'_1 + \cdots + w'_n$,

$$w'_0/\omega' = \lambda \text{ and } w'_i/\omega' = (1-\lambda)u_i/u \text{ for } i = 1, \ldots, n,$$

then $w'_j = \mu w_j$ for $j = 0, \ldots, n$ for some non-zero integer μ.

Equation (2) provides the weights that are to be assigned to the Centre and the States. Equation (3) shows the correctness of these weights, i.e., the Centre obtains a fraction λ of the total weight and the other $1-\lambda$ fraction is distributed among the States in proportion to their a priori weights. The second part of Theorem 9 shows that the weights obtained in (2) are unique up to scaling by a non-zero integer.

Let $G = (N, \mathbf{w}, q)$ with $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ and $G' = (N, \mathbf{w}', q)$ with $\mathbf{w}' = (w'_0, w'_1, \ldots, w'_n)$ and $w'_i = \mu w_i$ for some non-zero real number μ. Then G and G' have the same characteristic function and so, in particular, all of their voting game properties are exactly the same. So, from Theorem 9, it follows that the game arising out of the weights determined by (2) is essentially unique.

Corollary 1 *Let $u_1, \ldots, u_n$ and $u'_1, \ldots, u'_n$ be positive integers such that $u'_i = \nu u_i$ for some non-zero integer ν. Let $w_0, w_1, \ldots, w_n$ be obtained from $u_1, \ldots, u_n$ as in (2) and $w'_0, w'_1, \ldots, w'_n$ be obtained from $u'_1, \ldots, u'_n$ as in (2). Then $w_i = w'_i$ for $i = 0, 1, \ldots, n$.*

The interpretation of Corollary 1 is that scaling of the a priori weights of the States does not change the eventual weights assigned to the Centre and the States.

7 Voting Game in the GST Council

The voting game arising from the GST Act is formally captured by the following definition.

Definition 6 (GST Voting Game) *Let A_0 denote the Centre and $\{A_1, \ldots, A_m\}$ denote the m States. For $i = 1, \ldots, m$, suppose v_i is the* a priori *weight of State A_i and let $\mathbf{v} = (v_1, \ldots, v_m)$. Let $1 \leq n \leq m$ and suppose the set of States $V = \{A_{j_1}, \ldots, A_{j_n}\}$ participate in the voting. Let $u_1 = v_{j_1}, \ldots, u_n = v_{j_n}$ and $\mathbf{u} = (u_1, \ldots, u_n)$. Let $q \in (1/2, 1)$ and $\lambda = a/b \in (0, q)$ for some integers a and b. The GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ is defined to be a weighted majority voting game on $n+1$ players as follows.*

$$G_{m,n,\mathbf{v},V,\lambda,q} = (\{A_0\} \cup V, \mathbf{w}, q) \tag{4}$$

where $\mathbf{w} = (w_0, w_1, \ldots, w_n)$; w_0 is the weight of A_0; and $w_1, \ldots, w_n$ are the weights of $A_{j_1}, \ldots, A_{j_n}$ respectively. The components of $\mathbf{w}$ are obtained from the components of $\mathbf{u}$ as given by (2).

Remarks:

1. The definition of the GST Voting Game assumes that the Centre A_0 will always participate in every GST voting game while some of the States may not participate. Since $n \geq 1$, the definition assumes that at least one State will participate in the voting.

2. The condition $q \in (1/2, 1)$ ensures that the game is proper.

3. The condition $\lambda \in (0, q)$ has two implications. The lower bound $\lambda > 0$ ensures that λ is a positive fraction while the upper bound $\lambda < q$ ensures that the Centre is not a dictator and cannot pass a resolution by itself.

The Centre enjoys the confidence of the Lok Sabha which is the supreme legislative body in the country. As a result, it is appropriate to ensure that the GST Voting Game is designed in a manner such that Centre is a blocker, i.e., no resolution can be passed without the consent of the Centre. The following result characterises the condition under which the Centre becomes a blocker.

Theorem 10 *In a GST game $G_{m,n,\mathbf{v},V,\lambda,q}$, the Centre A_0 is a blocker if and only if $q > 1 - \lambda$.*

The quorum requirement specified in the GST Act stipulates that at least $\lceil m/2 \rceil$ of the States must be present in a meeting. It does not, however, stipulate that all such States have to participate in the voting. So, according to the present GST Act, it is possible that quorum is attained, but, the actual number n of States which participate in the voting is less than $\lceil m/2 \rceil$. In view of this, we distinguish between two types of quorum.

Meeting Quorum: This is the quorum requirement as specified in the GST Act. It requires at least $\lceil m/2 \rceil$ members to be present in a meeting.

Voting Quorum: We introduce the notion of voting quorum. The requirement is that $n \geq \tau m$ for some fixed $\tau \in (0, 1)$. This ensures that at least τm States participate in the voting. The GST game will be said to have τ as the threshold of voting quorum.

It is desirable that the weight of the Centre should be more than that of any of the States. This requirement is characterised in the following result.

Theorem 11 *Consider a GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ with $V = \{A_{j_1}, \ldots, A_{j_n}\}$. The State A_{j_i} has weight less than that of the Centre A_0, i.e., $w_i < w_0$ if and only if*

$$\frac{u_i}{u_1 + \cdots + u_n} < \frac{\lambda}{1 - \lambda}.$$

Further suppose $\lambda < 1/2$; $\mathbf{v} = (v_1, \ldots, v_m)$ is such that $v_1 \geq \cdots \geq v_m$; and τ is the threshold of voting quorum for G. Let $r = \lceil \tau m \rceil$ and

$$v_1 \quad < \quad \frac{\lambda}{1 - 2\lambda} \left(v_{m-r+2} + \cdots + v_m \right). \tag{5}$$

Then in G, the weight of any State is less than that of the Centre.

We now consider the possibility of whether a State can become a dictator or a blocker. These conditions are characterised as follows.

Theorem 12 *Consider a GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ with $V = \{A_{j_1}, \ldots, A_{j_n}\}$.*

1. *The State A_{j_i} is a dictator if and only if $\dfrac{u_i}{u_1 + \cdots + u_n} \geq \dfrac{q}{1 - \lambda}$.*

2. *The State A_{j_i} is a blocker if and only if $\dfrac{u_i}{u_1 + \cdots + u_n} > \dfrac{1 - q}{1 - \lambda}$.*

Further suppose $\mathbf{v} = (v_1, \ldots, v_m)$ is such that $v_1 \geq \cdots \geq v_m$ and τ is the threshold of voting quorum for G. Let $r = \lceil \tau m \rceil$ and

$$v_1 \quad < \quad \frac{1 - q}{q - \lambda} \left(v_{m-r+2} + \cdots + v_m \right). \tag{6}$$

Then in G, no State is a blocker.

If the Centre is a blocker (i.e., the condition $q > 1 - \lambda$ holds), then none of the States can be a dictator. It is still possible though, for a State to be a blocker.

Theorem 13 *Suppose a GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ is (α, β)-efficient with respect to the Centre. Then $\alpha < \beta$ if and only if $q \leq (\lambda + 1)/2$.*

The value of α denotes the minimum number of States that the Centre has to rope in to pass a resolution, while the value of β denotes the minimum number of States who need to get together to block a resolution. If $\alpha < \beta$, then the Centre's task of passing a resolution becomes easier than that of a coalition of States from trying to block a resolution. This would make the Centre-States balance of power more skewed towards the Centre. Theorem 13 characterises this scenario and it can be prevented by choosing λ and q such that $q > (\lambda + 1)/2$. The GST Act specifies that $\lambda = 1/3$ and $q = 3/4$ which satisfies the condition of Theorem 13.

7.1 Simple GST Games

Of special interest is the case when the a priori weights of all the States are equal.

Definition 7 *A GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ is said to be simple if $v_1 = \cdots = v_m$.*

In view of Corollary 1, in a simple GST game we may take $v_1 = \cdots = v_m = 1$ and consequently from (2), $w_1 = \cdots = w_n = (b - a)/d$.

Theorem 14 *Consider a simple GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ with $V = \{A_{j_1}, \ldots, A_{j_n}\}$.*

1. *State A_{j_i} has weight less than that of the Centre if and only if $n > \frac{1-\lambda}{\lambda}$.*
2. *State A_{j_i} is a dictator if and only if $n \leq (1 - \lambda)/q$.*
3. *State A_{j_i} is a blocker if and only if $n < (1 - \lambda)/(1 - q)$.*

Considering $\lambda = 1/3$ as specified in the GST Act, we see that for $1 \leq n \leq 2$, the States have weight more than that of the Centre. So, there is indeed a theoretical possibility that a State obtains greater weight than the Centre. Note that the requirement of meeting quorum as specified in the GST Act is not sufficient to rule out the possibility of n being as low as 1 or 2. The meeting quorum only requires that more than half of the members of the Council are present in the meeting. So, it is theoretically possible, that the quorum is attained but, only one or two of the States participate in the actual voting. In practice, though, this is quite unlikely. With $\lambda = 1/3$ and $q = 3/4$ as specified in the GST Act, the Centre is a blocker and so no State can be a dictator. Further, a State is a blocker if and only if n is either 1 or 2. Again, though this is a theoretical possibility (which is not ruled out by the requirement of meeting quorum), such a situation is unlikely to arise in practice.

The rest of this section is devoted to a detailed study of simple GST games. Apart from conceptual simplicity, the importance of such games arises from the following result.

Theorem 15 *Let $\mathbf{v} = (v_1, \ldots, v_m) = (1, \ldots, 1)$ and $\mathbf{v}' = (v'_1, \ldots, v'_m)$ where not all the v'_i's are equal. Suppose a GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ is (α, β)-efficient with respect to the Centre and a GST game $G'_{m,n,\mathbf{v}',V,\lambda,q}$ is (α', β')-efficient with respect to the Centre. Then $\alpha \geq \alpha'$ and $\beta \geq \beta'$.*

In other words, Theorem 15 shows that if a GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ is (α, β)-efficient with respect to the Centre then the values of α and β are maximum when the GST game is simple.

Recall that increasing the value of α ensures that the Centre requires more States to pass a resolution. The goal of decentralisation is to ensure a greater participation in the decision making process. Theorem 15 shows that maximum decentralisation is achieved by simple GST voting games. In particular, assigning unequal weights to the States based on some socio-economic parameter will result in a system which has a lower level of decentralisation. On the other hand, increasing the value of β means that the more States need to come together to block a resolution. Theorem 15 shows that β is maximum for simple GST voting games. Consequently, for such games it is the most difficult for a coalition of States to block a resolution. So, by assigning unequal weights to the States based on some socio-economic parameter will improve the blocking capability of the States. Later we discuss both the issues of decentralisation and blocking capability with concrete data arising from the prevalent political situation.

In the case of simple GST games, the influence of a State with respect to the Centre is essentially given by the notion of efficiency of the game with respect to the Centre. This is made precise in the following result.

Theorem 16 *Consider a simple GST game* $G_{m,n,\mathbf{v},V,\lambda,q}$ *which is* (α,β)*-efficient with respect to the Centre. Then any State* $A_{j_i} \in V$ *has* $(\alpha,\alpha,\beta,\beta)$*-influence with respect to the Centre.*

Given that Theorem 15 shows that simple GST games achieve maximum decentralisation, we next consider the possible values of α and β in a simple GST game which is (α,β)-efficient with respect to the Centre. One may wish to increase the values of both α and β by appropriately setting the values of λ and q. The next result shows that this is not possible.

Theorem 17 *A simple GST game* $G_{m,n,\mathbf{v},V,\lambda,q}$ *is* $(\mathfrak{p},\mathfrak{q})$*-efficient with respect to the Centre where*

$$\mathfrak{p} = n-\rho; \quad (7)$$

$$\mathfrak{q} = 1+\rho; \quad (8)$$

and $\rho = \left\lfloor \frac{n(1-q)}{(1-\lambda)} \right\rfloor$. *Consequently,*

$$\mathfrak{p}+\mathfrak{q} = n+1. \quad (9)$$

There are several consequences of Theorem 17.

