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Abstract

This paper examines the macroeconomic effects of asset price bubbles and crashes in an overlapping generations economy. The model highlights the effects of asset price fluctuations on labor supply decisions, and demonstrates how labor market adjustment can help propagate the effects of these fluctuations to the aggregate economy. It is shown that, under certain conditions, asset bubbles can crowd in productive investment and lead to an expansion in total employment, and the bursting of these bubbles can have an immediate negative impact on these variables.

Keywords: Asset Bubbles, Overlapping Generations, Endogenous Labor.

JEL classification: E22, E44.

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1 Introduction

In this paper, we present a stylized model of asset bubbles and crashes, and analyze the effects of these phenomena on the macroeconomy. The model is an extended version of the stochastic bubble model in Weil (1987) that takes into account the effects of asset bubbles on labor supply decisions. Using this model, we demonstrate how labor market responses to asset price fluctuations can help propagate the effects of bubbles and crashes to the aggregate economy.

Since the seminal work of Tirole (1985), it has been known that asset price bubbles — defined as substantial positive deviations of an asset's market price from its fundamental value — can emerge and grow indefinitely in an overlapping generations (OLG) economy. Weil (1987) generalizes the main results in this study to an environment in which asset bubbles may randomly crash in any period. These studies provide an important conceptual framework for understanding the effects of bubbles and crashes, based on rational expectations and general equilibrium analysis. There are, however, two features of these models that are at odd with empirical evidence. First, both Tirole (1985) and Weil (1987) assume that labor supply is exogenously given. Thus, the implicit assumption is that labor market variables, such as total employment and aggregate labor hours, are unrelated to and unaffected by fluctuations in asset prices. This assumption is at odd with the observation that total employment and aggregate labor hours tend to move closely with asset prices in the actual data. In particular, the bursting of asset bubbles is often followed by a noticeable decline in these labor market variables (see Section 2 for details). Second, both studies suggest that the formation of asset bubbles will crowd out investment in physical capital and impede economic growth, while the bursting of these bubbles will have the opposite effects. These predictions are also difficult to reconcile with empirical evidence. For instance, private nonresidential fixed investment in the U.S. has increased significantly during the formation of the internet bubble in the 1990s and the formation of the housing bubble in the 2000s, and has dropped markedly when these bubbles burst. Chirinko and Schaller (2001, 2011) and Gan (2007) provide formal empirical evidence showing that asset bubbles have positive effects on private investment in the U.S. and Japan. Martin and Ventura (2012) also observe that asset bubbles in these countries are often associated with robust economic growth.

In a previous study (Shi and Suen, 2014), we show that these conflicts between theory and evidence can potentially be resolved by relaxing the assumption of exogenous labor supply. More specifically, we show that when labor supply is endogenously determined in Tirole's (1985) model, asset bubbles can potentially lead to an expansion in steady-state capital, investment, employment and output. This happens when the inverse of the intertemporal elasticity of substitution (IES) for consumption is small and the Frisch elasticity of labor supply is large, so that individual labor supply will respond strongly and positively to changes in interest rate. This result highlights the importance of labor supply decisions in analyzing the effects of asset bubbles. This study, however, does not take into account one salient feature of asset bubbles, namely that they will crash at some point but the timing of this cannot be predicted with certainty. Allowing for bubble crashes is important for the issue at hand because, as history attests, these incidents can often lead to great disturbances in the aggregate economy. Motivated by this, the present study extends the analysis in Shi and Suen (2014) to the case of stochastic bubbles and explores the circumstances under which our model can account for the empirical evidence mentioned above.

Similar to our prior work, we consider a two-period OLG model in which consumers can choose how much time to work, and how much to save and consume in their first period of life. There are two types of assets in this economy: physical capital and an intrinsically worthless asset. The latter is similar in nature to fiat money and unbacked government debt. Asset bubble is said to occur when this type of asset is traded across generations at a positive price. The main point of departure from our previous study is the assumption that asset bubbles may randomly crash as in the model of Weil (1987).[1] A crash in this context refers to the situation in which the price of the intrinsically worthless asset falls abruptly and unexpectedly to its fundamental value which is zero. The prospect of this happening means that investment in asset bubbles is subject to considerable risks. A key question is whether this type of risk will spawn uncertainty at the aggregate level. We show that the answer to this question depends crucially on the endogeneity of labor supply. To see this, suppose an asset bubble exists in the current period and it will either survive or crash in the next period. Whether this type of uncertainty will affect the aggregate economy depends on the effects of asset bubbles on the inputs of production. Since the next-period stock of capital is determined by the savings in the current period, it is unaffected by the future state of the bubble. If labor supply is exogenous as in Weil's (1987) model, then both capital and labor inputs (as well as their marginal products and aggregate output) are independent of the state of the bubble. Thus, the bursting of asset bubble will have *no immediate impact* on aggregate quantities and factor prices, and the risky investment in asset bubbles will not generate aggregate uncertainty.[2] This

[1] This type of stochastic bubble is also considered in Caballero and Krishnamurthy (2006), Farhi and Tirole (2012, Section 4.2) and Ventura (2012, Section 3.3).

[2] In the present study, the factor markets are assumed to be competitive so that factor prices (i.e., the rental price of capital and wage rate) are determined by the marginal products of capital and labor.

implication of Weil's model is no longer valid once we allow for an endogenous labor supply. In this case, individual labor hours will in general depend on the state of the asset bubble. As a result, the uncertain prospect of the bubble will create uncertainty in future labor inputs and future prices, which will in turn affect consumers' choices in the current period. This provides a simple and intuitive mechanism through which bubbles and crashes can affect the wider economy. The present study provides the first attempt to analyze this mechanism in a rational bubble model. The main results of this paper are largely in line with those obtained from our previous work. Specifically, we show that the existence of stochastic bubbles can potentially crowd in productive investment, but this happens only if the bubbles can induce a significant expansion in labor supply. Again this scenario is likely to occur when the inverse of the IES for consumption is small and the Frisch elasticity of labor supply is large.

Several recent studies have explored other channels through which asset bubbles can crowd in productive investment and foster economic growth using OLG models. For instance, Martin and Ventura (2012) and Ventura (2012) present models in which asset bubbles can improve investment efficiency by shifting resources from less productive firms or countries to more productive ones. Caballero and Krishnamurthy (2006) and Farhi and Tirole (2012) develop models in which asset bubbles can facilitate investment by providing liquidity to financially constrained firms. These existing studies, however, choose to adopt some strongly simplifying assumptions on consumer preferences which thwart both the intertemporal substitution in consumption and the intratemporal substitution between consumption and labor.[3] The present study complements the existing literature by showing that these forces are important for understanding the macroeconomic impact of bubbles and crashes.

The rest of this paper is organized as follows. Section 2 provides evidence showing that total employment, aggregate labor hours and private investment tend to move closely with asset prices during episodes of asset bubbles. Section 3 describes the structure of the model. Section 4 defines the equilibrium concepts and investigates the main properties of the model. Section 5 concludes.

2 Recent Cases of Asset Bubbles in the U.S.

In this section, we use the two most recent episodes of asset bubbles in the United States as examples to show that total employment, aggregate labor hours and private investment tend to move closely with

[3] In addition to an exogenous labor supply, these studies also assume that consumers (or investors) are risk neutral and only care about their consumption at the old age. Thus, the consumers will save all their income when young which is completely determined by the wage rate.

asset prices during the course of these episodes. The first case that we consider is the "internet bubble" or "dot-com bubble" which formed during the second half of the 1990s. The second one is the housing price bubble which formed during the first half of the 2000s. Figure 1 shows the monthly data of the Dow Jones Industrial Average and the Standard & Poor's 500 index between January 1995 and December 2003. Unless otherwise stated, all the data reported in this section were obtained from the Federal Reserve Economic Data (FRED) website. Both the Dow Jones index and the S&P 500 have tripled between January 1995 and January 2000, and have dropped significantly afterward. Ofek and Richardson (2002) and LeRoy (2004) provide detailed account on why the surge in stock prices between 1995 and 2000 cannot be explained by the growth in fundamentals (e.g., corporate earnings and dividends), and thus suggest the existence of an asset bubble. Figure 2 shows the monthly data of the Case-Shiller 20-City Home Price Index between June 2003 and June 2010. From June 2003 to June 2006, this index has increased by 46 percent. According to Shiller (2007) and other subsequent studies, this surge in home prices represents a substantial deviation from the fundamentals (e.g., rent and construction costs) and is thus generally regarded as a bubble.

The next three diagrams show the relationship between stock prices, employment and private nonresidential fixed investment during the internet bubble episode. Figure 3 shows the monthly data of total employment between January 1995 and December 2003, and compares it to the Dow Jones index. Total employment refers to the total number of employees in all private industries in the Current Employment Statistics (CES) data. Figure 4 shows the monthly data of the aggregate weekly hours index in the CES data over the same time period.[4] These two diagrams show that total employment and aggregate labor hours have moved closely with stock prices during the internet bubble episode. Between January 1995 and January 2000, both total employment and aggregate labor hours have increased by 13 percent, which is equivalent to an average annual growth rate of 2.6 percent. This is significantly higher than the average annual growth rate of total employment between 1948 and 2013, which was 1.3 percent. The average annual growth rate of the aggregate hours index between 1964 and 2013 was 1.5 percent.[5] Figures 3 and 4 also show a noticeable decline in aggregate labor input after the bursting of the internet bubble. Figure 5 shows the quarterly data of private nonresidential fixed investment (deflated by the GDP deflator) between 1995Q1 and 2003Q4. These data were obtained from the National Income and

[4] The scale of these diagrams has been adjusted so as to highlight the timing of the rise and fall of these variables. This is necessary because otherwise the threefold increase in the Dow Jones index will dwarf the changes of employment in these diagrams.

[5] Data on this index are only available from January 1964 onward.

Product Accounts. Between 1995Q1 and 2000Q1, real nonresidential investment has increased by 41 percent which is equivalent to an average annual growth rate of 7.1 percent. As a point of reference, the average annual growth rate of the same variable between 1948 and 2012 was 3.5 percent.

Next, we turn to the relationship between home prices, employment and private nonresidential fixed investment during the housing price bubble episode. Figures 6 and 7 show the monthly data of total employment and aggregate labor hours between June 2003 and June 2010, and compare them to the Case-Shiller index. Between June 2003 and June 2006, total employment has increased by 5.3 percent while aggregate labor hours have increased by 7 percent. These are equivalent to an average annual growth rate of 1.7 percent and 2.4 percent, respectively, which are again higher than their long-term averages. Figure 8 shows the Case-Shiller index and private nonresidential fixed investment during the period 2003Q3 to 2010Q3. The starting value of these time series have been normalized to one so that the two are directly comparable. Between 2003Q3 and 2006Q3, real nonresidential investment has increased by 18 percent, which is equivalent to an average annual growth rate of 5.6 percent. This is again significantly higher than the average annual growth rate between 1948 and 2012.

To summarize, total employment and aggregate labor hours (and also private investment) have moved closely with asset prices during the two most recent cases of asset bubbles in the United States. This provides a direct justification for endogenizing labor supply in the rational bubble model.

3 The Model

3.1 The Environment

Consider an economy inhabited by an infinite sequence of overlapping generations. In each period $t \in \{0, 1, 2, ...\}$, a new generation of identical consumers is born. The size of generation t is given by $N_t = (1+n)^t$, with $n > 0$. Each consumer lives two periods, which we will refer to as the young age and the old age. In each period, each consumer has one unit of time which can be allocated between work and leisure. Retirement is mandatory in the old age, so the labor supply of old consumers is zero. Young consumers, on the other hand, can choose how much time to work, and how much to save and consume. There is a single commodity in this economy which can be used for consumption and capital accumulation. All prices are expressed in units of this commodity.

Consider a consumer who is born at time $t \geq 0$. Let $c_{y,t}$ and $c_{o,t+1}$ denote his consumption when young and old, respectively; and let l_t denote his labor supply when young. The consumer's expected lifetime utility is given by

$$E_t \left[\frac{c_{y,t}^{1-\sigma}}{1-\sigma} - A \frac{l_t^{1+\psi}}{1+\psi} + \beta \frac{c_{o,t+1}^{1-\sigma}}{1-\sigma} \right], \tag{1}$$

where $\sigma > 0$ is the coefficient of relative risk aversion and the inverse of the IES for consumption, $\psi \geq 0$ is the inverse of the Frisch elasticity of labor supply, $\beta \in (0, 1)$ is the subjective discount factor, and A is a positive constant.[6] The consumer can invest in two types of assets: the first one is physical capital and the second one is an intrinsically worthless asset. The latter is called "intrinsically worthless" because it has no consumption value and it cannot be used for production. The only motivation for holding this asset is to resell it at a higher price in the next period. The total supply of the intrinsically worthless asset is fixed and is denoted by $M > 0$.[7]

Let $\widetilde{p}_t \geq 0$ be the price of the intrinsically worthless asset in period t, which is a random variable. Since the fundamental value of this asset is zero, a strictly positive $\widetilde{p}_t$ signifies an overvaluation in period t, which we will refer to as an asset bubble. Following Weil (1987), we assume that $\widetilde{p}_t$ can be separated into a purely random component ε_t and a purely deterministic component p_t, so that $\widetilde{p}_t \equiv \varepsilon_t p_t$ for all t. The random component, or asset price shock, is assumed to follow a Markov chain with two possible states $\{0, 1\}$, transition probabilities

$$\Pr \{\varepsilon_{t+1} = 1 | \varepsilon_t = 1\} = q \in (0, 1),$$

$$\Pr \{\varepsilon_{t+1} = 0 | \varepsilon_t = 0\} = 1,$$

and initial value $\varepsilon_0 = 1$. The asset price shock is the only source of uncertainty in this economy. The time path of the deterministic component, $\{p_t\}_{t=0}^{\infty}$, is endogenously determined in equilibrium. At the beginning of each period t, the value of ε_t is revealed and publicly observed. Suppose $\varepsilon_t = 1$ and $p_t > 0$ so that an asset bubble exists in period t. Then, with probability q, the price of the intrinsically worthless asset will remain on the deterministic time path in period $t + 1$ (i.e., $\widetilde{p}_{t+1} = p_{t+1}$), and with probability $(1 - q)$, it will drop to zero in period $t + 1$. One can think of the latter case as the result of a sudden, unanticipated change in market sentiment which triggers a crash in the financial market. The parameter q can be interpreted as the persistence of asset bubbles.[8] Since the probability of moving from $\varepsilon_t = 1$

[6] If $A = 0$, then all consumers will supply one unit of labor inelastically when young. In this case, our model is essentially identical to the production economy in Weil (1987).

[7] At time 0, all assets are owned by a group of "initial-old" consumers. The decision problem of these consumers is trivial and does not play any role in the following analysis.

