Revisiting the Synthetic Control Estimator

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Abstract

We analyze the conditions under which the Synthetic Control (SC) estimator is unbiased. We show that the SC estimator is generally biased if treatment assignment is correlated with unobserved confounders, even when the number of pre-treatment periods goes to infinity, and in settings where one should expect an almost perfect pre-treatment fit. While our results suggest that researchers should be more careful in interpreting the identification assumptions required for the SC method, we show that, with a slight modification, the SC method can substantially improve in terms of bias and variance relative to standard methods.

Keywords: synthetic control; difference-in-differences; policy evaluation; linear factor model

JEL Codes: C13; C21; C23

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1 Introduction

In a series of influential papers, Abadie and Gardeazabal (2003), Abadie et al. (2010), and Abadie et al. (2015) proposed the Synthetic Control (SC) method as an alternative to estimate treatment effects in comparative case studies when there is only one treated unit. The main idea of the SC method is to use the pre-treatment periods to estimate weights such that a weighted average of the control units reconstructs the pre-treatment outcomes of the treated unit, and then use these weights to compute the counterfactual of the treated unit in case it were not treated. According to Athey and Imbens (2017), *"the simplicity of the idea, and the obvious improvement over the standard methods, have made this a widely used method in the short period of time since its inception"*, making it "*arguably the most important innovation in the policy evaluation literature in the last 15 years*". As one of the main advantages that helped popularize the method, Abadie et al. (2010) derive conditions under which the SC estimator would allow confounding unobserved characteristics with time-varying effects, as long as we can fit a long set of pre-intervention periods.[1]

In this paper, we revisit the conditions under which the SC estimator is unbiased in a linear factor model setting. We show that the SC estimator is generally biased if treatment assignment is correlated with unobserved confounders, even when the number of pre-treatment periods goes to infinity, and in settings where one should expect to have an almost perfect pre-treatment fit. While our results suggest that researchers should be more careful in interpreting the identification assumptions required for the SC method, we show that, with a slight modification, the SC method can substantially improve in terms of bias and variance relative to standard methods.

In a model with "stationary" common factors, we show that the SC weights converge in probability to weights that do *not*, in general, reconstruct the factor loadings of the treated unit when the number of pre-treatment periods (T_0) goes to infinity.[2,3] This happens because the SC weights converge to weights that simultaneously attempt to match the factor loadings of the treated unit *and* to minimize the variance of a linear combination of the transitory shocks. Therefore, weights that reconstruct the factor loadings of the treated unit are not generally the solution to this problem, even if such weights exist. While in many

[1] Abadie et al. (2010) derive this result based on a linear factor model for the potential outcomes. However, they point out that the SC estimator can be useful in more general contexts.

[2] We refer to "stationary" in quotation marks because we only need the assumption that pre-treatment averages of the first and second moments of the common factors converge when the number of pre-treatment periods goes to infinity for this result.

[3] We focus on the SC specification that uses all pre-treatment periods as economic predictors. Ferman et al. (2017) provide conditions under which the SC estimator using this specification is asymptotically equivalent to SC estimators using alternative specifications. We also consider the case of the average of the pre-treatment periods and the average of the pre-treatment periods plus other covariates as predictors in Appendix A.5.

SC applications T_0 may not be large enough to justify large-T_0 asymptotics (see, for example, Doudchenko and Imbens (2016)), our results should be interpreted as the SC weights not converging to weights that reconstruct the factor loadings of the treated unit, *even when T_0 is large*. We also show that the SC weights are biased estimators for weights that reconstruct the factor loadings of the treated unit when T_0 is finite. Moreover, based on our Monte Carlo (MC) simulations, the SC weights should be, on average, even farther from weights that reconstruct the factor loadings of the treated unit when T_0 is small.

As a consequence, the SC estimator is, in general, biased if treatment assignment is correlated with the unobserved heterogeneity, even when the number of pre-treatment periods goes to infinity.[4] The intuition is the following: if treatment assignment is correlated with common factors in the post-treatment periods, then we would need a SC unit that is affected in exactly the same way by these common factors as the treated unit, but did not receive the treatment. This would be attained with weights that reconstruct the factor loadings of the treated units. However, since the SC weights do not converge, in general, to weights that satisfy this condition, the distribution of the SC estimator will still depend on the common factors, implying in a biased estimator when selection depends on the unobserved heterogeneity.[5] These results do not rely on the fact that the SC unit is constrained to convex combinations of control units, which implies that they also apply to the panel data approach suggested by Hsiao et al. (2012).

One important implication of the SC restriction to convex combinations of the control units is that the SC estimator may be biased even if treatment assignment is only correlated with time-invariant unobserved variables, which is essentially the identification assumption of the difference-in-differences (DID) model. We therefore recommend a slight modification in the SC method, where we demean the data using information from the pre-intervention period, and then construct the SC estimator using the demeaned data.[6] If selection into treatment is only correlated with time-invariant common factors, then this demeaned SC estimator is unbiased. Assuming stability in the first and second moments of common factors and transitory shocks before and after the treatment, we also guarantee that this demeaned SC estimator has a lower asymptotic mean

[4] This is true whether we define asymptotic bias based on the expected value of the asymptotic distribution or on the limit of the expected value of the estimator. Details in Appendix A.4

[5] Ando and Sävje (2013) point out that the SC estimator can be biased if there is no set of weights that reconstructs the factor loadings of the treated unit. However, they do not analyze in detail the minimization problem that is used to estimate the SC weights. In contrast, we show that this minimization problem inherently leads to weights that do not reconstruct the factor loadings of the treated unit, *even if such weights exist*. Moreover, we show that this potential problem persists even when the number of pre-treatment periods is large.

[6] Demeaning the data before applying the SC estimator is equivalent to relaxing the non-intercept constraint, as suggested, in parallel to our paper, by Doudchenko and Imbens (2016). Unlike Doudchenko and Imbens (2016), we formally analyze the implication of this modification to the bias of the SC estimator.

squared error (MSE) relative to DID. However, if selection into treatment is correlated with time-varying common factors, then both the demeaned SC and the DID estimators would be asymptotically biased. For a particular class of linear factor models, we show that the demeaned SC estimator dominates the DID estimator in terms of asymptotic bias and asymptotic MSE.[7] Overall, while we show that the SC method is, in general, asymptotically biased if treatment assignment is correlated with time-varying confounders, we also show that it can still provide important improvement over DID. Our MC simulations suggest that such improvement relative to DID can be attained even when the pre-treatment match is imperfect and/or T_0 is small.[8]

Our results for models with "stationary" common factors are not as conflicting with the results from Abadie et al. (2010) as it might appear at first glance. The asymptotic bias of the SC estimator, in this case, goes to zero when the variance of the transitory shocks is small, in which case one should expect to have a close-to-perfect pre-treatment match and, therefore, Abadie et al. (2010) would recommend using the SC method.[9] When a subset of the common factors is non-stationary, however, we show that the asymptotic bias may not go to zero even in situations where one would expect a close-to-perfect pre-treatment fit. In a model with a combination of $I(1)$ common factors and/or deterministic polynomial trends in addition to $I(0)$ common factors, we show that the demeaned SC weights will converge to weights that reconstruct the factor loadings associated to the non-stationary common trends of the treated unit, but that will generally fail to reconstruct the factor loadings associated with the $I(0)$ common factors.[10] Therefore, in this setting, non-stationary common trends will not generate asymptotic bias in the demeaned SC estimator, but we need that treatment assignment is uncorrelated with the $I(0)$ common factors to guarantee asymptotic unbiasedness.[11] In this setting, a close-to-perfect pre-treatment match for a long set of pre-intervention

[7] This result is only valid for a particular set of linear factor models. We provide a very specific example in which the asymptotic bias and the MSE of the SC can be larger in Appendix A.3.

[8] We also provide in Appendix A.5.4 an instrumental variables estimator for the SC weights that generates an asymptotically unbiased SC estimator under additional assumptions on the error structure, which would be valid if, for example, the idiosyncratic error is serially uncorrelated *and* all the common factors are serially correlated. The idea behind this strategy is similar to the strategy outlined by Heckman and Scheinkman (1987).

[9] An important caveat is that the placebo test suggested by Abadie et al. (2010) may lead to over-rejection even when the variance of the transitory shocks is close to zero. This happens because, while the bias would be small in this scenario, the variance of the SC estimator would be small as well. See our companion paper Ferman and Pinto (2017) for details.

[10] We assume existence of weights that perfectly reconstructs the factor loadings of the treated unit associated with the non-stationary trends. In a setting with $\mathcal{I}(1)$ common factors, this is equivalent to assume that the vector of outcomes is cointegrated. If there were no set of weights that satisfies this condition, then the asymptotic distribution of the SC estimator would depend on the non-stationary common trends.

[11] The result that non-stationary common trends do not generate asymptotic bias is not valid for the original SC weights,

periods does not guarantee that the asymptotic bias of the SC estimator is close to zero. Given that, we recommend that researchers applying the SC method should also assess the pre-treatment fit of the SC estimator after de-trending the data.[12] We show that prominent SC applications that display a seemingly perfect pre-treatment fit in the original data does not provide such a perfect pre-treatment fit once the data is de-trended.

Our paper is related to a recent literature that analyzes the asymptotic properties of the SC estimator and of generalizations of the method. Gobillon and Magnac (2016) derive conditions under which the assumption of perfect match from Abadie et al. (2010) can be satisfied when both the number of pre-treatment periods *and* the number of control units go to infinity. Gobillon and Magnac (2016), Xu (2017), and Athey et al. (2017) provide alternative estimation methods that are asymptotically valid when the number of both pre-treatment periods and controls increase.[13] Unlike these papers, we consider the case with a finite number of control units, as do Abadie et al. (2010). We show that, in this case, the SC estimator can be biased even when $T_0 \to \infty$, and even when the pre-treatment fit is almost perfect. Carvalho et al. (2015) and Carvalho et al. (2016) also propose an alternative method that is related to the SC estimator, and derive conditions under which their estimator yields a consistent estimator. However, in a linear factor model as the one we consider, their assumptions would essentially exclude the possibility that treatment assignment is correlated with the unobserved heterogeneity, which is our main focus.[14] Amjad et al. (2017) suggest an interesting de-noising algorithm that leads to a consistent estimator even when the number of control units is fixed. Their method, however, relies on the assumption that transitory shocks are independent across units and

providing another reason to demean the data before applying the SC method.

[12] We do not imply that one should not use the SC method when the data is non-stationary. On the contrary, we show that the SC method is very efficient in dealing with non-stationary trends. The only caveat is that measures of pre-intervention fit could be misleading as diagnostic tests, as they may hide important discrepancies in the factor loadings associated to stationary common factors beyond these non-stationary trends. Given that, we recommend alternative diagnostic tests. Another possibility would be to apply the SC method on a transformation of the data that makes it stationary.

[13] Bai (2009) and Moon and Weidner (2015) consider the asymptotic properties of estimators in linear factor models when the number of time periods and the number of cross-sectional units jointly go to infinity, without restricting to the particular case of estimation of treatment effects.

[14] Their main assumption is that the outcomes of the control units are independent of treatment assignment. However, in our setting, if we assume that transitory shocks are uncorrelated with the treatment assignment, then the potential outcomes of the treated unit being correlated with treatment assignment implies that treatment assignment is correlated with the common factors. If this is the case, then it cannot be that the outcomes of the control units are independent of the treatment assignment. In an extension, Carvalho et al. (2015) consider the case in which the intervention also affects the control units. They model that as a structural change in the common factors after the treatment, in which case they find that their estimator would be biased. Note, however, that they do not treat such change in the common factors as selection on unobservables. Instead, they consider this as a case in which the intervention *affects* all units.

time. Under this assumption, an IV-like SC estimator we present in Appendix A.5.4 would also be valid. We do not focus on this strategy because it relies on the assumption that the time-series correlation of the outcome variable can only be driven by serial correlation in the common factors. Moreover, Amjad et al. (2017) assume that both common factors and factor loadings are deterministic, making it harder to model selection on unobservables. Finally, building on our paper, Powell (2017) proposes a 2-step estimation in which the SC unit is constructed based on the fitted values of the outcomes on unit-specific time trends. However, we show that the demeaned SC method is already very efficient in controlling for polynomial time trends, so the possibility of asymptotic bias in the SC estimator would come from correlation between treatment assignment and common factors beyond such time trends, which would not generally be captured in the strategy proposed by Powell (2017).

The remainder of this paper proceeds as follows. We start Section 2 with a brief review of the SC estimator. We highlight in this section that we rely on different assumptions relative to Abadie et al. (2010). In Section 3, we show that, in a model such that pre-treatment averages of the first and second moments of the common factors converge, the SC estimator is, in general, asymptotically biased. In Section 4, we contrast the SC estimator with the DID estimator, and propose the demeaned SC estimator. In Section 5, we consider a setting in which pre-treatment averages of the common factor diverge, and we show that, in this case, the SC estimator can be asymptotically biased even if we have a close-to-perfect pre-treatment match. We revisit the applications from Abadie and Gardeazabal (2003), Abadie et al. (2010), and Abadie et al. (2015) in light of these results. In Section 6, we present a particular class of linear factor models in which we consider the asymptotic properties of the SC estimator, and MC simulations with finite T_0. We conclude in Section 7.

2 Base Model

Suppose we have a balanced panel of $J+1$ units indexed by i observed on a total of T periods. We want to estimate the treatment effect of a policy change that affected only unit $j=1$, and we have information before and after the policy change. Let T_0 be the number of pre-intervention periods. Since we want to consider the asymptotic behavior of the SC estimator when $T_0 \rightarrow \infty$, we label the periods as $t \in \{-T_0+1, ..., -1, 0, 1, ..., T_1\}$, where $T_1 = T - T_0$ is the total number of post-treatment periods. The potential outcomes are given by

$$\begin{cases} y_{it}(0) = \delta_t + \lambda_t \mu_i + \varepsilon_{it} \\ y_{it}(1) = \alpha_{it} + y_{it}(0) \end{cases} \tag{1}$$

where δ_t is an unknown common factor with constant factor loadings across units, λ_t is a $(1 \times F)$ vector of common factors, μ_i is a $(F \times 1)$ vector of unknown factor loadings, and the error terms ε_{it} are unobserved transitory shocks. We only observe $y_{it} = d_{it} y_{it}(1) + (1 - d_{it}) y_{it}(0)$, where $d_{it} = 1$ if unit i is treated at time t. Since we hold the number of units $(J + 1)$ fixed and look at asymptotics when the number of pre-treatment periods goes to infinity, we treat the vector of unknown factor loads (μ_i) as fixed and the common factors (λ_t) as random variables. In order to simplify the exposition of our main results, we consider the model without observed covariates Z_i. In Appendix Section A.5.2 we consider the model with covariates. The main goal of the SC method is to estimate the effect of the treatment for unit 1 for each post-treatment t, that is $\{\alpha_{11}, ..., \alpha_{1T_1}\}$.

Since the SC estimator is only well defined if it actually happened that one unit received treatment in a given period, all results of the paper are conditional on that. Let $D(j, t)$ be a dummy variable equal to 1 if unit j starts to be treated after period t, while all other units do not receive treatment. Without loss of generality, we consider that unit 1 is treated and that treatment starts after $t = 0$, so $D(1, 0) = 1$. Assumption 1 defines the sample a researcher observes in the SC problem.

Assumption 1 (conditional sample) We observe a realization of $\{y_{1t}, ..., y_{J+1,t}\}_{t=-T_0+1}^{T_1}$ conditional on $D(1, 0) = 1$.

We also impose that the treatment assignment is not informative about the first moment of the transitory shocks.

Assumption 2 (transitory shocks) $\mathbb{E}[\varepsilon_{jt} | D(1, 0) = 1] = \mathbb{E}[\varepsilon_{jt}] = 0$ for all j and t.

Assumption 2 implies that transitory shocks are mean-independent from the treatment assignment. However, we still allow for the possibility that treatment assignment to unit 1 is correlated with the unobserved common factors. More specifically, we allow for $\mathbb{E}[\lambda_t | D(1, 0) = 1] \neq \mathbb{E}[\lambda_t]$. To better understand the implications of this possibility, suppose that the treatment is more likely to happen in unit j at time t if $\lambda_t \mu_j$ is high, and let λ_t^1 be a common factor that strongly affects unit 1.[15] Under these conditions, the fact that unit 1 is treated after $t = 0$ is informative about the common factor λ_t^1, because one should expect $\mathbb{E}[\lambda_t^1 | D(1, 0) = 1] > \mathbb{E}[\lambda_t^1]$. We allow for dependence between treatment assignment and common factors both before and after the start of the treatment. So we can consider, for example, a case in which treatment is triggered in unit 1 by a sequence of positive shocks on $\lambda_t \mu_1$ even before $t = 0$.

In order to present the main intuition of the SC estimator, we assume that there exists a stable linear combination of the control units that absorbs all time correlated shocks of unit 1, $\lambda_t \mu_1$. However, this

[15] That is, the factor loading of unit 1 associated with this common factor, μ_1^1, is large.

assumption is not necessary for any of our main results. Following the original SC papers, we restrict to convex combinations of the control units. We relax these constraints in Section 4.

Assumption 3 (existence of weights)

$$\exists\ \mathbf{w}^* \in \mathbb{R}^J \mid \mu_1 = \sum_{j\neq 1} w_j^* \mu_j,\ \sum_{j\neq 1} w_j^* = 1, \text{ and } w_j^* \geq 0$$

There is no guarantee that there is only one set of weights that satisfies Assumption 3, so we define $\Phi = \{\mathbf{w} \in \mathbb{R}^J \mid \mu_1 = \sum_{j\neq 1} w_j \mu_j,\ \sum_{j\neq 1} w_j = 1, \text{ and } w_j \geq 0\}$ as the set of weights that satisfy this condition. For all our main results, it may be that Assumption 3 does not hold, which implies $\Phi = \varnothing$.

If we knew $\mathbf{w}^* \in \Phi$, then we could consider an infeasible SC estimator using these weights, $\hat{\alpha}_{1t}^* = y_{1t} - \sum_{j\neq 1} w_j^* y_{it}$. For a given $t > 0$, we would have

$$\hat{\alpha}_{1t}^* = y_{1t} - \sum_{j\neq 1} w_j^* y_{it} = \alpha_{1t} + \left(\varepsilon_{1t} - \sum_{j\neq 1} w_j^* \varepsilon_{jt} \right). \tag{2}$$

We consider the expected value of $\hat{\alpha}_{1t}^*$ conditional on $D(1,0) = 1$ (Assumption 1). Therefore, under Assumption 2, $\mathbb{E}[\hat{\alpha}_{1t}^* | D(1,0) = 1] = \alpha_{1t}$, which implies that this infeasible SC estimator is unbiased. Intuitively, the infeasible SC estimator constructs a SC unit for the counterfactual of y_{1t} that is affected in the same way as unit 1 by each of the common factors (that is, $\mu_1 = \sum_{j\neq 1} w_j^* \mu_j$), but did not receive treatment. Therefore, the only difference between unit 1 and this SC unit, beyond the treatment effect, would be given by the transitory shocks, which we assumed are not related to the treatment assignment (Assumption 2). This guarantees that a SC estimator, using these infeasible weights, provides an unbiased estimator. Since there might be multiple weights in Φ, we define the infeasible SC estimator from equation 2 considering $\mathbf{w}^* \in \Phi$ that minimizes $var(\hat{\alpha}_{1t}^*)$.

It is important to note that Abadie et al. (2010) do note make any assumption on the existence of weights that reconstruct the factor loadings of the treated unit. Instead, they consider that there is a set of weights that satisfies $y_{1t} = \sum_{j\neq 1} w_j^* y_{jt}$ for all $t \leq 0$. While subtle, this reflects a crucial difference between our setting and the setting considered in the original SC papers. Abadie et al. (2010) and Abadie et al. (2015) consider the properties of the SC estimator conditional on having a good pre-intervention fit. As stated by Abadie et al. (2015), they *"do not recommend using this method when the pretreatment fit is poor or the number of pretreatment periods is small"*. Abadie et al. (2010) provide conditions under which $y_{1t} = \sum_{j\neq 1} w_j^* y_{jt}$ for all $t \leq 0$ (for large T_0) implies that Assumption 3 must hold approximately. In this case, the bias of the SC estimator would be bounded by a function that goes to zero when T_0 increases. We depart from the original SC setting in that we do not condition on having a perfect pre-intervention fit. The motivation to analyze the SC method in our setting is that, even if Assumption 3 is valid, in a model with only "stationary" factors, the probability that we find a perfect pre-intervention fit goes to zero when T_0 increases, unless the variance of the transitory shocks is equal to zero. Still, we show in Section 4 that the SC method can provide important improvement over the DID estimator, even if the pre-intervention fit is imperfect. Moreover, we also show in Section 5 that, if a subset of the common factors is non-stationary, then the SC estimator may be asymptotically biased even if the pre-treatment fit is almost perfect.