1. Equation (7) provides the minimum number $\mathfrak{p}$ of States that the Centre requires to form a winning coalition.

2. Equation (8) provides the minimum number $\mathfrak{q}$ of States that can form a blocking coalition without the Centre.

3. The quantities $\mathfrak{p}$ and $\mathfrak{q}$ satisfy the invariance condition $\mathfrak{p}+\mathfrak{q}=n+1$. As a result, $\mathfrak{p}$ increases if and only if $\mathfrak{q}$ decreases. In particular, it is not possible to increase or decrease both $\mathfrak{p}$ and $\mathfrak{q}$ by tweaking the parameters λ and q.

4. For fixed q and λ, both $\mathfrak{p}$ and $\mathfrak{q}$ decrease with decreasing n. So, as the number of States which participate in voting goes down, the Centre requires lesser number of States to win and also a lesser number of States can form a blocking coalition.

5. For fixed q and n, as λ increases, the value of ρ increases in unit steps from $n(1-q)$ to $2n(1-q)-1$.

The quantity λ denotes the Centre's fraction of the total weight. Suppose that q is chosen such that $q > 1-\lambda$ so that the Centre is a blocker. In this setting, for fixed n and q, it is of interest to determine the effect of changing λ on $\mathfrak{p}$ and $\mathfrak{q}$.

Combining Theorems 15 and 17, we obtain the following result for a general GST game.

Theorem 18 *Suppose a GST game* $G_{m,n,\mathbf{v},V,\lambda,q}$ *(which is not necessarily simple) is* (α,β)*-efficient with respect to the Centre. Then* $\alpha+\beta \le n+1$.

Theorem 19 *Let $G_{m,n,\mathbf{v},V,\lambda_1,q}$ be a simple GST game with $q > 1 - \lambda_1$ which is $(\mathfrak{p}_1, \mathfrak{q}_1)$-efficient with respect to the Centre and let $G_{m,n,\mathbf{v},V,\lambda_2,q}$ be a simple GST game with $q > 1-\lambda_2$ which is $(\mathfrak{p}_2, \mathfrak{q}_2)$-efficient with respect to the Centre. Then*

1. *$\lambda_1 \geq \lambda_2$ if and only if $\mathfrak{p}_1 \leq \mathfrak{p}_2$.*
2. *$\lambda_1 \geq \lambda_2$ if and only if $\mathfrak{q}_1 \geq \mathfrak{q}_2$.*

Theorem 19 shows that as the fraction of the Centre's weight increases, the Centre requires lesser number of partners to pass a resolution while a greater number of States must come together to block a resolution. Consequently, providing more weight to the Centre decreases its dependence on the States and so works against the principle of decentralisation. In the next section, we take up the question of determining how much weight should be given to the Centre.

Next we obtain the number of minimal winning coalitions and swings in a simple GST game.

Theorem 20 *In a simple GST game $G_{m,n,\mathbf{v},V,\lambda,q}$, with $q > 1 - \lambda$, the number of minimal winning coalitions is $\binom{n}{\mathfrak{p}}$ where $\mathfrak{p}$ is given by (7) and this number is equal to the number of swings for the Centre.*

Theorem 20 follows from the fact that any winning coalition will contain the Centre and the Centre requires at least $\mathfrak{t}$ States to win the simple GST game. It can form a winning coalition with any $\mathfrak{t}$ States from the n States that are present in the meeting and voting. By Definition 3, $\mathfrak{t}$ is the minimum number of States that need to be in the coalition for a win and hence any such coalition is minimal.

Theorem 21 *In a simple GST game $G_{m,n,\mathbf{v},V,\lambda,q}$, with $q > 1 - \lambda$, if the Centre and a designated set S of at least $\mathfrak{p}$ (where $\mathfrak{p}$ is given by (7)) of the States form a bloc then such a bloc is dictatorial.*

The Centre will have the support of the political parties that form a coalition for attaining majority in the Lok Sabha. Some or all of these parties will also be in power in some or all of the States by full majority or as a senior partner (owing to the largest number of seats they have in the respective Vidhan Sabha). When there is a coalition with a single party that has the majority by itself in the Lok Sabha (as is currently the case), all the States that are ruled by that party will play the simple GST game in complete alignment with the Centre. In such a scenario, the Centre and those States will form a bloc. If the number of such States is greater than or equal to $\mathfrak{p}$, the Centre's bloc becomes dictatorial in the GST game.

When the coalition that has formed the Central Government is such that no single party has attained a majority in the Lok Sabha, all the parties that are part of the coalition may not agree on a particular motion. Hence, the Centre's vote in the GST game will be with the support of only such parties in the coalition that agree with it on the motion. In such a case, a bloc is formed with the Centre and the States ruled by the political parties that decide to play the GST game in alignment with the Centre. As before, if the number of such States is greater than or equal to $\mathfrak{p}$, the Centre's bloc becomes dictatorial in the GST game.

When the number k of States that have decided to always vote in alignment with the Centre (by virtue of the respective ruling party's allegiance to the coalition that has attained a majority in the Lok Sabha) is less than $\mathfrak{p}$, the Centre and k such States form a bloc, but such a bloc is not dictatorial. So, States outside the bloc have power in the sense of being a swing voter in some winning coalition. The number of swings for States outside the Centre's bloc is given by the following result.

Theorem 22 *In a simple GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ where the Centre is a blocker (equivalently, $q > 1-\lambda$), suppose the Centre and k states form a bloc where $k < \mathfrak{p}$, where $\mathfrak{p}$ is given by (7). Then for any state outside the bloc, the number of swings is $\binom{n-k-1}{\mathfrak{p}-k-1}$.*

The number of minimal winning coalitions for the Centre's bloc is given by the following result.

Theorem 23 *In a simple GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ where the Centre is a blocker (equivalently, $q > 1-\lambda$), suppose the Centre and k states form a bloc where $k < \mathfrak{p}$, where $\mathfrak{p}$ is given by (7). Then the number of minimal winning coalitions in the game is $\binom{n-k}{\mathfrak{p}-k}$. This is also the number of swings for the bloc.*

A typical scenario in a multi-party democracy is the following. The political coalition that enjoys the confidence of the Lok Sabha will determine the vote of the Centre in the simple GST voting game. States that are ruled by this political coalition will vote in alignment with the Centre in the GST game. The Centre along with all the aligned States will form a bloc S_1. There will typically be a principal opposition political coalition. Such a coalition will be in control of some of the States and will generally vote as a bloc S_2. Typically, there will also be a few other States which do not belong to either of the blocs S_1 or S_2. For the bloc S_1, it is of interest to know the number of minimal winning coalitions and swings of which it is a part; for the bloc S_2, it is of interest to know the number of minimal blocking coalitions and blocking swings of which it is a part; and for any State which is neither in S_1 nor in S_2, it is of interest to know the number of coalitions in which it is a swing and the number of coalitions in which it is a blocking swing. All of these information is provided by the following result.

Theorem 24 *Let $G_{m,n,\mathbf{v},V,\lambda,q}$ be a simple GST game; $\mathfrak{p}$ and $\mathfrak{q}$ be given by (7) and (8) respectively; S_1 be a bloc containing the Centre and $k_1 < \mathfrak{p}$ states; S_2 is a bloc containing $k_2 < \mathfrak{q}$ states such that $S_1 \cap S_2 = \emptyset$; and S_1 and S_2 vote in opposition to each other, i.e., S_1 votes for a resolution if and only if S_2 votes against it.*

1. *The number of minimal winning coalitions for the bloc S_1 is $\binom{n-k_1-k_2}{\mathfrak{p}-k_1}$. This is also the number of swings for the bloc S_1.*

2. *The number of minimal blocking coalitions containing the bloc S_2 is $\binom{n-k_1-k_2}{\mathfrak{q}-k_2}$. This is also the number of blocking swings for the bloc S_2.*

3. *Suppose A_i is a player such that $A_i \in N \setminus (S_1 \cup S_2)$. Then the number of swings for A_i is $\binom{n-k_1-k_2-1}{\mathfrak{p}-k_1-1}$ and the number of blocking swings for A_i which also contains S_2 is $\binom{n-k_1-k_2-1}{\mathfrak{q}-k_2-1}$.*

8 Modified GST Voting Game

In the GST game, the Centre is assigned a fraction λ of the total weight while the winning threshold is q. From Theorem 10 we have that the Centre is a blocker if and only if $q > 1 - \lambda$. The GST Act specifies $q = 3/4$ and $\lambda = 1/3$ and so $q > 1 - \lambda$ resulting in the Centre being a blocker in the game. From Theorem 19 we have that for fixed values of q and n, as λ increases, the Centre requires lesser number of States to pass a resolution while more States have to get together to block a resolution. In other words, increasing the value of λ increases the centralisation of power. In a federal set-up, too much centralisation is undesirable.

The question that we consider is the following. What fraction of the total weight should be assigned to the Centre? As discussed earlier, since the Centre enjoys the confidence of the Lok Sabha which is the supreme legislative body in the country, it should not be possible to pass a resolution without the consent of the Centre. So, the Centre should certainly be a blocker. Consequently, the fraction λ of the total weight assigned to the Centre should be such that the Centre is a blocker. Is there any political/economic/social justification for using a value of λ which is more than the minimum required to make the Centre a blocker? We have not been able to come with any such justification. So, we proceed with the condition that λ should be the minimum value such that the Centre is a blocker.

From Theorem 10, any value of λ and q such that $q > 1-\lambda$ will ensure that the Centre is a blocker. Suppose q is fixed to some value. Then λ is to be greater than $1-q$ to ensure that the Centre is a blocker. This creates a problem. The condition λ greater than $1-q$ does not provide a unique value of λ. For any choice λ_1 of λ, it is possible to find a λ_2 such that $1-q < \lambda_2 < \lambda_1$. So, the problem of assigning the minimum possible weight to the Centre resulting in it being a blocker cannot be solved.

To tackle this problem, we make a slight change to the GST game to obtain the modified GST game. The modified GST game has at its core the modified weighted majority voting game given in Definition 2 where the winning condition for a coalition S is $w_S > q$ instead of the winning condition $w_S \geq q$ as required for the weighted majority voting game. Further, the modified GST game sets $q = 1-\lambda$. These points are made precise in the following definition.

Definition 8 (Modified GST Voting Game) *Let A_0 denote the Centre and $\{A_1, \ldots, A_m\}$ denote the m states. For $i = 1, \ldots, m$, suppose v_i is the* a priori *weight of state A_i and let $\mathbf{v} = (v_1, \ldots, v_m)$. Let $1 \leq n \leq m$ and suppose the set of States $V = \{A_{j_1}, \ldots, A_{j_n}\}$ participate in the voting. Let $u_1 = v_{j_1}, \ldots, u_n = v_{j_n}$ and $\mathbf{u} = (u_1, \ldots, u_n)$. Let $\lambda = a/b \in (0, 1/2)$ for some integers a and b and $q = 1-\lambda$. The GST game $G_{m,n,\mathbf{v},V,\lambda}$ is defined to be a* modified *weighted majority voting game on $n+1$ players as follows.*

$$G_{m,n,\mathbf{v},V,\lambda} = (\{A_0\} \cup V, \mathbf{w}, q) \quad (10)$$

where $\mathbf{w} = (w_0, w_1, \ldots, w_n)$; w_0 is the weight of A_0; and $w_1, \ldots, w_n$ are the weights of $A_{j_1}, \ldots, A_{j_n}$ respectively. The components of $\mathbf{w}$ are obtained from the components of $\mathbf{u}$ as given by (2).