[8] The deterministic model considered in Shi and Suen (2014) can be considered as a special case of this model with $q = 1$. In this case, an asset bubble will last forever.

to $\varepsilon_{t+1} = 0$ is strictly positive in every period t, every asset bubble is destined to crash in the long run (technically, this means $\widetilde{p}_t$ will converge in probability to zero as t tends to infinity). The timing of the crash, however, is uncertain. Figure 9 shows the probability tree diagram for the asset price shock. The dark line in the diagram traces the time path of ε_t before the crash. We will refer to this as the *pre-crash economy* and the other parts of the diagram as the *post-crash economy*. Once the bubble bursts, the asset price $\widetilde{p}_t$ will remain zero from that point on. Hence, there is no incentive for the consumers to hold the intrinsically worthless asset in the post-crash economy.

3.2 Consumer's Problem

In this section, we will analyze the consumer's problem before and after the crash. To distinguish between these two scenarios, we use a hat ($\hat{}$) to indicate variables in the post-crash economy. First, consider the case when $\varepsilon_t = 0$. A young consumer at time t now faces a deterministic problem, which is given by

$$\max_{\widehat{c}_{y,t}, \widehat{s}_t, \widehat{l}_t, \widehat{c}_{o,t+1}} \left[\frac{\widehat{c}_{y,t}^{1-\sigma}}{1-\sigma} - A\frac{\widehat{l}_t^{1+\psi}}{1+\psi} + \beta\frac{\widehat{c}_{o,t+1}^{1-\sigma}}{1-\sigma} \right]$$

subject to the budget constraints:

$$\widehat{c}_{y,t} + \widehat{s}_t = \widehat{w}_t \widehat{l}_t, \qquad \text{and} \qquad \widehat{c}_{o,t+1} = \widehat{R}_{t+1}\widehat{s}_t,$$

where $\widehat{s}_t$ denotes savings in physical capital, $\widehat{w}_t$ is the market wage rate, and $\widehat{R}_{t+1}$ is the gross return from physical capital between time t and $t+1$. The solution of this problem is characterized by

$$\widehat{c}_{y,t} = \left(\beta \widehat{R}_{t+1}\right)^{-\frac{1}{\sigma}} \widehat{c}_{o,t+1} = \frac{\widehat{w}_t \widehat{l}_t}{1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}}, \tag{2}$$

$$\widehat{l}_t = A^{-\frac{1}{\sigma+\psi}} \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}\right]^{\frac{\sigma}{\sigma+\psi}} \widehat{w}_t^{\frac{1-\sigma}{\sigma+\psi}}, \tag{3}$$

$$\widehat{s}_t = \Sigma\left(\widehat{R}_{t+1}\right) \widehat{w}_t \widehat{l}_t, \qquad \text{where } \Sigma\left(\widehat{R}_{t+1}\right) \equiv \frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}}. \tag{4}$$

The function $\Sigma : \mathbb{R}_+ \to [0, 1]$ defined in (4) summarizes the effects of interest rate on savings. First, a higher interest rate means that with the same amount of savings in the young age, there will be more interest income when old. This creates an income effect which encourages consumption when young and discourages saving. Second, an increase in interest rate also lowers the price of future consumption relative to current consumption. This creates an intertemporal substitution effect which discourages consumption when young and promotes saving. The relative strength of these two effects is determined by the value of σ. In particular, the intertemporal substitution effect dominates when $\sigma < 1$. In this case, $\Sigma(\cdot)$ is a strictly increasing function. When $\sigma > 1$, the income effect dominates so that $\Sigma(\cdot)$ is strictly decreasing. The two effects exactly cancel out when $\sigma = 1$. In this case, $\Sigma(\cdot)$ is a positive constant which means the consumer will save (and consume) a constant fraction of his labor income when young.

Next, consider the case when $\varepsilon_t = 1$. Let m_t be the consumer's demand for the intrinsically worthless asset at time t. The consumer now faces the following budget constraint in the young age

$$c_{y,t} + s_t + p_t m_t = w_t l_t. \tag{5}$$

The gross return from physical capital between time t and $t+1$ is now a random variable, which means its value depends on the realization of ε_{t+1} (except under some special cases which we will discuss below). Let R_{t+1} be the value when $\varepsilon_{t+1} = 1$, and $\widehat{R}_{t+1}$ be the value when $\varepsilon_{t+1} = 0$. The consumer's old-age consumption is now given by

$$c_{o,t+1} = \begin{cases} R_{t+1}s_t + p_{t+1}m_t & \text{with probability } q, \\ \widehat{R}_{t+1}s_t & \text{with probability } 1-q. \end{cases} \tag{6}$$

Taking $\left\{w_t, p_t, p_{t+1}, R_{t+1}, \widehat{R}_{t+1}\right\}$ as given, the consumer's problem is to choose an allocation $\{c_{y,t}, s_t, l_t, m_t, c_{o,t+1}\}$ so as to maximize his expected lifetime utility in (1), subject to the budget constraints in (5) and (6), and the non-negativity constraint: $m_t \geq 0$.[9] The first-order conditions regarding s_t and l_t are given by

$$c_{y,t}^{-\sigma} = \beta\left[qR_{t+1}\left(R_{t+1}s_t + p_{t+1}m_t\right)^{-\sigma} + (1-q)\,\widehat{R}_{t+1}\left(\widehat{R}_{t+1}s_t\right)^{-\sigma}\right], \tag{7}$$

$$w_t c_{y,t}^{-\sigma} = A l_t^{\psi}. \tag{8}$$

Equation (7) is the standard Euler equation for consumption in the presence of aggregate uncertainty.

[9] Given a constant-relative-risk-aversion (CRRA) utility function, it is never optimal for the consumer to choose $c_{y,t} = 0$ or $c_{o,t+1} = 0$, regardless of the existence of asset bubble. Hence, the non-negativity constraint for these variables is never binding. It is also never optimal to have $s_t \leq 0$ and $l_t = 0$. Suppose the contrary that $s_t \leq 0$, then the consumer will end up having $c_{o,t+1} \leq 0$ when $\varepsilon_{t+1} = 0$, which cannot be optimal. This, together with $m_t \geq 0$, means that consumers will never borrow. Finally, since labor income is the only source of income during the consumer's lifetime, it is never optimal to choose $l_t = 0$.

Equation (8) is the optimality condition for labor supply. Conditional on $\varepsilon_t = 1$, the optimal choice of m_t is determined by

$$p_t c_{y,t}^{-\sigma} \geq \beta E_t \left[\widetilde{p}_{t+1} \left(c_{o,t+1} \right)^{-\sigma} \right] = \beta q p_{t+1} \left(R_{t+1} s_t + p_{t+1} m_t \right)^{-\sigma}, \tag{9}$$

with equality holds in the first part if $m_t > 0$. This equation states that if the marginal cost of holding the intrinsically worthless asset (which is $p_t c_{y,t}^{-\sigma}$) is greater than the marginal benefit of doing so (which is $\beta E_t \left[\widetilde{p}_{t+1} \left(c_{o,t+1} \right)^{-\sigma} \right]$), then the consumer will choose to have $m_t = 0$. Equation (9) can be rewritten as

$$\Rightarrow p_t \geq E_t \left[\beta \left(\frac{c_{o,t+1}}{c_{y,t}} \right)^{-\sigma} \widetilde{p}_{t+1} \right],$$

which is the standard consumption-based asset pricing equation.

We now explore the conditions under which the optimal choice of m_t is strictly positive. Consider a young consumer who initially chooses $m_t = 0$. Suppose now he is considering increasing it to $\xi / p_t > 0$, where $\xi > 0$ is infinitesimal. In order to balance his budget, the consumer will simultaneously reduce s_t by ξ. Define $\pi_{t+1} \equiv p_{t+1}/p_t$ which is the gross return from the intrinsically worthless asset conditional on $\varepsilon_{t+1} = 1$. Increasing m_t from zero to ξ/p_t will generate an expected return of $q\pi_{t+1}\xi$, which will in turn increase expected future utility by $q\pi_{t+1} \left(R_{t+1} s_t \right)^{-\sigma} \xi$. At the same time, the reduction in s_t will lower expected future utility by

$$\left[q R_{t+1} \left(R_{t+1} s_t \right)^{-\sigma} + (1-q) \widehat{R}_{t+1} \left(\widehat{R}_{t+1} s_t \right)^{-\sigma} \right] \xi. \tag{10}$$

Such an increase in m_t is desirable if and only if the marginal benefit of doing so outweighs the marginal cost, i.e.,

$$q\pi_{t+1} \left(R_{t+1} s_t \right)^{-\sigma} \xi > \left[q R_{t+1} \left(R_{t+1} s_t \right)^{-\sigma} + (1-q) \widehat{R}_{t+1} \left(\widehat{R}_{t+1} s_t \right)^{-\sigma} \right] \xi.$$

This can be simplified to

$$q\pi_{t+1} > \left[q + (1-q) \left(\frac{\widehat{R}_{t+1}}{R_{t+1}} \right)^{1-\sigma} \right] R_{t+1}. \tag{11}$$

This means the consumer is willing to hold the intrinsically worthless asset if and only if the expected return $q\pi_{t+1}$ exceeds a certain threshold. The threshold level is determined by three factors: (i) the persistence of asset bubble q; (ii) the state-dependent returns from physical capital R_{t+1} and $\widehat{R}_{t+1}$; and (iii) the preference parameter σ. If the gross return from physical capital is *not* state-dependent, i.e.,

$R_{t+1} = \widehat{R}_{t+1}$, then the condition in (11) can be simplified to $q\pi_{t+1} > R_{t+1}$. If the utility function for consumption is logarithmic, i.e., $\sigma = 1$, then the expression in (10) can be simplified to $s_t^{-1}\xi$. In this case, both the marginal benefit and the marginal cost of increasing m_t are independent of $\widehat{R}_{t+1}$, and the condition in (11) can again be simplified to become $q\pi_{t+1} > R_{t+1}$.

Suppose the condition in (11) is valid. Then the optimal investment in the intrinsically worthless asset, denoted by $a_t \equiv p_t m_t$, is given by

$$a_t \equiv p_t m_t = \frac{p_t}{p_{t+1}} \left(\Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) s_t, \tag{12}$$

where

$$\Omega_{t+1} \equiv \left[\frac{q\left(\pi_{t+1} - R_{t+1}\right)}{\left(1-q\right) \widehat{R}_{t+1}} \right]^{\frac{1}{\sigma}}.$$

It is straightforward to show that $\Omega_{t+1} \widehat{R}_{t+1} > R_{t+1}$ is equivalent to (11). Further details of the consumer's problem in the pre-crash economy can be found in Appendix A.

3.3 Production

On the supply side of the economy, there are a large number of identical firms. In each period, each firm hires labor and physical capital from the competitive factor markets, and produces output according to a Cobb-Douglas production function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}, \qquad \text{with } \alpha \in (0,1),$$

where Y_t denotes output produced at time t, K_t and L_t denote capital input and labor input, respectively. Since the production function exhibits constant returns to scale, we can focus on the problem faced by a single price-taking firm. We assume that physical capital is fully depreciated after one period, so that R_t coincides with the rental price of physical capital at time $t \geq 0$. The representative firm's problem is given by

$$\max_{K_t, L_t} \left\{ K_t^{\alpha} L_t^{1-\alpha} - R_t K_t - w_t L_t \right\},$$

and the first-order conditions are

$$R_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha} \qquad \text{and} \qquad w_t = \left(1-\alpha\right) K_t^{\alpha} L_t^{-\alpha}. \tag{13}$$

Note that neither the production function nor the representative firm's problem is directly affected by the asset price shock, so the above equations are valid both before and after the asset bubble crashes.[10]

4 Equilibria

In this section, we will define and characterize an equilibrium in which the intrinsically worthless asset is valued at some point in time, i.e., $\widetilde{p}_t > 0$ for some t. We will refer to this as a *bubbly equilibrium*. Such an equilibrium will have to take into account the stochastic timing of the crash, and specify the conditions under which the economy is in equilibrium both before and after the crash. One crucial element of a bubbly equilibrium is the interactions between the pre-crash and the post-crash economies. First, given the chronological order of events, the equilibrium outcomes in the pre-crash economy will determine the initial state (more specifically, the initial value of physical capital) of the post-crash economy. Second, when consumers are making their decisions before the crash, say at some time t, the anticipated value of $\widehat{R}_{t+1}$ will have to be consistent with an equilibrium in the post-crash economy at time $t+1$. In other words, the equilibrium quantities and prices in the post-crash economy will also affect the equilibrium outcomes prior the crash.[11]

4.1 Bubbleless Equilibrium

Suppose the crash happens at time $T > 0$, i.e., $\varepsilon_{T-1} = 1$ and $\varepsilon_T = 0$. Then the economy is free of asset bubbles from time T onward. Given an initial value $\widehat{K}_T > 0$, a post-crash bubbleless equilibrium consists of sequences of allocation $\left\{\widehat{c}_{y,t}, \widehat{s}_t, \widehat{l}_t, \widehat{c}_{o,t}\right\}_{t=T}^{\infty}$, aggregate inputs $\left\{\widehat{K}_t, \widehat{L}_t\right\}_{t=T}^{\infty}$, and prices $\left\{\widehat{w}_t, \widehat{R}_t\right\}_{t=T}^{\infty}$ such that for all $t \geq T$, (i) the allocation $\left\{\widehat{c}_{y,t}, \widehat{s}_t, \widehat{l}_t, \widehat{c}_{o,t+1}\right\}$ solves the consumer's problem at time t given $\widehat{w}_t$ and $\widehat{R}_{t+1}$; (ii) the consumption of old consumers at time T is determined by

$$N_{T-1}\widehat{c}_{o,T} = \widehat{R}_T \widehat{K}_T;$$

(iii) the aggregate inputs $\left\{\widehat{K}_t, \widehat{L}_t\right\}$ solve the representative firm's problem at time t given $\widehat{w}_t$ and $\widehat{R}_t$; and (iv) all markets clear at time t, i.e., $\widehat{L}_t = N_t \widehat{l}_t$ and $\widehat{K}_{t+1} = N_t \widehat{s}_t$.

[10] In the post-crash economy, all the variables in the above equations will be decorated with a hat.

[11] For reasons that we will discuss below, the second type of interaction is not present in Weil's (1987) model.

Define $\widehat{k}_t \equiv \widehat{K}_t/N_t$. Then the equilibrium dynamics of $\widehat{k}_t$ and $\widehat{R}_t$ are determined by[12]

$$\widehat{k}_{t+1} = \frac{1-\alpha}{\alpha(1+n)} \left[\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_t \widehat{k}_t, \tag{14}$$

$$\widehat{R}_t^{\eta} \widehat{k}_t = \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}, \tag{15}$$

where $\eta \equiv \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha}\frac{1-\sigma}{\sigma+\psi} > 0$. The initial value $\widehat{k}_T = \widehat{K}_T/N_T$ is given. Once the equilibrium time path of $\widehat{k}_t$ and $\widehat{R}_t$ are known, all other variables in the bubbleless equilibrium can be uniquely determined.

For any $\sigma > 0$, the dynamical system in (14)-(15) has a unique steady state, which we will call a *bubbleless steady state*. This result is formally stated in Proposition 1. All proofs can be found in Appendix B.