In order to implement their method, Abadie et al. (2010) recommend a minimization problem using the pre-intervention data to estimate the SC weights. They define a set of K predictors where X_1 is a $(K \times 1)$ vector containing the predictors for the treated unit, and X_0 is a $(K \times J)$ matrix of economic predictors for the control units.[16] The SC weights are estimated by minimizing $||X_1 - X_0\mathbf{w}||_V$ subject to $\sum_{i=2}^{J+1} w_j = 1$ and $w_j \geq 0$, where V is a $(K \times K)$ positive semidefinite matrix. They discuss different possibilities for choosing the matrix V, including an iterative process where V is chosen such that the solution to the $||X_1 - X_0\mathbf{w}||_V$ optimization problem minimizes the pre-intervention prediction error. In other words, let $\mathbf{Y}_1^P$ be a $(T_0 \times 1)$ vector of pre-intervention outcomes for the treated unit, while $\mathbf{Y}_0^P$ be a $(T_0 \times J)$ matrix of pre-intervention outcomes for the control units. Then the SC weights would be chosen as $\widehat{\mathbf{w}}(V^*)$ such that V^* minimizes $||\mathbf{Y}_1^P - \mathbf{Y}_0^P\widehat{\mathbf{w}}(V)||$.

We focus on the case where one includes all pre-intervention outcome values as predictors. In this case, the matrix V that minimizes the second step of the nested optimization problem would be the identity matrix (see Kaul et al. (2015) and Doudchenko and Imbens (2016)), so the optimization problem suggested by Abadie et al. (2010) to estimate the weights simplifies to

$$\widehat{\mathbf{w}} \quad = \quad \underset{\mathbf{w} \in W}{\text{argmin}} \frac{1}{T_0} \sum_{t=-T_0+1}^{0} \left[y_{1t} - \sum_{j \neq 1} w_j y_{jt} \right]^2 \tag{3}$$

where $W = \{\widehat{\mathbf{w}} \in \mathbb{R}^J | w_j \geq 0 \text{ and } \sum_{j \neq 1} w_j = 1\}$.

Ferman et al. (2017) provide conditions under which the SC estimator using all pre-treatment outcomes as predictors will be asymptotically equivalent when $T_0 \to \infty$ to any alternative SC estimator such that the number of pre-treatment outcomes used as predictors goes to infinity with T_0.[17] Therefore, our results are also valid for these SC specifications under these conditions. In Appendix A.5 we also consider SC estimators

[16] Predictors can be, for example, linear combinations of the pre-intervention values of the outcome variable or other covariates not affected by the treatment.

[17] Ferman et al. (2017) show that this will be true if we assume that, for any subsequence $\{t_k\}_{k \in \mathbb{N}}$ with $t_k > t_{k-1}$, pre-treatment averages of second moments of the outcomes converge in probability to the same values.

using (1) the average of the pre-intervention outcomes as predictor, and (2) other time-invariant covariates in addition to the average of the pre-intervention outcomes as predictors.

3 Asymptotic Bias with "stationary" common factors

We start assuming that pre-treatment averages of the first and second moments of the common factors and the transitory shocks converge. Let $\epsilon_t = (\varepsilon_{1t}, ..., \varepsilon_{J+1,t})$.

Assumption 4 (convergence of pre-treatment averages) Conditional on $D(1,0)=1$, $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \lambda_t \overset{p}{\to} \omega_0$, $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \epsilon_t \overset{p}{\to} 0$, $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \lambda_t'\lambda_t \overset{p}{\to} \Omega_0$ positive semi-definite, $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \epsilon_t\epsilon_t' \overset{p}{\to} \sigma_\varepsilon^2 I_{J+1}$, and $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \epsilon_t\lambda_t \overset{p}{\to} 0$ when $T_0 \to \infty$.

Assumption 4 allows for serial correlation for both transitory shocks and common factors. We assume $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \epsilon_t\epsilon_t' \overset{p}{\to} \sigma_\varepsilon^2 I_{J+1}$ in order to simplify the exposition of our results. However, this can be easily replace by $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \epsilon_t\epsilon_t' \overset{p}{\to} \Sigma$ for any positive definite $(J+1)\times(J+1)$ matrix Σ, so that transitory shocks are correlated across j. Assumption 4 would be satisfied if the processes ϵ_t and λ_t are weakly stationary and second order ergodic in the pre-treatment period conditional on $D(1,0)=1$, and ϵ_t and λ_t are independent. However, such assumption would be too restrictive, and would not allow for important possibilities in the treatment selection process. Recall that Assumption 2 allows for $\mathbb{E}[\lambda_t|D(1,0)=1] \neq \mathbb{E}[\lambda_t]$, even for $t \leq 0$, which will happen if treatment assignment to unit 1 is correlated with common factors before treatment starts. In this case, it would be too restrictive to impose the assumption that, conditional on $D(1,0)=1$, λ_t is stationary, even if only for the pre-treatment periods.

We show first the convergence of $\widehat{\mathbf{w}}$.

Proposition 1 Under Assumptions 1, 2 and 4, we have that $\widehat{\mathbf{w}} \overset{p}{\to} \bar{\mathbf{w}}$ when $T_0 \to \infty$, where $\mu_1 \neq \sum_{j\neq 1} \bar{w}_j \mu_j$, unless $\sigma_\varepsilon^2 = 0$ or $\exists \mathbf{w} \in \Phi | \mathbf{w} \in \underset{\mathbf{w}\in W}{\text{argmin}} \left\{ \sum_{j\neq 1} (w_j)^2 \right\}$

Proof. Details in Appendix A.1.1 ■

The intuition of Proposition 1 is that we can treat the SC weights as an M-estimator, so we have that $\widehat{\mathbf{w}}$ converges in probability to $\bar{\mathbf{w}}$ such that

$$\bar{\mathbf{w}} = \underset{\mathbf{w}\in W}{\text{argmin}} \left\{ \sigma_\varepsilon^2 \left(1 + \sum_{j\neq 1} (w_j)^2 \right) + \left(\mu_1 - \sum_{j\neq 1} w_j \mu_j \right)' \Omega_0 \left(\mu_1 - \sum_{j\neq 1} w_j \mu_j \right) \right\} \tag{4}$$

which is the probability limit of the M-estimator objective function (equation 3).

This objective function has two parts. The first one reflects that different choices of weights will generate different weighted averages of the idiosyncratic shocks ε_{it}. In this simpler case, if we consider the specification that restricts weights to sum one, then this part would be minimized when we set all weights equal to $\frac{1}{J}$.[18] The second part reflects the presence of common factors λ_t that would remain after we choose the weights to construct the SC unit. If Assumption 3 is satisfied, then we can set this part equal to zero by choosing $\mathbf{w}^* \in \Phi$. Now start from $\mathbf{w}^* \in \Phi$ and move in the direction of weights that minimize the first part of this expression. Since $\mathbf{w}^* \in \Phi$ minimizes the second part, there is only a second order loss in doing so. On the contrary, since we are moving in the direction of weights that minimize the first part, there is a first order gain in doing so. This will always be true, unless $\sigma_\varepsilon^2 = 0$ or $\exists \mathbf{w} \in \Phi$ such that $\mathbf{w} \in \underset{\mathbf{w} \in W}{\text{argmin}} \left\{ \sum_{j \neq 1} (w_j)^2 \right\}$. Therefore, the SC weights will not generally converge to weights that reconstruct the factor loadings of the treated unit. If $\Phi = \varnothing$, then Proposition 1 trivially holds. Another intuition for this result is that the outcomes of the controls are proxy variables for the factor loadings, but they are measured with error. We present this interpretation in more detail in Appendix A.2.

For a given $t > 0$, the SC estimator is given by

$$\hat{\alpha}_{1t} = y_{1t} - \sum_{j \neq 1} \hat{w}_j y_{it} \xrightarrow{d} \alpha_{1t} + \left(\varepsilon_{1t} - \sum_{j \neq 1} \bar{w}_j \varepsilon_{jt} \right) + \lambda_t \left(\mu_1 - \sum_{j \neq 1} \bar{w}_j \mu_j \right) \text{ when } T_0 \to \infty. \quad (5)$$

Note that $\hat{\alpha}_{1t}$ converges in distribution to the parameter we want to estimate (α_{1t}) plus linear combinations of contemporaneous transitory shocks and common factors. Therefore, the SC estimator will be asymptotically unbiased if, conditional $D(1,0) = 1$, the expected values of these linear combinations of transitory shocks and common factors are equal to zero.[19] More specifically, we need that $\mathbb{E}[\varepsilon_{1t} - \sum_{j \neq 1} \bar{w}_j \varepsilon_{jt} | D(1,0) = 1] = 0$ and $\mathbb{E}[\lambda_t(\mu_1 - \sum_{j \neq 1} \bar{w}_j \mu_j) | D(1,0) = 1] = 0$. While the first equality is guaranteed by Assumption 2, the second one may not hold if treatment assignment is correlated with the unobserved heterogeneity.

Since $\mu_1 \neq \sum_{j \neq 1} \bar{w}_j \mu_j$, the SC estimator will only be asymptotically unbiased, in general, if we impose an additional assumption that $\mathbb{E}\left[\lambda_t^k | D(1,0) = 1\right] = 0$ for all common factors k such that $\mu_1^k \neq \sum_{j \neq 1} \bar{w}_j \mu_j^k$. In order to better understand the intuition behind this result, we consider a special case in which, conditional on $D(1,0) = 1$, λ_t is stationary for $t \leq 0$. In this case, we can assume, without loss of generality, that $\omega_0^1 = \mathbb{E}[\lambda_t^1] = 1$ and $\omega_0^k = \mathbb{E}[\lambda_t^k] = 0$ for $k > 0$. Therefore, the SC estimator will only be asymptotically unbiased if the weights turn out to recover unit 1 fixed effect (that is, $\mu_1^1 = \sum_{j \neq 1} \mu_j^1$) *and* treatment assignment is uncorrelated with time-varying unobserved common factors (that is, $\mathbb{E}[\lambda_t^k | D(1,0) = 1] = 0$

[18] If we do not impose this restriction, then this part would be minimized setting all weights equal to zero, and our main argument would remain valid.

[19] We consider the definition of asymptotic unbiasedness as the expected value of the asymptotic distribution of $\hat{\alpha}_{1t} - \alpha_{1t}$ equal to zero. An alternative definition is that $\mathbb{E}[\hat{\alpha}_{1t} - \alpha_{1t}] \to 0$. We show in Appendix A.4 that these two definitions are equivalent in this setting under standard assumptions.

for $t > 0$ and $k > 1$). Importantly, this implies that the SC estimator may be asymptotically biased even in settings in which the DID estimator is unbiased, as the DID estimator takes into account unobserved characteristics that are fixed over time, while the SC estimator would not necessarily do so. We discuss this issue in more detail in Section 4. We also discuss in Section 4 the implications of this result for the asymptotic MSE of the SC estimator.

While many SC applications does not have a large number of pre-treatment periods to justify large-T_0 asymptotics (see, for example, Doudchenko and Imbens (2016)), our results should be interpreted as the SC weights not converging to weights that reconstruct the factor loadings of the treated unit *even when* T_0 *is large*. In Appendix A.2, we show that, with finite T_0, the SC weights will be biased estimators for $\mathbf{w}^*$. The intuition for this result is that the SC method uses the vector of control outcomes as a proxy for the vector common factors. That is, we can write the potential outcome of the treated unit as a linear combination of the control units using a set of weights $\mathbf{w}^* \in \Phi$. However, in this case the control outcomes will be, by construction, correlated with the error in this model. The intuition is that the transitory shocks would behave as a measurement error in these proxy variables, which leads to bias. In Section 6, we show that, in our MC simulations, the SC weights are, on average, even further from weights that reconstruct the factor loadings of the treated unit when T_0 is finite.

The discrepancy of our results with the results from Abadie et al. (2010) arises because we rely on different assumptions. Abadie et al. (2010) consider the properties of the SC estimator conditional on having a perfect fit in the pre-treatment period in the data at hand. They do not consider the asymptotic properties of the SC estimator when T_0 goes to infinity. Instead, they provide conditions under which the bias of the SC estimator is bounded by a term that goes to zero when T_0 increases, *if the pre-treatment fit is perfect*. Our results are not as conflicting with the results from Abadie et al. (2010) as they may appear at first glance. In a model with "stationary" common factors, the probability that one would actually have a dataset at hand such that the SC weights provide a close-to-perfect pre-intervention fit with a moderate T_0 is close to zero, unless the variance of the transitory shocks is small. Therefore, our results agree with the theoretical results from Abadie et al. (2010) in that the aymptotic bias of the SC estimator should be small in situations where one would expect to have a close-to-perfect fit for a large T_0. An important caveat is that the placebo test suggested by Abadie et al. (2010) may lead to over-rejection even when the variance of the transitory shocks is close to zero. In this case, the asymptotic bias of the SC estimator will be close to zero. However, the variance of the SC estimator, which depends on a linear combination of the transitory shocks, will be close to zero as well. Therefore, even a small bias may lead to over-rejection under the null in this setting. We exploit in detail the implications of our results for the placebo test suggested by Abadie et al. (2010) in a companion paper (Ferman and Pinto (2017)). Moreover, in Section 5 we show that the SC estimator may remain biased even in settings where one would expect a close-to-perfect pre-treatment fit if we have non-stationary common factors.

In Appendix A.5 we consider alternative specifications used in the SC method to estimate the weights. In particular, we consider the specification that uses the pre-treatment average of the outcome variable as predictor, and the specification that uses the pre-treatment average of the outcome variable and other time-invariant covariates as predictors. In both cases, we show that the objective function used to calculate the weights converge in probability to a function that can, in general, have multiple minima. If Φ is non-empty, then $\mathbf{w} \in \Phi$ will be one solution. However, there might be $\mathbf{w} \notin \Phi$ that also minimizes this function, so there is no guarantee that the SC weights in these specifications will converge in probability to weights in Φ.

4 Comparison to DID & alternative SC estimators

We show in Section 3 that the SC estimator can be asymptotically biased even in situations where the DID estimator is unbiased. In contrast to the SC estimator, the DID estimator for the treatment effect in a given post-intervention period $t > 0$, under Assumption 4, would be given by

$$\begin{aligned} \hat{\alpha}_{1t}^{DID} &= y_{1t} - \frac{1}{J}\sum_{j\neq 1} y_{jt} - \frac{1}{T_0}\sum_{\tau=1}^{T_0}\left[y_{1\tau} - \frac{1}{J}\sum_{j\neq 1} y_{j\tau}\right] \\ &\overset{d}{\rightarrow} \alpha_{1t} + \left(\varepsilon_{1t} - \frac{1}{J}\sum_{j\neq 1}\varepsilon_{jt}\right) + (\lambda_t - \omega_0)\left(\mu_1 - \frac{1}{J}\sum_{j\neq 1}\mu_j\right) \text{ when } T_0 \rightarrow \infty. \end{aligned} \tag{6}$$

Therefore, the DID estimator will be asymptotically unbiased if $\mathbb{E}[\lambda_t | D(1,0) = 1] = \omega_0$, which means that the fact that unit 1 is treated after period $t = 0$ is not informative about the first moment of the common factors relative to their pre-treatment averages. Intuitively, the unit fixed effects control for any difference in unobserved variables that remain constant (in expectation) before and after the treatment. Moreover, the DID allows for arbitrary correlation between treatment assignment and δ_t (which is captured by the time effects). However, the DID estimator will be biased if the fact that unit 1 is treated after period $t = 0$ is informative about variations in the common factors relative to their pre-treatment mean.

As an alternative to the standard SC estimator, we suggest a modification in which we calculate the pre-treatment average for all units and demean the data. This is equivalent to a generalization of the SC method suggested, in parallel to our paper, by Doudchenko and Imbens (2016), which includes an intercept parameter in the minimization problem to estimate the SC weights and construct the counterfactual. Here we formally consider the implications of this alternative on the bias and MSE of the SC estimator.

The demeaned SC estimator is given by $\hat{\alpha}_{1t}^{SC'} = y_{1t} - \sum_{j\neq 1} \hat{w}_j^{\text{SC}'} y_{jt} - (\bar{y}_1 - \sum_{j\neq 1} \hat{w}_j^{\text{SC}'} \bar{y}_j)$, where $\bar{y}_j$ is the pre-treatment average of unit j, and the weights $\widehat{\mathbf{w}}^{\text{SC}'} = \{\hat{w}_j^{\text{SC}'}\}_{j=2}^{J+1}$ are given by

$$\widehat{\mathbf{w}}^{\text{SC}'} = \underset{\mathbf{w}\in W}{\text{argmin}} \frac{1}{T_0} \sum_{t=-T_0+1}^{0} \left[y_{1t} - \sum_{j\neq 1} w_j y_{jt} - \left(\bar{y}_1 - \sum_{j\neq 1} w_j \bar{y}_j \right) \right]^2 . \tag{7}$$

Proposition 2 Under Assumptions 1, 2 and 4, we have that $\widehat{\mathbf{w}}^{\text{SC}'} \overset{p}{\to} \bar{\mathbf{w}}^{\text{SC}'}$ when $T_0 \to \infty$, where $\mu_1 \neq \sum_{j\neq 1} \bar{w}_j^{\text{SC}'} \mu_j$, unless $\sigma_\varepsilon^2 = 0$ or $\exists \mathbf{w} \in \Phi | \mathbf{w} \in \underset{\mathbf{w}\in W}{\text{argmin}} \left\{ \sum_{j\neq 1} (w_j)^2 \right\}$. Moreover, for $t > 0$,

$$\hat{\alpha}_{1t}^{\text{SC}'} \overset{d}{\to} \alpha_{1t} + \left(\varepsilon_{1t} - \sum_{j\neq 1} \bar{w}_j^{\text{SC}'} \varepsilon_{jt} \right) + (\lambda_t - \omega_0) \left(\mu_1 - \sum_{j\neq 1} \bar{w}_j^{\text{SC}'} \mu_j \right) \text{ when } T_0 \to \infty. \tag{8}$$

Proof.

See details in Appendix A.1.2 ■

Therefore, the demeaned SC estimator is asymptotically unbiased under the same conditions as the DID estimator. Moreover, under the DID assumptions, both estimators are unbiased even for finite T_0. With additional assumptions on $(\varepsilon_{1t}, ..., \varepsilon_{J+1,t}, \lambda_t')$ in the post-treatment periods, we can also assure that the demeaned SC estimator is asymptotically more efficient than DID.

Assumption 5 (Stability in the pre- and post-treatment periods) For $t > 0$, $\mathbb{E}[\lambda_t | D(1,0) = 1] = \omega_0$, $\mathbb{E}[\epsilon_t | D(1,0) = 1] = 0$, $\mathbb{E}[\lambda_t' \lambda_t | D(1,0) = 1] = \Omega_0$, and $\mathbb{E}[\epsilon_t \epsilon_t' | D(1,0) = 1] = \sigma_\varepsilon^2 I_{J+1}$, $cov(\epsilon_t, \lambda_t | D(1,0) = 1) = 0$.

Assumptions 4 and 5 imply that transitory shocks and common factors have the same first and second moments in the pre- and post-treatment periods. From Proposition 2, Assumption 5 implies that the demeaned SC estimator is asymptotically unbiased. We now show that this assumption also implies that the demeaned SC estimator has lower asymptotic MSE than both the DID estimator and the infeasible SC estimator.

Proposition 3 Under Assumptions 1, 2, 4, and 5, the demeaned SC estimator $(\hat{\alpha}_{1t}^{\text{SC}'})$ dominates both the DID estimator $(\hat{\alpha}_{1t}^{DID})$ and the infeasible SC estimator $(\hat{\alpha}_{1t}^{*})$ in terms of asymptotic MSE when $T_0 \to \infty$.

Proof.

See details in Appendix A.1.3 ■

The intuition of this result is that, under Assumption 5, the demeaned SC weights converge to weights that minimize a function $\Gamma(\mathbf{w})$ such that $\Gamma(\bar{\mathbf{w}}^{\text{SC}'}) = a.var(\hat{\alpha}_{1t}^{\text{SC}'} | D(1,0) = 1)$, $\Gamma(\mathbf{w}^*) = a.var(\hat{\alpha}_{1t}^* | D(1,0) = 1)$, and $\Gamma(\{\frac{1}{J}, ..., \frac{1}{J}\}) = a.var(\hat{\alpha}_{1t}^{\text{DID}} | D(1,0) = 1)$. Therefore, it must be that the asymptotic variance of $\hat{\alpha}_{1t}^{\text{SC}'}$ is weakly lower than the variance of both $\hat{\alpha}^{*}_{1t}$ and $\hat{\alpha}_{1t}^{\text{DID}}$. Moreover, these three estimators are unbiased under these assumptions.