Remarks:

1. As in the case of GST voting game, we assume that the Centre A_0 will always participate in every modified GST voting game while some of the States may not do so. Since $n \geq 1$, at least one State participates in the voting.

2. In a modified GST voting game the value of q is fixed to $1-\lambda$. It is possible to define a more general notion of modified GST voting game where $q > 1-\lambda$. The reason we do not do this is that we wish to exclusively analyse the case $q = 1-\lambda$. On the other hand, our method of analysis can be easily extended to cover the more general case.

3. For the game to be proper, we require $q > 1/2$ which combined with $q = 1-\lambda$ implies that $\lambda < 1/2$. Since $q > 1/2$ and $\lambda < 1/2$, it follows that $\lambda < q$ and so the Centre is not a dictator.

In the following, we present a sequence of definitions and results for the modified GST voting game which are analogues of the corresponding results for the GST voting game. The interpretations of these results to the context of modified GST game are exactly the same as the interpretations of the corresponding results to the GST game. So, we do not repeat the discussions already provided in Section 7.

Theorem 25 *In a modified GST game $G_{m,n,\mathbf{v},V,\lambda}$, the Centre A_0 is a blocker.*

In Definition 6 of the GST voting game, the relationship between the winning threshold q and the weight fraction λ of the Centre A_0 is not specified. Theorem 10 subsequently characterises the relationship between q and λ so that the Centre is a blocker. Definition 8 of the modified GST game however assumes that $\lambda = 1-q$. Theorem 25 shows that the fact of the Centre being a blocker follows directly from Definition 8.

This requirement that the Centre has weight more than that of any State is characterised in the same manner as in the case of GST game.

Theorem 26 *Consider a modified GST game* $G_{m,n,\mathbf{v},V,\lambda}$ *with* $V = \{A_{j_1}, \ldots, A_{j_n}\}$*. The state* A_{j_i} *has weight less than that of the Centre* A_0*, i.e.,* $w_i < w_0$ *if and only if*

$$\frac{u_i}{u_1 + \cdots + u_n} < \frac{\lambda}{1-\lambda}.$$

Further suppose $\mathbf{v} = (v_1, \ldots, v_m)$ *is such that* $v_1 \geq \cdots \geq v_m$ *and* τ *is the threshold of voting quorum for* G*. Let* $r = \lceil \tau m \rceil$ *and*

$$v_1 \quad < \quad \frac{\lambda}{1-2\lambda}\left(v_{m-r+2} + \cdots + v_m\right). \tag{11}$$

Then in G*, the weight of any State is less than that of the Centre.*

Theorem 27 *Consider a modified GST game* $G_{m,n,\mathbf{v},V,\lambda,q}$ *with* $V = \{A_{j_1}, \ldots, A_{j_n}\}$*.*

1. *No State is a dictator.*
2. *The State* A_{j_i} *is a blocker if and only if* $\dfrac{u_i}{u_1 + \cdots + u_n} \geq \dfrac{\lambda}{1-\lambda}$*.*

Further suppose $\mathbf{v} = (v_1, \ldots, v_m)$ *is such that* $v_1 \geq \cdots \geq v_m$ *and* τ *is the threshold of voting quorum for* G*. Let* $r = \lceil \tau m \rceil$ *and*

$$v_1 \quad < \quad \frac{\lambda}{1-2\lambda}\left(v_{m-r+2} + \cdots + v_m\right). \tag{12}$$

Then in G*, no State is a blocker.*

Theorem 28 *Suppose a modified GST game* $G_{m,n,\mathbf{v},V,\lambda,q}$ *is* (α, β)*-efficient with respect to the Centre. Then* $\alpha < \beta$ *if and only if* $\lambda > 1/3$*.*

8.1 Simple Modified GST Games

In this section, we state results for simple modified GST games which are analogues of the result for simple GST games.

Definition 9 *A modified GST game* $G_{m,n,\mathbf{v},V,\lambda}$ *is said to be simple if* $v_1 = \cdots = v_m$*.*

In view of Corollary 1, in a simple GST game we may take $v_1 = \cdots = v_m = 1$ and consequently from (2), $w_1 = \cdots = w_n = (b-a)/d$.

Theorem 29 *Consider a simple modified GST game* $G_{m,n,\mathbf{v},V,\lambda,q}$ *with* $V = \{A_{j_1}, \ldots, A_{j_n}\}$*.*

1. *All States have weight less than that of the Centre.*
2. *State* A_{j_i} *is a blocker if and only if* $n \leq (1-\lambda)/\lambda$*.*

As in the case of simple GST games, there is a theoretical possibility that a State is a blocker. For example, if we take $\lambda = 1/4$, then for $n = 1, 2$ and 3, the State(s) is/are blocker(s).

Theorem 30 *Let* $\mathbf{v} = (v_1, \ldots, v_m) = (1, \ldots, 1)$ *and* $\mathbf{v}' = (v'_1, \ldots, v'_m)$ *where not all the* v'_i*'s are equal. Suppose a modified GST game* $G_{m,n,\mathbf{v},V,\lambda}$ *is* (α, β)*-efficient with respect to the Centre and a GST game* $G'_{m,n,\mathbf{v}',V,\lambda}$ *is* (α', β')*-efficient with respect to the Centre. Then* $\alpha \geq \alpha'$ *and* $\beta \geq \beta'$*.*

In other words, Theorem 30 shows that if a modified GST game $G_{m,n,\mathbf{v},V,\lambda}$ is (α,β)-efficient with respect to the Centre then the values of α and β are maximum when the GST game is simple.

Theorem 31 *Consider a modified simple GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ which is (α,β)-efficient with respect to the Centre. Then any State $A_{j_i} \in V$ has $(\alpha,\alpha,\beta,\beta)$-influence with respect to the Centre.*

Theorem 32 *A simple modified GST game $G_{m,n,\mathbf{v},V,\lambda}$ is $(\mathfrak{s},\mathfrak{t})$-efficient with respect to the Centre where*

$$\mathfrak{s} = n - \delta + 1; \tag{13}$$

$$\mathfrak{t} = \delta \tag{14}$$

and $\delta = \left\lceil \frac{n\lambda}{1-\lambda} \right\rceil$. Consequently, we have

$$\mathfrak{s} + \mathfrak{t} = n + 1. \tag{15}$$

Combining Theorems 30 and 32, we obtain the following result for a general modified GST game.

Theorem 33 *Suppose a modified GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ (which is not necessarily simple) is (α,β)-efficient with respect to the Centre. Then $\alpha + \beta \leq n + 1$.*

Theorem 34 *Let $G_{m,n,\mathbf{v},V,\lambda_1}$ be a simple modified GST game which is $(\mathfrak{s}_1,\mathfrak{t}_1)$-efficient with respect to the Centre and let $G_{m,n,\mathbf{v},V,\lambda_2}$ be a simple modified GST game which is $(\mathfrak{s}_2,\mathfrak{t}_2)$-efficient with respect to the Centre. Then*

1. *$\lambda_1 \geq \lambda_2$ if and only if $\mathfrak{s}_1 \leq \mathfrak{s}_2$.*
2. *$\lambda_1 \geq \lambda_2$ if and only if $\mathfrak{t}_1 \geq \mathfrak{t}_2$.*

Theorem 35 *In a simple modified GST game $G_{m,n,\mathbf{v},V,\lambda}$ the number of minimal winning coalitions is $\binom{n}{\mathfrak{s}}$ where $\mathfrak{s}$ is given by (13) and this number is equal to the number of swings for the Centre.*

Theorem 36 *In a simple modified GST game $G_{m,n,\mathbf{v},V,\lambda}$ if the Centre and a designated set S of $\mathfrak{s}$ (where $\mathfrak{s}$ is given by (13)) of the states form a bloc then such a bloc is dictatorial.*

Theorem 37 *In a simple modified GST game $G_{m,n,\mathbf{v},V,\lambda}$ suppose the Centre and k states form a bloc where $k < \mathfrak{s}$ where $\mathfrak{s}$ is given by (13). Then for any state outside the bloc, the number of swings is $\binom{n-k-1}{\mathfrak{s}-k-1}$.*

Theorem 38 *In a simple modified GST game $G_{m,n,\mathbf{v},V,\lambda}$ suppose the Centre and k states form a bloc where $k < \mathfrak{s}$ and $\mathfrak{s}$ is given by (13). Then the number of minimal winning coalitions in the game is $\binom{n-k}{\mathfrak{s}-k}$. This is also the number of swings for the bloc.*

Theorem 39 *Let $G_{m,n,\mathbf{v},V,\lambda}$ be a simple modified GST game; $\mathfrak{s}$ and $\mathfrak{t}$ be given by (13) and (14) respectively; S_1 be a bloc containing the Centre and $k_1 < \mathfrak{s}$ states; S_2 is a bloc containing $k_2 < \mathfrak{t}$ states such that $S_1 \cap S_2 = \emptyset$; and S_1 and S_2 vote in opposition to each other, i.e., S_1 votes for a resolution if and only if S_2 votes against it.*

1. *The number of minimal winning coalitions for the bloc S_1 is $\binom{n-k_1-k_2}{\mathfrak{s}-k_1}$. This is also the number of swings for the bloc S_1.*
2. *The number of minimal blocking coalitions containing the bloc S_2 is $\binom{n-k_1-k_2}{\mathfrak{t}-k_2}$. This is also the number of blocking swings for the bloc S_2.*
3. *Suppose A_i is a player such that $A_i \in N \setminus (S_1 \cup S_2)$. Then the number of swings for A_i is $\binom{n-k_1-k_2-1}{\mathfrak{s}-k_1-1}$ and the number of blocking swings for A_i which also contains S_2 is $\binom{n-k_1-k_2-1}{\mathfrak{t}-k_2-1}$.*

9 Concrete Analysis

There are two major national parties in India, namely the Bharatiya Janata Party (BJP) and the Indian National Congress (INC). The BJP is the senior partner in a political alliance called the National Democratic Alliance (NDA) while the INC is the senior partner in another political alliance called the United Progressive Alliance (UPA).

As mentioned earlier, the GST Council has representative from $m = 31$ States. There are two representatives of the Centre and for the purposes of voting, these two representatives are jointly considered to be the player representing the Centre.

Political situation as on 31 March, 2018. The members of the NDA and the States they control are available from[20]. The Chief Ministers of the various States are available from[21]. A summary of this data is as follows.

- The BJP has a majority in the Lok Sabha and the Central Government is formed by the NDA.
- The BJP has majority in the Vidhan Sabha of 13 States. In 2 other States, it is the senior partner in the respective ruling coalitions. Overall, there are 15 States which have Chief Ministers from the BJP.
- In 6 States, the Chief Minister is from a junior partner of the NDA.
- In 4 States, the Chief Minister is from the INC.
- In 6 States, the Chief Ministers are from neither the NDA nor the UPA.

So, the NDA has control of 21 States. It may be expected that if there is any voting in the GST Council, then the NDA will vote as a bloc. This, however, need not always be true. The junior partners of the NDA have their own political compulsions and under certain circumstances may not vote as part of the NDA. We may assume though that the States which have Chief Ministers from the BJP will always vote as a bloc along with the Centre. So, one voting bloc will certainly consist of the Centre and 15 other States. Mostly, this bloc will also consist of the 6 other States ruled by the junior partners of the NDA, but, this cannot be taken for granted. The States ruled by the INC can be assumed to vote as a bloc. The other 6 States which are neither part of the NDA nor the UPA are not a priori part of any bloc.

In the following subsections, we perform concrete analyses under the various settings outlined in the introduction.