Proposition 1 *A unique bubbleless steady state exists for any $\sigma > 0$. The steady-state values $\left(\widehat{R}^*, \widehat{k}^*\right)$ are determined by*

$$\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}^*\right)^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}} = \frac{(1+n)\alpha}{1-\alpha}, \tag{16}$$

$$\widehat{k}^* = (1-\alpha)^{\frac{1-\sigma}{\sigma+\psi}} A^{-\frac{1}{\sigma+\psi}} \left[1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left(\frac{\alpha}{\widehat{R}^*} \right)^{\eta}. \tag{17}$$

Next, we consider the stability property of the bubbleless steady state. This type of property is crucial in determining the uniqueness of non-stationary bubbleless equilibrium. When the utility function for consumption is logarithmic, i.e., $\sigma = 1$, the dynamical system in (14)-(15) is independent of $\widehat{R}_{t+1}$. In this case, (14) can be simplified to become $\widehat{k}_{t+1} = B\widehat{k}_t^{\alpha}$, where B is a positive constant, and the unique bubbleless steady state is globally stable. When $\sigma < 1$, the bubbleless steady state can be shown to be globally saddle-path stable. In both cases, any non-stationary bubbleless equilibrium that originates from a given initial value $\widehat{k}_T > 0$ must be unique and converges to the bubbleless steady state. In addition, if the post-crash economy begins with an initial value $\widehat{k}_T$ that is greater than the steady-state value $\widehat{k}^*$, then $\widehat{k}_t$ will decline monotonically during the transition and $\widehat{R}_t$ will rise monotonically towards $\widehat{R}^*$. In other words, $\widehat{R}_t$ and $\widehat{k}_t$ will always move in opposite directions on the saddle path. These results are summarized in Proposition 2.

[12] The derivation of these equations can be found in Appendix A.

Proposition 2 *Suppose $\sigma \leq 1$. Then any non-stationary bubbleless equilibrium that originates from a given initial value $\widehat{k}_T > 0$ must be unique and converges monotonically to the bubbleless steady state. In particular, the value of $\widehat{R}_T$ is uniquely determined by $\widehat{R}_T = \Phi\left(\widehat{k}_T\right)$, where $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a strictly decreasing function. In the transitional dynamics, $\widehat{R}_t$ and $\widehat{k}_t$ will move in opposite directions so that $\left(\widehat{k}_t - \widehat{k}^*\right)\left(\widehat{R}_t - \widehat{R}^*\right) \leq 0$ for all $t \geq T$.*

When $\sigma > 1$, the bubbleless steady state can be either a sink or a saddle (see Appendix A for more details). If it is a sink, then there exist multiple sets of equilibrium time paths that originate from the same initial value $\widehat{k}_T > 0$ and converge to the bubbleless steady state. In other words, local indeterminacy may occur when $\sigma > 1$. In this study, we confine our attention to bubbleless equilibria that are determinate. In particular, we focus on the case when $\sigma \leq 1$, which means the intertemporal substitution effect of a higher interest rate is no weaker than the income effect. This assumption is not uncommon in OLG models. For instance, Galor and Ryder (1989) show that this assumption plays an important role in establishing the existence, uniqueness and global stability of stationary equilibrium in a model with exogenous labor supply. Fuster (1999) uses this assumption to establish the existence and uniqueness of non-stationary equilibrium in a model with uncertain lifetime and accidental bequest. More recently, Andersen and Bhattacharya (2013) adopt the same assumption to analyze the welfare implications of unfunded pensions in a model with endogenous labor supply. In the rational bubble literature, Weil (1987, Section 2) focuses on equilibria in which the interest elasticity of savings is non-negative. Under a constant-relative-risk-aversion utility function, this assumption holds if and only if $\sigma \leq 1$. Other studies allow the per-period utility function to be different across age, and assume that the coefficient of relative risk aversion is no greater than one in the old age. For instance, Azariadis and Smith (1993) adopt this assumption to study the general equilibrium implications of credit rationing in a model with adverse selection. Morand and Reffett (2007) and Hillebrand (2014) use this assumption to establish the uniqueness of Markov equilibrium in a model with productivity shocks.

4.2 Bubbly Equilibrium

We now provide the formal definition of a bubbly equilibrium. Given the initial values $K_0 > 0$ and $\varepsilon_0 = 1$, a bubbly equilibrium consists of two sets of sequences $\{c_{y,t}, c_{o,t}, l_t, s_t, m_t, R_t, w_t, p_t, K_t, L_t\}_{t=0}^{\infty}$ and $\left\{\widehat{c}_{y,t}, \widehat{c}_{o,t}, \widehat{l}_t, \widehat{s}_t, \widehat{R}_t, \widehat{w}_t, \widehat{K}_t, \widehat{L}_t\right\}_{t=0}^{\infty}$ that satisfy the following conditions in every period $t \geq 0$.

1. If $\varepsilon_t = 0$, then $\left\{\widehat{c}_{y,\tau}, \widehat{c}_{o,\tau}, \widehat{l}_\tau, \widehat{s}_\tau, \widehat{R}_\tau, \widehat{w}_\tau, \widehat{K}_\tau, \widehat{L}_\tau\right\}_{\tau=t}^{\infty}$ constitutes a non-stationary bubbleless equilib-

rium with initial condition $\widehat{K}_t$.

2. If $\varepsilon_t = 1$, then

   (i) given $\left\{w_t, p_t, p_{t+1}, R_{t+1}, \widehat{R}_{t+1}\right\}$, the allocation $\{c_{y,t}, c_{o,t+1}, l_t, s_t, m_t\}$ solves the consumer's problem at time t, i.e., (5)-(9) are satisfied;

   (ii) given R_t and w_t, the aggregate inputs K_t and L_t solve the firm's problem at time t, i.e., (13) is satisfied;

   (iii) all markets clear at time t, i.e., $L_t = N_t l_t$, $K_{t+1} = N_t s_t$ and $N_t m_t = M$;

   (iv) if $\varepsilon_{t+1} = 0$, then $\widehat{K}_{t+1} = K_{t+1}$.

The last condition states that if the asset bubble crashes at time $t+1$, then K_{t+1} will provide the initial condition for the ensuing bubbleless equilibrium.

Regardless of the existence of asset bubbles, the labor market clears when the total supply of labor by young consumers equals the total demand by firms (i.e., $\widehat{L}_t = N_t \widehat{l}_t$ when $\varepsilon_t = 0$, and $L_t = N_t l_t$ when $\varepsilon_t = 1$); and the market for physical capital clears when the productive savings made by young consumers equal the stock of aggregate capital in the next period (i.e., $\widehat{K}_{t+1} = N_t \widehat{s}_t$ when $\varepsilon_t = 0$, and $K_{t+1} = N_t s_t$ when $\varepsilon_t = 1$). Note that, regardless of the state of the asset bubble, the stock of capital at time $t+1$ is predetermined at time t, and is thus independent of ε_{t+1}. This brings us back to one of the major differences between the present study and Weil (1987) that we have mentioned in the introduction. In the production economy of Weil (1987), every young consumer provides one unit of labor inelastically regardless of the existence of asset bubble. Thus, the equilibrium quantity of labor input at time $t+1$ is always determined by N_{t+1}, i.e., $L_{t+1} = \widehat{L}_{t+1} = N_{t+1}$. Suppose the asset bubble crashes at time $t+1$. Since neither K_{t+1} nor L_{t+1} depends on ε_{t+1}, *the crash will have no effect on aggregate output and factor prices at time* $t+1$. Thus, in Weil's (1987) model, the gross return from physical capital is not contingent on the realization of the asset price shock, i.e., $R_{t+1} = \widehat{R}_{t+1}$ for all t. When labor supply is endogenous, the equilibrium quantity of L_{t+1} will also depend on individual's choice of l_{t+1}. If this choice is contingent on the realization of ε_{t+1}, then this will open up a channel through which the asset price shock can affect the aggregate economy. Our next result shows that this channel is operative only if $\sigma \neq 1$.

Proposition 3 *Suppose the utility function for consumption is logarithmic, i.e.,* $\sigma = 1$*. Then the optimal labor supply is constant over time and is identical before and after the crash. Specifically,*

$$l_t = \widehat{l}_t = \left(\frac{1+\beta}{A}\right)^{\frac{1}{1+\psi}}, \qquad \text{for all } t \geq 0.$$

This result can be explained as follows: Regardless of the existence of asset bubble, the optimal choice of l_t is determined by (8). The expression $w_t c_{y,t}^{-\sigma}$ on the left captures both the income and substitution effects of a higher wage rate on labor supply. Holding $c_{y,t}$ constant, an increase in w_t raises the opportunity cost of leisure. This creates a substitution effect which discourages leisure and promotes labor supply. On the other hand, an increase in w_t also generates an income effect which promotes consumption and discourages labor supply. These two effects exactly offset each other when $\sigma = 1$. This happens because in this case, the consumers will save (and consume) a constant fraction of their labor income in the young age. Consequently, the expression $w_t c_{y,t}^{-1}$ in (8) is independent of w_t, which means individual labor supply is not affected by changes in wage rate. Thus, when $\sigma = 1$, our model is essentially identical to the production economy in Weil (1987).

When $\sigma < 1$, the optimal choice of l_t will not be a constant in general, and it will depend on the realization of the asset price shock. The rest of this paper is devoted to analyzing the effects of bubbles and crashes under this value of σ. To simplify the analysis, suppose the economy is in a *conditional bubbly steady state* before the crash happens. Formally, a conditional bubbly steady state is a set of stationary values $\mathcal{S} \equiv \left\{c_y^*, c_o^*, l^*, s^*, a^*, R^*, \widehat{R}_0^*, w^*, \pi^*, k^*\right\}$ such that conditional on $\varepsilon_t = 1$, we have $p_{t+1}/p_t = \pi^*$, $K_t = N_t k^*$, $L_t = N_t l^*$, $p_t m_t = a^* > 0$, and $(c_{y,t}, c_{o,t}, s_t, l_t, R_t, w_t) = \left(c_y^*, c_o^*, s^*, l^*, R^*, w^*\right)$ in a bubbly equilibrium.[13] The main ideas behind this definition are as follows: Before the crash happens, the consumers face a stationary environment in which (i) the probability of having a crash in the next period is constant over time; (ii) the market wage rate (w^*) and the expected return from the bubbly asset ($q\pi^*$) are identical in every period; and (iii) the state-contingent returns for physical capital are also identical in every period (specifically the return is R^* if the asset bubble persists in the next period and $\widehat{R}_0^*$ otherwise). Thus, the consumers will make the same choices in every period before the crash happens. In particular, they will invest an amount $a^* > 0$ in the asset bubble in the conditional steady state. Once the asset bubble crashes, the economy will follow the transition paths described in Proposition 2 and converge to the bubbleless steady state $\left(\widehat{R}^*, \widehat{k}^*\right)$. Note that, regardless of the timing of the crash, the

[13] The concept of "conditional steady state" is not new in macroeconomics. For instance, Cole and Rogerson (1999) and Galor and Weil (2000) have defined a similar notion in different contexts.

dynamical system in (14)-(15) will always begin with the same initial values: k^* and $\widehat{R}_0^* \equiv \Phi(k^*)$.[14]

We now summarize some of the main properties of a conditional bubbly steady state. Conditional on $\varepsilon_t = 1$, the market for the intrinsically worthless asset clears when $N_t m_t = M$. Using this and the stationary conditions $p_{t+1}/p_t = \pi^*$ and $p_t m_t = p_{t+1} m_{t+1} = a^*$, we can obtain

$$\frac{p_{t+1}}{p_t} = \pi^* = \frac{m_t}{m_{t+1}} = \frac{N_{t+1}}{N_t} = 1 + n.$$

Thus, before the crash happens, the price of the intrinsically worthless asset is growing deterministically at rate n. Given $\widehat{R}_0^* > 0$, the steady-state values $\{R^*, w^*, l^*, k^*, a^*\}$ are uniquely determined by[15]

$$1 + \left[1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}}\right] \left(\frac{q}{1-q}\right)^{\frac{1}{\sigma}} \left(\frac{\widehat{R}_0^*}{1+n}\right)^{1-\frac{1}{\sigma}} \left(1 - \frac{R^*}{1+n}\right)^{\frac{1}{\sigma}} = \frac{1}{\alpha} \frac{R^*}{1+n}, \tag{18}$$

$$w^* = (1-\alpha) \left(\frac{\alpha}{R^*}\right)^{\frac{\alpha}{1-\alpha}}, \tag{19}$$

$$A (l^*)^{\psi+\sigma} = \beta q \left[(1+n) w^*\right]^{1-\sigma} \left[\frac{(1-\alpha) R^*}{\alpha \Omega^* \widehat{R}_0^*}\right]^{\sigma}, \tag{20}$$

$$k^* = l^* \left(\frac{\alpha}{R^*}\right)^{\frac{1}{1-\alpha}}, \tag{21}$$

$$a^* = \left(\Omega^* \widehat{R}_0^* - R^*\right) k^*. \tag{22}$$

Once these values are known, the value of $\left\{c_y^*, c_o^*, s^*\right\}$ can be uniquely determined from the consumer's budget constraints. Equations (18)-(21) essentially define a one-to-one mapping between $\widehat{R}_0^*$ and k^*, which we will denote by $k^* = \Gamma\left(\widehat{R}_0^*\right)$. We now have a pair of equations, $\widehat{R}_0^* = \Phi(k^*)$ and $k^* = \Gamma\left(\widehat{R}_0^*\right)$, which can be used to solve for k^* and $\widehat{R}_0^*$. The first equation determines the initial value of $\widehat{R}_t$ in the post-crash bubbleless equilibrium. The actual form of $\Phi(\cdot)$ depends on the transitional dynamics in the bubbleless economy. The second equation states that, given $\widehat{R}_0^*$, $k^* = \Gamma\left(\widehat{R}_0^*\right)$ is the value of per-worker capital in the conditional bubbly steady state. The mapping $\Gamma(\cdot)$ is determined by (18)-(21). These two equations can be combined to form a one-dimensional fixed point equation $\widehat{R}_0^* = \Phi \circ \Gamma\left(\widehat{R}_0^*\right)$, which provides the basis for computing the bubbly equilibrium.

[14] The variable $\widehat{R}_0^*$ is not to be confused with the bubbleless steady-state value $\widehat{R}^*$ defined in Proposition 1. In the post-crash economy, $\widehat{R}_0^*$ is the initial value of $\widehat{R}_t$ while $\widehat{R}^*$ is the long-run value.

[15] The derivation of these equations can be found in Appendix A.

Our next proposition states that when $\sigma < 1$, the gross return from physical capital in the conditional bubbly steady state (R^*) is higher than the one in the bubbleless steady state $\left(\widehat{R}^*\right)$. This result is due to the combination of two factors. First, since aggregate uncertainty exists before the crash happens, consumers will require a higher return from savings in the conditional bubbly steady state. Second, even without any uncertainty, the existence of asset bubble tends to lower the capital-labor ratio and drives up the steady-state interest rate [see Shi and Suen (2014) Proposition 2].[16]

Proposition 4 *Suppose $\sigma < 1$. Then the existence of asset bubble is associated with a higher level of steady-state interest rate, i.e., $R^* > \widehat{R}^*$.*

Our last set of results concerns the expansionary effects of asset bubbles. Specifically, we seek conditions under which the conditional bubbly steady state has more physical capital per worker and a higher labor supply than the bubbleless steady state, i.e., $k^* > \widehat{k}^*$ and $l^* > \widehat{l}^*$. Note that $k^* > \widehat{k}^*$ implies that there is more physical capital per worker before the crash than after, i.e., $k^* \geq \widehat{k}_t$ for all t. To see this, suppose the post-crash economy begins at time T so that $\widehat{k}_T = k^*$. As shown in Proposition 2, if $\widehat{k}_T = k^* > \widehat{k}^*$, then $\widehat{k}_t$ is strictly decreasing along the transition path so that $\widehat{k}_T = k^* > \widehat{k}_t$ for all $t > T$.