The demeaned SC estimator dominates the infeasible one, in terms of MSE, because the infeasible SC estimator focuses on eliminating the common factors, even if this means using a linear combination of the transitory shocks with higher variance. In contrast, the demeaned SC estimator provides a better balance in terms of the variance of the common factors and transitory shocks. This dominance of the demeaned SC estimator, however, relies crucially on the assumption that the first and second moments of the common factors and transitory shocks remain stable before and after the treatment. If we had that $\mathbb{E}[\lambda_t'\lambda_t|D(1,0)=1] \neq \Omega_0$ for $t>0$, then $\Gamma(\mathbf{w})$ would not provide the variance of the estimators with weights $\mathbf{w}$. Therefore, it would not be possible to guarantee that the demeaned SC estimator has lower variance, even if the three estimators are unbiased.

If we had that $\mathbb{E}[\lambda_t|D(1,0)=1] \neq \omega_0$ for $t>0$, then both the demeaned SC and the DID estimators would be asymptotically biased, while the infeasible SC estimator would remain unbiased. The asymptotic bias of $\hat{\alpha}_{1t}^{\text{SC}'}$ would be given by $(\mathbb{E}[\lambda_t|D(1,0)=1]-\omega_0)(\mu_1 - \sum_{j\neq 1} \bar{w}_j^{\text{SC}'}\mu_j)$. Therefore, provided $\mu_1 \neq \sum_{j\neq 1} \bar{w}_j^{\text{SC}'}\mu_j$ (which, in general, will happen), the infeasible SC estimator will dominate the demeaned SC estimator in terms of asymptotic MSE if $(\mathbb{E}[\lambda_t|D(1,0)=1]-\omega_0)$ is large enough. In other words, once we relax Assumption 5, we cannot guarantee that the demeaned SC estimator provides a better prediction in terms of MSE relative to the infeasible one. Moreover, if the bias of the demeaned SC estimator is large enough, then the infeasible SC estimator will be better in terms of MSE relative to the demeaned SC estimator.

In general, it is not possible to rank the demeaned SC and the DID estimators in terms of bias and MSE if treatment assignment is correlated with time-varying common factors. We provide in Appendix A.3 an example in which the DID can have a smaller bias and MSE relative to the demeaned SC estimator. This might happen when selection into treatment depends on common factors with low variance, and it happens that a simple average of the controls provides a good match for the factor loadings associated with these common factors. For the particular class of linear factor models we present in Section 6, however, the asymptotic bias and the MSE of the demeaned SC estimator will always be lower relative to the DID estimator, provided that there is stability in the variance of common factors and transitory shocks before and after the treatment.

In addition to including an intercept, Doudchenko and Imbens (2016) also consider the possibility of relaxing the non-negative and the adding-up constraints in the SC model. We show in Appendix A.5.3 that our main result that the SC estimator will be asymptotically biased if there is selection on time-varying unobservables still apply if we relax these conditions.[20] The panel data approach suggested by Hsiao et al. (2012) is essentially the same as the SC estimator using all outcome lags as predictors, and relaxing the no-intercept, adding-up, and non-negativity constraints. Therefore, our result on asymptotic bias is also valid for the estimator suggested by Hsiao et al. (2012). Note also that relaxing the adding-up constraint implies that the SC estimator may be biased if the time effect δ_t is correlated with the treatment assignment.

We also present in Appendix A.5.4 an instrumental variables estimator for the SC weights that generates an asymptotically unbiased SC estimator under additional assumptions on the error structure, which would be valid if, for example, the idiosyncratic error is serially uncorrelated and all the common factors are serially correlated. The main idea is that, under these assumptions, one could use the lag outcome of the control units as instrumental variables to estimate parameters that reconstruct the factor loadings of the treated unit.

5 Model with "explosive" common factors

Many SC applications present time-series patterns that are not consistent with Assumption 4, including the applications considered by Abadie and Gardeazabal (2003), Abadie et al. (2010), and Abadie et al. (2015). This will be the case whenever we consider outcome variables that exhibit non-stationarities, such as GDP and average wages. We consider now the case in which the first and second moments of a subset of the common factors diverge. We modify the model to

$$\begin{cases} y_{it}(0) = \lambda_t \mu_i + \gamma_t \theta_i + \varepsilon_{it} \\ y_{it}(1) = \alpha_{it} + y_{it}(0) \end{cases} \tag{9}$$

where $\lambda_t = (\lambda_t^1, ..., \lambda_t^{F_0})$ is a $(1 \times F_0)$ vector of $I(0)$ common factors, and $\gamma_t = (\gamma_t^1, ..., \gamma_t^{F_1})$ is a $(1 \times F_1)$ vector of common factors that are $\mathcal{I}(1)$ and/or polynomial time trends t^f, while μ_i and θ_i are the vectors of factor loadings associated with these common factors. The time effect δ_t can be either included in vector λ_t or γ_t. Differently from the previous sections, in order to consider the possibility that treatment starts after a large number of periods in which some common factors may be $\mathcal{I}(1)$ and/or polynomial time trends, we label the periods as $t = 1, ..., T_0, T_0 + 1, ..., T$. We modify Assumption 4 to determine the behavior of the common factors and the transitory shocks in the pre-treatment periods.

Assumption 4′ (stochastic processes) Conditional on $D(1, T_0) = 1$, the process $z_t = (\varepsilon_{1t}, ..., \varepsilon_{J+1,t}, \lambda_t)$

[20] In this case, since we do not constraint the weights to sum 1, we need to adjust Assumption 4 so that it also includes convergence of the pre-treatment averages of the first and second moments of δ_t.

is $I(0)$ and weakly stationary with finite fourth moments, while the components of γ_t are $I(1)$ and/or polynomial time trends t^f for $t = 1, ..., T_0$.

Assumption 4′ restricts the behavior of the common factors in the pre-treatment periods. However, this assumption allows for correlation between treatment assignment and common factors in the post-intervention periods. For example, if $\gamma_t^k = \gamma_{t-1}^k + \eta_t$, then Assumption 4′ implies that η_t has mean zero for all $t \leq T_0$. However, it may be that $\mathbb{E}[\eta_t | D(1, T_0)] \neq 0$ for $t > T_0$. This assumption could be easily relaxed to allow for $\mathbb{E}[\eta_t | D(1, T_0)] \neq 0$ for a fixed number of periods prior to the start of the treatment.

We also modify Assumption 3 to state that there are weights that reconstruct the factor loadings of unit 1 associated with the non-stationary common trends.

Assumption 3′ (existence of weights)

$$\exists\ \mathbf{w}^* \in W \mid \theta_1 = \sum_{j \neq 1} w_j^* \theta_j$$

where W is the set of possible weights given the constrains on the weights the researcher is willing to consider. For example, Abadie et al. (2010) suggest $W = \{\mathbf{w} \in \mathbb{R}^J \mid \sum_{j \neq 1} w_j^* = 1, \text{ and } w_j^* \geq 0\}$, while Hsiao et al. (2012) allows for $W = \mathbb{R}^J$. Let Φ_1 be the set of weights in W that reconstruct the factor loadings of unit 1 associated with the $I(1)$ common factors. Assumption 3′ implies that $\Phi_1 \neq \varnothing$.

In a setting in which γ_t is a vector of $I(1)$ common factors, Assumption 3′ implies that the vector of outcomes $\mathbf{y}_t = (y_{1t}, ..., y_{J+1,t})'$ is co-integrated. Importantly, differently from our results in Section 3, Assumption 3′ is key for our results in this section. However, we do *not* need to assume existence of weights in Φ_1 that also reconstruct the factor loadings of unit 1 associated with the $I(0)$ common factors, so it may be that $\Phi = \varnothing$, where Φ is the set of weights that reconstruct *all* factor loadings.

We consider an asymptotic exercise where $T_0 \to \infty$ with "explosive" common factors, so it is not possible to fix the label of the post-treatment periods, as we do in Sections 3 and 4. Instead, we consider the asymptotic distribution of the estimator for the effect of the treatment τ periods after the start of the treatment.

Proposition 4 Under Assumptions 1, 2, 3′, and 4′, we have that, for $t = T_0 + \tau$, $\tau > 0$,

$$\hat{\alpha}_{1t}^{\text{SC}'} \xrightarrow{d} \alpha_{1t} + \left(\varepsilon_{1t} - \sum_{j \neq 1} \bar{w}_j \varepsilon_{jt} \right) + (\lambda_t - \omega_0) \left(\mu_1 - \sum_{j \neq 1} \bar{w}_j \mu_j \right) \text{ when } T_0 \to \infty \tag{10}$$

where $\mu_1 \neq \sum_{j \neq 1} \bar{w}_j \mu_j$, unless $\sigma_\varepsilon^2 = 0$ or $\exists \mathbf{w} \in \Phi | \mathbf{w} \in \underset{\mathbf{w} \in W}{\text{argmin}} \left\{ \sum_{j \neq 1} (w_j)^2 \right\}$.

Proof.

Details in Appendix A.1.4. ■

Proposition 4 has two important implications. First, if Assumption 3′ is valid, then the asymptotic distribution of the demeaned SC estimator does not depend on the non-stationary common trends. The intuition of this result is the following. The demeaned SC weights will converge to weights that reconstruct the factor loadings of the treated unit associated with the non-stationary common trends. Interestingly, while $\widehat{\mathbf{w}}$ will generally be only $\sqrt{T_0}-$consistent when Φ_1 is not a singleton, we show that there are linear combinations of $\widehat{\mathbf{w}}$ that will converge at a faster rate, implying that $\gamma_t(\theta_1 - \sum_{j\neq 1} \hat{w}_j \theta_j) \xrightarrow{p} 0$, despite the fact that γ_t explodes when $T_0 \rightarrow \infty$. Therefore, such non-stationary common trends will not lead to asymptotic bias in the SC estimator. Second, the demeaned SC estimator will be biased if there is correlation between treatment assignment and the $I(0)$ common factors. The intuition is that the demeaned SC weights will converge in probability to weights in Φ_1 that minimize the variance of the $I(0)$ process $u_t = y_{1t} - \sum_{j\neq 1} w_j y_{jt} = \lambda_t(\mu_1 - \sum_{j\neq 1} w_j \mu_j) + (\varepsilon_{1t} - \sum_{j\neq 1} w_j \varepsilon_{jt})$. Following the same arguments as in Proposition 1, $\widehat{\mathbf{w}}$ will not eliminate the $I(0)$ common factors, unless we have that $\sigma^2_\varepsilon = 0$ or it coincides that there is a $\mathbf{w} \in \Phi$ that also minimizes the linear combination of transitory shocks.

The result that the asymptotic distribution of the SC estimator does not depend on the non-stationary common trends depends crucially on Assumption 3′. If there were no linear combination of the control units that reconstruct the factor loadings of the treated unit associated to the non-stationary common trends, then the asymptotic distribution of the SC estimator would trivially depend on these common trends, which might lead to bias in the SC estimator.

Proposition 4 remains valid when we relax the adding-up and/or the non-negativity constraints, with minor variations in the conditions for unbiasedness.[21] However, these results are not valid when we consider the no-intercept constraint, as the original SC estimator does. When the intercept is not included, it remains true that $\widehat{\mathbf{w}} \xrightarrow{p} \bar{\mathbf{w}} \in \Phi_1$. However, in this case, the weights will not converge fast enough to compensate the fact that γ_t explodes. See an example in Appendix A.6.2. This provides another reason to use the demeaned instead of the original SC estimator.

An important feature of this setting is that, as $T_0 \rightarrow \infty$, the pre-treatment fit will become close to perfect, which is the case in which Abadie et al. (2010) recommend that the SC method should be used. As a measure of goodness of pre-treatment fit, we consider a pre-treatment normalized mean squared error

[21] Relaxing the adding-up constraint makes the estimator biased if δ_t is correlated with treatment assignment and if it is $I(0)$. If δ_t is $I(1)$, then the weights will converge to sum one even when such restriction is not imposed, so this would not generate bias. Including or not the non-negative constraint does not alter the conditions for unbiasedness, although it may be that assumption 3′ is valid in a model without the non-negativity constraints, but not valid in a model with these constraints.

index, as suggested by Ferman et al. (2017):

$$\widetilde{R}^2 = 1 - \frac{\frac{1}{T_0}\sum_{t=1}^{T_0}\left(y_{1t} - \widehat{y}_{1t}\right)^2}{\frac{1}{T_0}\sum_{t=1}^{T_0}\left(y_{1t} - \overline{y}_1\right)^2} \tag{11}$$

where $\widehat{y}_{1t}$ is the outcome of the SC unit and $\overline{y}_1 = \frac{\sum_{t=1}^{T_0} y_{1t}}{T_0}$. This measure is always lower than one, and it is close to one when the pre-treatment fit is close to perfect.[22] In this setting with non-stationary common trends, the numerator will converge to the variance of an $I(0)$ process, while the denominator will diverge as $T_0 \to \infty$. Therefore, in these cases, we show that the SC estimator can be asymptotically biased *even conditional on a close-to-perfect pre-treatment fit.*[23]

Therefore, in a setting with non-stationary trends, a seemingly perfect pre-treatment fit might hide important possibilities for asymptotic bias in the SC method. While this perfect pre-treatment fit would be indicative that the SC estimator was able to eliminate potential bias coming from correlations between treatment assignment and non-stationary common factors, this would not guarantee unbiasedness if there is a correlation between treatment assignment and common factors beyond such non-stationary trends. Therefore, we recommend that researchers should also present the pre-treatment fit after eliminating non-stationary trends as a diagnosis test for the SC estimator.[24] To illustrate this point, we consider the three influential applications presented by Abadie and Gardeazabal (2003), Abadie et al. (2010) and Abadie et al. (2015). We present in Figure 1.A the per capita GDP time series for the Basque Country and for other Spanish regions, while in Figure 1.B we replicate Figure 1 from Abadie and Gardeazabal (2003), which displays per capita GDP of the Basque Country contrasted with the per capita GDP of a synthetic control unit constructed to provide a counterfactual for the Basque Country without terrorism. The pre-treatment fit in this case is seemingly perfect, with an $\tilde{R}^2 = 0.99$. However, the per capita GDP series is clearly non-stationary, with all regions displaying similar trends before the intervention. Therefore, based on our results presented in Propositions 4, despite the seemingly perfect pre-treatment fit, it may still be that the SC estimator is biased if there is a correlation between treatment assignment and common factors beyond this non-stationary trend.

[22] Differently from the R^2 measure, this measure can be negative, which would suggest a poor pre-treatment fit.

[23] Note that, in their proof, Abadie et al. (2010) assume that there exists a constant $\bar{\lambda}$ such that $|\lambda_t^f| \leq \bar{\lambda}$ for all $t = 1, ..., T$ and $f = 1, ..., F$, where $\lambda_t = (\lambda_t^1, ..., \lambda_t^F)$ is the vector of common factors. Under this and other additional assumptions, they show that the bias of the SC estimator can be bounded by a function that depends on $\bar{\lambda}$ and T_0 if we have a perfect match in the pre-treatment outcomes. In order to guarantee that this function goes to zero when T_0 increases, however, we need to assume that the condition on $\bar{\lambda}$ remains valid when T_0 increases. This will not be the case if some components of λ_t increase without bound when T_0 increases. Therefore, our result does not contradicts the result from Abadie et al. (2010).

[24] Wooldridge (1991) provides a similar solution for computing the R^2 in regressions with trending and/or seasonal data.

In order to assess this possibility, we consider two different ways to de-trend the data, so we can have a better assessment on whether factor loadings associated with stationary common factors are well matched. In both cases, we subtract the outcome of the treated and control units by constant terms $\{a_t\}_{t=1}^{T}$. Note that, under the adding-up constraint ($\sum_{j\neq 1} w_j = 1$), the SC weights and the SC estimator will be numerically the same whether we estimate with the original data or with $\tilde{y}_{jt} = y_{jt} - a_t$. We first subtract the average of the control units at time t ($a_t = \frac{1}{J}\sum_{j\neq 1} y_{jt}$) for both treated and control units. Therefore, if the non-stationarity comes from a common factor δ_t that affects every unit in the same way, then the series $\tilde{y}_{jt} = y_{jt} - \frac{1}{J}\sum_{j'\neq 1} y_{j't}$ would not display non-stationary trends. As shown in Figure 1.C, in this case, the treated and SC units do not display a non-stationary trend, and the pre-treatment fit for this de-trended series would not be as good as in the previous case, with an $\tilde{R}^2 = 0.65$. We get similar results if we de-trend by fitting a polynomial $a(t)$ to the synthetic control series, with an $\tilde{R}^2 = 0.67$ (Figure 1.D).[25]

We consider in Figure 2 the application from Abadie et al. (2010), who estimate the effects of California's tobacco control program. This empirical application also presents a seemingly perfect pre-treatment fit, with an $\tilde{R}^2 = 0.96$, but with a highly non-stationary trend. Our first strategy to de-trend the data by subtracting the controls' average outcomes still leads to a non-stationary series, suggesting that the non-stationary common factors cannot be resumed into a simple time effect δ_t. When we consider a polynomial $a(t)$, then the pre-treatment fit for the de-trended series is very low. Finally, we consider in Figure 3 the study on the economic impact of the 1990 German reunification on West Germany, by Abadie et al. (2015). Again, this application displays a seemingly perfect pre-treatment fit ($\tilde{R}^2 = 0.99$), but a more modest pre-treatment fit when we de-trend the data using a time polynomial ($\tilde{R}^2 = 0.70$). Overall, our results point out that the diagnosis based on the pre-treatment fit for non-stationary series should be considered with caution, as they may hide discrepancies in common factors beyond non-stationary trends that may lead to asymptotic bias in the SC estimator.

Importantly, our results do not imply that one should not use the SC method when the data is non-stationary. On the contrary, we show that the SC method is very efficient in dealing with non-stationary trends. Indeed, in these three applications, the seemingly perfect pre-treatment fit when we consider the outcomes in level suggest that the method is being highly successful in taking into account non-stationary trends, which is an important advantage of the method relative to alternatives such as DID. The only caveat is that measures of pre-intervention fit could be misleading as diagnostic tests, as they may hide important discrepancies in the factor loadings associated to the stationary common factors. One suggestion is to calculate the measures of pre-intervention fit with a de-trended series. Another possibility would be to

[25] We used a polynomial of order 5 to fit the entire time series of the synthetic control unit (including both pre- and post-periods. Then we consider the de-trended series $\tilde{y}_{jt} = y_{jt} - \hat{a}(t)$.

apply the SC method on a transformation of the data that makes it stationary. In this case, however, the estimator would not be numerically the same as the estimator using the original data.

6 Particular Class of Linear Factor Models & Monte Carlo Simulations

We consider now in detail a particular class of linear factor models in which all units are divided into groups that follow different times trends. We present both theoretical and MC simulations for these models. In Section 6.1 we consider the case with stationary common factors, while in Section 6.2 we consider a case in which there are both $I(1)$ and $I(0)$ common factors.

6.1 Model with stationary common factors

We consider first a model in which the $J+1$ units are divided into K groups, where for each j we have that

$$y_{jt}(0) = \delta_t + \lambda_t^k + \varepsilon_{jt} \tag{12}$$

for some $k = 1, ..., K$. As in Section 3, let $t = -T_0+1, ..., 0, 1, ..., T_1$. We assume that $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \lambda_t^k \xrightarrow{p} 0$, $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} (\lambda_t^k)^2 \xrightarrow{p} 1$, $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \varepsilon_{jt} \xrightarrow{p} 0$, $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \varepsilon_{jt}^2 \xrightarrow{p} \sigma_\varepsilon^2$ and $\frac{1}{T_0}\sum_{t=-T_0+1}^{0} \lambda_t^k \varepsilon_{jt} \xrightarrow{p} 0$.

6.1.1 Asymptotic Results

Consider first an extreme case in which $K = 2$, so the first half of the $J+1$ units follows the parallel trend given by λ_t^1, while the other half follows the parallel trend given by λ_t^2. In this case, an infeasible SC estimator would only assign positive weights to units in the first group.

We calculate, for this particular class of linear factor models, the asymptotic proportion of misallocated weights of the SC estimator using all pre-treatment lags as predictors. From the minimization problem 4, we have that, when $T_0 \rightarrow \infty$, the proportion of misallocated weights converges to

$$\gamma_2(\sigma_\varepsilon^2, J) = \sum_{j=\frac{J+1}{2}+1}^{J+1} \bar{w}_j = \frac{J+1}{J^2 + 2 \times J \times \sigma_\varepsilon^2 - 1} \times \sigma_\varepsilon^2 \tag{13}$$

where $\gamma_K(\sigma_\varepsilon^2, J)$ is the proportion of misallocated weights when the $J+1$ groups are divided in K groups.

We present in Figure 4.A the relationship between asymptotic misallocation of weights, variance of the transitory shocks, and number of control units. For a fixed J, the proportion of misallocated weights converges to zero when $\sigma_\varepsilon^2 \to 0$, while this proportion converges to $\frac{J+1}{2J}$ (the proportion of misallocated weights of DID) when $\sigma_\varepsilon^2 \to \infty$. This is consistent with the results we have in Section 3. Moreover, for a given σ_ε^2, the proportion of misallocated weights converges to zero when the number of control units goes to infinity. This is consistent with Gobillon and Magnac (2016), who derive support conditions so that the assumptions from Abadie et al. (2010) for unbiasedness are satisfied when both T_0 and J go to infinity.