9.1 Simple GST Game

In this section we consider the Setting-A mentioned in the introduction. This is the setting of the simple GST voting game $G_{m,n,\mathbf{v},V,\lambda,q}$ with $\lambda = 1/3$ and $q = 3/4$ as specified in the GST Act. Since we are considering simple GST games, the a priori weights of the States as given by $\mathbf{v}$ are all taken to be equal. The value of n depends upon the number of States which participate in the actual voting. The values of n, λ and q determine the values of $\mathfrak{p}$ and $\mathfrak{q}$ such that $G_{m,n,\mathbf{v},V,\lambda,q}$ is $(\mathfrak{p}, \mathfrak{q})$-efficient with respect to the Centre. For $\lambda = 1/3$ and $q = 3/4$, the various values of $\mathfrak{p} = n - \rho$ and $\mathfrak{q} = \rho + 1$ with $\rho = \lfloor n(1-q)/(1-\lambda) \rfloor$ depending on n are shown in Table 1. To pass a resolution, the Centre needs the support of at least $\mathfrak{p}$ States. As n decreases, the values of $\mathfrak{p}$ are non-increasing, i.e., they either

[20] https://en.wikipedia.org/wiki/National_Democratic_Alliance_(India)

[21] https://en.wikipedia.org/wiki/List_of_current_Indian_chief_ministers

Table 1: Values of $\mathfrak{p}$ and $\mathfrak{q}$ such that the simple GST game $G_{m,n,\mathbf{v},V,\lambda,q}$ is $(\mathfrak{p},\mathfrak{q})$-efficient with respect to the Centre, for $\lambda = 1/3$ and $q = 3/4$.

n	ρ	$\mathfrak{p}$	$\mathfrak{q}$
31	11	20	12
30	11	19	12
29	10	19	11
28	10	18	11
27	10	17	11
26	9	17	10
25	9	16	10

decrease or remain unchanged. If $\mathfrak{p}$ decreases, then the Centre's task of mustering a winning coalition becomes easier. On the other hand, if $\mathfrak{p}$ remains unchanged with decrease in n, then the Centre has to find $\mathfrak{p}$ States among $n-1$ States and so its task of mustering a winning coalition becomes relatively a little more difficult. The actual difficulty would depend upon which of the States actually drop out of the voting.

To block a resolution, a coalition of at least $\mathfrak{q}$ States need to form. Again, as n decreases, the values of $\mathfrak{q}$ are non-increasing. If $\mathfrak{q}$ decreases as n decreases, then it becomes easier to block a resolution; if $\mathfrak{q}$ remains unchanged, it is required to find at least $\mathfrak{q}$ States among $n-1$ States and so the task of blocking becomes more difficult.

Since the a priori weights of all the States are equal, the influence vector of all the States are also equal. Theorem 16 shows that the influence vector for the States is determined from the values of $\mathfrak{p}$ and $\mathfrak{q}$.

Application to the political situation as on 31 March, 2018. Suppose $n = m = 31$, i.e., all the States participate in the voting. From Table 1, the value of $\mathfrak{p}$ in the row corresponding to $n = 31$ is 20. This means that if all States participate in the voting, the Centre needs at least 20 States to pass a resolution.

1. The NDA rules the Centre and 21 of the States. So, if the NDA votes as a bloc, then such a bloc is dictatorial. All the opposing States cannot get together to block any resolution.

2. The BJP by itself has Chief Ministers in 15 of the States. So, the Centre and these 15 States together do not form a dictatorial coalition. This bloc will require the support of at least 5 other States to pass a resolution.

If all the NDA ruled States participate in the voting and vote as a bloc, then the Centre's task of mustering a winning coalition actually becomes easier as n decreases. On the other hand, if some of the NDA ruled States do not necessarily vote along with the Centre, the situation becomes more complex. Depending on the scenario, the values in Table 1 can be used to analyse such possibilities. Suppose two of the NDA States do not necessarily vote along with the Centre. Then various possibilities arise.

1. If these two States participate in the voting and vote against the Centre's motion, then the Centre can count upon the support of at most 19 of the States ruled by the NDA and must look for support outside the NDA.

2. If these two States do not participate in the voting and all other States do, then we have to consider the game with $n = 29$ players. In such a situation, the Centre along with the 19 other

Table 2: Values of $\mathfrak{s}$ and $\mathfrak{t}$ such that the simple modified GST game $G_{m,n,\mathbf{v},V,\lambda}$ is $(\mathfrak{s},\mathfrak{t})$-efficient with respect to the Centre, for $\lambda = 1/4$.

n	δ	$\mathfrak{s}$	$\mathfrak{t}$
31	11	21	11
30	10	21	10
29	10	20	10
28	10	19	10
27	9	19	9
26	9	18	9
25	9	17	9

States ruled by the NDA again form a dictatorial coalition. So, by not participating in the voting, two such States are unable to influence the outcome.

9.2 Simple Modified GST Game

In this section we consider the Setting-B mentioned in the introduction. This is the setting of the simple modified GST voting game $G_{m,n,\mathbf{v},V,\lambda}$. In this setting, the goal is to have $q = 3/4$ as specified in the GST Act. Then according to the definition of the modified GST game the value of λ is set to $\lambda = 1-q = 1/4$. The values of n and λ determine the values of $\mathfrak{s}$ and $\mathfrak{t}$ such that $G_{m,n,\mathbf{v},V,\lambda}$ is $(\mathfrak{s},\mathfrak{t})$-efficient with respect to the Centre. For $\lambda = 1/4$, the various values of $\mathfrak{s} = n - \delta + 1$ and $\mathfrak{t} = \delta$ with $\delta = \lceil n\lambda/(1-\lambda)\rceil$ depending on n are shown in Table 2.

An analysis based on the values in Table 2 can be carried out in a manner similar to the analysis which has been done based on the values in Table 1. We do not repeat the discussion. Theorem 31 shows that the influence vector for the States is determined from the values of $\mathfrak{s}$ and $\mathfrak{t}$.

Application to the political situation as on 31 March, 2018. We note a few points.

1. If the Centre and the 21 States of the NDA vote as a bloc, then such a bloc is dictatorial as in the case of simple GST game.

2. The Centre and the 15 States having Chief Ministers from BJP do not form a dictatorial block. Such a bloc requires at least 6 more States to pass a resolution.

3. For each value of n, comparing the values in Table 1 and 2, we find that the Centre requires one more State to form a winning coalition. In other words, Setting-B requires a greater participation in the decision making process. So, we may consider Setting-B to be more decentralised than Setting-A.

4. For each value of n, comparing the values in Table 1 and 2, we find that a group of one less State can form a blocking coalition. So, in Setting-B, the States have a comparatively higher blocking capability compared to Setting-A.

9.3 GST Game with RS Weights

In this section we consider the Setting-C mentioned in the introduction. This is the setting of the GST voting game $G_{m,n,\mathbf{v},V,\lambda,q}$ arising when the a priori weight vector $\mathbf{v} = (v_1, \ldots, v_m)$ is determined by setting v_i to be equal to the number of seats in the Rajya Sabha for State A_i. The parameters λ and q are set to $\lambda = 1/3$, $q = 3/4$ as specified in the GST Act.

For the analysis, we only consider the case $n = m = 31$. Analysis of any case with $n < m$ can be carried out in a manner similar to that of $n = m$. It turns out that in this setting G is $(10, 5)$-efficient with respect to the Centre. This means that the Centre needs the support of at least 10 States to pass a resolution and at least 5 States must come together to block a resolution. For each State A_i, $i = 1, \ldots, m$, the values of $\gamma_i, \kappa_i, \mu_i, \nu_i$ such that A_i has $(\gamma_i, \kappa_i, \mu_i, \nu_i)$-influence with respect to the Centre are shown in Table 3.

We provide explanation for some of the entries in Table 3. Consider the row corresponding to the State Uttar Pradesh (UP). The number of seats in the RS for UP is 31. Based on the RS weights and with $n = m = 31$, the voting weight of UP is computed using (2) to be 62. With respect to the Centre, UP has $(10, 12, 5, 6)$-influence, i.e., any winning coalition containing UP has to contain at least 10 States; any winning coalition not containing UP has to contain at least 12 States; any blocking coalition containing UP has to contain at least 5 States; and any blocking coalition not containing UP has to contain at least 6 States. Similar interpretation holds for the other States. Note that for the last 12 States in the row, it turns out that the value of μ is more than the value of ν, i.e., if any of these States is part of a blocking coalition, then such a blocking coalition will need to be larger than a minimum size blocking coalition not containing this State. So, such States do not have any influence in the blocking process.

Application to the political situation as on 31 March, 2018. Voting in the GST Council is most likely to take place in a bloc-wise manner, i.e., the States which belong to a single political party or coalition will behave as a bloc. So, it is of interest to perform a bloc-wise analysis. For carrying out this exercise, we consider two situations. In both situations, it is assumed that all the States participate in the voting. Situations where some States do not participate in the voting can be analysed in a similar manner.

1. The Centre and the 21 NDA ruled States form a bloc. The States ruled by the INC form a bloc. The corresponding game turns out to be $(0, -)$-efficient with respect to the Centre-led bloc. This means that the Centre-led coalition is a dictatorial coalition. The influence vectors of the players outside the Centre-led coalition and the bloc-wise blocking powers of the players are shown in Table 4.

2. The Centre and the 15 States having Chief Ministers from the BJP form a bloc. The States ruled by the INC form a bloc. The game is $(1, 6)$-efficient with respect to the Centre-led bloc. This means that the Centre-led coalition cannot pass a resolution on its own; it needs the support of at least one other player in the voting game. On the other hand, a coalition of 6 players (excluding than the Centre-led bloc) can block a resolution. The influence vectors of the players outside the Centre-led coalition and the bloc-wise blocking powers of the players are shown in Table 5.

9.4 Modified GST Game with RS Weights

In this section we consider the Setting-D mentioned in the introduction. This is the setting of the modified GST game $G_{m,n,\mathbf{v},V,\lambda}$ arising when the a priori weight vector $\mathbf{v} = (v_1, \ldots, v_m)$ is determined by setting v_i to be equal to the number of seats in the Rajya Sabha for State A_i. We set $\lambda = 1/4$ so that $q = 1 - \lambda = 3/4$ corresponds to the GST Act.

As in Section 9.3, we only consider the case $n = m = 31$ and note that analysis of any case with $n < m$ can be carried out in a manner similar to that of $n = m$. It turns out that in this setting G is

Table 3: For GST game specified in the GST Act, the table provides the influence vector $(\gamma, \kappa, \mu, \nu)$ for all the States where $\mathbf{v}$ is given by the number of seats in the Rajya Sabha. Here $\lambda = 1/3$, $q = 3/4$ and $n = m = 31$, i.e., it is assumed that all States participate in the voting.

Player Name	$\mathbf{v}$	$\mathbf{w}$	γ	κ	μ	ν
Centre		233	–	–	–	–
Uttar Pradesh	31	62	10	12	5	6
Maharashtra	19	38	10	10	5	5
Tamil Nadu	18	36	10	10	5	5
Bihar	16	32	10	10	5	5
West Bengal	16	32	10	10	5	5
Karnataka	12	24	10	10	5	5
Andhra Pradesh	11	22	10	10	5	5
Gujarat	11	22	10	10	5	5
Madhya Pradesh	11	22	10	10	5	5
Odisha	10	20	10	10	5	5
Rajasthan	10	20	10	10	5	5
Kerala	9	18	10	10	5	5
Assam	7	14	10	10	5	5
Punjab	7	14	10	10	5	5
Telangana	7	14	10	10	5	5
Jharkhand	6	12	10	10	5	5
Chhattisgarh	5	10	10	10	5	5
Haryana	5	10	10	10	5	5
Jammu & Kashmir	4	8	10	10	5	5
Delhi	3	6	10	10	6	5
Himachal Pradesh	3	6	10	10	6	5
Uttarakhand	3	6	10	10	6	5
Arunachal Pradesh	1	2	10	10	6	5
Goa	1	2	10	10	6	5
Manipur	1	2	10	10	6	5
Meghalaya	1	2	10	10	6	5
Mizoram	1	2	10	10	6	5
Nagaland	1	2	10	10	6	5
Puducherry	1	2	10	10	6	5
Sikkim	1	2	10	10	6	5
Tripura	1	2	10	10	6	5

Table 4: For the GST game, the table provides the influence vector $(\gamma, \kappa, \mu, \nu)$ and the blocking power $\mathfrak{M}$ where voting is assumed to be bloc-wise. The Centre and the NDA-ruled States form a single bloc while the INC ruled States form also form a single bloc. The values of $\mathbf{v}$ are given by the number of seats in the Rajya Sabha belonging to the States in the respective blocs. Here $\lambda = 1/3$ and $q = 3/4$.