Using (21), which is valid both before and after the crash, we can obtain

$$k^* = l^* \left(\frac{\alpha}{R^*}\right)^{\frac{1}{1-\alpha}} > \widehat{l}^* \left(\frac{\alpha}{\widehat{R}^*}\right)^{\frac{1}{1-\alpha}} = \widehat{k}^* \qquad \Leftrightarrow \qquad \frac{l^*}{\widehat{l}^*} > \left(\frac{R^*}{\widehat{R}^*}\right)^{\frac{1}{1-\alpha}} > 1. \tag{23}$$

This shows that asset bubbles can potentially crowd in productive investment in the current framework, but this happens only if these bubbles can induce a sufficiently large expansion in labor supply.

Regardless of the existence of asset bubbles, individual labor supply is determined by equation (8), which can be rewritten as

$$Al_t^{\psi+\sigma} = w_t^{1-\sigma} \left(\frac{c_{y,t}}{w_t l_t}\right)^{-\sigma}. \tag{24}$$

The above equation shows how individual labor supply is determined by the current wage rate and the propensity to consume when young. Holding other things constant, labor supply increases when wage rate increases (as $\sigma < 1$). Since $R^* > \widehat{R}^*$ implies $w^* < \widehat{w}^*$, this effect in itself will *lower* labor supply in the presence of asset bubble. On the other hand, labor supply increases when the consumers allocate a smaller fraction of their labor income to young-age consumption. This captures the intratemporal

[16] This result is also consistent with the findings in other rational bubble models. For instance, the models of Tirole (1985), Weil (1987), Olivier (2000), and Farhi and Tirole (2012) all predict that the long-run interest rate is higher in the presence of asset bubble.

substitution between consumption and labor. Thus, $l^* > \widehat{l}^*$ is possible *only if* the consumers have a lower propensity to consume in the conditional bubbly steady state, i.e.,

$$\frac{\widehat{c}_y^*}{\widehat{w}^* \widehat{l}^*} > \frac{c_y^*}{w^* l^*}.$$

In the bubbleless steady state, this propensity is determined by

$$\frac{\widehat{c}_y^*}{\widehat{w}^* \widehat{l}^*} = \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}\right]^{-1}, \tag{25}$$

which is strictly decreasing in the long-run interest rate when $\sigma < 1$. A similar expression can be obtained for its counterpart in the conditional bubbly steady state, which is

$$\frac{c_y^*}{w^* l^*} = \left[1 + \beta^{\frac{1}{\sigma}} \left(\rho^*\right)^{\frac{1}{\sigma}-1}\right]^{-1}, \tag{26}$$

where

$$\left(\rho^*\right)^{\frac{1}{\sigma}-1} \equiv \frac{\left[q\left(1+n\right)\right]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left[1 + \frac{1}{1+n}\left(\Omega^* \widehat{R}_0^* - R^*\right)\right].$$

The variable ρ^* can be interpreted as the certainty equivalent return from investment in the conditional bubbly steady state. Specifically, this means a consumer in the conditional bubbly steady state will have the same amount of consumption $\left(c_y^*, c_o^*\right)$ and labor supply (l^*) as a consumer in a deterministic bubbleless steady state where the gross return from savings is ρ^*. Under the assumption of $\sigma < 1$, an increase in interest rate will induce the consumers to save more and consume less when young. Thus, the consumers will have a lower propensity to consume in the conditional bubbly steady state if and only if $\rho^* > \widehat{R}^*$. After some manipulations, we can derive the following equivalent condition:

$$\frac{\widehat{c}_y^*}{\widehat{w}^* \widehat{l}^*} > \frac{c_y^*}{w^* l^*} \quad \Leftrightarrow \quad \left[\frac{q\left(1+n\right)}{\widehat{R}^*}\right]^{\frac{1}{\sigma}} > \frac{\Omega^* \widehat{R}_0^*}{R^*} > 1. \tag{27}$$

Finally, using (19) and (23)-(27), we can derive a necessary and sufficient condition for $l^* > \widehat{l}^*$ and one for $k^* > \widehat{k}^*$. The results are stated in Proposition 5.

Proposition 5 *Suppose $\sigma < 1$. Then $l^* > \widehat{l}^*$ if and only if*

$$\left[\frac{q\left(1+n\right)}{\widehat{R}^*}\right]^{\frac{1}{\sigma}} \left(\frac{R^*}{\widehat{R}^*}\right)^{-\frac{\alpha(1-\sigma)}{(1-\alpha)\sigma}} > \frac{\Omega^* \widehat{R}_0^*}{R^*},$$

and the asset bubble can crowd in productive investment, i.e., $k^* > \widehat{k}^*$*, if and only if*

$$\left[\frac{q(1+n)}{\widehat{R}^*}\right]^{\frac{1}{\sigma}} \left(\frac{R^*}{\widehat{R}^*}\right)^{-\left[1+\frac{\psi+\sigma}{(1-\alpha)\sigma}\right]} > \frac{\Omega^* \widehat{R}_0^*}{R^*}.$$

4.3 Numerical Examples

We now present a set of numerical examples to illustrate how the key variables in our model respond to an asset bubble crash. Through these examples, we also want to highlight the importance of σ in determining the macroeconomic effects of asset bubbles. We stress at the outset that these examples are only intended to demonstrate the working of the model and the results in the previous sections. For this reason, some of the parameter values are specifically chosen so that asset bubbles can crowd in productive investment in some cases.

Suppose one model period takes 30 years. Set the annual subjective discount factor to 0.9950 and the annual employment growth rate to 1.6 percent.[17] These values imply $\beta = (0.9950)^{30} = 0.8604$ and $n = (1.0160)^{30} - 1 = 0.6099$. In addition, we set $q = 0.90$, $\alpha = 0.30$ so that the share of capital income in total output is 30 percent, and $\psi = 0$ so that the utility function in (1) is quasi-linear in labor hours. As shown in Hansen (1985), this type of utility function is consistent with the assumption of indivisible labor. Our choice of q and n implies that the expected return from the intrinsically worthless asset is $q(1+n) = 1.4490$. To highlight the importance of σ, we consider four different values of this parameter between 0.10 and 0.30. For each value of σ, the parameter A is chosen so that $\widehat{l}^*$ is 0.50.[18] For each set of parameter values, we solve for the equilibrium time paths under the following scenario: Suppose the economy starts from a conditional bubbly steady state at time $t = 0$, and suppose the bubble bursts unexpectedly at time $t = 3$.[19] We then solve for the conditional bubbly steady state and the bubbleless steady state, and compute the transition path in the post-crash economy using backward shooting method.

[17] The latter is consistent with the average annual growth rate of U.S. employment over the period 1953-2008.

[18] Under the assumption of indivisible labor, the variable l_t is more suitably interpreted as the labor force participation rate at time t. Thus, we choose a target value of $\widehat{l}^*$ based on the average labor force participation rate in the United States during the postwar period, which is about 0.50.

[19] In other words, we consider a particular sequence of asset price shocks in which $\varepsilon_t = 1$ for $t \in \{0, 1, 2\}$ and $\varepsilon_t = 0$ for $t \geq 3$. As explained earlier, the non-stationary bubbleless equilibrium will always begin with the same initial values k^* and $\widehat{R}_0^*$ regardless of the timing of the crash. Thus, the exact time period when the crash happens is immaterial.

Table 1

Conditional Bubbly Steady State vs Bubbleless Steady State

	$\sigma = 0.10$		$\sigma = 0.15$		$\sigma = 0.20$		$\sigma = 0.30$	
Steady State	Bubbleless	Bubbly	Bubbleless	Bubbly	Bubbleless	Bubbly	Bubbleless	Bubbly
R	1.2176	1.4671	1.2416	1.4548	1.2637	1.4485	1.3036	1.4434
ρ	—	1.4402	—	1.4382	—	1.4381	—	1.4395
c_y	0.0832	0.0374	0.0846	0.0538	0.0858	0.0640	0.0878	0.0758
l	0.5000	0.7306	0.5000	0.5862	0.5000	0.5416	0.5000	0.5132
k	0.0676	0.0757	0.0657	0.0614	0.0641	0.0571	0.0613	0.0544
y	0.2743	0.3701	0.2720	0.2980	0.2700	0.2758	0.2664	0.2617
a	0	0.0998	0	0.0559	0	0.0371	0	0.0198

Note: The notation y denotes per-worker output, i.e., $y = k^{\alpha} l^{1-\alpha}$.

Table 1 shows the key variables in the conditional bubbly steady state and the bubbleless steady state under different values of σ. In the first row, we report the value of $\widehat{R}^*$ and R^* in each case. In the second row, we report the certainty equivalent return from savings in the conditional bubbly steady state. In all four cases, we have $\rho^* > \widehat{R}^*$ and $l^* > \widehat{l}^*$. In particular, the gap between l^* and $\widehat{l}^*$ widens as the value of σ decreases. This captures the effects of a stronger intertemporal substitution effect. When $\sigma = 0.1$, the difference between l^* and $\widehat{l}^*$ is sufficiently large so that asset bubble can crowd in productive investment (i.e., $k^* > \widehat{k}^*$).

Figures 10-12 show the time path of interest rate (R), labor supply (l) and per-worker capital (k) before and after the crash happens at $t = 3$. In all four cases, the crash induces an immediate reduction in interest rate and labor supply. During the transition, $\widehat{R}_t$ and $\widehat{k}_t$ move in opposite directions as predicted by Proposition 2. In the more interesting case where asset bubble crowds in physical capital (i.e., $\sigma = 0.1$), labor supply and productive investment fall markedly at the time of the crash and continue to decline afterward. These patterns are qualitatively similar to those observed in the United States after the bursting of the internet bubble and the housing price bubble.

5 Concluding Remarks

The present study joins a growing body of literature that examines the effects of asset price bubbles and crashes on the aggregate economy. We contribute to this literature by demonstrating the importance of intratemporal and intertemporal substitution effects to the issue at hand. In particular, we show that the existence of asset bubbles can crowd in productive investment and induce an expansion in aggregate employment when these effects are sufficiently strong. We remark that the present study is mainly theoretical in nature and more effort is needed in order to generate realistic quantitative results. In particular, expanding the consumer's planning horizon (and thus reducing the length of each model period) is crucial for matching the model to the data. Introducing other model features, such as financial market imperfections and heterogeneity in firm productivity as in Martin and Ventura (2012) and Farhi and Tirole (2012), may also help expand the range of parameter values under which asset bubbles can crowd in productive investment. We leave these intriguing possibilities for future research.

Appendix A: Mathematical Derivations

Post-Crash Equilibrium

In this section, we provide a detailed characterization of a post-crash equilibrium. Since the consumer's problem in the post-crash economy is standard, the derivations of (2)-(4) are omitted. The dynamical system in (14)-(15) can be derived as follows. In equilibrium, the market wage rate and the gross return from physical capital are determined by $\widehat{w}_t = (1-\alpha)\widehat{K}_t^{\alpha}\widehat{L}_t^{-\alpha}$ and $\widehat{R}_t = \alpha\widehat{K}_t^{\alpha-1}\widehat{L}_t^{1-\alpha}$, respectively. Using these, we can obtain

$$\widehat{w}_t\widehat{l}_t = \frac{1-\alpha}{\alpha}\widehat{R}_t\widehat{k}_t, \tag{28}$$

$$\widehat{w}_t = (1-\alpha)\left(\frac{\alpha}{\widehat{R}_t}\right)^{\frac{\alpha}{1-\alpha}}, \tag{29}$$

$$\widehat{l}_t = \left(\frac{\widehat{R}_t}{\alpha}\right)^{\frac{1}{1-\alpha}}\widehat{k}_t. \tag{30}$$

where $\widehat{k}_t \equiv \widehat{K}_t/N_t$ and $\widehat{l}_t \equiv \widehat{L}_t/N_t$. Then we can rewrite the capital market clearing condition as

$$(1+n)\,\widehat{k}_{t+1} = \left[\frac{\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}}\right]\widehat{w}_t\widehat{l}_t \equiv \Sigma\left(\widehat{R}_{t+1}\right)\widehat{w}_t\widehat{l}_t.$$

Substituting (28) into the above expression gives (14). Next, substituting (29) and (30) into (3) gives

$$\left(\frac{\widehat{R}_t}{\alpha}\right)^{\frac{1}{1-\alpha}}\widehat{k}_t = A^{-\frac{1}{\sigma+\psi}}\left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}\right]^{\frac{\sigma}{\sigma+\psi}}\left[(1-\alpha)\left(\frac{\alpha}{\widehat{R}_t}\right)^{\frac{\alpha}{1-\alpha}}\right]^{\frac{1-\sigma}{\sigma+\psi}}$$

$$\Rightarrow \left(\frac{\widehat{R}_t}{\alpha}\right)^{\eta}\widehat{k}_t = \left[\frac{(1-\alpha)^{1-\sigma}}{A}\right]^{\frac{1}{\sigma+\psi}}\left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}\right]^{\frac{\sigma}{\sigma+\psi}}, \tag{31}$$

where

$$\eta \equiv \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha}\frac{1-\sigma}{\sigma+\psi} = \frac{\psi+\alpha+\sigma(1-\alpha)}{(1-\alpha)(\sigma+\psi)} > 0,$$

$$\eta - 1 = \frac{\alpha}{1-\alpha}\frac{1+\psi}{\sigma+\psi} > 0,$$

for any $\sigma > 0$. Equation (15) can be obtained by rearranging terms in (31).