In this example, the SC estimator, for $t > 0$, converges to

$$\hat{\alpha}_{1t} \xrightarrow{d} \alpha_{1t} + \left(\varepsilon_{1t} - \sum_{j \neq 1} \bar{w}_j \varepsilon_{jt} \right) + \lambda_t^1 \times \gamma_2(\sigma_\varepsilon^2, J) - \lambda_t^2 \times \gamma_2(\sigma_\varepsilon^2, J) \tag{14}$$

so the potential bias due to correlation between treatment assignment and common factors (for example, $\mathbb{E}[\lambda_t^1 | D(1,0) = 1] \neq 0$ for $t > 0$) will directly depend on the proportion of misallocated weights.

We consider now another extreme case in which the $J+1$ units are divided into $K = \frac{J+1}{2}$ groups that follow the same parallel trend. In this case, each unit has a pair that follows its same parallel trend, while all other units follow different parallel trends. The proportion of misallocated weights converges to

$$\gamma_{\frac{J+1}{2}}(\sigma_\varepsilon^2, J) = \sum_{j=3}^{J+1} \bar{w}_j = \frac{J-1}{J(1+\sigma_\varepsilon^2)+1} \times \sigma_\varepsilon^2. \tag{15}$$

We present the relationship between misallocation of weights, variance of the transitory shocks, and number of control units in Figure 4.B. Again, the proportion of misallocated weights converges to zero when $\sigma_\varepsilon^2 \to 0$ and to the proportion of misallocated weights of DID when $\sigma_\varepsilon^2 \to \infty$ (in this case, $\frac{J-1}{J}$). Differently from the previous case, however, for a given σ_ε^2, the proportion of misallocated weights converges to $\frac{\sigma_\varepsilon^2}{1+\sigma_\varepsilon^2}$ when $J \to \infty$. Therefore, the SC estimator would remain asymptotically biased even when the number of control units is large. This happens because, in this model, the number of common factors increases with J, so the conditions derived by Gobillon and Magnac (2016) are not satisfied.

In both cases, the proportion of misallocated weights is always lower than the proportion of misallocated weights of DID. Therefore, in this particular class of linear factor models, the asymptotic bias of the SC estimator will always be lower than the asymptotic bias of DID. If we further assume that the variance of common factors and transitory shocks remain constant in the pre- and post-intervention periods, then we also have that the SC estimator will have lower variance and, therefore, lower MSE relative to the DID estimator. However, this is not a general result, as we show in Appendix A.3.

Finally, we compare the asymptotic MSE between the feasible and the infeasible SC estimator in this particular class of linear factor models. As outlined in Section 4, assuming that common factors and transitory shocks are stable before and after the intervention, the feasible SC estimator has a lower asymptotic MSE relative to the infeasible one. However, if the feasible SC estimator is asymptotically biased, and the bias is large enough, then it will have a higher asymptotic MSE relative to the infeasible SC estimator. We illustrate these features in Table 1. Considering 20 units divided in 10 groups of 2 (columns 1 to 3), the feasible SC estimator has a lower asymptotic MSE for the estimator of α_{1t}, for $t > 0$, when $\mathbb{E}[\lambda_t | D(1,0) = 1] = 1$. However, when the correlation between treatment assignment and common factors is larger, then the feasible SC estimator has a higher asymptotic MSE relative to the infeasible one. When the number of post-treatment periods is greater than one (that is, $T_1 > 1$), if we consider estimators for the average treatment effect across all post-treatment periods, then the ratio of asymptotic MSE for the feasible and infeasible SC estimators would be substantially higher. In this case, the infeasible SC estimator dominates the feasible one in terms of asymptotic MSE even when $\mathbb{E}[\lambda_t | D(1,0) = 1] = 1$ (panel ii of Table 1). This happens because averaging across post-treatment periods does not affect the asymptotic bias, while it reduces the variance of both estimators. In columns 4 to 6, we present the case in which 20 units are divided in 2 groups of 10. In this case, the difference between the two estimators is much smaller, although it also shows that the feasible SC estimator has a higher asymptotic MSE when its bias is large enough. While, of course, the infeasible SC estimator would not be available in real applications, these results highlight that researchers applying the SC estimator should be aware that it may have a non-trivial asymptotic MSE if there is correlation between treatment assignment and unobserved common factors.

6.1.2 Monte Carlo Simulations

The results presented in Section 6.1.1 are based on large-T_0 asymptotics. We now consider, in MC simulations, the finite T_0 properties of the SC estimator, both unconditional and conditional on a good pre-treatment fit. We present MC simulations using a data generating process (DGP) based on equation 12. We consider in our MC simulations $J + 1 = 20$, λ_t^k normally distributed following an AR(1) process with 0.5 serial correlation parameter, $\varepsilon_{jt} \sim N(0, \sigma_\varepsilon^2)$, and $T - T_0 = 10$. We also impose that there is no treatment effect, i.e., $y_{jt} = y_{jt}(0) = y_{jt}(1)$ for each time period $t \in \{-T_0 + 1, ..., 0, 1, ..., T_1\}$. We consider variations in DGP in the following dimensions:

- The number of pre-intervention periods: $T_0 \in \{5, 20, 50, 100\}$.
- The variance of the transitory shocks: $\sigma_\varepsilon^2 \in \{0.1, 0.5, 1\}$.
- The number of groups with different λ_t^k: $K = 2$ (2 groups of 10) or $K = 10$ (10 groups of 2)

For each simulation, we calculate the SC estimator that uses all pre-treatment outcome lags as predictors, and calculate the proportion of misallocated weights. We also evaluate whether the SC method provides a good pre-intervention fit and calculate the proportion of misallocated weights conditional on a good pre-intervention fit. In order to determine that the SC estimator provided a good fit, we consider a pre-treatment normalized mean squared error index, presented in equation 11. For each scenario, we generate 20,000 simulations.

In columns 1 to 3 of Table 2, we present the proportion of misallocated weights when $K = 10$ for different values of T_0 and σ^2_ε. Consistent with our analytical results from Section 6.1.1, the misallocation of weights is increasing with the variance of the transitory shocks. With $T_0 = 100$, the proportion of misallocated weights is close to the asymptotic values, while the proportion of misallocated weights is substantially higher when T_0 is small. From equation 14, there is a direct link between misallocation of weights and the bias of the SC estimator (for a given $\mathbb{E}[\lambda_t | D(1,0) = 1]$). Therefore, if there is correlation between treatment assignment and common factors, then the bias of the SC estimator should be expected to be much larger than its asymptotic values when T_0 is small.

We present in columns 4 to 6 of Table 2 the probability that the SC method provides a good fit when we define good fit as $\widetilde{R}^2 > 0.8$. As expected, with a large T_0 the SC method only provides a good pre-intervention fit if the variance of the transitory shock is low. If the variance of the transitory shocks is higher, then the probability that the SC method provides a good match is approximately zero, unless the number of pre-treatment periods is rather low. These results suggest that, in a model with stationary factors, the SC estimator would only provide a close-to-perfect pre-treatment fit with a moderate number of pre-treatment periods if the variance of the transitory shocks is low, in which case the bias of the SC estimator would be relatively small. With $T_0 = 20$ and $\sigma^2_\varepsilon = 0.5$ or $\sigma^2_\varepsilon = 1$, the probability of having a good fit is, respectively, equal to 1.3% and 0.1%. Interestingly, when we condition on having a good pre-treatment fit the proportion of misallocated weights reduces, but still remains quite high (for example, it goes from 50% to 33% when $\sigma^2_\varepsilon = 0.5$ and from 65% to 45% when $\sigma^2_\varepsilon = 1$). These results are presented in Table 2, columns 7 to 9. In Appendix Table A.1 we replicate Table 2 using a more stringent definition of good fit, which is equal to one if $\widetilde{R}^2 > 0.9$. In this case, conditioning has a larger effect in reducing the discrepancy of factor loadings between the treated and the SC units, but at the expense of having a lower probability of accepting that the pre-treatment fit is good. These results suggest that, with stationary data, the SC estimator would only provide a close-to-perfect match with a moderate T_0, and therefore be close to unbiased, when the variance of the transitory shocks converges to zero. In Appendix Table A.2 we also consider the case with 2 groups of 10 units each ($K = 2$). All results are qualitatively the same.

In this particular class of linear factor models, the proportion of misallocated weights is always lower than the proportion of misallocated weights of the DID estimator, which implies in a lower bias if treatment assignment is correlated with common factors. This is true even when the pre-treatment match is not perfect and when the number of pre-treatment periods is very small. From Section 4, we also know that, if common factors are stationary for both pre- and post-treatment periods, then a demeaned SC estimator is unbiased and has a lower asymptotic variance than DID. Since this DGP has no time-invariant factor, this is true for the standard SC estimator as well. We present in Table 3 the DID/SC ratio of standard errors. With $T_0 = 100$, the DID standard error is 2.4 times higher than the SC standard errors when $\sigma^2_\varepsilon = 0.1$. When σ^2_ε is higher, the advantage of the SC estimator is reduced, although the DID standard error is still 1.3 (1.1) times higher when σ^2_ε is equal to 0.5 (1). This is expected given that, in this model, the SC estimator converges to the DID estimator when $\sigma^2_\varepsilon \rightarrow \infty$. More strikingly, the variance of the SC estimator is lower than the variance of DID even when the number of pre-treatment periods is small. These results suggest that the SC estimator can still improve relative to DID even when the number of pre-treatment periods is not large and when the pre-treatment fit is not perfect, situations in which Abadie et al. (2015) suggest the method should not be used. However, a very important qualification is that, in these cases, the SC estimator requires stronger identification assumptions than stated in the original SC papers. More specifically, it is generally asymptotically biased if treatment assignment is correlated with time-varying confounders.

6.2 Model with "explosive" common factors

We consider now a model in which a subset of the common factors is $I(1)$. We consider the following DGP:

$$y_{jt}(0) = \delta_t + \lambda^k_t + \gamma^r_t + \varepsilon_{jt} \tag{16}$$

for some $k = 1, ..., K$ and $r = 1, ..., R$. We maintain that λ^k_t is stationary, while γ^r_t follows a random walk.

6.2.1 Asymptotic results

Based on our results from Section 5, the SC weights will converge to weights in Φ_1 that minimize the second moment of the $I(0)$ process that remains after we eliminate the $I(1)$ common factor. Consider the case $K = 10$ and $R = 2$. Therefore, units $j = 2, ..., 10$ follow the same non-stationary path γ^1_t as the treated unit, although only unit $j = 2$ also follows the same stationary path λ^1_t as the treated unit. In this case, asymptotically, all weights would be allocated among units 2 to 10, eliminating the relevance of the $I(1)$ common factor. However, the allocation of weights within these units will not assign all weights to unit 2, so the $I(0)$ common factor will remain relevant.

6.2.2 Monte Carlo simulations

In our MC simulations, we maintain that λ_t^k is normally distributed following an AR(1) process with 0.5 serial correlation parameter, while γ_t^r follows a random walk. We consider the case $K = 10$ and $R = 2$.

The proportion of misallocated weights (in this case, weights not allocated to unit 2) is very similar to the proportion of misallocated weights in the stationary case (columns 1 to 3 of Table 4). If we consider the misallocation of weights only for the $I(1)$ factors, then the misallocation of weights is remarkably low with moderate T_0, even when the variance of the transitory shocks is high (columns 4 to 6 of Table 4). The reason is that, with a moderate T_0, the $I(1)$ common factors dominate the transitory shocks, so the SC method is extremely efficient selecting control units that follow the same non-stationary trend as the treated unit. For the same reason, the probability of having a dataset with a close-to-perfect pre-treatment fit is also very high if a subset of the common factors is $I(1)$ (columns 7 to 9 of Table 4). Finally, we show in columns 10 to 12 of Table 4 that conditioning on a close-to-perfect match makes virtually no difference in the proportion of misallocated weights for the stationary factor.

These results suggest that the SC method works remarkably well to control for $I(1)$ common factors. In this scenario, one would usually have a close-to-perfect fit, and there would be virtually no bias associated to the $I(1)$ factors. However, we might have a substantial misallocation of weights for the $I(0)$ common factors *even conditional on a close-to-perfect pre-treatment match*. Taken together, these results suggest that the SC method provides substantial improvement relative to DID in this scenario, as the SC estimator is extremely efficient in capturing the $I(1)$ factors. Also, if the DID and SC estimators are unbiased, then the variance of the DID relative to the variance of the SC estimator would be substantially higher, as presented in Table 5. However, one should be aware that, in this case, the identification assumption only allows for correlation of treatment assignment with the $I(1)$ factors. Still, this potential bias of the SC estimator due to a correlation between treatment assignment and the $I(0)$ common shocks, in this particular class of linear factor models, would be lower than the bias of DID.

7 Conclusion

We revisit the theory behind the SC method in a linear factor model setting. If the model has "stationary" common factors, in the sense that pre-treatment averages of the first and second moments of the common factors converge, we show that the SC estimator is biased if treatment assignment is correlated with unobserved confounders, even when weights that reconstruct the factor loadings of the treated unit exist and when $T_0 \to \infty$. Our simulations suggest that the bias may be larger when T_0 is finite. The asymptotic bias goes to zero when the variance of the transitory shocks goes to zero, which is exactly the case in which one would expect to find a good pre-treatment fit. Therefore, our results, under these conditions on the common factors, are consistent with the results from Abadie et al. (2010). However, if pre-treatment averages of a subset of the common factors diverge, then we show that the SC estimator can be asymptotically biased even conditional on a close-to-perfect pre-treatment match.

Our results suggest that researchers should be more careful in interpreting the identification assumptions required for the SC method. Moreover, we suggest that, in addition to the standard graph comparing treated and SC units, researchers should also present a graph comparing the treated and SC units after de-trending the data, so that it is possible to assess whether there might be relevant possibilities for bias arising due to a correlation between treatment assignment and common factors beyond non-stationary trends. Our results also have implications for the placebo test suggested in Abadie et al. (2010), as we explore in a companion paper (Ferman and Pinto (2017)).

Despite these caveats, we show that a demeaned SC estimator can substantially improve relative to currently available methods, even if the pre-treatment fit is not close to perfect and if T_0 is not large. This is particularly true when a subset of the common factors is non-stationary.

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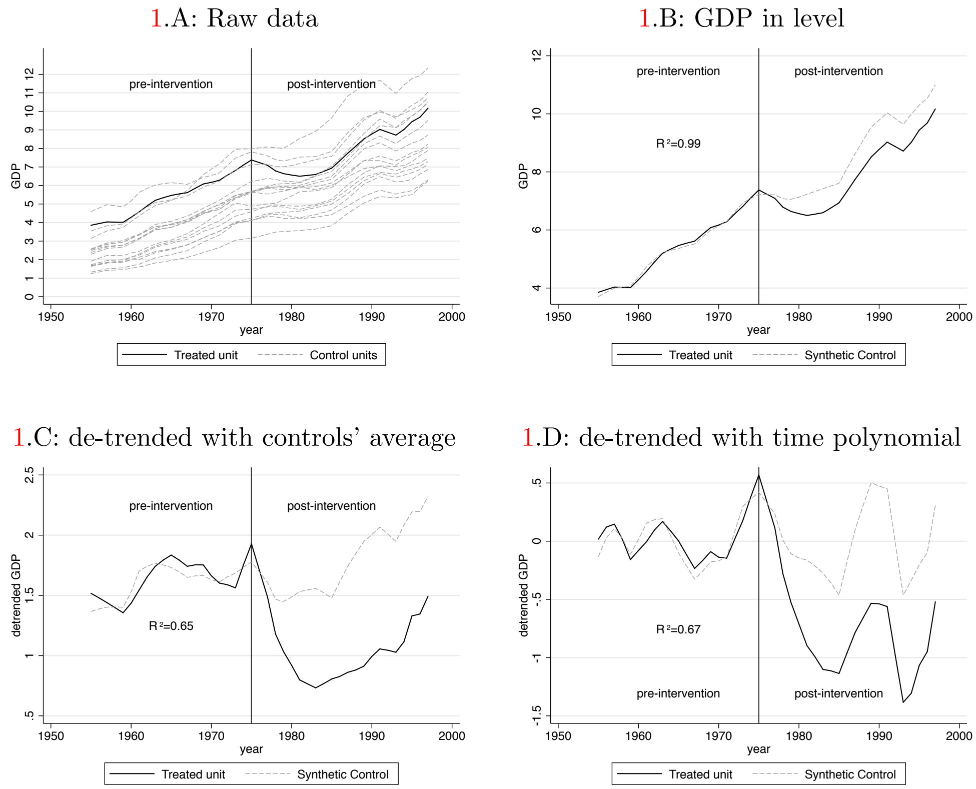
Figure 1: **Abadie and Gardeazabal (2003) application**



1.A: Raw data

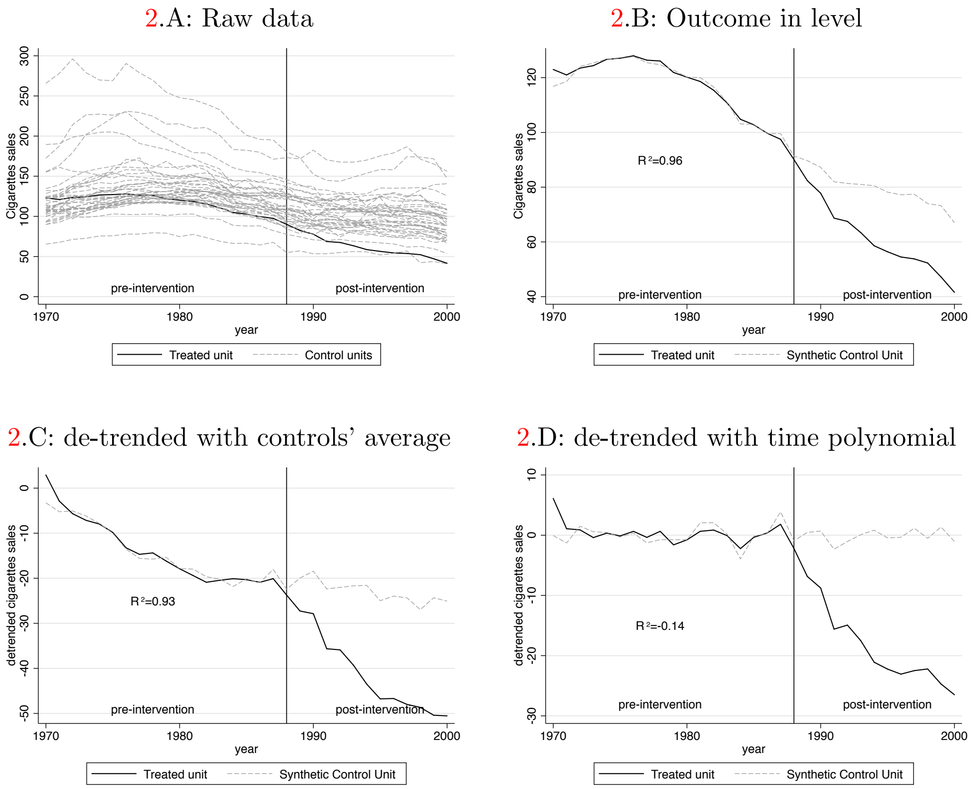
1.B: GDP in level

1.C: de-trended with controls' average

1.D: de-trended with time polynomial

Notes: Figure A presents time series for the treated and for the control units used in the empirical application from Abadie and Gardeazabal (2003). In Figure B we present the time series for the treated and for the SC units. In Figure C we present the same information as in Figure B after subtracting the control groups' averages for each time period. In Figure C we present the same information as in Figure B after subtracting a time trend estimated by fitting a 5th order polynomial on the SC series. Figures B to D we also report the measure of pre-treatment fit defined in equation 11.