Bloc Name	$\mathbf{v}$	$\mathbf{w}$	γ	κ	μ	ν	$\mathfrak{M}$
Centre with NDA	149 (NDA States only)	531	–	–	–	–	1
INC	21	42	–	0	–	–	0
Tamil Nadu	18	36	–	0	–	–	0
West Bengal	16	32	–	0	–	–	0
Odisha	10	20	–	0	–	–	0
Kerala	9	18	–	0	–	–	0
Telangana	7	14	–	0	–	–	0
Delhi	3	6	–	0	–	–	0

Table 5: For the GST game, the table provides the influence vector $(\gamma, \kappa, \mu, \nu)$ and the blocking power $\mathfrak{M}$ where voting is assumed to be bloc-wise. The Centre and the BJP-ruled States form a single bloc while the INC ruled States form also form a single bloc. The values of $\mathbf{v}$ are given by the number of seats in the Rajya Sabha belonging to the States in the respective blocs. Here $\lambda = 1/3$ and $q = 3/4$.

Bloc Name	$\mathbf{v}$	$\mathbf{w}$	γ	κ	μ	ν	$\mathfrak{M}$
Centre with BJP	126 (BJP States only)	485	–	–	–	–	0.9594727
INC	21	42	–	2	6	–	0.0405273
Tamil Nadu	18	36	2	1	6	10	0.0385742
West Bengal	16	32	2	1	6	8	0.0361328
Bihar	16	32	2	1	6	8	0.0361328
Odisha	10	20	2	1	6	7	0.0214844
Kerala	9	18	2	1	6	6	0.0200195
Telangana	7	14	2	1	6	6	0.0170898
Jammu & Kashmir	4	8	2	1	7	6	0.0097656
Delhi	3	6	2	1	7	6	0.0083008
Meghalaya	1	2	2	1	7	6	0.0019531
Nagaland	1	2	2	1	7	6	0.0019531
Sikkim	1	2	2	1	7	6	0.0019531

$(11, 4)$-efficient with respect to the Centre. This means that the Centre needs the support of at least 11 States to pass a resolution and at least 4 States must come together to block a resolution. For each State A_i, $i = 1, \ldots, m$, the values of $\gamma_i, \kappa_i, \mu_i, \nu_i$ such that A_i has $(\gamma_i, \kappa_i, \mu_i, \nu_i)$-influence with respect to the Centre are shown in Table 6.

Application to the political situation as on 31 March, 2018. As in Section 9.3, we analyse the situation arising when voting takes place in bloc-wise manner. For carrying out this exercise, we consider two situations. In both situations, it is assumed that all the States participate in the voting.

1. The Centre and the 21 NDA rules States form a bloc. The States ruled by the INC form a bloc. The corresponding game turns out to be $(1, 6)$-efficient with respect to the Centre-led bloc. This means that the Centre-led coalition needs the support of another player to pass a resolution while a coalition of at least 6 players (excluding the Centre-led bloc) is required to block a resolution. The influence vectors of the players outside the Centre-led coalition and the bloc-wise blocking powers of the players are shown in Table 7.

2. The Centre and the 15 States having Chief Ministers from the BJP form a bloc. The States ruled by the INC form a bloc. The game is $(2, 5)$-efficient with respect to the Centre-led bloc. This means that the Centre-led coalition requires the support of at least 2 other players to pass a resolution while a coalition of at least 5 players (excluding the Centre-led bloc) is required to block a resolution. The influence vectors of the players outside the Centre-led coalition and the bloc-wise blocking powers of the players are shown in Table 8.

9.5 Summary

The detailed analysis provided in the previous sections leads to the following two top-level conclusions.

1. Comparison of GST game with $\lambda = 1/3$ and $q = 3/4$ with the modified GST game with $\lambda = 1/4$ and $q = 3/4$. In the second case, the Centre needs the support of more States to pass a resolution while a smaller group of States can form a blocking coalition. In view of this, we conclude that the second case is more democratic than the first.

2. Comparison of using equal a priori weights with that of using a priori weights based on the number of seats in the RS. As predicted by Theorems 15 and 30, using equal a priori weights leads to a more decentralised decision making where the Centre requires the support of more States to pass a resolution. On the other hand, as again predicted by Theorems 15 and 30, using equal a priori weights results in requiring a higher number of States to come together to block a resolution. For example, based on the political situation on 31 March, 2018, if equal a priori weights are used, then the Centre and the NDA States form a dictatorial coalition in both the cases of the GST game with $\lambda = 1/3$, $q = 3/4$ and the modified GST game with $\lambda = 1/4$ and $q = 3/4$. This remains true for the GST game with the number of RS seats as the a priori weights. On the other hand, for the modified GST game with the number of RS seats as the a priori weights, a coalition of 4 States can block a resolution.

Given the dichotomy between the requirement of a higher size of winning coalition versus the higher size of a blocking coalition, we do not find any clear quantifiable preference for choosing the a priori weights to be equal or choosing them to be the number of RS seats. Given that the GST Act does not explicitly specify how the voting weights are to be distributed among the States, the actual decision of choosing these weights would have to be taken at an appropriate political level possibly after absorbing the intricacies of the problem as uncovered by our analysis.

Table 6: For modified GST game, the table provides the influence vector $(\gamma, \kappa, \mu, \nu)$ for all the States where $\mathbf{v}$ is given by the number of seats in the Rajya Sabha. Here $\lambda = 1/4$ and so $q = 3/4$; $n = m = 31$, i.e., it is assumed that all States participate in the voting.

Player Name	$\mathbf{v}$	$\mathbf{w}$	γ	κ	μ	ν
Centre		233	–	–	–	–
Uttar Pradesh	31	93	11	13	4	5
Maharashtra	19	57	11	12	4	4
Tamil Nadu	18	54	11	11	4	4
Bihar	16	48	11	11	4	4
West Bengal	16	48	11	11	4	4
Karnataka	12	36	11	11	4	4
Andhra Pradesh	11	33	11	11	4	4
Gujarat	11	33	11	11	4	4
Madhya Pradesh	11	33	11	11	4	4
Odisha	10	30	11	11	4	4
Rajasthan	10	30	11	11	4	4
Kerala	9	27	11	11	5	4
Assam	7	21	11	11	5	4
Punjab	7	21	11	11	5	4
Telangana	7	21	11	11	5	4
Jharkhand	6	18	11	11	5	4
Chhattisgarh	5	15	11	11	5	4
Haryana	5	15	11	11	5	4
Jammu & Kashmir	4	12	11	11	5	4
Delhi	3	9	11	11	5	4
Himachal Pradesh	3	9	11	11	5	4
Uttarakhand	3	9	11	11	5	4
Arunachal Pradesh	1	3	11	11	5	4
Goa	1	3	11	11	5	4
Manipur	1	3	11	11	5	4
Meghalaya	1	3	11	11	5	4
Mizoram	1	3	11	11	5	4
Nagaland	1	3	11	11	5	4
Puducherry	1	3	11	11	5	4
Sikkim	1	3	11	11	5	4
Tripura	1	3	11	11	5	4

Table 7: For the modified GST game, the table provides the influence vector $(\gamma, \kappa, \mu, \nu)$ and the blocking power $\mathfrak{M}$ where voting is assumed to be bloc-wise. The Centre and the NDA-ruled States form a single bloc while the INC ruled States form also form a single bloc. The values of $\mathbf{v}$ are given by the number of seats in the Rajya Sabha belonging to the States in the respective blocs. Here $\lambda = 1/4$ and $q = 3/4$.

Bloc Name	$\mathbf{v}$	$\mathbf{w}$	γ	κ	μ	ν	$\mathfrak{M}$
Centre with NDA	149 (NDA States only)	680	–	–	–	–	0.9843750
INC	21	63	1	1	6	–	0.0156250
Tamil Nadu	18	54	1	1	6	–	0.0156250
West Bengal	16	48	1	1	6	–	0.0156250
Odisha	10	30	1	1	6	–	0.0156250
Kerala	9	27	1	1	6	–	0.0156250
Telangana	7	21	1	1	6	–	0.0156250
Delhi	3	9	2	1	7	6	0

Table 8: For the modified GST game, the table provides the influence vector $(\gamma, \kappa, \mu, \nu)$ and the blocking power $\mathfrak{M}$ where voting is assumed to be bloc-wise. The Centre and the BJP-ruled States form a single bloc while the INC ruled States form also form a single bloc. The values of $\mathbf{v}$ are given by the number of seats in the Rajya Sabha belonging to the States in the respective blocs. Here $\lambda = 1/4$ and $q = 3/4$.

Bloc Name	$\mathbf{v}$	$\mathbf{w}$	γ	κ	μ	ν	$\mathfrak{M}$
Centre with BJP	126 (BJP States only)	611	–	–	–	–	0.8840332
INC	21	63	2	2	5	7	0.1003418
Tamil Nadu	18	54	2	2	5	6	0.0886230
West Bengal	16	48	2	2	5	6	0.0783691
Bihar	16	48	2	2	5	6	0.0783691
Odisha	10	30	2	2	5	5	0.0539551
Kerala	9	27	2	2	5	5	0.0485840
Telangana	7	21	3	2	5	5	0.0339355
Jammu & Kashmir	4	12	3	2	6	5	0.0236816
Delhi	3	9	3	2	6	5	0.0168457
Meghalaya	1	3	3	2	6	5	0.0051270
Nagaland	1	3	3	2	6	5	0.0051270
Sikkim	1	3	3	2	6	5	0.0051270

9.6 More Granular Analysis

The bloc-wise analysis carried out in Sections 9.3 and 9.4 assumes that the INC bloc behave independently of the Centre-led bloc. As a result, the analysis covers the possibility that the INC votes along with the Centre-led bloc. Considering that the INC is the principal opposition party, this is unlikely in the present political scenario. A more granular analysis would be to consider the scenario where the INC always votes in opposition to the Centre-led coalition. While we do not carry out such an analysis, we note that our techniques can be extended to do this.

9.7 Voting Weights Based on GDP

It may be argued that since the affairs of the GST Council pertain to financial issues, the voting weights of the States should also reflect the contribution of the States to the national GDP. Considering voting weights based only on population (as captured by the number of seats in the Rajya Sabha) does not incentivise the States which contribute more to the national economy. This is a debate which is really out of the scope of the current work. The concepts and methods of analysis that we have developed can be applied to any vector $\mathbf{v}$ of a priori weights. So, in particular, it can also be applied to $\mathbf{v}$ obtained from state-wise GDP numbers. This can be an interesting task for policy makers and possible future research work.

10 A Proposal for GST Voting

We list down a set of basic desiderata and their rationale that a voting procedure in the GST Council should satisfy.

Desideratum 1: The Centre should be a blocker. The Centre enjoys the confidence of the Lok Sabha which is the apex legislative body in the country. So, any resolution that is passed should have the approval of the Centre and by implication the approval of the Lok Sabha.