Local Analysis

We now explore the local stability property of the unique bubbleless steady state under different values of σ. To achieve this, we consider a linearized version of the dynamical system in (14)-(15). First, taking logarithms of both sides of these equations gives

$$\ln \widehat{k}_{t+1} - \ln \Sigma\left(\widehat{R}_{t+1}\right) = \ln\left[\frac{1-\alpha}{\alpha\left(1+n\right)}\right] + \ln \widehat{R}_t + \ln \widehat{k}_t,$$

$$\ln\left\{\alpha^{\eta}\left[\frac{\left(1-\alpha\right)^{1-\sigma}}{A}\right]^{\frac{1}{\sigma+\psi}}\right\} + \frac{\sigma}{\sigma+\psi}\ln\left(1+\beta^{\frac{1}{\sigma}}\widehat{R}_{t+1}^{\frac{1}{\sigma}-1}\right) = \eta\ln \widehat{R}_t + \ln \widehat{k}_t.$$

Next, taking the first-order Taylor expansion of these equations around $\left(\widehat{k}^*, \widehat{R}^*\right)$ gives

$$\widehat{\mathbf{k}}_{t+1} - \frac{\widehat{R}^*\Sigma'\left(\widehat{R}^*\right)}{\Sigma\left(\widehat{R}^*\right)}\widehat{\mathbf{R}}_{t+1} = \widehat{\mathbf{k}}_t + \widehat{\mathbf{R}}_t,$$

$$\frac{1-\sigma}{\sigma+\psi}\left[\frac{\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}}\right]\widehat{\mathbf{R}}_{t+1} = \widehat{\mathbf{k}}_t + \eta\widehat{\mathbf{R}}_t,$$

where $\widehat{\mathbf{k}}_t \equiv \left(\widehat{k}_t - \widehat{k}^*\right)/\widehat{k}^*$ and $\widehat{\mathbf{R}}_t \equiv \left(\widehat{R}_t - \widehat{R}^*\right)/\widehat{R}^*$ represent the percentage deviations of $\widehat{k}_t$ and $\widehat{R}_t$ from their steady-state values. Finally, rewrite the linearized system in matrix form

$$\underbrace{\begin{bmatrix} 1 & b_{12} \\ 0 & b_{22} \end{bmatrix}}_{B}\begin{bmatrix} \widehat{\mathbf{k}}_{t+1} \\ \widehat{\mathbf{R}}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \eta \end{bmatrix}\begin{bmatrix} \widehat{\mathbf{k}}_t \\ \widehat{\mathbf{R}}_t \end{bmatrix}, \tag{32}$$

where

$$b_{12} = -\frac{\widehat{R}^*\Sigma'\left(\widehat{R}^*\right)}{\Sigma\left(\widehat{R}^*\right)} = \left(1-\frac{1}{\sigma}\right)\left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}\right]^{-1},$$

$$b_{22} = \frac{1-\sigma}{\sigma+\psi}\left[\frac{\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}}\right].$$

The inverse of the matrix B is given by

$$B^{-1} = \frac{1}{b_{22}} \begin{bmatrix} b_{22} & -b_{12} \\ 0 & 1 \end{bmatrix}.$$

Using this, we can rewrite (32) as

$$\begin{bmatrix} \widehat{\mathbf{k}}_{t+1} \\ \widehat{\mathbf{R}}_{t+1} \end{bmatrix} = \underbrace{\frac{1}{b_{22}} \begin{bmatrix} b_{22} - b_{12} & b_{22} - \eta b_{12} \\ 1 & \eta \end{bmatrix}}_{J} \begin{bmatrix} \widehat{\mathbf{k}}_{t} \\ \widehat{\mathbf{R}}_{t} \end{bmatrix}, \tag{33}$$

where J is the Jacobian matrix of the linearized system. Let ρ_1 and ρ_2 be the characteristic roots of the linearized system. These can be obtained by solving

$$\Xi(\rho) \equiv \rho^2 - \left(1 - \frac{b_{12}}{b_{22}} + \frac{\eta}{b_{22}}\right)\rho + \frac{\eta - 1}{b_{22}} = 0.$$

If $\sigma < 1$, then we have $b_{12} < 0$ and $b_{22} > 0$ which imply

$$\Xi(\rho) > 0, \qquad \text{for all } \rho < 0,$$

$$\Xi(0) = \frac{\eta - 1}{b_{22}} > 0, \qquad \text{as } \eta > 1,$$

$$\Xi(1) \equiv 1 - \left(1 - \frac{b_{12}}{b_{22}} + \frac{\eta}{b_{22}}\right) + \frac{\eta - 1}{b_{22}} = \frac{b_{12} - 1}{b_{22}} < 0.$$

The last two inequalities ensure that one of the characteristic roots can be found within the interval of $(0, 1)$. This rules out the possibility of complex roots. Since $\Xi(\rho) > 0$ for all $\rho \leq 0$, both ρ_1 and ρ_2 must be strictly positive. Finally, if both ρ_1 and ρ_2 are within the interval of $(0, 1]$, then we should have $\Xi(1) \geq 0$ instead. Thus, the second root must be greater than one. This proves that the system in (33) is saddle-path stable within the neighborhood of the bubbleless steady state when $\sigma < 1$. Proposition 2 strengthens this result by showing that this steady state is globally saddle-path stable when $\sigma < 1$.

If $\sigma > 1$, then we have $b_{12} \in (0, 1)$ and $b_{22} < 0$ which imply $\Xi(0) < 0 < \Xi(1)$. Hence, one of the characteristic roots must lie within the interval of $(0, 1)$. Since the product of roots $\Xi(0)$ is strictly negative, the second characteristic root must be strictly negative. If $\Xi(-1) > 0$, then the second root must lie within the interval of $(-1, 0)$. In this case, the linearized system has two stable roots which means the bubbleless steady state is a sink. If $\Xi(-1) < 0$, then the absolute magnitude of the second root is greater than one. In this case, the bubbleless steady state is again saddle-path stable. The value of $\Xi(-1)$ is determined by

$$\Xi(-1) = \underbrace{2 - \frac{b_{12}}{b_{22}}}_{(+)} + \underbrace{\frac{2\eta - 1}{b_{22}}}_{(-)}.$$

Unfortunately, the sign of this expression cannot be readily determined. Hence, the local stability property of the post-crash equilibrium is ambiguous when $\sigma > 1$.

Bubbly Equilibrium

In this section, we will provide a detailed characterization of the consumer's problem in the pre-crash economy, and present the derivation of (18)-(22). Substituting (5) and (6) into the consumer's expected lifetime utility gives

$$\mathcal{L} = \frac{(w_t l_t - s_t - p_t m_t)^{1-\sigma}}{1-\sigma} - A\frac{l_t^{1+\psi}}{1+\psi} + \beta\left[\frac{q\left(R_{t+1}s_t + p_{t+1}m_t\right)^{1-\sigma} + (1-q)\left(\widehat{R}_{t+1}s_t\right)^{1-\sigma}}{1-\sigma}\right].$$

The first-order conditions with respect to s_t, m_t and l_t are, respectively, given by

$$(w_t l_t - s_t - p_t m_t)^{-\sigma} = \beta\left[qR_{t+1}\left(R_{t+1}s_t + p_{t+1}m_t\right)^{-\sigma} + (1-q)\widehat{R}_{t+1}\left(\widehat{R}_{t+1}s_t\right)^{-\sigma}\right], \tag{34}$$

$$(w_t l_t - s_t - p_t m_t)^{-\sigma} = \beta q\left(\frac{p_{t+1}}{p_t}\right)\left(R_{t+1}s_t + p_{t+1}m_t\right)^{-\sigma}, \tag{35}$$

$$Al_t^{\psi} = w_t\left(w_t l_t - s_t - p_t m_t\right)^{-\sigma}. \tag{36}$$

Here we only focus on interior solutions of m_t. Define $\pi_{t+1} \equiv p_{t+1}/p_t$. Combining (34) and (35) gives

$$q\pi_{t+1}\left(R_{t+1}s_t + p_{t+1}m_t\right)^{-\sigma} = qR_{t+1}\left(R_{t+1}s_t + p_{t+1}m_t\right)^{-\sigma} + (1-q)\widehat{R}_{t+1}\left(\widehat{R}_{t+1}s_t\right)^{-\sigma},$$

$$\Rightarrow q\left(\pi_{t+1} - R_{t+1}\right)\left(R_{t+1}s_t + p_{t+1}m_t\right)^{-\sigma} = (1-q)\widehat{R}_{t+1}\left(\widehat{R}_{t+1}s_t\right)^{-\sigma},$$

$$\Rightarrow R_{t+1}s_t + p_{t+1}m_t = \underbrace{\left[\frac{q\left(\pi_{t+1} - R_{t+1}\right)}{(1-q)\widehat{R}_{t+1}}\right]^{\frac{1}{\sigma}}}_{\Omega_{t+1}}\left(\widehat{R}_{t+1}s_t\right), \tag{37}$$

$$\Rightarrow m_t = \frac{1}{p_{t+1}} \left(\Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) s_t,$$

$$\Rightarrow s_t + p_t m_t = \left[1 + \frac{R_{t+1}}{\pi_{t+1}} \left(\Lambda_{t+1} - 1 \right) \right] s_t, \tag{38}$$

where $\Lambda_{t+1} \equiv \Omega_{t+1} \widehat{R}_{t+1} / R_{t+1}$. Using (35) and (37), we can get

$$R_{t+1} s_t + p_{t+1} m_t = \left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}} \left(w_t l_t - s_t - p_t m_t \right) = \Omega_{t+1} \widehat{R}_{t+1} s_t,$$

$$\Rightarrow s_t = \left\{ \frac{\left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + \left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}} \left[1 + \frac{R_{t+1}}{\pi_{t+1}} \left(\Lambda_{t+1} - 1 \right) \right]} \right\} w_t l_t. \tag{39}$$

Using this and (38), we can obtain

$$c_{y,t} = w_t l_t - \left(s_t + p_t m_t \right) = \left\{ \frac{\Omega_{t+1} \widehat{R}_{t+1}}{\Omega_{t+1} \widehat{R}_{t+1} + \left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}} \left[1 + \frac{R_{t+1}}{\pi_{t+1}} \left(\Lambda_{t+1} - 1 \right) \right]} \right\} w_t l_t. \tag{40}$$

Substituting this into (36) and rearranging terms give

$$A l_t^{\psi + \sigma} = \left(w_t \right)^{1-\sigma} \left\{ \frac{\Omega_{t+1} \widehat{R}_{t+1} + \left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}} \left[1 + \frac{R_{t+1}}{\pi_{t+1}} \left(\Lambda_{t+1} - 1 \right) \right]}{\Omega_{t+1} \widehat{R}_{t+1}} \right\}^{\sigma}. \tag{41}$$

These equations characterize the optimal choice of $c_{y,t}$, l_t, s_t and m_t before the crash.

We now provide the derivation of (18)-(22). In equilibrium, the market for physical capital clears when

$$(1+n) k_{t+1} = s_t = \left\{ \frac{\left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + \left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}} \left[1 + \frac{R_{t+1}}{\pi_{t+1}} \left(\Lambda_{t+1} - 1 \right) \right]} \right\} w_t l_t$$

$$\Rightarrow (1+n) k_{t+1} = \left\{ \frac{\left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + \left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}} \left[1 + \frac{R_{t+1}}{\pi_{t+1}} \left(\Lambda_{t+1} - 1 \right) \right]} \right\} \left(\frac{1-\alpha}{\alpha} \right) R_t k_t. \tag{42}$$

The second line uses the fact that $\alpha w_t l_t = (1-\alpha) R_t k_t$. Combining (41) and (42) gives

$$A l_t^{\psi + \sigma} = \left(w_t \right)^{1-\sigma} \left\{ \frac{\left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1}} \left[\frac{1-\alpha}{\alpha (1+n)} \right] \frac{R_t k_t}{k_{t+1}} \right\}^{\sigma}. \tag{43}$$

Upon setting $k_{t+1} = k_t = k^*$, $R_t = R_{t+1} = R^*$, $\widehat{R}_{t+1} = \widehat{R}_0^*$ and $\pi_{t+1} = 1+n$, equation (42) becomes

$$1+n = \left\{ \frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^* + [\beta q (1+n)]^{\frac{1}{\sigma}} \left[1 + \frac{R^*}{1+n} (\Lambda^* - 1)\right]} \right\} \left(\frac{1-\alpha}{\alpha} \right) R^*, \tag{44}$$

where $\Lambda^* = \Omega^* \widehat{R}_0^* / R^*$. Rearranging terms in this equation gives

$$1 + \left[1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}}\right] \left(\frac{\Omega^* \widehat{R}_0^*}{1+n} \right) = \frac{1}{\alpha} \frac{R^*}{1+n}$$

$$\Rightarrow 1 + \left[1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}}\right] \left(\frac{q}{1-q} \right)^{\frac{1}{\sigma}} \left(\frac{\widehat{R}_0^*}{1+n} \right)^{1-\frac{1}{\sigma}} \left(1 - \frac{R^*}{1+n} \right)^{\frac{1}{\sigma}} = \frac{1}{\alpha} \frac{R^*}{1+n},$$

which is equation (18) in the text. Similarly, after substituting the stationarity conditions into (43), we can obtain

$$A (l^*)^{\psi+\sigma} = (w^*)^{1-\sigma} \left\{ \frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left(\frac{1-\alpha}{\alpha} \right) \frac{R^*}{1+n} \right\}^{\sigma}.$$

Equation (20) follows immediately from this equation. Equations (19) and (21) can be obtained from (13). Finally, equation (22) can be obtained from (12).

Define $\theta^* \equiv R^*/(1+n)$. Then we can rewrite (18) as

$$\Psi(\theta^*) \equiv 1 + \left[1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}}\right] \left(\frac{q}{1-q} \right)^{\frac{1}{\sigma}} \left(\frac{\widehat{R}_0^*}{1+n} \right)^{1-\frac{1}{\sigma}} (1-\theta^*)^{\frac{1}{\sigma}} = \frac{\theta^*}{\alpha}. \tag{45}$$

For any $\widehat{R}_0^* > 0$ and $\sigma > 0$, $\Psi : [0,1] \to \mathbb{R}_+$ is a strictly decreasing function that satisfies $\Psi(0) > 0$ and $\Psi(1) = 1 < 1/\alpha$. Meanwhile, the right-hand side of the above equation is a straight line that passes through the origin and $1/\alpha$ (when $\theta^* = 1$). Thus, for any $\widehat{R}_0^* > 0$ and $\sigma > 0$, there exists a unique $\theta^* \in (0,1)$ that solves (45). Once θ^* is determined, the value of $\{k^*, w^*, l^*, a^*\}$ can be uniquely determined using (19)-(22).