Figure 2: **Abadie et al. (2010) application**


2.A: Raw data

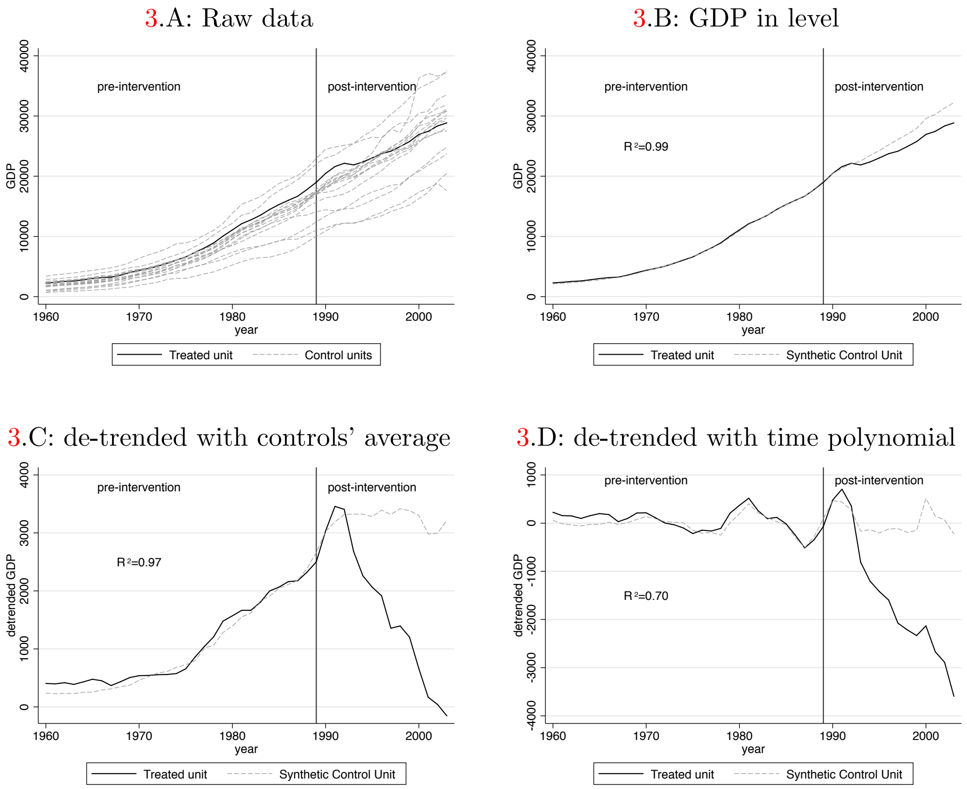
2.B: Outcome in level

2.C: de-trended with controls' average

2.D: de-trended with time polynomial

Notes: Figure A presents time series for the treated and for the control units used in the empirical application from Abadie et al. (2010). In Figure B we present the time series for the treated and for the SC units. In Figure C we present the same information as in Figure B after subtracting the control groups' averages for each time period. In Figure C we present the same information as in Figure B after subtracting a time trend estimated by fitting a 5th order polynomial on the SC series. Figures B to D we also report the measure of pre-treatment fit defined in equation 11.

Figure 3: **Abadie et al. (2015) application**



3.A: Raw data

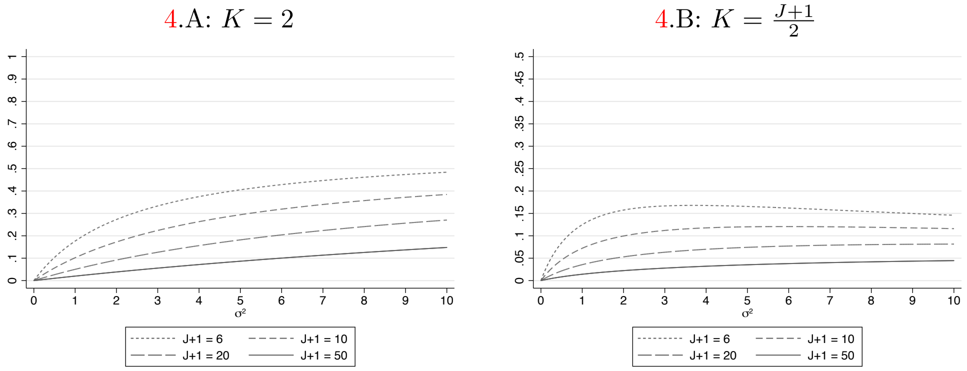
3.B: GDP in level

3.C: de-trended with controls' average

3.D: de-trended with time polynomial

Notes: Figure A presents time series for the treated and for the control units used in the empirical application from Abadie et al. (2015). In Figure B we present the time series for the treated and for the SC units. In Figure C we present the same information as in Figure B after subtracting the control groups' averages for each time period. In Figure C we present the same information as in Figure B after subtracting a time trend estimated by fitting a 5th order polynomial on the SC series. Figures B to D we also report the measure of pre-treatment fit defined in equation 11.

Figure 4: **Asymptotic Misallocation of Weights**



4.A: $K = 2$

4.B: $K = \frac{J+1}{2}$

Notes: these figures present the asymptotic misallocation of weights of the SC estimator as a function of the variance of the transitory shocks for different numbers of control units. Figures 4.A presents results when there are 2 groups of $\frac{J+1}{2}$ units each, while Figure 4.B presents results when there are $\frac{J+1}{2}$ groups of 2 units each. The misallocation of weights is defined as the proportion of weight allocated to units that do not belong to the group of treated unit.

Table 1: **Asymptotic MSE (Feasible SC estimator / Infeasible SC estimator)**

	$K = \frac{J+1}{2} = 10$			$K = 2$		
	$\sigma^2_\varepsilon = 0.1$	$\sigma^2_\varepsilon = 0.5$	$\sigma^2_\varepsilon = 1$	$\sigma^2_\varepsilon = 0.1$	$\sigma^2_\varepsilon = 0.5$	$\sigma^2_\varepsilon = 1$
$\mathbb{E}[\lambda_t \| D(1,0) = 1]$	(1)	(2)	(3)	(4)	(5)	(6)
			Panel i: $T_1 = 1$			
1	0.99	0.94	0.88	1.00	1.00	1.00
2	1.09	1.22	1.20	1.00	1.00	1.00
4	1.50	2.34	2.47	1.00	1.02	1.03
			Panel ii: $T_1 = 10$			
1	1.07	1.14	1.11	1.00	1.00	1.00
2	1.39	2.02	2.12	1.00	1.01	1.02
4	2.67	5.56	6.16	1.01	1.06	1.11

Notes: this table presents the ration of the asymptotic MSE of the feasible and infeasible SC estimator for the model presented in Section 6.1. We set $J + 1 = 20$. Columns 1 to 3 present the case in which these 20 units are divided in 10 groups of 2 units each, while columns 4 to 6 present the case in which units are divided in 2 groups of 10. Different columns present different values of σ^2_ϵ, while $\sigma^2_\lambda = 1$. Different rows present different values of $\mathbb{E}[\lambda_t|D(1,0)]$ for $t > 0$ (that is, in the post-treatment periods. Panel i displays the results when there is only one post-treatment periods. Panel ii assumes 10 post-treatment periods, considering an estimator for the average treatment effect across all post-treatment periods.

Table 2: **Misallocation of weights and probability of perfect match - stationary model**

	Misallocation of weights			Probability of perfect match ($\widetilde{R}^2 > 0.8$)			Misallocation conditional on perfect match		
	$\sigma^2_\varepsilon = 0.1$	$\sigma^2_\varepsilon = 0.5$	$\sigma^2_\varepsilon = 1$	$\sigma^2_\varepsilon = 0.1$	$\sigma^2_\varepsilon = 0.5$	$\sigma^2_\varepsilon = 1$	$\sigma^2_\varepsilon = 0.1$	$\sigma^2_\varepsilon = 0.5$	$\sigma^2_\varepsilon = 1$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$T_0 = 5$	0.418	0.714	0.807	0.729	0.510	0.469	0.425	0.743	0.833
	[0.002]	[0.002]	[0.002]	[0.003]	[0.004]	[0.004]	[0.003]	[0.003]	[0.002]
$T_0 = 20$	0.197	0.495	0.653	0.639	0.013	0.001	0.174	0.331	0.445
	[0.001]	[0.001]	[0.001]	[0.003]	[0.001]	[0.000]	[0.001]	[0.008]	[0.040]
$T_0 = 50$	0.150	0.415	0.573	0.701	0.000	0.000	0.137	-	-
	[0.000]	[0.001]	[0.001]	[0.003]	[0.000]	[0.000]	[0.000]	-	-
$T_0 = 100$	0.130	0.384	0.539	0.766	0.000	0.000	0.122	-	-
	[0.000]	[0.001]	[0.001]	[0.003]	[0.000]	[0.000]	[0.000]	-	-

Notes: this table presents MC simulations results from a stationary model. We consider the SC estimator that uses all pre-treatment outcome lags as economic predictors for a given $(T_0, \sigma^2_\varepsilon)$. In all simulations, we set $J + 1 = 20$ and $K = 10$, which means that the 20 units are divided into 10 groups of 2 units that follow the same common factor λ^k_t. Columns 1 to 3 present the proportion of misallocated weights, which is given by the sum of weights allocated to units 3 to 20. Columns 4 to 6 present the probability that the pre-treatment match is close to perfect, defined as a $\widetilde{R}^2 > 0.8$. Columns 7 to 9 present the proportion of misallocated weights conditional on a perfect match.

Table 3: **DID/SC ratio of standard errors - stationary model**

	$\sigma_{\varepsilon}^2 = 0.1$	$\sigma_{\varepsilon}^2 = 0.5$	$\sigma_{\varepsilon}^2 = 1$
	(1)	(2)	(3)
$T_0 = 5$	1.585	1.082	1.005
	[0.011]	[0.007]	[0.005]
$T_0 = 20$	2.232	1.231	1.074
	[0.014]	[0.005]	[0.003]
$T_0 = 50$	2.327	1.294	1.101
	[0.010]	[0.005]	[0.004]
$T_0 = 100$	2.389	1.314	1.123
	[0.012]	[0.005]	[0.003]

Notes: this table presents MC simulations results from a stationary model as in Table 2. We present the ratio of standard errors of the DID estimator vs. the SC estimator for different $(T_0, \sigma_{\varepsilon}^2)$ scenarios.

Table 4: **Misallocation of weights and probability of perfect match - non-stationary model**

	Misallocation of weights			Misallocation of weights (non-stationary factors)		
	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$
	(1)	(2)	(3)	(4)	(5)	(6)
$T_0 = 5$	0.372	0.661	0.762	0.107	0.192	0.232
	[0.002]	[0.002]	[0.002]	[0.001]	[0.002]	[0.002]
$T_0 = 20$	0.176	0.441	0.589	0.029	0.069	0.095
	[0.001]	[0.001]	[0.001]	[0.000]	[0.001]	[0.001]
$T_0 = 50$	0.136	0.373	0.518	0.015	0.036	0.050
	[0.001]	[0.001]	[0.001]	[0.000]	[0.000]	[0.000]
$T_0 = 100$	0.120	0.346	0.489	0.009	0.022	0.030
	[0.000]	[0.001]	[0.001]	[0.000]	[0.000]	[0.000]

	Probability of perfect match ($\widetilde{R}^2 > 0.8$)			Misallocation conditional on perfect match		
	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$
	(7)	(8)	(9)	(10)	(11)	(12)
$T_0 = 5$	0.846	0.618	0.542	0.377	0.683	0.784
	[0.003]	[0.003]	[0.004]	[0.002]	[0.003]	[0.003]
$T_0 = 20$	0.984	0.556	0.296	0.175	0.427	0.571
	[0.001]	[0.004]	[0.003]	[0.001]	[0.002]	[0.003]
$T_0 = 50$	1.000	0.835	0.550	0.136	0.371	0.515
	[0.000]	[0.003]	[0.004]	[0.001]	[0.001]	[0.001]
$T_0 = 100$	1.000	0.973	0.822	0.120	0.346	0.487
	[0.000]	[0.001]	[0.003]	[0.000]	[0.001]	[0.001]

Notes: this table presents MC simulations results from a model with non-stationary and stationary common factors. We consider the SC estimator that uses all pre-treatment outcome lags as economic predictors for a given $(T_0, \sigma_\varepsilon^2, K)$. In all simulations, we set $J + 1 = 20$, $K = 10$ (which means that the 20 units are divided into 10 groups of 2 units each that follow the same stationary common factor λ_t^k) and $R = 2$ (which means that the 20 units are divided into 2 groups of 10 units each that follow the same non-stationary common factor γ_t^r). Columns 1 to 3 present the proportion of misallocated weights, which is given by the sum of weights allocated to units 3 to 20. Columns 4 to 6 present the proportion of misallocated weights considering only the non-stationary common factor, which is given by the sum of weights allocated to units 11 to 20. Columns 7 to 9 present the probability that the pre-treatment match is close to perfect, defined as a $\widetilde{R}^2 > 0.8$. Columns 10 to 12 present the proportion of misallocated weights conditional on a perfect match. Standard errors in brackets.

Table 5: **DID/SC ratio of standard errors - non-stationary model**

	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$
	(1)	(2)	(3)
$T_0 = 5$	2.072	1.263	1.115
	[0.016]	[0.007]	[0.005]
$T_0 = 20$	4.374	2.155	1.680
	[0.029]	[0.011]	[0.010]
$T_0 = 50$	6.649	3.190	2.420
	[0.040]	[0.021]	[0.016]
$T_0 = 100$	9.462	4.494	3.369
	[0.057]	[0.027]	[0.022]

Notes: this table presents MC simulations results from a non-stationary model as in Table 4. We present the ratio of standard errors of the DID estimator vs. the SC estimator for different $(T_0, \sigma_\varepsilon^2)$ scenarios. Standard errors in brackets.

A Supplemental Appendix: Revisiting the Synthetic Control Estimator (For Online Publication)

A.1 Proof of the Main Results

A.1.1 Proposition 1

Proof.

Let $\mathbf{y}_{0t} = (y_{2t}, ..., y_{J+1,t})'$, $\varepsilon_{0t} = (\varepsilon_{2t}, ..., \varepsilon_{J+1,t})'$, and $\mu_0 = (\mu_2, ..., \mu_{J+1})$. The SC weights $\widehat{\mathbf{w}} \in \mathbb{R}^J$ are given by?

$$\widehat{\mathbf{w}} = \arg\min_{\mathbf{w} \in W} \frac{1}{T_0} \sum_{t=-T_0+1}^{0} \left(y_{1t} - \mathbf{y}_{0t}'\mathbf{w}\right)^2 \tag{17}$$

where $W = \{\mathbf{w} \in \mathbb{R}^J | w_j \geq 0 \text{ and } \sum_{j \neq 1} w_j = 1\}$.[26]

Under Assumptions 1 and 4, the objective function $\widehat{Q}_{T_0}(\mathbf{w}) \equiv \frac{1}{T_0}\sum_{t=-T_0+1}^{0}\left(y_{1t} - \mathbf{y}_{0t}'\mathbf{w}\right)^2$ converges pointwise in probability to

$$Q_0(\mathbf{w}) \equiv \sigma_\varepsilon^2(1 + \mathbf{w}'\mathbf{w}) + (\mu_1 - \mu_0\mathbf{w})'\,\Omega_0\,(\mu_1 - \mu_0\mathbf{w}) \tag{18}$$

which is a continuous and strictly convex function. Therefore, $Q_0(\mathbf{w})$ is uniquely minimized over W, and we define its minimum as $\bar{\mathbf{w}} \in W$.

We show that this convergence in probability is uniform over $\mathbf{w} \in W$. Define $\tilde{y}_{1t} = y_{1t} - \delta_t$ and $\tilde{\mathbf{y}}_{0t} = \mathbf{y}_{0t} - \delta_t\mathbf{i}$, where $\mathbf{i}$ is a $J \times 1$ vector of ones. For any $\mathbf{w}', \mathbf{w} \in W$, using the mean value theorem, we can find a $\widetilde{\mathbf{w}} \in W$ such that

$$\begin{aligned}\left|\widehat{Q}_{T_0}(\mathbf{w}') - \widehat{Q}_{T_0}(\mathbf{w})\right| &= \left|2\left(\frac{1}{T_0}\sum_{t=-T_0+1}^{0}\tilde{\mathbf{y}}_{0t}\tilde{y}_{1t} - \frac{1}{T_0}\sum_{t=-T_0+1}^{0}\tilde{\mathbf{y}}_{0t}\tilde{\mathbf{y}}_{0t}'\widetilde{\mathbf{w}}\right)\cdot(\mathbf{w}' - \mathbf{w})\right| \\ &\leq \left[\left(2\left|\left|\frac{1}{T_0}\sum_{t=-T_0+1}^{0}\tilde{\mathbf{y}}_{0t}\tilde{y}_{1t}\right|\right| + \left|\left|\frac{1}{T_0}\sum_{t=-T_0+1}^{0}\tilde{\mathbf{y}}_{0t}\tilde{\mathbf{y}}_{0t}'\right|\right| \times ||\widetilde{\mathbf{w}}||\right)||\mathbf{w}' - \mathbf{w}||\right]^{\frac{1}{2}}.\end{aligned} \tag{19}$$

Define $B_{T_0} = 2\left|\left|\frac{1}{T_0}\sum_{t=-T_0+1}^{0}\tilde{\mathbf{y}}_{0t}\tilde{y}_{1t}\right|\right| + \left|\left|\frac{1}{T_0}\sum_{t=-T_0+1}^{0}\tilde{\mathbf{y}}_{0t}\tilde{\mathbf{y}}_{0t}'\right|\right| \times C$. Since W is compact, $||\widetilde{\mathbf{w}}||$ is bounded, so we can find a constant C such that $\left|\widehat{Q}_{T_0}(\mathbf{w}') - \widehat{Q}_{T_0}(\mathbf{w})\right| \leq B_{T_0}\left(||\mathbf{w}' - \mathbf{w}||\right)^{\frac{1}{2}}$. Since $\tilde{y}_{1t}\tilde{\mathbf{y}}_{0t}$

[26] If the number of control units is greater than the number of pre-treatment periods, then the solution to this minimization problem might not be unique. However, since we consider the asymptotics with $T_0 \to \infty$, then we guarantee that, for large enough T_0, the solution will be unique.

and $\tilde{\mathbf{y}}_{0t}\tilde{\mathbf{y}}_{0t}'$ are linear combinations of cross products of λ_t and ε_{it}, from Assumptions 1 and 4 we have that B_{T_0} converges in probability to a positive constant, so $B_{T_0} = O_p(1)$. Note also that $Q_0(\mathbf{w})$ is uniformly continuous on W. Therefore, from Corollary 2.2 of Newey (1991), we have that $\widehat{Q}_{T_0}$ converges uniformly in probability to Q_0. Since Q_0 is uniquely minimized at $\bar{\mathbf{w}}$, W is a compact space, Q_0 is continuous and $\widehat{Q}_{T_0}$ converges uniformly to Q_0, from Theorem 2.1 of Newey and McFadden (1994), $\widehat{\mathbf{w}}$ exists with probability approaching one, and $\widehat{\mathbf{w}} \xrightarrow{p} \bar{\mathbf{w}}$.

Now we show that $\bar{\mathbf{w}}$ does not generally reconstruct the factor loadings. Note that Q_0 has two parts. The first one reflects that different choices of weights will generate different weighted averages of the idiosyncratic shocks ε_{it}. In this simpler case, this part would be minimized when we set all weights equal to $\frac{1}{J}$. Let the $J \times 1$ vector $\mathbf{j_J} = \left(\frac{1}{J}, ..., \frac{1}{J}\right)' \in W$. The second part reflects the presence of common factors λ_t that would remain after we choose the weights to construct the SC unit. This part is minimized if we choose a $\mathbf{w}^* \in \Phi = \{\mathbf{w} \in W \mid \mu_1 = \mu_0\mathbf{w}\}$. Suppose that we start at $\mathbf{w}^* \in \Phi$ and move in the direction of $\mathbf{j_J}$, with $\mathbf{w}(\Delta) = \mathbf{w}^* + \Delta(\mathbf{j_J} - \mathbf{w}^*)$. Note that, for all $\Delta \in [0,1]$, these weights will continue to satisfy the constraints of the minimization problem. If we consider the derivative of function 18 with respect to Δ at $\Delta = 0$, we have that:

$$\Gamma'(\mathbf{w}^*) = 2\sigma_\varepsilon^2 \left(\frac{1}{J} - \mathbf{w}^{*\prime}\mathbf{w}^*\right) < 0 \text{ unless } \mathbf{w}^* = \mathbf{j_J} \text{ or } \sigma_\varepsilon^2 = 0$$

Therefore, $\mathbf{w}^*$ will not, in general, minimize Q_0. This implies that, when $T_0 \to \infty$, the SC weights will converge in probability to weights $\bar{\mathbf{w}}$ that does not reconstruct the factor loadings of the treated unit, unless it turns out that $\mathbf{w}^*$ also minimizes the variance of this linear combination of the idiosyncratic errors or if $\sigma_\varepsilon^2 = 0$. ■

A.1.2 Proposition 2

Proof.

The demeaned SC estimator is given by $\widehat{\mathbf{w}}^{\text{SC}'} = \underset{\mathbf{w}\in W}{\text{argmin}}\ \widehat{Q}'_{T_0}(\mathbf{w})$, where

$$\begin{aligned}\widehat{Q}'_{T_0}(\mathbf{w}) &= \frac{1}{T_0}\sum_{t=-T_0+1}^{0}\left(y_{1t} - \mathbf{y}_{0t}'\mathbf{w} - \left(\frac{1}{T_0}\sum_{t=-T_0+1}^{0} y_{1t} - \frac{1}{T_0}\sum_{t=-T_0+1}^{0}\mathbf{y}_{0t}'\mathbf{w}\right)\right)^2 \\ &= \widehat{Q}_{T_0}(\mathbf{w}) - \left(\frac{1}{T_0}\sum_{t=-T_0+1}^{0} y_{1t} - \frac{1}{T_0}\sum_{t=-T_0+1}^{0}\mathbf{y}_{0t}'\mathbf{w}\right)^2. \qquad (20)\end{aligned}$$

$\widehat{Q}'_{T_0}(\mathbf{w})$ converges pointwise in probability to

$$Q'_0(\mathbf{w}) \equiv \sigma^2_\varepsilon(1+\mathbf{w}'\mathbf{w}) + (\mu_1 - \mu_0\mathbf{w})'\left(\Omega_0 - \omega'_0\omega_0\right)(\mu_1 - \mu_0\mathbf{w}) \tag{21}$$

where $\Omega_0 - \omega'_0\omega_0$ is positive semi-definite, so $Q'_0(\mathbf{w})$ is a continuous and convex function.