Desideratum 2: The Centre's weight should be enough to ensure it is a blocker and no more. While the Centre should certainly be a blocker, we do not find any socio-political justification to assign more weight to the Centre beyond the minimum necessary to ensure that it is a blocker.

As discussed in the motivation for modified GST game, if the GST game is followed, then the problem of assigning minimum weight to the Centre cannot be solved. It is solved by considering the modified GST game where the Centre has weight λ, the winning threshold is $q = 1 - \lambda$ and the winning condition is changed from "$\geq$" to "$>$". From Theorem 25, in the modified GST game, the Centre is ensured to be a blocker.

In view of the requirement of satisfying Desiderata 1 and 2, we suggest that the modified GST game be adopted. The actual value of λ needs to be appropriately chosen. Choosing $\lambda = 1/4$ ensures that $q = 3/4$ which matches the value of q specified in the GST Act.

Desideratum 3: The weight of any State should be less than that of the Centre. While any State represents only a segment of the population, the Centre represents the whole population. In view of this, it is reasonable to require that the Centre has weight more than any of the States.

Desideratum 4: No State should be a blocker. Since a State represents only a particular geographical segment of the population, it should not be the case that one State alone can stand in the way of a resolution being adopted.

Note that the requirement of meeting quorum is not sufficient to ensure that Desiderata 3 and 4 hold. In view of Desiderata 3 and 4, we suggest that the notion of meeting quorum be replaced or supplemented by the condition of voting quorum. The choice of the value of the threshold τ of voting quorum depends upon the value of λ and the choice of the a priori weights. We consider two situations.

The modified simple GST game with $\lambda = 1/4$. Theorem 29 shows that all States have weight less than that of the Centre and for $\lambda = 1/4$, the value of n has to be at least 4 to ensure that no State is a blocker. In other words, if at least 4 States participate in the voting, then no State is a blocker. So, the value of the threshold τ of voting quorum can be fixed to any value greater than 3/31. For example, specifying $\tau = 1/10$ will suffice.

With $\lambda = 1/4$, the total voting weight of the States is 3/4. If 3 States participate in the voting, then since the voting weights of the States will be equal, each State will have 1/4 fraction of the total voting weight. In other words, each of the States will have the same fraction of the total voting weight as the Centre and so, each State will be a blocker. This shows the tightness of the lower bound which requires that at least 4 States to participate in the voting to ensure that no State is a blocker.

The modified GST game with $\lambda = 1/4$ and the a priori weights of the States are obtained from the number of RS seats. Further, suppose that the a priori weight vector $\mathbf{v} = (v_1, \ldots, v_m)$ is ordered such that $v_1 \geq \cdots \geq v_m$. Then the condition in (11) ensures that the weight of any State is less than that of the Centre and the condition in (12) ensures that no State is a blocker. Both of these conditions are the same and is the following:

$$v_1 < \frac{\lambda}{1 - 2\lambda}(v_{m-r+2} + \cdots + v_m) \tag{16}$$

where $m = 31$ and $r = \lceil \tau m \rceil$. From Table 6, we see that the highest value of the a priori weight is 31, i.e., $v_1 = 31$. So, to satisfy (16) τ has to be chosen so that the sum $v_{m-r+2} + \cdots + v_m$ is greater than $2v_1 = 62$. Again, from Table 6, the number of RS seats of the last 19 States equals 62 and that of the last 20 States equals 71. So, to satisfy (16), we need $r - 1 = 20$ implying $r = 21$. Consequently, the threshold τ of voting quorum has to be set to be greater than 20/31. For example, the choice $\tau = 2/3$ will satisfy (16) while the choice $\tau = 1/2$ will not satisfy (16).

UP has the maximum number of RS seats which is $v_1 = 31$. Suppose UP and the last 19 States participate in the voting along with the Centre. The total number of RS seats of the last 19 States is 62 which is twice that of UP. Since $\lambda = 1/4$, the voting weight of all the States will be 3/4 of the total voting weight. Of this, UP will have one-third fraction and so the voting weight of UP will be $1/3 \times 3/4 = 1/4$ fraction of the total voting weight. So, UP will have the same weight as the Centre and hence will be a blocker. This shows the tightness of the lower bound which requires at least 21 States to participate in the voting to ensure that the weights of all the States are less than that of the Centre and that no State is a blocker.

The GST Act already sets the meeting quorum to be at least half of the total number of members of the GST Council. Presently, there are 33 members out of which 2 are representatives of the Centre and one representative from each of the 31 States. So, the meeting quorum requirement is that at least 17 members should attend a meeting. This could lead to a situation where both the representatives of the Centre attend the meeting and exactly 15 representatives of the States attend the meeting. In such a situation, the meeting quorum would be satisfied. Assuming that the representatives of all the 15 States take part in voting we will have $n = r = 15$ which would not satisfy (16) and so it is not guaranteed that Desiderata 3 and 4 will hold.

Desideratum 5: The game should be (α, β)-efficient with respect to the Centre where $\alpha \geq \beta$. This condition will ensure that the Centre will require at least as many States to pass a resolution as the minimum number of States who need to come together to block a resolution. In other words, the Centre's job of mustering a supporting coalition should not be any easier than the possible role of a coalition of States to block a resolution. From Theorem 28, this condition is ensured by choosing $\lambda \leq 1/3$.

In view of our analysis, we put forward the following suggestions for modifying the voting procedure in the GST Council.

1. A resolution will be considered to be passed if it receives *more than* a fraction q of the total voting weight.

2. The Centre is assigned a fraction λ of the total voting weight and the winning threshold q is set to be equal to $1 - \lambda$.

3. Regarding the choice of the voting weights of the States, we have analysed the following two options without finding a clear quantifiable preference of the one over the other.

   (a) All States have equal voting weights.

   (b) The voting weights of the States are proportional to the number of seats in the RS.

   Our personal preference would be to have the voting weights of the States to be proportional to the number of RS seats. This is due to the fact that with this option, it becomes easier for the States to block a resolution. This will force the Centre to hold wider consultations with the States which is of crucial importance in a federal set-up like India.

4. A threshold τ of voting quorum should be specified. If the a priori weights are taken to be equal, then $\tau = 1/10$ is sufficient while if the a priori weights are taken to be the number of RS seats, then $\tau = 2/3$ is sufficient. This will ensure that Desiderata 3 and 4 hold. Additionally, a meeting quorum may be specified where the proceedings of a meeting are considered to be valid if at least $\lceil m/2 \rceil$ States participate in a meeting. This will cover cases where a decision is taken unanimously without any actual voting.
   Note that specifying a threshold of voting quorum provides a default blocking capability to the States. If more than $m - \lceil \tau m \rceil$ States decline to take part in the voting, then the result of the voting will not be valid. As discussed in Section 3, a similar default blocking power arises from the specification of a meeting quorum.

5. To satisfy Desideratum 5, λ needs to be at most $1/3$. We suggest the value of λ be chosen to be equal to $1/4$ which corresponds to $q = 3/4$. This is the value of winning threshold specified in the GST Act.

11 Conclusion

This work carried out an in-depth analysis of the voting rule specified in the GST Act. The analysis has been done using the formal framework of voting games. In the process, we introduced new notions to capture blocking phenomenon in such games. The theoretical study has been complemented by application to the concrete setting arising in the present political context. Finally, we identify a set of desiderata and put forward suggestions for modifying the voting rule in the GST Act.

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A Proofs

Proof of Theorem 2: Suppose $N = \{A_1, \ldots, A_n\}$, $\mathbf{w} = (w_1, \ldots, w_n)$ and let $\omega = \sum_{i=1}^{n} w_i$. S is a minimal blocking coalition in G if and only if the following two conditions hold:

$$w_S > (1-q) \cdot \omega \text{ and for any player } A_i \in S,\ w_S - w_i \leq (1-q) \cdot \omega. \quad (*)$$

By definition of modified weighted majority voting game, S is a minimal winning coalition in G'.

The proof of the converse is similar. □

Proof of Theorem 4: Since A_i is a blocker, $w_i > (1-q)\omega$ and since A_j is a non-blocker, $w_j \leq (1-q)\omega$ and so $w_i > w_j$. Consider a coalition S for which A_j is a blocking swing. If A_i is in S, then since $w_i > w_j$, it follows that A_i is a blocking swing in S. Now suppose that A_i is not in S. Since A_j is a blocking swing for S, it follows that $w_S > (1-q)\omega$ and $w_{S'} \leq (1-q)\omega$ where $S' = S \setminus \{A_j\}$. Let $T = S' \cup \{A_i\}$. Since $w_i > w_j$, we have $w_T > w_S > (1-q)\omega$. So, it follows that A_i is a blocking swing for T. Consequently, $\eta_G(A_j) \leq \eta_G(A_i)$. Since A_i is a blocker and A_j is not a blocker, A_i is a blocking swing in $\{A_i\}$, while there is no corresponding coalition where A_j is a blocking swing. So, $\eta_G(A_j) < \eta_G(A_i)$.

The argument for modified weighted majority game is similar. □

Proof of Theorem 5: Note that $\alpha = 0$ if and only if A_0 needs support from no other player to pass a resolution. In other words, A_0 is a dictator. This shows (1) $\Leftrightarrow$ (2). A_0 is a dictator if and only if no subset of $\{A_1, \ldots, A_n\}$ is a blocking coalition which is equivalent to β begin undefined. This shows (2) $\Leftrightarrow$ (3).

Consider any coalition S which contains A_0. Since S contains A_0, it is a blocking coalition. Further, since A_0 is a dictator, $S \setminus \{A_0\}$ is not a blocking coalition. So, A_0 is a blocking dictator.

For any $A_i \neq A_0$, suppose S contains A_i and S is a blocking coalition. Since A_0 is a dictator, S must contain A_0. Further, since A_0 is a blocker, $S \setminus \{A_i\}$ is still a blocking coalition. So, there is no coalition S such that A_i is a blocking swing in S. Consequently, A_i is a blocking dummy. □

Proof of Theorem 6: Let S be a non-empty coalition not containing A_0. Since $\alpha = 0$, A_0 is a dictator and so the only minimal winning coalition in the game is $\{A_0\}$. So, there cannot be any minimal winning coalition containing S. So, γ_S is undefined.

Since $\{A_0\}$ is the only minimal winning coalition, it follows from the definition of κ_S that $\kappa_S = 0$. The only minimal blocking coalition is $\{A_0\}$ and so both μ_S and ν_S are undefined. □

Proof of Theorem 9: Note that $\omega = w_0 + w_1 + \cdots + w_n = ub/d$ and so $w_0/\omega = a/b = \lambda$ and $w_i/\omega = u_i(b-a)/(ub) = (1-a/b)u_i/u = (1-\lambda)u_i/u$. This shows that $w_0, w_1, \ldots, w_n$ determined by (2) satisfy (3).

The following computation shows the second part of the result.

$$
\begin{aligned}
[w'_0 : w'_1 : \cdots : w'_n] &= \left[\frac{w'_0}{\omega'} : \frac{w'_1}{\omega'} : \cdots : \frac{w'_n}{\omega'}\right] \\
&= \left[\lambda : (1-\lambda)\frac{u_1}{u} : \cdots : (1-\lambda)\frac{u_n}{u}\right] \\
&= \left[\frac{a}{b} : \frac{(b-a)u_1}{bu} : \cdots : \frac{(b-a)u_n}{bu}\right] \\
&= [au : (b-a)u_1 : \cdots : (b-a)u_n] \\
&= \left[\frac{au}{d} : \frac{(b-a)u_1}{d} : \cdots : \frac{(b-a)u_n}{d}\right] \\
&= [w_0 : w_1 : \cdots : w_n].
\end{aligned}
$$

□

Proof of Theorem 10: A_0 is a blocker if and only if any coalition not containing A_0 is a losing coalition. The sum of weights of all the players other than the Centre is $w_1 + \cdots + w_n$ and $\omega = w_0 + w_1 + \cdots + w_n$. So, $(w_1 + \cdots + w_n)/\omega = (\omega - w_0)/\omega = 1 - \lambda < q$. So, any coalition not containing the Centre is necessarily a losing coalition. □

Proof of Theorem 11: From (2), we have $w_i < w_0$ if and only if $u_i/u < a/(b-a)$ where $u = u_1 + \cdots + u_n$. Using $\lambda = a/b$ we obtain the required result.