Propensity to Consumer When Young

Using (40), we can get

$$\begin{aligned}\frac{c_y^*}{w^*l^*} &= \frac{\Omega^*\widehat{R}_0^*}{\Omega^*\widehat{R}_0^* + \beta^{\frac{1}{\sigma}}\left[q\left(1+n\right)\right]^{\frac{1}{\sigma}}\left[1+\frac{R^*}{1+n}\left(\Lambda^*-1\right)\right]} \\ &= \left\{1+\beta^{\frac{1}{\sigma}}\frac{\left[q\left(1+n\right)\right]^{\frac{1}{\sigma}}}{\Omega^*\widehat{R}_0^*}\left[1+\frac{R^*}{1+n}\left(\Lambda^*-1\right)\right]\right\}^{-1} \equiv \left[1+\beta^{\frac{1}{\sigma}}\left(\rho^*\right)^{\frac{1}{\sigma}-1}\right]^{-1},\end{aligned}$$

where ρ^* is the certainty equivalent return defined in the text. An alternative expression for the propensity to consume can be obtained as follows. First, rewrite the above expression as

$$\frac{c_y^*}{w^*l^*} = \frac{\Omega^*\widehat{R}_0^*}{\left[\beta q\left(1+n\right)\right]^{\frac{1}{\sigma}}}\left\{\frac{\left[\beta q\left(1+n\right)\right]^{\frac{1}{\sigma}}}{\Omega^*\widehat{R}_0^* + \left[\beta q\left(1+n\right)\right]^{\frac{1}{\sigma}}\left[1+\frac{R^*}{1+n}\left(\Lambda^*-1\right)\right]}\right\}. \tag{46}$$

Using (44), we can obtain

$$\frac{\left[\beta q\left(1+n\right)\right]^{\frac{1}{\sigma}}}{\Omega^*\widehat{R}_0^* + \left[\beta q\left(1+n\right)\right]^{\frac{1}{\sigma}}\left[1+\frac{R^*}{1+n}\left(\Lambda^*-1\right)\right]} = \frac{\alpha\left(1+n\right)}{1-\alpha}\frac{1}{R^*}.$$

Substituting this into (46) gives

$$\frac{c_y^*}{w^*l^*} = \frac{\Omega^*\widehat{R}_0^*}{\left[\beta q\left(1+n\right)\right]^{\frac{1}{\sigma}}}\left[\frac{\alpha\left(1+n\right)}{1-\alpha}\frac{1}{R^*}\right].$$

On the other hand, in the bubbleless steady state, we have

$$\frac{\widehat{c}_y^*}{\widehat{w}^*\widehat{l}^*} = \left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}\right]^{-1} = \frac{\alpha\left(1+n\right)}{1-\alpha}\left(\beta\widehat{R}^*\right)^{-\frac{1}{\sigma}}.$$

The second equality follows from (16). Hence, we have

$$\frac{\widehat{c}_y^*}{\widehat{w}^*\widehat{l}^*} > \frac{c_y^*}{w^*l^*} \quad\Leftrightarrow\quad \left(\widehat{R}^*\right)^{-\frac{1}{\sigma}} > \frac{\Omega^*\widehat{R}_0^*}{\left[q\left(1+n\right)\right]^{\frac{1}{\sigma}}}\frac{1}{R^*} \quad\Leftrightarrow\quad \left[\frac{q\left(1+n\right)}{\widehat{R}^*}\right]^{\frac{1}{\sigma}} > \frac{\Omega^*\widehat{R}_0^*}{R^*}.$$

Appendix B: Proofs

Proof of Proposition 1

In any bubbleless steady state, we have $\widehat{k}_{t+1} = \widehat{k}_t = \widehat{k}^*$ and $\widehat{R}_{t+1} = \widehat{R}_t = \widehat{R}^*$ for all t. Substituting these into (4) and rearranging terms gives

$$\Gamma\left(\widehat{R}^*\right) \equiv \frac{\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}} = \frac{(1+n)\,\alpha}{1-\alpha}. \tag{47}$$

Substituting the steady state conditions into (15) and rearranging terms gives (17). Note that the function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ defined in (47) is continuously differentiable and satisfies $\Gamma(0) = 0$. Straightforward differentiation gives

$$\Gamma'\left(\widehat{R}\right) = \frac{\beta^{\frac{1}{\sigma}}\widehat{R}^{\frac{1}{\sigma}-1}\left(\frac{1}{\sigma}+\beta^{\frac{1}{\sigma}}\widehat{R}^{\frac{1}{\sigma}-1}\right)}{\left(1+\beta^{\frac{1}{\sigma}}\widehat{R}^{\frac{1}{\sigma}-1}\right)^2} > 0, \qquad \text{for any } \sigma > 0.$$

Hence, there exists a unique value of $\widehat{R}^* > 0$ that solves (47). Using (17), one can obtain a unique value of $\widehat{k}^* > 0$. This establishes the existence and uniqueness of bubbleless steady state.

Proof of Proposition 2

First, consider the case when $\sigma = 1$. Equations (14) and (15) now become

$$\widehat{k}_{t+1} = \frac{1-\alpha}{\alpha(1+n)}\left(\frac{\beta}{1+\beta}\right)\widehat{R}_t\widehat{k}_t, \qquad \text{and} \qquad \widehat{R}_t^{\frac{1}{1-\alpha}}\widehat{k}_t = \alpha^{\frac{1}{1-\alpha}}\left(\frac{1+\beta}{A}\right)^{\frac{1}{1+\psi}}. \tag{48}$$

Combining the two gives

$$\widehat{k}_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\left(\frac{1+\beta}{A}\right)^{\frac{1-\alpha}{1+\psi}}\widehat{k}_t^{\alpha}.$$

Since $\alpha \in (0,1)$, there exists a unique non-trivial steady state $\widehat{k}^* > 0$ which is globally stable. The second equation in (48) can be rewritten as

$$\widehat{R}_t = \alpha\left(\frac{1+\beta}{A}\right)^{\frac{1-\alpha}{1+\psi}}\left(\widehat{k}_t\right)^{\alpha-1} \equiv \Phi\left(\widehat{k}_t\right),$$

where $\Phi(\cdot)$ is a strictly decreasing function.

Next, consider the case when $\sigma < 1$. To prove that the bubbleless steady state is globally saddle-path stable, we will use the same "phase diagram" approach as in Tirole (1985) and Weil (1987). To start, define a function $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}_+$ according to

$$\mathcal{F}(R) = \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left(1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1} \right)^{\frac{\sigma}{\sigma+\psi}} R^{-\eta}. \tag{49}$$

Note that the unique bubbleless steady state must satisfy $\widehat{k}^* = \mathcal{F}\left(\widehat{R}^*\right)$. Taking the logarithm of both sides of (49) and differentiating the resultant expression with respect to R gives

$$\frac{R\mathcal{F}'(R)}{\mathcal{F}(R)} = \frac{1-\sigma}{\sigma+\psi} \left(\frac{\beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}} - \widetilde{\eta} \right) = \frac{1-\sigma}{\sigma+\psi} \left[\Sigma(R) - \widetilde{\eta} \right],$$

where $\widetilde{\eta} \equiv (\sigma+\psi)\eta/(1-\sigma)$ and $\Sigma(\cdot)$ is the function defined in (4). There are two possible scenarios: (i) $\widetilde{\eta} \geq 1$ and (ii) $\widetilde{\eta} < 1$. Since $\Sigma(\cdot)$ is strictly increasing and bounded above by one, in the first scenario we have $\mathcal{F}'(R) < 0$ for all $R \geq 0$, $\lim_{R\to 0} \mathcal{F}(R) = +\infty$ and $\lim_{R\to\infty} \mathcal{F}(R) = 0$. In the second scenario, $\mathcal{F}(\cdot)$ is a U-shaped function. Figures B1 and B2 provide a graphical illustration of these two scenarios. In both diagrams, the function $\mathcal{F}(\cdot)$ and the vertical line representing $R = \widehat{R}^*$ divide the (R,k)-space into four quadrants:

$$Q_1 \equiv \left\{ (R,k) : k \leq \mathcal{F}(R),\ R \leq \widehat{R}^*, \text{ and } (R,k) \neq \left(\widehat{R}^*, \widehat{k}^*\right) \right\},$$

$$Q_2 \equiv \left\{ (R,k) : k > \mathcal{F}(R) \text{ and } R < \widehat{R}^* \right\},$$

$$Q_3 \equiv \left\{ (R,k) : k \geq \mathcal{F}(R),\ R \geq \widehat{R}^*, \text{ and } (R,k) \neq \left(\widehat{R}^*, \widehat{k}^*\right) \right\},$$

$$Q_4 \equiv \left\{ (R,k) : k < \mathcal{F}(R) \text{ and } R > \widehat{R}^* \right\}.$$

The rest of the proof is divided into a number of intermediate steps. These steps are valid both when $\widetilde{\eta} \geq 1$ and when $\widetilde{\eta} < 1$.

Step 1 For any initial value $\left(\widehat{R}_T, \widehat{k}_T\right) > 0$, there exists a unique sequence $\left\{\widehat{R}_{T+1}, \widehat{k}_{T+1}, \widehat{R}_{T+2}, \widehat{k}_{T+2}, ...\right\}$ that solves the dynamical system in (14)-(15). Whether this is part of a non-stationary bubbleless equilibrium depends on the location of $\left(\widehat{R}_T, \widehat{k}_T\right)$ on the (R,k)-space. A solution $\left\{\widehat{R}_{T+1}, \widehat{k}_{T+1}, \widehat{R}_{T+2}, \widehat{k}_{T+2}, ...\right\}$ is said to originate from Q_n if $\left(\widehat{R}_T, \widehat{k}_T\right) \in Q_n$, for $n \in \{1,2,3,4\}$. In the first step of the proof, it is shown that any solution that originates from Q_1 or Q_3 cannot be part of a bubbleless equilibrium.

Suppose $\left(\widehat{R}_t, \widehat{k}_t\right)$ is in Q_1 for some $t \geq T$. This means either (i) $\widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right)$ and $\widehat{R}_t \leq \widehat{R}^*$, or (ii) $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$ and $\widehat{R}_t < \widehat{R}^*$. First consider the case when $\widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right)$ and $\widehat{R}_t \leq \widehat{R}^*$. Using (15), we can obtain

$$\begin{aligned}\widehat{R}_t^{\eta}\widehat{k}_t &= \alpha^{\eta}\left[\frac{(1-\alpha)^{1-\sigma}}{A}\right]^{\frac{1}{\sigma+\psi}}\left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}\right]^{\frac{\sigma}{\sigma+\psi}} \\ &< \alpha^{\eta}\left[\frac{(1-\alpha)^{1-\sigma}}{A}\right]^{\frac{1}{\sigma+\psi}}\left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t}\right)^{\frac{1}{\sigma}-1}\right]^{\frac{\sigma}{\sigma+\psi}},\end{aligned}$$

which implies $\widehat{R}_{t+1} < \widehat{R}_t \leq \widehat{R}^*$. Recall that the function $\Sigma(\cdot)$ defined in (4) is strictly increasing when $\sigma < 1$. Then it follows from (14) that

$$\begin{aligned}\widehat{k}_{t+1} &= \frac{1-\alpha}{\alpha(1+n)}\Sigma\left(\widehat{R}_{t+1}\right)\widehat{R}_t\widehat{k}_t \\ &< \frac{1-\alpha}{\alpha(1+n)}\Sigma\left(\widehat{R}^*\right)\widehat{R}_t\widehat{k}_t \leq \frac{1-\alpha}{\alpha(1+n)}\Sigma\left(\widehat{R}^*\right)\widehat{R}^*\widehat{k}_t = \widehat{k}_t.\end{aligned}$$

The last equality follows from equation (16). This result implies $\widehat{k}_{t+1} < \widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right) < \mathcal{F}\left(\widehat{R}_{t+1}\right)$. Next, consider the case when $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$ and $\widehat{R}_t < \widehat{R}^*$. Equation (15) and $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$ together imply $\widehat{R}_{t+1} = \widehat{R}_t < \widehat{R}^*$. This, together with (14), implies $\widehat{k}_{t+1} < \widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right) = \mathcal{F}\left(\widehat{R}_{t+1}\right)$. This proves the following: Any solution that originates from Q_1 is a strictly decreasing sequence and is confined in Q_1, i.e., $\left(\widehat{R}_t, \widehat{k}_t\right) \in Q_1$ for all $t \geq T$. Since both $\widehat{k}_t$ and $\widehat{R}_t$ are strictly decreasing over time, in the long run we will have either $\widehat{k}_t = 0$ or $\widehat{R}_t = 0$, which cannot happen in equilibrium.

Using a similar argument, we can show that any solution that originates from Q_3 is a strictly increasing sequence and is confined in Q_3. Using the young consumer's budget constraint and the capital market clearing condition, we can obtain the following condition

$$\widehat{s}_t = \frac{\widehat{k}_{t+1}}{1+n} < \widehat{w}_t\widehat{l}_t \leq \widehat{w}_t = (1-\alpha)\left(\frac{\alpha}{\widehat{R}_t}\right)^{\frac{\alpha}{1-\alpha}}.$$

Obviously, this will be violated at some point if both $\widehat{k}_t$ and $\widehat{R}_t$ are strictly increasing over time. Hence, any solution that originates from Q_3 cannot be part of a bubbleless equilibrium.

Step 2 We now show that any solution that originates from Q_2 will never enter Q_4, i.e., $\left(\widehat{R}_T, \widehat{k}_T\right) \in Q_2$ implies $\left(\widehat{R}_t, \widehat{k}_t\right) \notin Q_4$, for all $t > T$; likewise, any solution that originates from Q_4 will never enter Q_2.

Suppose $\left(\widehat{R}_t, \widehat{k}_t\right)$ is in Q_2 for some $t \geq T$. Then we have

$$\begin{aligned}
\widehat{R}_t^{\eta} \widehat{k}_t &= \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \\
&> \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}},
\end{aligned}$$

which implies $\widehat{R}_{t+1} > \widehat{R}_t$. Suppose the contrary that $\left(\widehat{R}_{t+1}, \widehat{k}_{t+1}\right)$ is in Q_4, so that $\widehat{R}_{t+1} > \widehat{R}^* > \widehat{R}_t$ and $\widehat{k}_{t+1} < \mathcal{F}\left(\widehat{R}_{t+1}\right)$. Then, using (14) we can get

$$\begin{aligned}
\widehat{R}_{t+1} \widehat{k}_{t+1} &= \frac{1-\alpha}{\alpha(1+n)} \left[\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_t \widehat{k}_t \\
&> \frac{1-\alpha}{\alpha(1+n)} \left[\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}^* \right)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}^* \right)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_t \widehat{k}_t = \widehat{R}_t \widehat{k}_t. \qquad (50)
\end{aligned}$$

The second line uses the fact that $\Sigma(\cdot)$ is strictly increasing and $\widehat{R}_{t+1} > \widehat{R}^*$. The last equality follows from the steady-state condition in (16). Since $\eta > 1$, we also have $\widehat{R}_{t+1}^{\eta-1} > \widehat{R}_t^{\eta-1}$. This, together with (15) and (50), implies

$$\widehat{R}_{t+1}^{\eta} \widehat{k}_{t+1} > \widehat{R}_t^{\eta} \widehat{k}_t = \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}$$

$$\Rightarrow \widehat{k}_{t+1} > \mathcal{F}\left(\widehat{R}_{t+1}\right),$$

which gives rise to a contradiction. Hence, any solution that originates from Q_2 will never enter Q_4. Using similar arguments, we can show that any solution that originates from Q_4 will never enter Q_2.