The proof that $\widehat{\mathbf{w}}^{\text{SC}'} \xrightarrow{p} \bar{\mathbf{w}}^{\text{SC}'}$ where $\bar{\mathbf{w}}^{\text{SC}'}$ will generally not reconstruct the factor loadings of the treated unit follows exactly the same steps as the proof of Proposition 1. Therefore

$$\hat{\alpha}^{\text{SC}'}_{1t} = y_{1t} - \mathbf{y}_{0t}\widehat{\mathbf{w}}^{\text{SC}'} - \left[\frac{1}{T_0}\sum_{t'=-T_0+1}^{0} y_{1t} - \frac{1}{T_0}\sum_{t'=-T_0+1}^{0} \mathbf{y}'_{0t}\widehat{\mathbf{w}}^{\text{SC}'}\right] \xrightarrow{d} \alpha_{1t} + \left(\varepsilon_{1t} - \varepsilon'_{0t}\bar{\mathbf{w}}^{\text{SC}'}\right) + (\lambda_t - \omega_0)\left(\mu_1 - \mu_0\bar{\mathbf{w}}^{\text{SC}'}\right). \tag{22}$$

■

A.1.3 Proposition 3

Proof.

For any estimator $\hat{\alpha}_{1t}(\widetilde{\mathbf{w}}) = y_{1t} - \mathbf{y}_{0t}\widetilde{\mathbf{w}} - \left[\frac{1}{T_0}\sum_{t'=-T_0+1}^{0} y_{1t} - \frac{1}{T_0}\sum_{t'=-T_0+1}^{0} \mathbf{y}'_{0t}\widetilde{\mathbf{w}}\right]$ such that $\widetilde{\mathbf{w}} \xrightarrow{p} \mathbf{w}$, we have that, under Assumptions 1, 2, 4 and 5,

$$a.var(\hat{\alpha}_{1t}(\widetilde{\mathbf{w}})|D(1,0)=1) = \sigma^2_\varepsilon(1+\mathbf{w}'\mathbf{w}) + (\mu_1 - \mu_0\mathbf{w})'\left(\Omega_0 - \omega'_0\omega_0\right)(\mu_1 - \mu_0\mathbf{w}) = Q'_0(\mathbf{w}), \tag{23}$$

which implies that $a.var(\hat{\alpha}^{\text{SC}'}_{1t}|D(1,0)=1) = Q'_0(\hat{\alpha}^{\text{SC}'}_{1t})$, $a.var(\hat{\alpha}^{DID}_{1t}|D(1,0)=1) = Q'_0(\hat{\alpha}^{DID}_{1t})$, and $a.var(\hat{\alpha}^{*}_{1t}|D(1,0)=1) = Q'_0(\hat{\alpha}^{*}_{1t})$. By definition of $\hat{\alpha}^{\text{SC}'}_{1t}$, it must be that $Q'_0(\hat{\alpha}^{\text{SC}'}_{1t}) \leq Q'_0(\hat{\alpha}^{DID}_{1t})$ and $Q'_0(\hat{\alpha}^{\text{SC}'}_{1t}) \leq Q'_0(\hat{\alpha}^{*}_{1t})$. ■

A.1.4 Proposition 4

Proof.

We show this result for the case without the adding-up, non-negativity, and no intercept constraints. In Appendix A.6.1 we extend these results for the cases with the adding-up and/or non-negativity constraints. In Appendix A.6.2 we show that this result is not valid when we use the no intercept constraint.

Note first that we can re-write model 9 as

$$\mathbf{Y}_t = \begin{bmatrix} \theta'_1 \\ \vdots \\ \theta'_{J+1} \end{bmatrix} \gamma'_t + \tilde{\epsilon}_t = \Theta\gamma'_t + \tilde{\epsilon}_t, \tag{24}$$

where $\gamma_t = (\gamma_t^1, ..., \gamma_t^{F_1})$, and Θ is a $J+1 \times F$ matrix with the factor loadings associated with γ_t for all units and $\tilde{\epsilon}_t$ is an $\mathcal{I}(0)$ vector that includes the stationary common factors and the transitory shocks. Without loss of generality, we assume that the elements of γ_t are ordered so that its first element of γ_t is the deterministic polynomial trend with highest power, and the last elements are the $\mathcal{I}(1)$ common factors.

Suppose there are h linearly independent vectors $\mathbf{b} \in \mathbb{R}^{J+1}$ such that $\mathbf{b}'\Theta = 0$. In this case, we can consider the triangular representation

$$\mathbf{y}_{1t} = \Gamma'\mathbf{y}_{2t} + \mu_1^* + \mathbf{z}_t^*, \tag{25}$$

where $\mathbf{y}_{1t}$ is $h \times 1$, $\mathbf{y}_{2t}$ is $g \times 1$, and Γ' is $h \times g$; $\mathbf{z}_t^*$ is a $h \times 1$ $\mathcal{I}(0)$ series with mean zero and μ_1^* is an $h \times 1$ vector of constants. Given Assumption 3$'$, we can write this representation with unit 1 in the vector $\mathbf{y}_{1t}$. Without loss of generality, we consider the case where $\mathbf{y}_{1t} = (y_{1t}, ..., y_{ht})'$ and $\mathbf{y}_{2t} = (y_{h+1,t}, ..., y_{J+1,t})'$. We define the matrix Θ_i^j as a submatrix with the lines i to j of matrix Θ. Importantly, note that equation 25 implies that $\Theta_1^h = \Gamma'\Theta_{h+1}^{J+1}$.

From the definition of $\mathbf{y}_{2t}$, we have that $rank(\Theta_{h+1}^{J+1}) = g$. Otherwise, it would be possible to find another linearly independent vector $v \in \mathbb{R}^{J+1}$ such that $v'\mathbf{y}_t$ is stationary, which contradicts the fact that the dimension of such space is h. We consider a linear transformation $\tilde{\mathbf{y}}_{2t} \equiv A\mathbf{y}_{2t}$ for some invertible $g \times g$ matrix A such that the matrix $\tilde{\Theta}_{h+1}^{J+1} \equiv A\Theta_{h+1}^{J+1}$ with elements $\tilde{\theta}_{j,f}$ has the following property: there exist integers $1 = f_1 < ... < f_g \leq F_1$ such that $\tilde{\theta}_{j,f_j} \neq 0$ and $\tilde{\theta}_{j,f} = 0$ if $f > f_j$. In words, this transformed vector $\tilde{\mathbf{y}}_{2t}$ is such that its n^{th} element does not contain a common factor of higher order than the highest order common factors for any element $j < n$ of $\tilde{\mathbf{y}}_{2t}$.

We show that it is possible to construct such matrix given the definition of $\mathbf{y}_{2t}$. We start setting $\tilde{y}_{1,t} = y_{j,t}$ for some $j \in \{h+1, .., J+1\}$ such that $\theta_{j,1} \neq 0$. For the second row, consider linear combinations $b'\mathbf{y}_{2t}$ for some $b \in \mathbb{R}^g$ and let $\tilde{\theta}_f(b)$ be the f-component of the $(1 \times F_1)$ row vector $b'\Theta_{h+1}^{J+1}$. Consider now the set of all linear combinations $b'\mathbf{y}_{2t}$ such that $\tilde{\theta}_1(b) = 0$, and let f_2 be the largest $f \in \{1, ..., F_1\}$ such that $\tilde{\theta}_{f_2}(b) \neq 0$ for some b in this set. We pick one b such that $\tilde{\theta}_1(b) = 0$ and $\tilde{\theta}_{f_2}(b) \neq 0$ and set $\tilde{y}_{2,t} = b'\mathbf{y}_{2t}$. For the third row, we consider linear combinations of $\mathbf{y}_{2t}$ such that $\tilde{\theta}_f(b) = 0$ for all $f \leq f_2$, and choose $\tilde{y}_{3,t}$ as a linear combination $b'\mathbf{y}_{2t}$ such that $\tilde{\theta}_{f_3}(b) \neq 0$. Since, $rank(\Theta_{h+1}^{J+1}) = g$, we can continue this construction until we get $\tilde{y}_{g,t} = b'\mathbf{y}_{2t}$ for a linear combination b such that $\tilde{\theta}_f(b) = 0$ for all $f \leq f_{g-1}$ with $\tilde{\theta}_f(b) \neq 0$ for at least one $f > f_{g-1}$.

Therefore, we have that

$$\mathbf{y}_{1t} = \Gamma' A^{-1}\tilde{\mathbf{y}}_{2t} + \mu_1^* + \mathbf{z}_t^*. \tag{26}$$

Now closely following the proof of proposition 19.3 in Hamilton (1994), we consider the OLS regression

$$z_{1t}^{*} = \alpha + \boldsymbol{\beta}'\mathbf{z}_{2t}^{*} + \phi'\tilde{\mathbf{y}}_{2t} + u_t \tag{27}$$

where z_{1t}^{*} is the first element of $\mathbf{z}_t^{*}$, and $\mathbf{z}_{2t}^{*} = (z_{2t}^{*}, ..., z_{ht}^{*})'$.

Now let $\tilde{f}_k$ be equal to the order of the polynomial common factor $\gamma_t^{f_k}$ or equal to $\frac{1}{2}$ is $\gamma_t^{f_k}$ is an $\mathcal{I}(1)$ common factor. Then OLS estimator for this model is

$$\begin{bmatrix} \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \\ \widehat{\alpha} \\ T_0^{\tilde{f}_1}\widehat{\phi}_1 \\ \vdots \\ T_0^{\tilde{f}_g}\widehat{\phi}_g \end{bmatrix} = \begin{bmatrix} \frac{\sum \mathbf{z}_{2t}^{*}\mathbf{z}_{2t}^{*}{}'}{T_0} & \frac{\sum \mathbf{z}_{2t}^{*}}{T_0} & \frac{\sum \mathbf{z}_{2t}^{*}\tilde{y}_{1,t}}{T_0^{\tilde{f}_1+1}} & \cdots & \frac{\sum \mathbf{z}_{2t}^{*}\tilde{y}_{g,t}}{T_0^{\tilde{f}_g+1}} \\ \frac{\sum \mathbf{z}_{2t}^{*}{}'}{T_0} & 1 & \frac{\sum \tilde{y}_{1,t}}{T_0^{\tilde{f}_1+1}} & \cdots & \frac{\sum \tilde{y}_{g,t}}{T_0^{\tilde{f}_g+1}} \\ \frac{\sum \tilde{y}_{1,t}\mathbf{z}_{2t}^{*}{}'}{T_0^{\tilde{f}_1+1}} & \frac{\sum \tilde{y}_{1,t}}{T_0^{\tilde{f}_1+1}} & \frac{\sum \tilde{y}_{1,t}^2}{T_0^{2\tilde{f}_1+1}} & \cdots & \frac{\sum \tilde{y}_{1,t}\tilde{y}_{g,t}}{T_0^{\tilde{f}_1+\tilde{f}_g+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sum \tilde{y}_{g,t}\mathbf{z}_{2t}^{*}{}'}{T_0^{\tilde{f}_g+1}} & \frac{\sum \tilde{y}_{g,t}}{T_0^{\tilde{f}_g+1}} & \frac{\sum \tilde{y}_{g,t}\tilde{y}_{1,t}}{T_0^{\tilde{f}_1+\tilde{f}_g+1}} & \cdots & \frac{\sum \tilde{y}_{g,t}^2}{T_0^{2\tilde{f}_g+1}} \end{bmatrix}^{-1} \times \begin{bmatrix} T_0^{-1}\sum \mathbf{z}_{2t}^{*}u_t \\ T_0^{-1}\sum u_t \\ T_0^{-(1+\tilde{f}_1)}\sum \tilde{y}_{1,t}u_t \\ \vdots \\ T_0^{-(1+\tilde{f}_g)}\sum \tilde{y}_{g,t}u_t \end{bmatrix}. \tag{28}$$

Suppose that $\tilde{y}_{jt}$ has non-negative coefficients for at least one polynomial common factor for $j = 1, ..., g'$, while $\tilde{y}_{jt}$ has non-negative coefficients only for $\mathcal{I}(1)$ common factors for $j = g'+1, ..., g$. We start showing that the first matrix in the right hand side of equation 28 converges to a matrix that is almost surely non-singular. Note that the terms $T_0^{-1}\sum \mathbf{z}_{2t}^{*}$ and $T_0^{-(\tilde{f}_j+1)}\sum \mathbf{z}_{2t}^{*}\tilde{y}_{j,t}$ converge in probability to zero, while $T_0^{-1}\sum \mathbf{z}_{2t}^{*}\mathbf{z}_{2t}^{*}{}' \xrightarrow{p} \mathbb{E}[\mathbf{z}_{2t}^{*}\mathbf{z}_{2t}^{*}{}']$. Also, for $j \in \{1, ..., g'\}$, $\sum \tilde{y}_{j,t}$ is dominated by $\sum \tilde{\theta}_{j,f_j}t^{\tilde{f}_j}$, which implies that $T_0^{-(\tilde{f}_j+1)}\sum \tilde{y}_{j,t} \xrightarrow{p} \tilde{\theta}_{j,f_j}/(\tilde{f}_j+1)$. Similarly, for $(i,j) \in \{1, ..., g'\}$, $\sum \tilde{y}_{j,t}\tilde{y}_{i,t}$ is dominated by $\sum \tilde{\theta}_{j,f_j}\tilde{\theta}_{i,f_i}t^{\tilde{f}_i+\tilde{f}_j}$, which implies that $T_0^{-(\tilde{f}_j+\tilde{f}_i+1)}\sum \tilde{y}_{j,t}\tilde{y}_{i,t} \xrightarrow{p} \tilde{\theta}_{j,f_j}\tilde{\theta}_{i,f_i}/(\tilde{f}_i+\tilde{f}_j+1)$. Finally, the terms that include interactions with $\tilde{y}_{j,t}$ for $j \in \{g'+1, ..., g\}$ will converge in law to functions of an $(g-g')$-dimensional Brownian motion (with exception of those interacted with $\mathbf{z}_{2t}^{*}$, which, in this case, converge in probability to zero).[27] Putting these results together, we have that

$$\begin{bmatrix} \frac{\sum \mathbf{z}_{2t}^{*}\mathbf{z}_{2t}^{*}{}'}{T_0} & \frac{\sum \mathbf{z}_{2t}^{*}}{T_0} & \frac{\sum \mathbf{z}_{2t}^{*}\tilde{y}_{1,t}}{T_0^{\tilde{f}_1+1}} & \cdots & \frac{\sum \mathbf{z}_{2t}^{*}\tilde{y}_{g,t}}{T_0^{\tilde{f}_g+1}} \\ \frac{\sum \mathbf{z}_{2t}^{*}{}'}{T_0} & 1 & \frac{\sum \tilde{y}_{1,t}}{T_0^{\tilde{f}_1+1}} & \cdots & \frac{\sum \tilde{y}_{g,t}}{T_0^{\tilde{f}_g+1}} \\ \frac{\sum \tilde{y}_{1,t}\mathbf{z}_{2t}^{*}{}'}{T_0^{\tilde{f}_1+1}} & \frac{\sum \tilde{y}_{1,t}}{T_0^{\tilde{f}_1+1}} & \frac{\sum \tilde{y}_{1,t}^2}{T_0^{2\tilde{f}_1+1}} & \cdots & \frac{\sum \tilde{y}_{1,t}\tilde{y}_{g,t}}{T_0^{\tilde{f}_1+\tilde{f}_g+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sum \tilde{y}_{g,t}\mathbf{z}_{2t}^{*}{}'}{T_0^{\tilde{f}_g+1}} & \frac{\sum \tilde{y}_{g,t}}{T_0^{\tilde{f}_g+1}} & \frac{\sum \tilde{y}_{g,t}\tilde{y}_{1,t}}{T_0^{\tilde{f}_1+\tilde{f}_g+1}} & \cdots & \frac{\sum \tilde{y}_{g,t}^2}{T_0^{2\tilde{f}_g+1}} \end{bmatrix} \xrightarrow{L} \begin{bmatrix} \mathbb{E}[\mathbf{z}_{2t}^{*}\mathbf{z}_{2t}^{*}{}']_{h\times h} & \mathbf{0}_{h\times(g'+1)} & \mathbf{0}_{h\times(g-g')} \\ \mathbf{0}_{(g'+1)\times h} & \mathbf{C}_{(g'+1)\times(g'+1)} & \mathbf{D}'_{(g'+1)\times(g-g')} \\ \mathbf{0}_{(g-g')\times h} & \mathbf{D}_{(g-g')\times(g'+1)} & \mathbf{E}_{(g-g')\times(g-g')} \end{bmatrix} \equiv \mathbf{V} \tag{29}$$

where $\mathbf{C}$ is a non-random matrix with the limits of the terms $T_0^{-(\tilde{f}_j+\tilde{f}_i+1)}\sum \tilde{y}_{j,t}\tilde{y}_{i,t}$ and $T_0^{-(\tilde{f}_i+1)}\sum \tilde{y}_{i,t}$ for

[27] See the proof of proposition 19.3 in Hamilton (1994) for details.

$(i,j) \in \{1,...,g'\}$, $\mathbf{E}$ is a random matrix for where the terms $T_0^{-(\tilde{f}_j+\tilde{f}_i+1)} \sum \tilde{y}_{j,t}\tilde{y}_{i,t}$ for $(i,j) \in \{g'+1,...,g\}$ converge in law, and $\mathbf{D}$ is a random matrix for where the terms $T_0^{-(\tilde{f}_j+\tilde{f}_i+1)} \sum \tilde{y}_{j,t}\tilde{y}_{i,t}$ and $T_0^{-(\tilde{f}_j+1)} \sum \tilde{y}_{j,t}$ for $i \in \{1,...,g'+1\}$ and $j \in \{g'+1,...,g\}$ converge in law. Note that $\mathbb{E}[\mathbf{z}^*_{2t}\mathbf{z}^{*\prime}_{2t}]$ is non-singular by definition of $\mathbf{z}^*_{2t}$. It is also easy to show that $\mathbf{C}$ is non-singular.[28] Following the proof of Proposition 19.3 in Hamilton (1994), we also have that $\mathbf{E}$ is nonsingular with probability one. Therefore, we have that $\mathbf{V}$ is non-singular with probability one.[29]

Now we show that the second matrix in the right hand side of equation 28 converges in probability to zero. In this case, note that $\sum \tilde{y}_{j,t}u_t$ for $j = g'+1,...,g$ is dominated by terms $\sum \xi_t u_t$ where ξ_t is $\mathcal{I}(1)$, which implies that $T_0^{-\frac{3}{2}} \sum \tilde{y}_{j,t}u_t \xrightarrow{p} 0$. For $j \in \{1,...,g'\}$, note that $\sum \tilde{y}_{j,t}u_t$ is dominated by a term $\sum t^{\tilde{f}_j} u_t$. Therefore, $T_0^{-(1+\tilde{f}_j)} \sum \tilde{y}_{j,t}u_t$ converges in probability to zero. Finally, we also have that $T^{-1}\sum u_t$ and $T^{-1}\sum \mathbf{z}^*_{2t}u_t$ converge in probability to zero. Therefore, $\hat{\alpha} \xrightarrow{p} 0$, $\widehat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$, and $T^{\tilde{f}_i}\widehat{\phi}'_i \xrightarrow{p} 0$. From equations 26 and 27, we have that OLS estimator of y_{1t} on a constant and $y_{2t},...,y_{ht},\tilde{y}_{h+1,t},...,\tilde{y}_{J+1,t}$ is given by $(\hat{\beta}' \quad \widehat{\phi}' + [1 \quad \hat{\beta}']\Gamma' A^{-1})$.[30] This implies that the OLS estimator of y_{1t} on a constant and $y_{2t},...,y_{J+1,t}$ is given by $\widehat{\mathbf{w}}' = (\hat{\beta}' \quad \widehat{\phi}' A + [1 \quad \hat{\beta}']\Gamma')$.