By the first part, the weight w_i of a State A_{j_i} in the game $G_{m,n,\mathbf{v},V,\lambda,q}$ is less than that of the Centre if $u_i/(u_1 + \cdots + u_n) < \lambda/(1-\lambda)$. We show that this inequality holds under the given conditions.

Note that from the definition of the GST game, $u_i = v_{j_i}$. Since the game has τ to be the threshold for voting quorum, we have $n \geq r = \lceil \tau m \rceil$. Under the given condition on the weight vector $\mathbf{v}$

$$
\begin{aligned}
u_1 + \cdots + u_n &= u_1 + \cdots + u_{i-1} + u_i + u_{i+1} + \cdots + u_n \\
&\geq v_{j_i} + v_{m-n+2} + \cdots + v_m \\
&\geq v_{j_i} + v_{m-r+2} + \cdots + v_m.
\end{aligned}
$$

So,

$$
\frac{u_i}{u_1 + \cdots + u_n} \leq \frac{v_{j_i}}{v_{j_i} + v_{m-r+2} + \cdots + v_m}. \tag{17}
$$

From (5) we have $v_1 < (\lambda/(1-2\lambda))(v_{m-r+2} + \cdots + v_m)$. Also, it is given that $v_{j_i} \leq v_1$ and so $v_{j_i} < (\lambda/(1-2\lambda))(v_{m-r+2} + \cdots + v_m)$. Since $\lambda < 1/2$ is given, $1 - 2\lambda > 0$ and so rearranging terms, we obtain

$$
\frac{v_{j_i}}{v_{j_i} + v_{m-r+2} + \cdots + v_m} < \frac{\lambda}{1-\lambda}. \tag{18}
$$

Combining (17) with (18), we obtain $u_i/(u_1 + \cdots + u_n) < \lambda/(1-\lambda)$ as required. □

Proof of Theorem 12: From (2), the State A_{j_i} has weight $w_i = u_i(b-a)/d$ while the sum of the weights of the Centre and the States in V is

$$
\begin{aligned}
\omega &= w_0 + w_1 + \cdots + w_n \\
&= \frac{ua}{d} + \frac{u_1(b-a)}{d} + \cdots + \frac{u_n(b-a)}{d} \\
&= \frac{ub}{d}
\end{aligned}
$$

since $u = u_1 + \cdots + u_n$.

A_{j_i} is a dictator if and only if $w_i/\omega > q$ which holds if and only if $u_i(b-a)/(ub) > q$. Using $\lambda = a/b$ provides the desired result for dictator.

Further, A_{j_i} is a blocker if and only if $w_i/\omega > 1-q$ which holds if and only if $u_i(b-a)/(ub) > 1-q$. Again using $\lambda = a/b$ we obtained the desired result for blocker.

The proof of the condition under which no State is a blocker is similar to the argument in the proof of Theorem 11 for the condition ensuring that all States have weights less than that of the Centre. □

Proof of Theorem 13: By the definition of GST games, $\lambda < q$ and so the Centre is not a blocker. Consequently, using Theorem 5, we have that $\alpha > 0$ and $\beta > 0$.

Let the weight of the Centre be w_0 and that of the n states be $w_1, \ldots, w_n$ where we assume without loss of generality that $w_1 \geq w_2 \geq \cdots \geq w_n$. Let $\omega = w_0 + w_1 + \cdots + w_n$. Note that $w_0/\omega = \lambda$. Since G is (α, β)-efficient with respect to the Centre, it follows that the minimum size of a winning coalition is α and the minimum size of a blocking coalition is β. Using the order condition on the w_i's, we have the following relations.

$$w_0 + w_1 + \cdots + w_\alpha \geq q\omega; \tag{19}$$

$$w_1 + \cdots + w_\beta > (1-q)\omega. \tag{20}$$

Now $\alpha < \beta$ if and only if the sum of the first α weights is at most $(1-q)\omega$, i.e., if and only if

$$w_1 + \cdots + w_\alpha \leq (1-q)\omega. \tag{21}$$

Combining (19) with (21), we have

$$q\omega - w_0 \leq w_1 + \cdots + w_\alpha \leq (1-q)\omega.$$

So, $q\omega - w_0 \leq (1-q)\omega$ which is equivalent to $2q - 1 \leq w_0/\omega = \lambda$. Rearranging, we obtain the desired inequality. □

Proof of Theorem 14: In a simple GST game, $u_1 = \cdots = u_n$ and so $u_i/(u_1 + \cdots + u_n) = 1/n$. Combining this with Theorem 11 and 12 we obtain the desired results. □

Proof of Theorem 15: Let $G_{m,n,\mathbf{v},V,\lambda,q}$ be a simple GST game where the u_i's are all equal. The weight w_0 of the Centre and the weights $w_1, \ldots, w_n$ of the n States which participate in the voting are obtained from (2). Since the u_i's are equal, so are the w_i's. Let $\omega = w_0 + w_1 + \cdots + w_n$. So, $w_0 = \lambda\omega$ and $w_i = (1-\lambda)\omega/n$ for $i = 1, \ldots, n$.

Now consider a GST game $G'_{n,\mathbf{v}',V,\lambda,q}$ where the a priori weights v'_i of the States are not equal. The weights in the voting game are again obtained from (2). In G', let the weight of the Centre be w'_0 and the weights of the States be $w'_1, \ldots, w'_n$. Without loss of generality, we assume that $w'_1 \geq w'_2 \geq \cdots \geq w'_n$. Let $\omega' = w'_0 + w'_1 + \cdots + w'_n$. So, $w'_0 = \lambda\omega'$ and $w'_1 + \cdots + w'_n = (1-\lambda)\omega'$.

Claim 1: The minimum of the sizes of the minimal winning coalitions in G' is at most the minimum of the sizes of the minimal winning coalitions in G.

Proof of Claim 1: Suppose G has a winning coalition consisting of the Centre and k of the States. Then

$$\frac{\lambda\omega + k\omega(1-\lambda)/n}{\omega} = \lambda + \frac{k(1-\lambda)}{n} \geq q. \tag{22}$$

Consider the coalition S consisting of the Centre and the States having weights $w'_1, \dots, w'_k$. We show that S is a winning coalition.

$$\begin{aligned}
\frac{w_S}{\omega'} &= \frac{w'_0 + w'_1 + \cdots + w'_k}{\omega'} \\
&= \frac{\lambda\omega' + w'_1 + \cdots + w'_k}{\omega'} \\
&= \lambda + \frac{w'_1 + \cdots + w'_k}{\omega'} \\
&= \lambda + (1-\lambda)\frac{w'_1 + \cdots + w'_k}{w'_1 + \cdots + w'_n}. \qquad (23)
\end{aligned}$$

From $w'_1 \geq w'_2 \geq \cdots \geq w'_n$, we have

$$\begin{aligned}
(n-k)(w'_1 + \cdots + w'_k) &\geq (n-k)kw'_k \\
&\geq k(n-k)w'_{k+1} \\
&\geq k(w'_{k+1} + \cdots + w'_1).
\end{aligned}$$

This shows $n(w'_1 + \cdots + w'_k) \geq k(w'_1 + \cdots + w'_n)$ and consequently

$$\frac{w'_1 + \cdots + w'_k}{w'_1 + \cdots + w'_n} \geq \frac{k}{n}. \qquad (24)$$

Combining (23) with (22) and (24) we obtain

$$\begin{aligned}
\frac{w_S}{\omega'} &= \lambda + (1-\lambda)\frac{w'_1 + \cdots + w'_k}{w'_1 + \cdots + w'_n} \\
&\geq \lambda + \frac{k(1-\lambda)}{n} \\
&\geq q.
\end{aligned}$$

This shows that if there is a winning coalition of size k in G, then there is also a winning coalition of size k in G'. So, the minimum of the sizes of the minimal winning coalitions in G' is at most the minimum of the sizes of the minimal winning coalitions in G. This establishes Claim 1.

Claim 2: The minimum of the sizes of the minimal blocking coalitions not containing the Centre in G' is at most the minimum of the sizes of the minimal blocking coalitions not containing the Centre in G.
Proof of Claim 2: In G, suppose that ℓ of the States form a blocking coalition. Then

$$\frac{\ell(1-\lambda)\omega/n}{\omega} = \frac{\ell(1-\lambda)}{n} > 1-q. \qquad (25)$$

In G', consider the coalition T consisting of the States having the weights $w'_1, \dots, w'_\ell$. Then

$$\begin{aligned}
\frac{w_T}{\omega'} &= \frac{w'_1 + \cdots + w'_\ell}{\omega'} \\
&= (1-\lambda)\frac{w'_1 + \cdots + w'_\ell}{w'_1 + \cdots + w'_n}. \qquad (26)
\end{aligned}$$

As in the proof of Claim 1, it can be shown that

$$\frac{w'_1 + \cdots + w'_\ell}{w'_1 + \cdots + w'_n} \geq \frac{\ell}{n}$$

and so from (26) and (25), we obtain

$$\begin{aligned} \frac{w_T}{\omega'} &\geq \lambda + \frac{k(1-\lambda)}{n} \\ &\geq 1-q. \end{aligned}$$

This shows that if there is a blocking coalition not containing the Centre of size ℓ in G, then there is a blocking coalition not containing the Centre of size ℓ in G'. So, the minimum of the sizes of the minimal blocking coalitions not containing the Centre in G' is at most the minimum of the sizes of the minimal blocking coalitions not containing the Centre in G. This establishes Claim 2.

From Claims 1 and 2, it follows that if G is (α, β)-efficient with respect to the Centre, then G' is (α', β')-efficient with respect to the Centre where $\alpha' \leq \alpha$ and $\beta' \leq \beta$. □

Proof of Theorem 16: Since G is (α, β)-efficient with respect to the Centre, there is a minimal winning coalition of minimum size containing the Centre and α other States and a minimal blocking coalition of minimum size containing β States. Suppose A_{j_i} has $(\gamma_i, \kappa_i, \mu_i, \nu_i)$-influence with respect to the Centre.

Since G is symmetric with respect to the States, for any State A_{j_i} there is a minimal winning coalition of minimum size consisting of the Centre and α States of which A_{j_i} is a member and a minimal blocking coalition of minimum size β containing A_{j_i}. This shows $\gamma_i = \alpha$ and $\mu_i = \beta$. Again using the symmetry of G with respect to the players, leaving out A_{j_i}, then there is a minimal winning coalition of minimum size containing the Centre and α States other than A_{j_i} and a minimal blocking coalition of minimum size not containing either A_{j_i} or the Centre and having size β. This shows $\kappa_i = \alpha$ and $\nu_i = \beta$.