Step 3 Consider a solution that originates from Q_2. As shown in Step 2, $\left(\widehat{R}_T, \widehat{k}_T\right) \in Q_2$ implies $\widehat{R}_{T+1} > \widehat{R}_T$. If $\widehat{R}_{T+1} \geq \widehat{R}^*$, then the economy is in Q_3 at time $T+1$ and by the results in Step 1, we know that $\widehat{R}_t$ will diverge to infinity in the long run. If $\widehat{R}_{T+1} < \widehat{R}^*$, then using (14) we can obtain

$$\begin{aligned}\widehat{k}_{T+1} &= \frac{1-\alpha}{\alpha(1+n)}\left[\frac{\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{T+1}\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{T+1}\right)^{\frac{1}{\sigma}-1}}\right]\widehat{R}_T\widehat{k}_T \\ &< \frac{1-\alpha}{\alpha(1+n)}\left[\frac{\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}}\right]\widehat{k}_T = \widehat{k}_T.\end{aligned}$$

There are two possible scenarios: First, if $\widehat{R}_{T+1} < \widehat{R}^*$ and $\widehat{k}_{T+1} \leq \mathcal{F}\left(\widehat{R}_{T+1}\right)$, then the economy is in Q_1 at time $T+1$. By the results in Step 1, we know that all subsequent values of $\widehat{R}_t$ will be strictly less than $\widehat{R}^*$. Second, if $\widehat{R}_{T+1} < \widehat{R}^*$ and $\mathcal{F}\left(\widehat{R}_{T+1}\right) < \widehat{k}_{T+1}$, then that means the economy remains in Q_2 at time $T+1$. In addition, we have $\widehat{R}_{T+1} > \widehat{R}_T$ and $\widehat{k}_T > \widehat{k}_{T+1}$ which means the economy is now getting closer to the steady state $\left(\widehat{R}^*, \widehat{k}^*\right)$. Thus, any solution that originates from Q_2 has three possible fates: (i) It will enter Q_3 at some point and $\widehat{R}_t$ will then diverge to infinity. (ii) It will enter Q_1 at some point and $\widehat{R}_t$ will be strictly less than $\widehat{R}^*$ afterward. (iii) It will converge to the bubbleless steady state. For reasons explained above, the first two types of solutions cannot be part of an equilibrium. Hence, a solution originating from Q_2 is an equilibrium path only if it converges to the steady state $\left(\widehat{R}^*, \widehat{k}^*\right)$. The above argument also shows that, along the convergent path, $\widehat{k}_t$ is decreasing towards $\widehat{k}^*$ while $\widehat{R}_t$ is increasing towards $\widehat{R}^*$.

Using a similar argument, we can show that any solution originating from Q_4 is an equilibrium path only if it converges to the steady state $\left(\widehat{R}^*, \widehat{k}^*\right)$, and that along the convergent path, $\widehat{k}_t$ is increasing towards $\widehat{k}^*$ while $\widehat{R}_t$ is decreasing towards $\widehat{R}^*$.

Step 4 We now establish the uniqueness of saddle path. Fix $\widehat{k}_T > 0$. Suppose the contrary that there exists two saddle paths, denoted by $\left\{\widehat{R}'_t, \widehat{k}'_t\right\}_{t=T}^{\infty}$ and $\left\{\widehat{R}''_t, \widehat{k}''_t\right\}_{t=T}^{\infty}$, with $\widehat{k}'_T = \widehat{k}''_T = \widehat{k}_T$ and $\widehat{R}'_T > \widehat{R}''_T > 0$. By the results in Step 3, we know that $\lim_{t\to\infty}\widehat{R}'_t = \lim_{t\to\infty}\widehat{R}''_t = \widehat{R}^*$. Substituting $\widehat{k}'_T = \widehat{k}''_T$ and $\widehat{R}'_T > \widehat{R}''_T$ into (15) gives

$$\left(\frac{\widehat{R}'_T}{\widehat{R}''_T}\right)^{\eta} = \left[\frac{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}'_{T+1}\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}''_{T+1}\right)^{\frac{1}{\sigma}-1}}\right]^{\frac{\sigma}{\sigma+\psi}} > 1,$$

which implies $\widehat{R}'_{T+1} > \widehat{R}''_{T+1} > 0$. Using (14), we can get

$$\frac{\widehat{k}'_{T+1}}{\widehat{k}''_{T+1}} = \frac{\Sigma\left(\widehat{R}'_{T+1}\right)}{\Sigma\left(\widehat{R}''_{T+1}\right)} \frac{\widehat{R}'_T}{\widehat{R}''_T} > 1.$$

Using (15) again, but now for $t = T+1$, gives

$$\left(\frac{\widehat{R}'_{T+1}}{\widehat{R}''_{T+1}}\right)^{\eta} \left(\frac{\widehat{k}'_{T+1}}{\widehat{k}''_{T+1}}\right) = \left[\frac{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}'_{T+2}\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}''_{T+2}\right)^{\frac{1}{\sigma}-1}}\right]^{\frac{\sigma}{\sigma+\psi}} > 1,$$

which implies $\widehat{R}'_{T+2} > \widehat{R}''_{T+2}$. By an induction argument, we can show that $\widehat{R}'_{T+j} > \widehat{R}''_{T+j}$ implies $\widehat{k}'_{T+j} > \widehat{k}''_{T+j}$, and $\widehat{R}'_{T+j+1} > \widehat{R}''_{T+j+1}$, for all $j \geq 1$. The last result contradicts $\lim_{t\to\infty} \widehat{R}'_t = \lim_{t\to\infty} \widehat{R}''_t = \widehat{R}^*$. Hence, we can rule out the possibility of multiple saddle paths.

In sum, we have shown that any equilibrium path that originates from a given value of $\widehat{k}_T > 0$ must be unique and converge to the bubbleless steady state. Hence, the dynamical system in (14)-(15) is globally saddle-path stable. The one-to-one relationship between $\widehat{R}_T$ and $\widehat{k}_T$ can be captured by a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. Since the saddle path is downward sloping in the (R,k)-space, $\Phi(\cdot)$ must be strictly decreasing. This completes the proof of Proposition 2.

Proof of Proposition 3

In the post-crash economy, optimal labor supply is determined by (3). Setting $\sigma = 1$ gives $\widehat{l}_t = \left(\frac{1+\beta}{A}\right)^{\frac{1}{1+\psi}}$ for all t. In the pre-crash economy, optimal labor supply is determined by

$$Al_t^{\psi+\sigma} = (w_t)^{1-\sigma} \left\{\frac{\Omega_{t+1}\widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}}\left[1+\frac{R_{t+1}}{\pi_{t+1}}(\Lambda_{t+1}-1)\right]}{\Omega_{t+1}\widehat{R}_{t+1}}\right\}^{\sigma},$$

which is equation (41) in Appendix A, where

$$\Omega_{t+1} \equiv \left[\frac{q(\pi_{t+1}-R_{t+1})}{(1-q)\widehat{R}_{t+1}}\right]^{\frac{1}{\sigma}} \qquad \text{and} \qquad \Lambda_{t+1} \equiv \frac{\Omega_{t+1}\widehat{R}_{t+1}}{R_{t+1}}.$$

When $\sigma = 1$, the right-hand side (RHS) of the above equation becomes

$$\begin{aligned} RHS &= 1 + \left(\Omega_{t+1}\widehat{R}_{t+1}\right)^{-1}(\beta q \pi_{t+1})\left[1 + \frac{R_{t+1}}{\pi_{t+1}}(\Lambda_{t+1} - 1)\right] \\ &= 1 + \frac{\beta(1-q)}{\pi_{t+1} - R_{t+1}}\left(\pi_{t+1} + \Omega_{t+1}\widehat{R}_{t+1} - R_{t+1}\right) \\ &= 1 + \frac{\beta(1-q)}{\pi_{t+1} - R_{t+1}}\left[\pi_{t+1} - R_{t+1} + \frac{q(\pi_{t+1} - R_{t+1})}{1-q}\right] = 1 + \beta. \end{aligned}$$

Hence, we have $Al_t^{\psi+1} = 1 + \beta$ for all t. The desired result follows immediately from this expression. This completes the proof of Proposition 3.

Proof of Proposition 4

The main ideas of the proof are as follows. In any conditional bubbly steady state, we have $a^* > 0$ which is equivalent to $\Lambda^* > 1$. This, together with $\sigma < 1$ and $R^* \leq \widehat{R}^*$, implies two things: $k^* > \widehat{k}^*$ and $\widehat{R}_0^* \equiv \Phi(k^*) > \widehat{R}^*$. But as we have seen in Proposition 2, these two results cannot be both true which means we have reached a contradiction. Hence, it must be the case that $R^* > \widehat{R}^*$.

The main task of the proof is to verify the following two claims:

Claim #1 *Suppose $\sigma < 1$ and $\Lambda^* > 1$. Then $R^* \leq \widehat{R}^*$ implies $l^* > \widehat{l}^*$ and $k^* > \widehat{k}^*$.*

Claim #2 *Suppose $\sigma < 1$ and $\Lambda^* > 1$. Then $R^* \leq \widehat{R}^*$ implies $\widehat{R}^* < \widehat{R}_0^*$.*

Proof of Claim #1 Suppose $R^* \leq \widehat{R}^*$ and $l^* > \widehat{l}^*$ are both true. Then using (21), we can get

$$k^* = l^*\left(\frac{\alpha}{R^*}\right)^{\frac{1}{1-\alpha}} > \widehat{l}^*\left(\frac{\alpha}{\widehat{R}^*}\right)^{\frac{1}{1-\alpha}} = \widehat{k}^*.$$

Hence, it suffice to show that $R^* \leq \widehat{R}^*$ implies $l^* > \widehat{l}^*$.

When evaluated in a recurring bubbly equilibrium, equation (41) becomes

$$\begin{aligned} A(l^*)^{\psi+\sigma} &= (w^*)^{1-\sigma}\left\{1 + \frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^*\widehat{R}_0^*}\left[1 + \frac{R^*}{1+n}(\Lambda^* - 1)\right]\right\}^{\sigma} \\ &= \left[(1-\alpha)\left(\frac{\alpha}{R^*}\right)^{\frac{\alpha}{1-\alpha}}\right]^{1-\sigma}\left\{1 + \frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^*\widehat{R}_0^*}\left[1 + \frac{R^*}{1+n}(\Lambda^* - 1)\right]\right\}^{\sigma}. \end{aligned}$$

On the other hand, the value of $\widehat{l}^*$ in the bubbleless steady state is determined by

$$A\left(\widehat{l}^*\right)^{\psi+\sigma}=\left[(1-\alpha)\left(\frac{\alpha}{\widehat{R}^*}\right)^{\frac{\alpha}{1-\alpha}}\right]^{1-\sigma}\left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}\right]^{\sigma}.$$

Combining the two gives

$$\left(\frac{l^*}{\widehat{l}^*}\right)^{\psi+\sigma}=\left(\frac{\widehat{R}^*}{R^*}\right)^{\frac{\alpha(1-\sigma)}{1-\alpha}}\left\{\frac{1+\frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^*\widehat{R}_0^*}\left[1+\frac{R^*}{1+n}\left(\Lambda^*-1\right)\right]}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}}\right\}^{\sigma}.$$

Since $\sigma<1$ and $R^*\leq\widehat{R}^*$, we have $\left(\widehat{R}^*/R^*\right)^{\frac{\alpha(1-\sigma)}{1-\alpha}}\geq 1$. Thus, it suffice to show that $\frac{[\beta q(1+n)]^{\frac{1}{\sigma}}}{\Omega^*\widehat{R}_0^*}>\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}$.

Define $\theta^*\equiv R^*/\left(1+n\right)$, $\widehat{\theta}_0^*\equiv\widehat{R}_0^*/(1+n)$ and $\widehat{\theta}^*\equiv\widehat{R}^*/(1+n)$. As shown in Proposition 1, the value of $\widehat{R}^*$ is determined by (16), which can be rewritten as

$$\left(\frac{1-\alpha}{\alpha}\right)\widehat{\theta}^*=1+\beta^{-\frac{1}{\sigma}}\left(1+n\right)^{1-\frac{1}{\sigma}}\left(\widehat{\theta}^*\right)^{1-\frac{1}{\sigma}}. \tag{51}$$

On the other hand, the relationship between R^* and $\widehat{R}_0^*$ is characterized by (18), which is derived from (44) in Appendix A. The latter can be rewritten as

$$\begin{aligned}\left(\frac{1-\alpha}{\alpha}\right)\theta^* &= 1+\theta^*\left(\Lambda^*-1\right)+\left[\beta q\left(1+n\right)\right]^{-\frac{1}{\sigma}}\Omega^*\widehat{R}_0^* \\ &= 1+\theta^*\left(\Lambda^*-1\right)+\beta^{-\frac{1}{\sigma}}q^{-\frac{1}{\sigma}}\left(1+n\right)^{1-\frac{1}{\sigma}}\Omega^*\widehat{\theta}_0^*. \end{aligned}\tag{52}$$

Combining (51) and (52) gives

$$\left(\frac{1-\alpha}{\alpha}\right)\left(\theta^*-\widehat{\theta}^*\right)=\theta^*\left(\Lambda^*-1\right)+\beta^{-\frac{1}{\sigma}}\left(1+n\right)^{1-\frac{1}{\sigma}}\left[q^{-\frac{1}{\sigma}}\Omega^*\widehat{\theta}_0^*-\left(\widehat{\theta}^*\right)^{1-\frac{1}{\sigma}}\right]. \tag{53}$$

Under the conditions $\sigma<1$, $\Lambda^*>1$ and $R^*\leq\widehat{R}^*$ (i.e., $\theta^*\leq\widehat{\theta}^*$), we can get

$$q^{-\frac{1}{\sigma}}\Omega^*\widehat{\theta}_0^*<\left(\widehat{\theta}^*\right)^{1-\frac{1}{\sigma}} \tag{54}$$

$$\Leftrightarrow q^{-\frac{1}{\sigma}}\Omega^*\left(\frac{\widehat{R}_0^*}{1+n}\right)<\left(\frac{\widehat{R}^*}{1+n}\right)^{1-\frac{1}{\sigma}}$$

$$\Leftrightarrow q^{-\frac{1}{\sigma}} \Omega^* \widehat{R}_0^* < \left(\widehat{R}^*\right)^{1-\frac{1}{\sigma}} (1+n)^{\frac{1}{\sigma}}$$

$$\Leftrightarrow \left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1} < \frac{[q(1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*}.$$

This establishes Claim #1.

Proof of Claim #2 First, note that $\Lambda^* > 1$ is true if and only if

$$q(1+n) > \left[q + (1-q)\left(\frac{\widehat{R}_0^*}{R^*}\right)^{1-\sigma}\right] R^*$$

$$\Leftrightarrow \frac{q(1-\theta^*)}{\theta^*} > (1-q)\left(\frac{\widehat{\theta}_0^*}{\theta^*}\right)^{1-\sigma}. \tag{55}$$

Next, rewrite (54) as

$$q^{-\frac{1}{\sigma}} \left[\frac{q(1-\theta^*)}{(1-q)\widehat{\theta}_0^*}\right]^{\frac{1}{\sigma}} \widehat{\theta}_0^* < \left(\widehat{\theta}^*\right)^{1-\frac{1}{\sigma}}$$

$$\Leftrightarrow 1-\theta^* < (1-q)\left(\frac{\widehat{\theta}_0^*}{\widehat{\theta}^*}\right)^{1-\sigma}. \tag{56}$$

Using (55) and (56), and the assumptions of $\theta^* \leq \widehat{\theta}^*$ and $\sigma < 1$, we can get

$$\frac{q(1-\theta^*)}{\theta^*} > (1-q)\left(\frac{\widehat{\theta}_0^*}{\theta^*}\right)^{1-\sigma} \geq (1-q)\left(\frac{\widehat{\theta}_0^*}{\widehat{\theta}^*}\right)^{1-\sigma} > 1-\theta^*,$$

which implies $q > \theta^*$. Using (56) and $q > \theta^*$, we can get

$$(1-q)\left(\frac{\widehat{\theta}_0^*}{\widehat{\theta}^*}\right)^{1-\sigma} > 1-\theta^* > 1-q \Rightarrow \widehat{\theta}_0^* > \widehat{\theta}^*.$$

This establishes Claim #2.