We are interested in the limiting distribution of $\hat{\alpha}_{1t}$, which is the effect of the treatment $\tau = t - T_0$ periods after the treatment started $(t > T_0)$. Note that

$$\begin{aligned}\hat{\alpha}^{\text{SC}'}_{1t} \quad = \quad & \alpha_{1t} + \lambda_t \left(\mu_1 - \sum_{j\neq 1} \hat{w}_j \mu_j \right) + \gamma_t \left(\theta_1 - \sum_{j\neq 1} \hat{w}_j \theta_j \right) + \left(\varepsilon_{1t} - \sum_{j\neq 1} \hat{w}_j \varepsilon_{jt} \right) \\ & - \frac{1}{T_0} \sum_{t'=1}^{T_0} \left[\lambda'_t \left(\mu_1 - \sum_{j\neq 1} \hat{w}_j \mu_j \right) + \gamma'_t \left(\theta_1 - \sum_{j\neq 1} \hat{w}_j \theta_j \right) + \left(\varepsilon_{1t'} - \sum_{j\neq 1} \hat{w}_j \varepsilon_{jt'} \right) \right]. \qquad (30)\end{aligned}$$

[28] When $\tilde{\theta}_{j,f_j} \neq 0$ and $0 < f_1 < ... < f_{g'}$, which will be the case by construction, it is possible to diagonalize this matrix. For each row $j = 2,...,g'+1$, we can subtract it by row 1 multiplied by $\frac{\theta_j}{1+f_j}$, and then divide that by $\frac{-f_j}{1+f_j}$. This will result in a matrix with the same entries as the original one, except that rows 2 to $g'+1$ in the first column will be equal to zero. Then for each row $j = 3,...,g'+1$ we can subtract it by row 2 multiplied by $\frac{\theta_j}{\theta_1}\frac{1+2f_1}{1+f_1+f_j}$, and then divide it by $-\frac{f_j - f_1}{1+f_1+f_j}$. This will transform rows 3 to $g'+1$ in column 2 to zero. Continuing this procedure, we have an upper triangular matrix with diagonal elements different from zero.

[29] Note that $det(\mathbf{V}) = det(\mathbb{E}[\mathbf{z}^*_{2t}\mathbf{z}^{*\prime}_{2t}])det(\mathbf{C} - \mathbf{D}'\mathbf{E}^{-1}\mathbf{D})det(\mathbf{E})$. We have that $det(\mathbb{E}[\mathbf{z}^*_{2t}\mathbf{z}^{*\prime}_{2t}]) \neq 0$ and that $det(\mathbf{E}) \neq 0$ with probability one (which also implies that $\mathbf{E}^{-1}$ exists with probability one). Therefore, we only need that $det(\mathbf{C} - \mathbf{D}'\mathbf{E}^{-1}\mathbf{D}) \neq 0$ to guarantee that $\mathbf{V}$ is non-singular. Since $\mathbf{C}$ is non-singular, the realizations of $\mathbf{D}'\mathbf{E}^{-1}\mathbf{D}$ such that $\mathbf{C} - \mathbf{D}'\mathbf{E}^{-1}\mathbf{D}$ is singular will have measure zero, which implies that $\mathbf{V}$ is non-singular with probability one.

[30] Those are the estimators associated with $\mathbf{z}^*_{2t}$ and $\tilde{\mathbf{y}}_{2t}$. The estimator for the constant is given by $\hat{\alpha} + [1 \quad -\hat{\beta}']\mu^*_1$.

For the term $\gamma_t\left(\theta_1 - \sum_{j\neq 1}\hat{w}_j\theta_j\right)$, note that

$$\begin{aligned}\sum_{j\neq 1}\hat{w}_j\theta_j &= \begin{bmatrix}\Theta_2^{h\prime} & \Theta_{h+1}^{J+1\prime}\end{bmatrix}\widehat{\mathbf{w}} = \begin{bmatrix}\Theta_2^{h\prime} & \Theta_{h+1}^{J+1\prime}\end{bmatrix}\begin{bmatrix}\hat{\beta} \\ A'\hat{\phi} + \Gamma\begin{bmatrix}1 \\ -\hat{\beta}\end{bmatrix}\end{bmatrix} \\ &= \Theta_2^{h\prime}\hat{\beta} + \Theta_{h+1}^{J+1\prime}A'\hat{\phi} + \Theta_{h+1}^{J+1\prime}\Gamma\begin{bmatrix}1 \\ -\hat{\beta}\end{bmatrix} \qquad (31) \\ &= \Theta_2^{h\prime}\hat{\beta} + \Theta_{h+1}^{J+1\prime}A'\hat{\phi} + \Theta_1^{h\prime}\begin{bmatrix}1 \\ -\hat{\beta}\end{bmatrix} = \theta_1 + \Theta_{h+1}^{J+1\prime}A'\hat{\phi}.\end{aligned}$$

Let $\Lambda = diag(T_0^{a_1}, ..., T_0^{a_F})$, where a_k is defined such that $\gamma_{T_0}^k T_0^{-a_k}$ converge either to a constant (when γ_t^k is a deterministic time trend) or to a distribution (when γ_t^k is an $\mathcal{I}(1)$ common factor). Then

$$\gamma_t\left(\theta_1 - \sum_{j\neq 1}\hat{w}_j\theta_j\right) = -\gamma_t\Theta_{h+1}^{J+1\prime}A'\hat{\phi} = -\gamma_t\Lambda^{-1}\Lambda\Theta_{h+1}^{J+1\prime}A'\hat{\phi}. \qquad (32)$$

If $\gamma_t = t^k$, then $\gamma_t = (T_0 + (t - T_0))^k$, which implies that $T_0^{-k}\gamma_t = (1 + \frac{(t-T_0)}{T_0})^k \rightarrow 1$ when $T_0 \rightarrow \infty$. If γ_t is $\mathcal{I}(1)$, then $\gamma_t = \gamma_{T_0} + \sum_{t'=T_0+1}^{t}\eta_t$, which implies that $T_0^{-\frac{1}{2}}\gamma_t$ converges in distribution to a normal variable when $T_0 \rightarrow \infty$. Using the properties of $A\Theta_{h+1}^{J+1}$, we also have that the n^{th} row of $\Lambda\Theta_{h+1}^{J+1\prime}A'\hat{\phi}$ will be given by $T_0^{a_n}$ multiplied by a linear combination of elements $\hat{\phi}_j$ such that $f_j \geq a_n$. Therefore, the random variables $\hat{\phi}_j$ that are present in row n converge to zero at a faster rate than $T_0^{a_n}$, so $\Lambda\Theta_{h+1}^{J+1\prime}A'\hat{\phi} \xrightarrow{p} 0$. That is, we show that the SC weights will converge to weights that reconstruct the factor loadings of the treated unit associated with the non-stationary common factors, and the convergence in this case will be fast enough to compensate the fact that the non-stationary factors explode. Similarly, we have that $\frac{1}{T_0}\sum_{t'=1}^{T_0}\gamma_t'(\theta_1 - \sum_{j\neq 1}\hat{w}_j\theta_j) \xrightarrow{p} 0$.

Finally, by definition of u_t in equation 27, the OLS estimator converges to weights that minimize $var[u_t^2]$ subject to $\mathbf{w} \in \Phi_1$, where $u_t = \lambda_t(\mu_1 - \sum_{j\neq 1} w_j\mu_j) + (\varepsilon_{1t} - \sum_{j\neq 1} w_j\varepsilon_{jt})$. Therefore, the proof that $\widehat{\mathbf{w}} \xrightarrow{p} \bar{\mathbf{w}} \notin \Phi$ is essentially the same as the proof of Proposition 1.

Combining these results, we have that:

$$\hat{\alpha}_{1t} \xrightarrow{d} \alpha_{1t} + \left(\varepsilon_{1t} - \sum_{j\neq 1}\bar{w}_j\varepsilon_{jt}\right) + (\lambda_t - \omega_0)\left(\mu_1 - \sum_{j\neq 1}\bar{w}_j\mu_j\right) \qquad (33)$$

where $\omega_0 = plim_{T_0\rightarrow\infty}\frac{1}{T_0}\sum_{t'=1}^{T_0}\lambda_t$. ■

A.2 Case with finite T_0

We consider here the case with T_0 fixed. For weights $\{w_j^*\}_{j\neq 1} \in \Phi$, note that:

$$y_{1t} = \sum_{j=1}^{J+1} w_j^* y_{jt} + \eta_t, \text{ for } t \leq 0, \text{ where } \eta_t = \varepsilon_{1t} - \sum_{j=1}^{J+1} w_j^* \varepsilon_{jt} \tag{34}$$

Since $\sum_{j=2}^{J+1} w_j^* = 1$, we can write:

$$\tilde{y}_{1t} = \sum_{j=1}^{J} w_j^* \tilde{y}_{jt} + \eta_t \tag{35}$$

where $\tilde{y}_{jt} = y_{jt} - y_{J+1,t}$. The SC weights will be given by the OLS regression in 35 with the non-negativity constraints. We ignore for now the non-negativity constraints. If we let $\tilde{y}_{0t} = (\tilde{y}_{2t}, ..., \tilde{y}_{Jt})'$, $\mathbf{w}_0^* = (w_2^*, ..., w_J^*)'$ and $\widehat{\mathbf{w}}_0 = (\widehat{w}_2, ..., \widehat{w}_J)'$, then we have that $\widehat{\mathbf{w}}_0 = \left(\sum_{t=-T_0+1}^{0} \tilde{y}_{0t}\tilde{y}_{0t}'\right)^{-1} \sum_{t=-T_0+1}^{0} \tilde{y}_{0t}\tilde{y}_{1t}$. We assume that T_0 is large enough so that $\sum_{t=-T_0+1}^{0} \tilde{y}_{0t}\tilde{y}_{0t}'$ has full rank. Therefore:

$$\mathbb{E}[\widehat{\mathbf{w}}_0 | \tilde{y}_{0,-T_0+1}, ..., \tilde{y}_{0,0}] = \mathbf{w}_0^* + \left(\sum_{t=-T_0+1}^{0} \tilde{y}_{0t}\tilde{y}_{0t}'\right)^{-1} \sum_{t=-T_0+1}^{0} \tilde{y}_{0t}\mathbb{E}[\eta_t | \tilde{y}_{0,-T_0+1}, ..., \tilde{y}_{0,0}] \tag{36}$$

By definition of η_t, we have that $\mathbb{E}[\eta_t | \tilde{y}_{0,-T_0+1}, ..., \tilde{y}_{0,0}] \neq 0$ for $t \leq 0$, which implies that $\widehat{\mathbf{w}}_0$ is a biased estimator of $\mathbf{w}_0^*$. Intuitively, the transitory shocks behave as a measurement error when we use the control outcomes as a proxy for the common factors. Considering the non-negativity constraints would affect the distribution of $\widehat{\mathbf{w}}_0$ because, with finite T_0, there will be a positive probability that the solution to the unrestricted OLS problem will not satisfy the non-negativity constraints. However, this would not change the conclusion that $\widehat{\mathbf{w}}_0$ is a biased estimator of $\mathbf{w}_0^*$.

A.3 Example: SC Estimator vs DID Estimator

We provide an example in which the asymptotic bias of the SC estimator can higher than the asymptotic bias of the DID estimator. Assume we have 1 treated and 4 control units in a model with 2 common factors. For simplicity, assume that there is no additive fixed effects and that $\mathbb{E}[\lambda_t] = 0$. We have that the factor loadings are given by:

$$\mu_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mu_2 = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}, \mu_3 = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix}, \mu_4 = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}, \mu_5 = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix} \tag{37}$$

Note that the linear combination $0.5\mu_2 + w_1^3\mu_3 + w_1^5\mu_5 = \mu_1$ with $w_1^3 + w_1^5 = 0.5$ satisfy Assumption

3. Note also that DID equal weights would set the first factor loading to 1, which is equal to μ_1^1, but the second factor loading would be equal to $0.75 \neq \mu_1^2$. We want to show that the SC weights would improve the construction of the second factor loading but it will distort the combination for the first factor loading. If we set $\sigma_\varepsilon^2 = \mathbb{E}[(\lambda_t^1)^2] = \mathbb{E}[(\lambda_t^2)^2] = 1$, then the factor loadings of the SC unit would be given by $(1.038, 0.8458)$. Therefore, there is small loss in the construction of the first factor loading and a gain in the construction of the second factor loading. Therefore, if selection into treatment is correlated with the common shock λ_t^1, then the SC estimator would be more asymptotically biased than the DID estimator.

A.4 Definition: Asymptotically Unbiased

We now show that the expected value of the asymptotic distribution will be the same as the limit of the expected value of the SC estimator in the setting described in Section 3. Let γ be the expected value of the asymptotic distribution of $\hat{\alpha}_{1t} - \alpha_{1t}$. Therefore, we have that:

$$\begin{aligned}
\mathbb{E}[\hat{\alpha}_{1t} - \alpha_{1t}] &= \gamma + E\left[\sum_{j\neq 1}(\bar{w}_j - \hat{w}_j)\varepsilon_{jt}\right] + E\left[\lambda_t \sum_{j\neq 1}(\bar{w}_j - \hat{w}_j)\mu_j\right] \\
&= \gamma + \sum_{j\neq 1} E\left[(\bar{w}_j - \hat{w}_j)\varepsilon_{jt}\right] + \sum_{j\neq 1} E\left[\lambda_t(\bar{w}_j - \hat{w}_j)\right]\mu_j
\end{aligned}$$

Therefore:

$$\left|E\left[(\bar{w}_j - \hat{w}_j)\varepsilon_{jt}\right]\right| \leq E\left[\left|(\bar{w}_j - \hat{w}_j)\varepsilon_{jt}\right|\right] \leq \sqrt{E\left[(\bar{w}_j - \hat{w}_j)^2\right]E\left[(\varepsilon_{jt})^2\right]}$$

Now note that $\hat{w}_j$ is a consistent estimator for $\bar{w}_j$ and the random variable $(\bar{w}_j - \hat{w}_j)^2$ is bounded, because W is compact. Therefore, the sequence $(\bar{w}_j - \hat{w}_j)^2$ is asymptotically uniformly integrable, which implies that $E\left[(\bar{w}_j - \hat{w}_j)^2\right] \to 0$. If we also assume that ε_{it} and λ_t^f for all $f = 1, ..., F$ have finite variance, then $\mathbb{E}[\hat{\alpha}_{1t} - \alpha_{1t}] \to \gamma$ when $T_0 \to \infty$.

A.5 Alternatives specifications and alternative estimators

A.5.1 Average of pre-intervention outcome as economic predictor

We consider now another very common specification in SC applications, which is to use the average pre-treatment outcome as the economic predictor. Note that if one uses only the average pre-treatment outcome as the economic predictor then the choice of matrix V would be irrelevant. In this case, the minimization problem would be given by:

$$\begin{aligned}\{\hat{w}_j\}_{j\neq 1} &= \text{argmin}_{w\in W}\left[\frac{1}{T_0}\sum_{t=-T_0+1}^{0}\left(y_{1t}-\sum_{j\neq 1}w_j y_{jt}\right)\right]^2 \\ &= \text{argmin}_{w\in W}\left[\frac{1}{T_0}\sum_{t=-T_0+1}^{0}\left(\varepsilon_{1t}-\sum_{j\neq 1}w_j\varepsilon_{jt}+\lambda_t\left(\mu_1-\sum_{j\neq 1}w_j\mu_j\right)\right)\right]^2 \end{aligned} \tag{38}$$

where $W=\{\{w_j\}_{j\neq 1}\in\mathbb{R}^J|w_j\geq 0 \text{ and } \sum_{j\neq 1}w_j=1\}$.

Therefore, under Assumptions 1, 2 and 4, the objective function converges in probability to:

$$\Gamma(\mathbf{w})=\left[E\left[\lambda_t|D(1,0)=1\right]\left(\mu_1-\sum_{j\neq 1}w_j\mu_j\right)\right]^2 \tag{39}$$

Assuming that there is a time-invariant common factor (that is, $\lambda_t^1=1$ for all t) and that the pre-treatment average of the conditional process λ_t converges to $\mathbb{E}[\lambda_t^k]=0$ for $k>1$, the objective function collapses to:

$$\Gamma(\mathbf{w})=\left[\left(\mu_1^1-\sum_{j\neq 1}w_j\mu_j^1\right)\right]^2 \tag{40}$$

Therefore, even if we assume that there exists at least one set of weights that reproduces all factor loadings (Assumption 3), the objective function will only look for weights that approximate the first factor loading. This is problematic because it might be that assumption 3 is satisfied, but there are weights $\{\tilde{w}_j\}_{j\neq 1}\notin\Phi$ that satisfy $\mu_1^1=\sum_{j\neq 1}\tilde{w}_j\mu_j^1$. In this case, there is no guarantee that the SC control method will choose weights that are close to the correct ones. This result is consistent with the MC simulations in Ferman et al. (2017), who show that this specification performs particularly bad in allocating the weights correctly.

A.5.2 Adding other covariates as predictors

Most SC applications that use the average pre-intervention outcome value as economic predictor also consider other time invariant covariates as economic predictors. Let Z_i be a $(R\times 1)$ vector of observed covariates (not affected by the intervention). Model 45 changes to:

$$\begin{cases} y_{it}(0)=\delta_t+\theta_t Z_i+\lambda_t\mu_i+\varepsilon_{it} \\ y_{it}(1)=\alpha_{it}+y_{it}(0) \end{cases} \tag{41}$$

We also modify Assumption 3 so that the weights reproduce both μ_1 and Z_1.

Assumption 3$''$ (existence of weights)

$$\exists \ \mathbf{w} \in W \mid \mu_1 = \sum_{j \neq 1} w_j^* \mu_j, \ Z_1 = \sum_{j \neq 1} w_j^* Z_j$$

Let X_1 be an $(R+1 \times 1)$ vector that contains the average pre-intervention outcome and all covariates for unit 1, while X_0 is a $(R+1 \times J)$ matrix that contains the same information for the control units. For a given V, the first step of the nested optimization problem suggested in Abadie et al. (2010) would be given by:

$$\widehat{\mathbf{w}}(V) \in \text{argmin}_{\mathbf{w} \in W} ||X_1 - X_0 \mathbf{w}||_V \tag{42}$$

where $W = \{\{w_j\}_{j \neq 1} \in \mathbb{R}^J | w_j \geq 0 \text{ and } \sum_{j \neq 1} w_j = 1\}$. Assuming again that there is a time-invariant common factor (that is, $\lambda_t^1 = 1$ for all t) and that the pre-treatment average of the unconditional process λ_t converges to $\mathbb{E}[\lambda_t^k] = 0$ for $k > 1$, objective function of this minimization problem converges to $||\bar{X}_1 - \bar{X}_0 \mathbf{w}||_V$, where:

$$\bar{X}_1 - \bar{X}_0 \mathbf{w} = \begin{bmatrix} \mathbb{E}[\theta_t | D(1, T_0) = 1] \left(Z_1 - \sum_{j \neq 1} w_j Z_j \right) + \left(\mu_1^1 - \sum_{j \neq 1} w_j \mu_j^1 \right) \\ \left(Z_1^1 - \sum_{j \neq 1} w_j Z_j^1 \right) \\ \vdots \\ \left(Z_1^R - \sum_{j \neq 1} w_j Z_j^R \right) \end{bmatrix} \tag{43}$$

Similarly to the case with only the average pre-intervention outcome value as economic predictor, it might be that Assumption 3$''$ is satisfied, but there are weights $\{\tilde{w}_j\}_{j \neq 1}$ that satisfy $\mu_1^1 = \sum_{j \neq 1} \tilde{w}_j \mu_j^1$ and $Z_1 = \sum_{j \neq 1} \tilde{w}_j Z_j$, although $\mu_1^k \neq \sum_{j \neq 1} \tilde{w}_j \mu_j^k$ for some $k > 1$. Therefore, there is no guarantee that an estimator based on this minimization problem would converge to weights that satisfy Assumption 3$''$ for any given matrix V.

The second step in the nested optimization problem is to choose V such that $\widehat{\mathbf{w}}(V)$ minimizes the pre-intervention prediction error. Note that this problem is essentially given by:

$$\widehat{\mathbf{w}} \quad = \quad \text{argmin}_{w \in \widetilde{W}} \left[\frac{1}{T_0} \sum_{t=-T_0+1}^{0} \left(y_{1t} - \sum_{j \neq 1} w_j y_{jt} \right) \right]^2 \tag{44}$$

where $\widetilde{W} \subseteq W$ is the set of $\mathbf{w}$ such that $\mathbf{w}$ is the solution to problem 42 for some positive semidefinite matrix

V. Similarly to the SC estimator that includes all pre-treatment outcomes, there is no guarantee that this minimization problem will choose weights that satisfy Assumption 3$''$ even when $T_0 \to \infty$. More specifically, if the variance of ε_{it} is large, then the SC estimator would tend to choose weights that are uniform across the control units in detriment of weights that satisfy Assumption 3$''$. Therefore, it is not possible to guarantee that this SC estimator would be asymptotically unbiased. MC simulation results in Ferman et al. (2017) confirm that this SC specification systematically misallocates more weight than alternatives that use a large number of pre-treatment outcome lags as predictors.