Proof of Theorem 17: Let $\lambda = a/b$. Since all the *a priori* weights are equal, from (2) we have $u = n$, $w_0 = an/d$ and $w_i = (b-a)/d$ for $i = 1, \ldots, n$ and so $\omega = bu/d = bn/d$. Suppose the Centre and k other states form a coalition. Such a coalition is a winning coalition if and only if

$$\begin{aligned} \frac{k(b-a)}{d} + \frac{an}{d} &\geq q\frac{bn}{d} \\ \Leftrightarrow k &\geq \frac{n(qb-a)}{b-a} = n\frac{q-\lambda}{1-\lambda} = n\frac{(1-\lambda)-(1-q)}{1-\lambda} = n - \frac{n(1-q)}{1-\lambda} \\ \Leftrightarrow k &\geq n - \left\lfloor \frac{n(1-q)}{1-\lambda} \right\rfloor = n - \rho \\ \Leftrightarrow k &\geq \mathfrak{p}. \end{aligned}$$

A coalition is blocking if and only if the sum of the weights of the players in the coalition is greater than $(1-q)\omega$. A coalition consisting of ℓ states has weight $\ell(b-a)/d$ and so it is blocking if and only if

$$\begin{aligned}
\frac{\ell(b-a)}{d} &> \frac{(1-q)bn}{d} \\
\Leftrightarrow \ell &> \frac{n(1-q)b}{(b-a)} = \frac{n(1-q)}{1-\lambda} \\
\Leftrightarrow \ell &> \left\lfloor \frac{n(1-q)}{1-\lambda} \right\rfloor \\
\Leftrightarrow \ell &\geq 1 + \left\lfloor \frac{n(1-q)}{1-\lambda} \right\rfloor = 1+\rho \\
\Leftrightarrow \ell &\geq \mathfrak{q}.
\end{aligned}$$

As a result, we have $\mathfrak{p}+\mathfrak{q} = n+1$. □

Proof of Theorem 19: From (7), we have $\mathfrak{p}_1 = n-\rho_1$ and $\mathfrak{p}_2 = n-\rho_2$ where

$$\rho_1 = \left\lfloor \frac{n(1-q)}{(1-\lambda_1)} \right\rfloor \text{ and } \rho_2 = \left\lfloor \frac{n(1-q)}{(1-\lambda_2)} \right\rfloor.$$

So,

$$\begin{aligned}
\lambda_1 \geq \lambda_2 &\Leftrightarrow \rho_1 \geq \rho_2 \\
&\Leftrightarrow \mathfrak{p}_1 \leq \mathfrak{p}_2.
\end{aligned}$$

The inequality cannot be made strict since it is possible that $\lambda_1 > \lambda_2$ but, $\rho_1 = \rho_2$.

From (8), we have $\mathfrak{q}_1 = 1+\rho_1$ and $\mathfrak{p}_2 = 1+\rho_2$.

$$\begin{aligned}
\lambda_1 \geq \lambda_2 &\Leftrightarrow \rho_1 \geq \rho_2 \\
&\Leftrightarrow \mathfrak{q}_1 \geq \mathfrak{q}_2.
\end{aligned}$$

Again, the inequality cannot be made strict. □

Proof of Theorem 20: Since $q > 1-\lambda$, any winning coalition must contain the Centre. For a winning coalition to be minimal, apart from the Centre it must contain exactly $\mathfrak{p}$ of the states. There are a total of n states and so the number of minimal winning coalitions is the number of ways in which $\mathfrak{p}$ states can be chosen from n states. □

Proof of Theorem 21: Since $q > 1-\lambda$, the Centre must be part of any winning coalition. Since the Centre and the $\mathfrak{p}$ states in S vote as a bloc, any winning coalition must also contain all the states in S. The Centre along with $\mathfrak{p}$ states already form a minimal winning coalition. So, the presence or absence of any other state in the coalition does not make a difference. Consequently, there is no minimal winning coalition containing any of the other states. □

Proof of Theorem 22: Consider any A_j state outside the bloc. Since $q > 1-\lambda$, any winning coalition must contain the Centre and so the whole bloc. For the state A_j to be a swing in such a winning coalition, it must be part of the coalition and its removal from the coalition results in a losing coalition. So, a swing containing A_j also contains the Centre and exactly $\mathfrak{p}$ states. Out of these $\mathfrak{p}$ states, k states of the bloc are necessarily present in the coalition. It also has A_j. So, an additional $\mathfrak{p}-k-1$ states are required to form a coalition in which A_j is a swing. Apart from A_j and k states in the bloc, there are a total of $n-k-1$ other states. So, the $\mathfrak{p}-k-1$ states are to be chosen from $n-k-1$ states and this can be done in $\binom{n-k-1}{\mathfrak{p}-k-1}$ ways giving rise to these many swings for A_j. □

Proof of Theorem 23: Apart from the bloc containing the Centre, any minimal winning coalition must contain exactly $\mathfrak{p} - k$ of the other states. These states can be chosen from $n - k$ states outside the bloc. □

Proof of Theorem 24: There are a total of n states in $G_{m,n,\mathbf{v},V,\lambda,q}$. Out of these k_1 states are part of S_1 and k_2 states are part of S_2. A minimal winning coalition must contain S_1 and $\mathfrak{p} - k_1$ other states. Since S_1 and S_2 vote in opposition to each other, a minimal winning coalition cannot contain any member of S_2. Apart from the states in S_1 and S_2 there are a total of $n - k_1 - k_2$ other states and a total of $\mathfrak{p} - k_1$ states can be selected from these states in $\binom{n-k_1-k_2}{\mathfrak{p}-k_1}$ ways. This proves the first point.

Any minimal blocking coalition must have $\mathfrak{q}$ states. If such a minimal blocking coalition contains S_2, then it must contain $\mathfrak{q} - k_2$ other states. Since the states in S_1 and S_2 vote in opposition, the $\mathfrak{q} - k_2$ states can be chosen from $n - k_1 - k_2$ states which gives the required count.

Any swing S containing A_i is a winning coalition and hence contains S_1. Since blocs S_1 and S_2 are in opposition, S cannot contain any element of S_2. To be a swing, apart from the Centre, S must have $\mathfrak{p}$ states. Out of these, k_1 states are accounted for by the bloc S_1 and one state is accounted for by A_i. So, there are $\mathfrak{p} - k_1 - 1$ other states in S that need to be accounted for. These states can be selected from the $n - k_1 - k_2 - 1$ states other than those in $S_1 \cup S_2 \cup \{A_i\}$. This gives the count of the number of swings for A_i. The argument for the number of blocking swings is similar. □

Proof of Theorem 25: A_0 is a blocker if and only if any coalition not containing A_0 is a losing coalition. The sum of weights of all the players other than the Centre is $w_1 + \cdots + w_n$ and $\omega = w_0 + w_1 + \cdots + w_n$. So, $(w_1 + \cdots + w_n)/\omega = (\omega - w_0)/\omega = 1 - \lambda = q$. So, in the modified GST game, any coalition not containing the Centre is necessarily a losing coalition. □

Proof of Theorem 27: We proceed as in the proof of Theorem 12. From (2), the State A_{j_i} has weight $w_i = u_i(b - a)/d$ while the sum of the weights of the Centre and the States in V is $\omega = ub/d$. A_{j_i} is a dictator if $w_i/\omega > q = 1 - \lambda$ which holds if and only if $u_i(b - a)/(ub) > 1 - \lambda$. Using $\lambda = a/b$, A_{j_i} is a dictator if and only if $u_i > u$. Since $u = u_1 + \cdots + u_n$ and each of the u_j's are positive real numbers, the condition $u_i > u$ cannot arise. Hence, no State can be a dictator.

A_{j_i} is a blocker if $w_i/\omega \geq 1 - q$ which holds if and only if $u_i(b - a)/(ub) \geq 1 - q$. Again using $\lambda = a/b$ we obtain the desired result for blocker.

The proof of the condition ensuring that no State is a blocker is similar to the argument in the proof of Theorem 11 for the condition ensuring that all States have weights less than that of the Centre. □

Proof of Theorem 28: From the definition of modified GST games, the Centre is not a blocker. Consequently, using Theorem 5, we have that $\alpha > 0$ and $\beta > 0$.

Let the weight of the Centre be w_0 and that of the n states be $w_1, \ldots, w_n$ where we assume without loss of generality that $w_1 \geq w_2 \geq \cdots \geq w_n$. Let $\omega = w_0 + w_1 + \cdots + w_n$. Note that $w_0/\omega = \lambda$ and $q = 1 - \lambda$. Since G is (α, β)-efficient with respect to the Centre, it follows that the minimum size of a winning coalition is α and the minimum size of a blocking coalition is β. Using the order condition on the w_i's, we have the following relations.

$$w_0 + w_1 + \cdots + w_\alpha > q\omega; \quad (27)$$
$$w_1 + \cdots + w_\beta \geq (1 - q)\omega. \quad (28)$$

Now $\alpha < \beta$ if and only if the sum of the first α weights is less than $(1 - q)\omega$, i.e., if and only if

$$w_1 + \cdots + w_\alpha < (1 - q)\omega. \quad (29)$$

Combining (27) with (29), we have

$$q\omega - w_0 < w_1 + \cdots + w_\alpha < (1-q)\omega.$$

So, $q\omega - w_0 < (1-q)\omega$ which is equivalent to $2q-1 < w_0/\omega = \lambda$. Using $q = 1-\lambda$, the condition $2q-1 < \lambda$ becomes $\lambda > 1/3$. □

Proof of Theorem 29: In a simple modified GST game, $u_1 = \cdots = u_n$ and so $u_i/(u_1 + \cdots + u_n) = 1/n$. Combining this with Theorem 11 and 12 we obtain the desired results. □

Proof of Theorem 30: With $q = 1-\lambda$, this proof is similar to that of Theorem 15. □

Proof of Theorem 31: This proof is similar to the proof of Theorem 16.

Proof of Theorem 32: Let $\lambda = a/b$. Since all the *a priori* weights are equal, from (2) we have $u = n$, $w_0 = an/d$ and $w_i = (b-a)/d$ for $i = 1, \ldots, n$ and so $\omega = bu/d = bn/d$. Suppose the Centre and k other states form a coalition. Such a coalition is a winning coalition if and only if

$$\begin{aligned}
\frac{k(b-a)}{d} + \frac{an}{d} &> (1-\lambda)\frac{bn}{d} \\
\Leftrightarrow k &> \frac{n((1-\lambda)b - a)}{b-a} = n\frac{(1-2\lambda)}{1-\lambda} = n - \frac{n\lambda}{1-\lambda} \\
\Leftrightarrow k &> n - \left\lceil \frac{n\lambda}{1-\lambda} \right\rceil = n - \delta \\
\Leftrightarrow k &\geq n - \left\lceil \frac{n\lambda}{1-\lambda} \right\rceil + 1 = n - \delta + 1 \\
\Leftrightarrow k &\geq \mathfrak{s}.
\end{aligned}$$

A coalition is blocking if and only if the sum of the weights of the players in the coalition is greater than or equal to $\lambda\omega$. A coalition consisting of ℓ states has weight $\ell(b-a)/d$ and so it is blocking if and only if

$$\begin{aligned}
\frac{\ell(b-a)}{d} &\geq \frac{\lambda bn}{d} \\
\Leftrightarrow \ell &\geq \frac{n\lambda b}{(b-a)} = \frac{n\lambda}{1-\lambda} \\
\Leftrightarrow \ell &\geq \left\lceil \frac{n\lambda}{1-\lambda} \right\rceil = \delta \\
\Leftrightarrow \ell &\geq \mathfrak{t}.
\end{aligned}$$

As a result, we have $\mathfrak{s} + \mathfrak{t} = n + 1$. □

Proof of Theorem 34: The proof is similar to that of Theorem 19. □

Proof of Theorem 35: With $q = 1-\lambda$, this proof is similar to that of Theorem 20. □

Proof of Theorem 36: With $q = 1-\lambda$, this proof is similar to that of Theorem 36. □

Proof of Theorem 37: With $q = 1 - \lambda$, this proof is similar to that of Theorem 37. $\square$

Proof of Theorem 38: With $q = 1 - \lambda$, this proof is similar to that of Theorem 38. $\square$

Proof of Theorem 39: With $q = 1 - \lambda$, this proof is similar to that of Theorem 39. $\square$