A.5.3 Relaxing constraints on the weights

If we assume that $W = \mathbb{R}^J$ instead of the compact set $\{\widehat{\mathbf{w}} \in \mathbb{R}^J | w_j \geq 0 \text{ and } \sum_{j \neq 1} w_j = 1\}$, then we can still guarantee consistency of the SC weights. The only difference is that we also need to assume convergence of the pre-treatment averages of δ_t. In Proposition 1 this was not necessary because the adding-up restriction implies that δ_t was always eliminated. Consider the model

$$y_{it}(0) = \dot{\lambda}_t \dot{\mu}_i + \varepsilon_{it} \tag{45}$$

where $\dot{\lambda}_t = (\delta_t, \lambda_t)$ and $\dot{\mu}_i = (1, \mu_i)'$. We modify Assumption 4 to include assumptions on the convergence of δ_t.

Assumption 4$''$ (convergence of pre-treatment averages) Conditional on $D(1,0) = 1$, $\frac{1}{T_0} \sum_{t=-T_0+1}^{0} \dot{\lambda}_t \xrightarrow{p} \dot{\omega}_0$, $\frac{1}{T_0} \sum_{t=-T_0+1}^{0} \dot{\lambda}_t' \dot{\lambda}_t \xrightarrow{p} \dot{\Omega}_0$, $\frac{1}{T_0} \sum_{t=-T_0+1}^{0} \epsilon_t \xrightarrow{p} 0$, $\frac{1}{T_0} \sum_{t=-T_0+1}^{0} \epsilon_t \epsilon_t' \xrightarrow{p} \sigma_\varepsilon^2 I_{J+1}$, $\varepsilon_{jt} \perp \dot{\lambda}_s$, and $\frac{1}{T_0} \sum_{t=-T_0+1}^{0} \epsilon_t \dot{\lambda}_t \xrightarrow{p} 0$ when $T_0 \to \infty$.

Note first that, under assumptions 1 and 4$''$, the objective function converges in probability to

$$\widehat{Q}_{T_0}(\mathbf{w}) \xrightarrow{p} \dot{Q}_0(\mathbf{w}) \equiv \sigma_\varepsilon^2 (1 + \mathbf{w}'\mathbf{w}) + (\dot{\mu}_1 - \dot{\mu}_0 \mathbf{w})' \dot{\Omega}_0 (\dot{\mu}_1 - \dot{\mu}_0 \mathbf{w}), \tag{46}$$

where $\dot{Q}_0(\mathbf{w})$ is continuous and strictly convex. Since W is a convex space, $\dot{Q}_0(\mathbf{w})$ has a unique minimum that is in the interior of W. Therefore, by Theorem 2.7 of Newey and McFadden (1994), $\widehat{\mathbf{w}}$ exists with probability approaching one and $\widehat{\mathbf{w}} \xrightarrow{p} \mathbf{w}_0$.

For the case $W = \{\mathbf{w} \in \mathbb{R}^J \mid \sum_{j=2}^{J+1} w_j = 1\}$, note that the transformed model with $y_{1t} - y_{2t}$ as the outcome of the treated unit and $y_{3t} - y_{2t}, ..., y_{J+1,t} - y_{2t}$ as the outcomes of the control units is equivalent to the original model. Then we can use the same arguments on this modified model.

Consistency when we impose only the non-negativity constraint follows from the same arguments as in Appendix A.6.1.

Given that we assure convergence of $\widehat{\mathbf{w}}$, the fact that $\widehat{\mathbf{w}}$ does not converge to weights that reconstruct the factor loadings of the treated unit follows from the same arguments as the proof of Proposition 1. Note that, without the adding-up constraint, it might be that the asymptotic distribution of the SC estimator depends on δ_t.

A.5.4 IV-Like SC Estimator

Consider again equation 34. The key problem is that η_t is correlated with y_{jt}, which implies that the restricted OLS estimators are biased and inconsistent. Imposing strong assumptions on the structure of the idiosyncratic error and the common factors, we show that it is possible to consider moment equations that will be equal to zero if, and only if, $\{w_j\}_{j \neq 1} \in \Phi$.

Let $\mathbf{y}_{0t} = (y_{2,t}, ..., y_{J+1,t})'$, $\mu_{\mathbf{0}}$ be a $(F \times J)$ matrix with columns μ_j, $\epsilon_{0t} = (\varepsilon_{2,t}, ..., \varepsilon_{J+1,t})$, and $\mathbf{w} = (w_2, ..., w_{J+1})'$. In this case, we can look at

$$\begin{aligned}
\mathbf{y}_{t-1}(y_{1t} - \mathbf{y}_{0t}'\mathbf{w}) &= (\mu_0'\lambda_{t-1}' + \epsilon_{0,t-1})\lambda_t\,(\mu_1 - \mu_{\mathbf{0}}\mathbf{w}) + (\mu_0'\lambda_{t-1}' + \epsilon_{0,t-1})(\varepsilon_{1t} - \epsilon_{0t}'\mathbf{w}) \qquad (47)\\
&= \mu_0'\lambda_{t-1}'\lambda_t\,(\mu_1 - \mu_{\mathbf{0}}\mathbf{w}) + \epsilon_{0,t-1}\lambda_t\,(\mu_1 - \mu_{\mathbf{0}}\mathbf{w}) + \mu_0'\lambda_{t-1}'(\varepsilon_{1t} - \epsilon_{0t}'\mathbf{w}) + \epsilon_{0,t-1}(\varepsilon_{1t} - \epsilon_{0t}'\mathbf{w}).
\end{aligned}$$

Under Assumptions 1 and 4, and assuming further that ε_{it} is independent across t, then the objective function given by $\frac{1}{T_0}\sum_{t=-T_0+1}^{0}\mathbf{y}_{t-1}(y_{1t} - \mathbf{y}_{0t}'\mathbf{w})$ converges uniformly to $\mathbb{E}[\mathbf{y}_{0,t-1}(y_{1t} - \mathbf{y}_{0t}'\mathbf{w})] = \mu_0'\mathbb{E}[\lambda_{t-1}'\lambda_t](\mu_1 - \mu_{\mathbf{0}}\mathbf{w})$

Therefore, if the $(J \times F)$ matrix $\mu_0'\mathbb{E}[\lambda_{t-1}'\lambda_t]$ has full rank, then the moment conditions equal to zero if, and only if, $\mathbf{w} \in \Phi$. One particular case in which this assumption is valid is if λ_t^f and $\lambda_t^{f'}$ are uncorrelated and λ_t^f is serially correlated for all $f = 1, ..., F$. Intuitively, under these assumptions, we can use the lagged outcome values of the control units as instrumental variables for the control units' outcomes.[31] One challenge to analyze this method is that there might be multiple solutions to the moment condition. Based on the results by Chernozhukov et al. (2007), it is possible to consistently estimate this set. Therefore, it is possible to generate an IV-like SC estimator that is, under additional assumptions, asymptotically unbiased.

[31] The idea of SC-IV is very similar to the IV estimator used in dynamic panel data. In the dynamic panel models, lags of the outcome are used to deal with the endogeneity that comes from the fact the idiosyncratic errors are correlated with the lagged depend variable included in the model as covariates. The number of lags that can be used as instruments depends on the serial correlation of the error terms.

A.6 Extensions on Proposition 4

A.6.1 Relaxing the adding-up and non-negativity constraints

To show that this result is also valid for the case with adding-up constraint we just have to consider the OLS regression of $y_{1t} - y_{2t}$ on a constant and $y_{3t} - y_{2t}, ..., y_{J+1,t} - y_{2t}$. Under assumption 3′, this transformed model is also cointegrated, so we can apply our previous result.

We now consider the case with the non-negative constraints. We prove the case $W = \{\mathbf{w} \in \mathbb{R}^J \mid w_j \geq 0\}$. Including an adding-up constraint then follows directly from a change in variables as we did for the case without non-negative constraints.

We first show that $\widehat{\mathbf{w}} \xrightarrow{p} \bar{\mathbf{w}}$ where $\bar{\mathbf{w}}$ minimizes $\mathbb{E}[u_t^2]$ subject to $\mathbf{w} \in \Phi_1 \cap W$. Suppose that $\bar{\mathbf{w}} \in int(W)$. This implies that $\bar{\mathbf{w}} \in int(\Phi_1 \cap W)$ relative to Φ_1. By convexity of $E[u_t^2]$, $\bar{\mathbf{w}}$ also minimizes $E[u_t^2]$ subject to Φ_1. We know that OLS without the non-negativity constraints converges in probability to $\bar{\mathbf{w}}$. Let $\widehat{\mathbf{w}}_u$ be the OLS estimator without the non-negativity constraints and $\widehat{\mathbf{w}}_r$ be the OLS estimator with the non-negativity constraint. Since $\bar{\mathbf{w}} \in int(W)$, then it must be that, for all $\varepsilon > 0$, $Pr(|\widehat{\mathbf{w}}_u - \bar{\mathbf{w}}| > \varepsilon) = 0$ with probability approaching to 1 (w.p.a.1). Since $\widehat{\mathbf{w}}_u = \widehat{\mathbf{w}}_r$ when $\widehat{\mathbf{w}}_u \in int(W)$ (due to convexity of the OLS objective function), these two estimators are asymptotically equivalent.

Consider now the case in which $\bar{\mathbf{w}}$ is on the boundary of W. This means that $\bar{w}_j = 0$ for at least one j. Let $A = \{j | w_j^* = 0\}$. Note first that $\bar{\mathbf{w}}$ also minimizes $E[u_t^2]$ subject to $\mathbf{w} \in \Phi \cap \{\mathbf{w} | w_j = 0 \ \forall j \in A\}$. That is, if we impose the restriction $w_j = 0$ for all j such that $\bar{w}_j = 0$, then we would have the same minimizer, even if we ignore the other non-negative constraints. Suppose there is an $\tilde{\mathbf{w}} \neq \bar{\mathbf{w}}$ that minimizes $E[u_t^2]$ subject to $\mathbf{w} \in \Phi \cap \{\mathbf{w} | w_j = 0 \ \forall j \in A\}$. By convexity of the objective function and the fact that $\bar{\mathbf{w}}$ is in the interior of $\Phi \cap W \cap \{\mathbf{w} | w_j = 0 \ \forall j \in A\}$ relative to $\Phi \cap \{\mathbf{w} | w_j = 0 \ \forall j \in A\}$, there must be $\mathbf{w}' \in \Phi \cap W \cap \{\mathbf{w} | w_j = 0 \ \forall j \in A\} \subset \Phi \cap W$ that attains a lower value in the objective function than $\bar{\mathbf{w}}$. However, this contradicts the fact that $\bar{\mathbf{w}} \in \Phi \cap W$ is the minimum.

Now let $\widehat{\mathbf{w}}'$ be the OLS estimator subject to $\{\mathbf{w} | w_j = 0 \ \forall j \in A\}$. We have that $\widehat{\mathbf{w}}'$ is consistent for $\bar{\mathbf{w}}$ (Lemma **??**). Now we show that $\widehat{\mathbf{w}}'$ is asymptotically equivalent to $\widehat{\mathbf{w}}''$, the OLS estimator subject to $\{\mathbf{w} | w_j \geq 0 \ \forall j \in A\}$. We prove the case in which $A = \{j\}$ (there is only one restriction that binds). The general case follows by induction. Suppose these two estimators are not asymptotically equivalent. Then there is $\varepsilon > 0$ such that $LimPr(|\widehat{\mathbf{w}}' - \widehat{\mathbf{w}}''| > \varepsilon) \neq 0$. There are two possible cases.

First, suppose that $\text{Lim}Pr\left(|\widehat{w}_j''| > \varepsilon'\right) = 0$ for all $\varepsilon' > 0$ (that is, the OLS subject to $\{\mathbf{w} | w_j \geq 0 \ \forall j \in A\}$ converges in probability to $\bar{\mathbf{w}}$ such that $\bar{w}_j = 0$). However, since the two estimators are not asymptotically equivalent, for all T_0', we can always find a $T_0 > T_0'$ such that, with positive probability, $|\widehat{\mathbf{w}}' - \widehat{\mathbf{w}}''| > \varepsilon$. Since $\{\mathbf{w} | w_j = 0 \ \forall j \in A\} \subset \{\mathbf{w} | w_j \geq 0 \ \forall j \in A\}$ and $\widehat{\mathbf{w}}' \neq \widehat{\mathbf{w}}''$, then $Q_{T_0}(\widehat{\mathbf{w}}'') < Q_{T_0}(\widehat{\mathbf{w}}')$, where $Q_{T_0}()$ is the OLS objective function. Now using the continuity of the OLS objective function and the fact that $\widehat{w}_j''$ converges in probability to zero, we can always find T_0' such that there will be a positive probability that $Q_{T_0}(\widehat{\mathbf{w}}'' - e_j \hat{w}_j'') < Q_{T_0}(\widehat{\mathbf{w}}')$. Since $\widehat{\mathbf{w}}'' - e_j \hat{w}_j'' \in \{\mathbf{w}|w_j = 0 \ \forall j \in A\}$, this contradicts $\widehat{\mathbf{w}}'$ being OLS subject to $\{\mathbf{w}|w_j = 0 \ \forall j \in A\}$.

Alternatively, suppose that there exists $\varepsilon' > 0$ such that $\text{Lim}Pr\left(|\widehat{w}_j''| > \varepsilon'\right) \neq 0$. This means that, for all T_0', we can find $T_0 > T_0'$ such that there is a positive probability that the solution to OLS on $\{\mathbf{w}|w_j \geq 0 \ \forall j \in A\}$ is in an interior point $\widehat{\mathbf{w}}''$ with $\hat{w}_j'' > \varepsilon' > 0$. By convexity of $Q_{T_0}()$, this would imply that $\widehat{\mathbf{w}}''$ is also the solution to the OLS without any restriction. However, this contradicts the fact that OLS without non-negativity restriction is consistent (see proof of Proposition 4).

Finally, we show that $\widehat{\mathbf{w}}''$ and $\widehat{\mathbf{w}}_r$ are asymptotically equivalent. Note that $\bar{\mathbf{w}}$ is in the interior of W relative to $\{\mathbf{w}|w_j \geq 0 \ \forall j \in A\}$. Therefore, w.p.a.1, $\widehat{\mathbf{w}}'' \in W$, which implies that $\widehat{\mathbf{w}}'' = \widehat{\mathbf{w}}_r$.

We still need to show that linear combinations of $\widehat{\mathbf{w}}^r$ converge fast enough to reconstruct the factor loadings of the treated unit associated with the non-stationary common factors, so that $\gamma_t(\theta_1 - \sum_{j\neq 1} \hat{w}_j^r \theta_j) \xrightarrow{p} 0$. Let $Q_{T_0}()$ be the OLS objective function, and let $\widetilde{\mathcal{W}} = \{\widetilde{\mathbf{w}}_1, ..., \widetilde{\mathbf{w}}_{2^J}\}$ be the set of all possible OLS estimators when we consider some of the non-negative constraints as equality and ignore the other ones. Let $\widetilde{\mathcal{W}}' \subset \widetilde{\mathcal{W}}$ be the set of estimators in $\widetilde{\mathcal{W}}$ such that all non-negative constraints are satisfied. Then we know that $\widehat{\mathbf{w}}^r = argmin_{\mathbf{w}\in\widetilde{\mathcal{W}}'} Q_{T_0}(\mathbf{w})$.

Suppose first that, for any of the 2^J combinations of restrictions, there is at least one $\mathbf{w} \in \Phi_1$ that satisfy these restrictions. In this case, we know from the first part of the proof that $\gamma_t\left(\theta_1 - \sum_{j\neq 1} \widetilde{w}_j^h \theta_j\right) \xrightarrow{p} 0$ for all $h = 1, ..., 2^J$, where $\widetilde{\mathbf{w}}_h = (\widetilde{w}_2^h, ..., \widetilde{w}_{J+1}^h)'$. Moreover, since $\widetilde{\mathcal{W}}$ is finite, then this convergence is uniform in $\widetilde{\mathcal{W}}$. Therefore, it must be that $\gamma_t(\theta_1 - \sum_{j\neq 1} \hat{w}_j^r \theta_j) \xrightarrow{p} 0$. Suppose now that for the combination of restrictions considered for $\widetilde{\mathbf{w}}_h$, with $h \in \{1, ..., 2^J\}$, there is no $\mathbf{w} \in \Phi_1$ that satisfies these restrictions. Since the parameter space with this combination of restrictions is closed, then $\exists \eta > 0$ such that $||\theta_1 - \sum_{j\neq 1} w_j \theta_0|| > \eta$ for all $\mathbf{w}$ that satisfy this combinations of restrictions.[32] Therefore, $Q_{T_0}(\widetilde{\mathbf{w}}_h)$ diverge when $T_0 \to \infty$, implying that, w.p.a.1, $\widehat{\mathbf{w}}^r \neq \widetilde{\mathbf{w}}_h$.

A.6.2 Example with no intercept

We consider now a very simple example to show that it is not possible to guarantee that $\gamma_t\left(\theta_1 - \sum_{j\neq 1} \hat{w}_j \theta_j\right) \xrightarrow{p} 0$ if we do not include the intercept. Consider the case in which there are only one treated and one control unit, and $y_{1t} = \mu_1 + t + u_{1t}$ while $y_{2t} = \mu_2 + t + u_{2t}$. We consider a regression of y_{1t} on y_{2t} without the

[32] Otherwise, there would be $\mathbf{w} \in \Phi_1$ that satisfies this combination of restrictions.

intercept. Note that $y_{1t} = (\mu_1 - \mu_2) + y_{2t} + u_{1t} - u_{2t} = \mu + y_{2t} + u_t$. Then we have that:

$$\hat{\beta} = \frac{\sum_{t=1}^{T_0} y_{2t} y_{1t}}{\sum_{t=1}^{T_0} y_{2t}^2} = 1 + \frac{\sum_{t=1}^{T_0} (\mu\mu_2 + \mu t + \mu u_{2t} + \mu_2 u_t + t u_t + u_t u_{2t})}{\sum_{t=1}^{T_0} (t^2 + \mu_2^2 + u_{2t}^2 + \text{“cross terms”})} \tag{48}$$

which implies that:

$$T(\hat{\beta} - 1) = \frac{\frac{1}{T^2} \sum_{t=1}^{T_0} (\mu\mu_2 + \mu t + \mu u_{2t} + \mu_2 u_t + t u_t + u_t u_{2t})}{\frac{1}{T^3} \sum_{t=1}^{T_0} (t^2 + \mu_2^2 + u_{2t}^2 + \text{“cross terms”})} \xrightarrow{p} \frac{\frac{1}{2}\mu}{\frac{1}{3}} \tag{49}$$

Therefore, while $\hat{\beta} \xrightarrow{p} 1$, it does not converge fast enough so that $T(\hat{\beta} - 1) \xrightarrow{p} 0$, except when $\mu_1 = \mu_2$.

A.7 Appendix Tables

Table A.1: **Misallocation of weights and probability of perfect match - alternative definition of perfect match**

	Misallocation of weights			Probability of perfect match ($\widetilde{R}^2 > 0.9$)			Misallocation conditional on perfect match		
	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$T_0 = 5$	0.418	0.714	0.807	0.490	0.319	0.296	0.448	0.771	0.848
	[0.002]	[0.002]	[0.002]	[0.004]	[0.003]	[0.003]	[0.003]	[0.003]	[0.003]
$T_0 = 20$	0.197	0.495	0.653	0.128	0.000	0.000	0.143	-	-
	[0.001]	[0.001]	[0.001]	[0.002]	[0.000]	[0.000]	[0.002]	-	-
$T_0 = 50$	0.150	0.415	0.573	0.032	0.000	0.000	0.102	-	-
	[0.000]	[0.001]	[0.001]	[0.001]	[0.000]	[0.000]	[0.002]	-	-
$T_0 = 100$	0.130	0.384	0.539	0.005	0.000	0.000	0.088	-	-
	[0.000]	[0.001]	[0.001]	[0.000]	[0.000]	[0.000]	[0.003]	-	-

Notes: this table replicates the results from Table 2 using a more stringent definition of perfect match.

Table A.2: **Misallocation of weights and probability of perfect match - stationary model ($K = 2$)**

	Misallocation of weights			Probability of perfect match ($\widetilde{R}^2 > 0.8$)			Misallocation conditional on perfect match		
	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$	$\sigma_\varepsilon^2 = 0.1$	$\sigma_\varepsilon^2 = 0.5$	$\sigma_\varepsilon^2 = 1$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$T_0 = 5$	0.092	0.199	0.266	0.842	0.631	0.555	0.086	0.198	0.268
	[0.001]	[0.001]	[0.002]	[0.003]	[0.003]	[0.004]	[0.001]	[0.002]	[0.002]
$T_0 = 20$	0.066	0.140	0.191	0.921	0.167	0.030	0.063	0.100	0.121
	[0.000]	[0.001]	[0.001]	[0.002]	[0.003]	[0.001]	[0.000]	[0.002]	[0.004]
$T_0 = 50$	0.053	0.110	0.155	0.987	0.024	0.000	0.052	0.066	-
	[0.000]	[0.000]	[0.001]	[0.001]	[0.001]	[0.000]	[0.000]	[0.003]	-
$T_0 = 100$	0.044	0.095	0.134	0.999	0.001	0.000	0.044	-	-
	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	-	-

Notes: this table replicates the results from Table 2 using a DGP with $K = 2$.