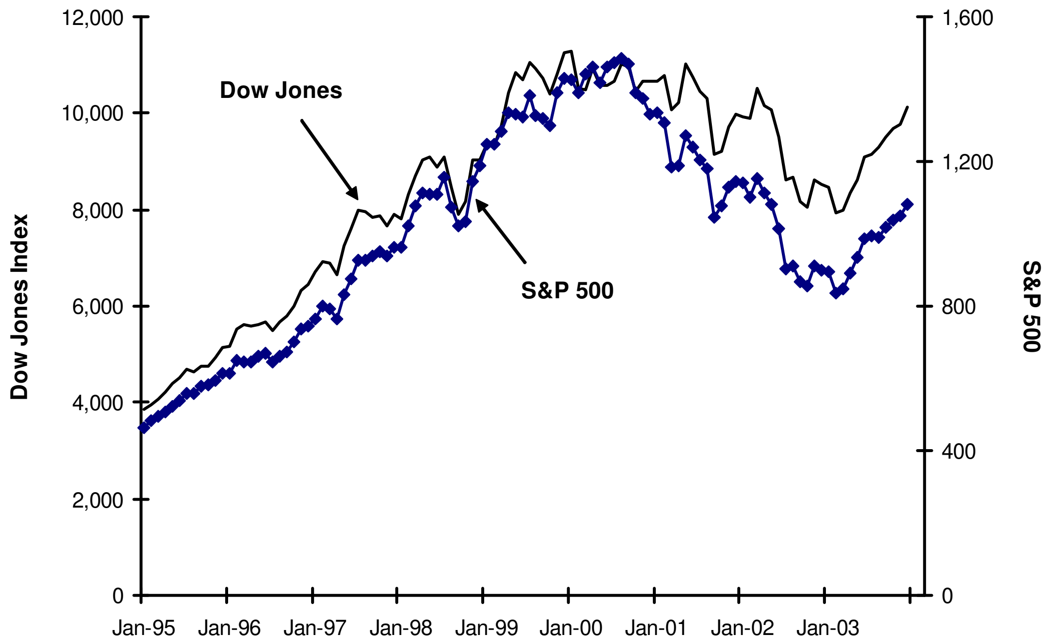

Figure 1: Dow Jones Industrial Average and S&P 500, 1995-2003.

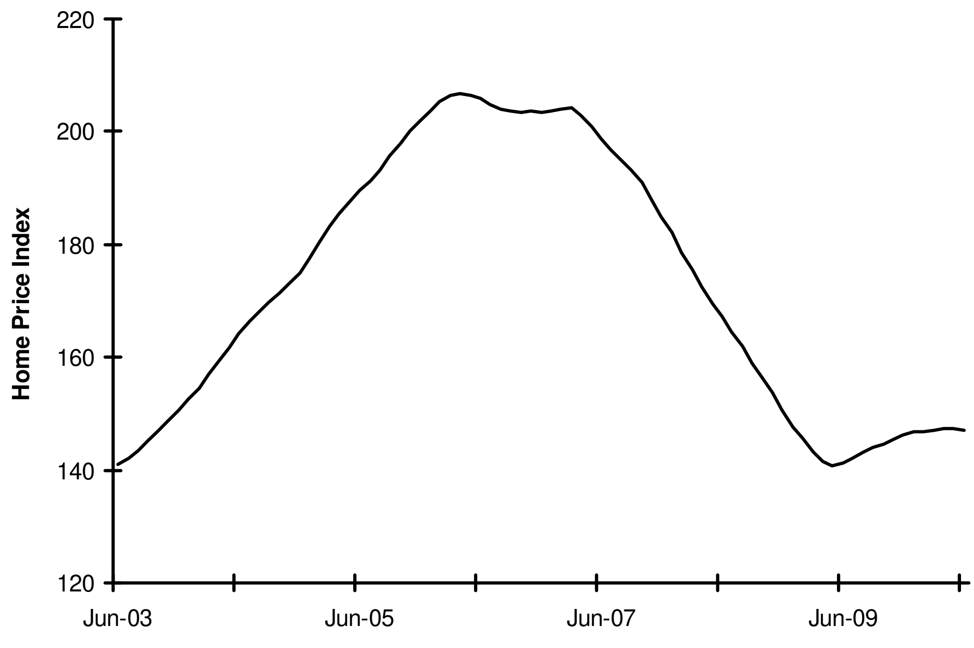

Figure 2: Case-Shiller 20-City Home Price Index, June 2003 to June 2010.

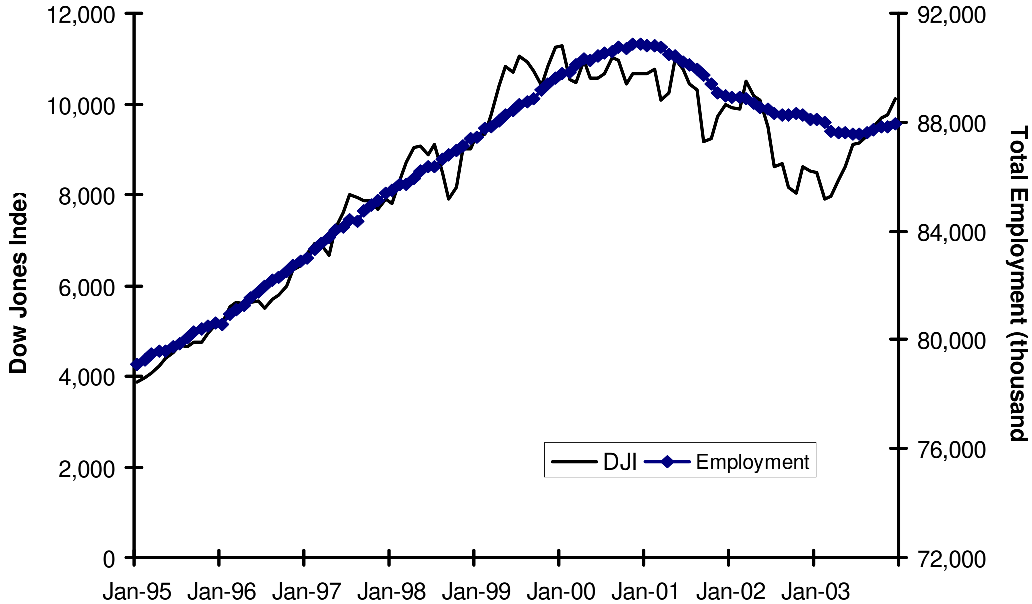

Figure 3: Total Employment and Dow Jones Index, 1995-2003.

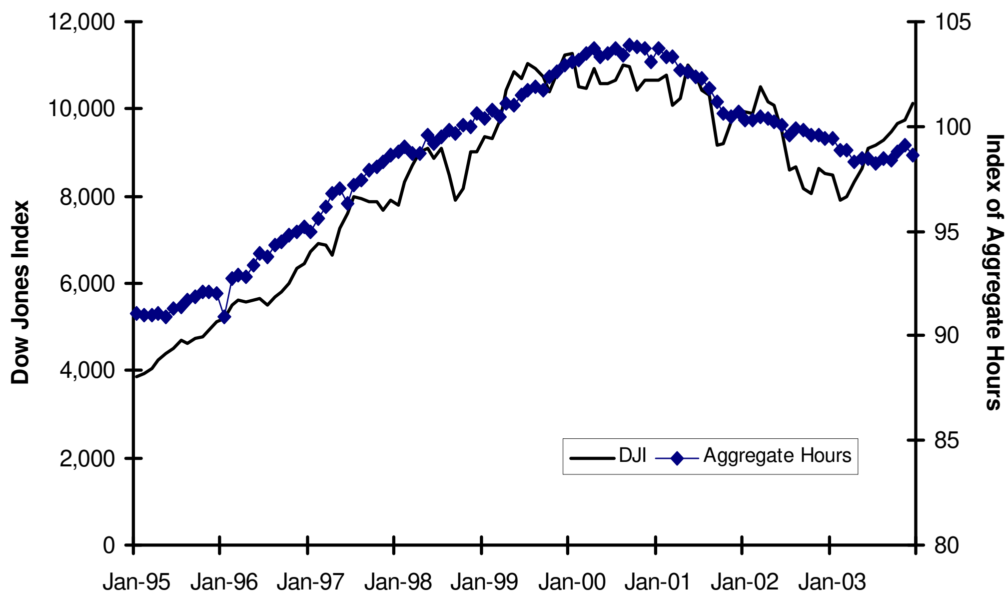

Figure 4: Aggregate Hours and Dow Jones Index, 1995-2003.

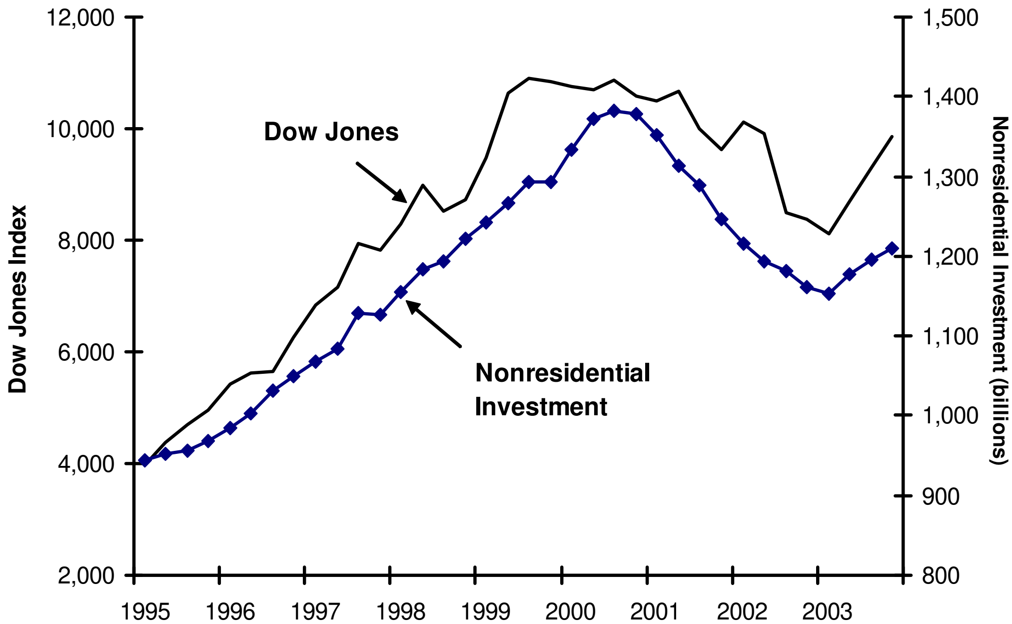

Figure 5: Private Nonresidential Fixed Investment and Dow Jones Index, 1995Q1 to 2003Q4.

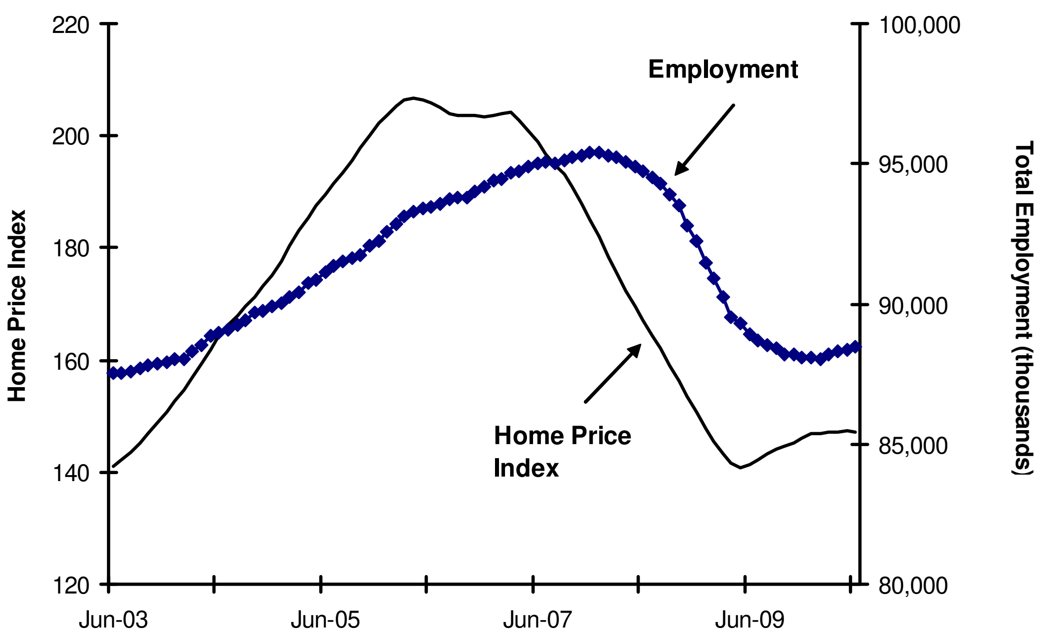

Figure 6: Total Employment and Home Price Index, June 2003 to June 2010.

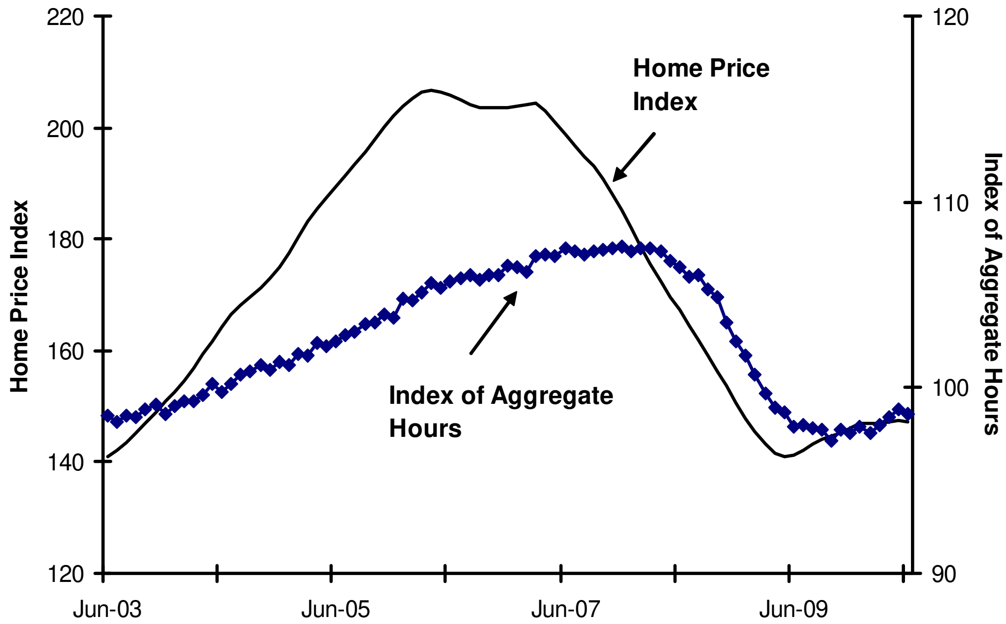

Figure 7: Aggregate Hours and Home Price Index, June 2003 to June 2010.

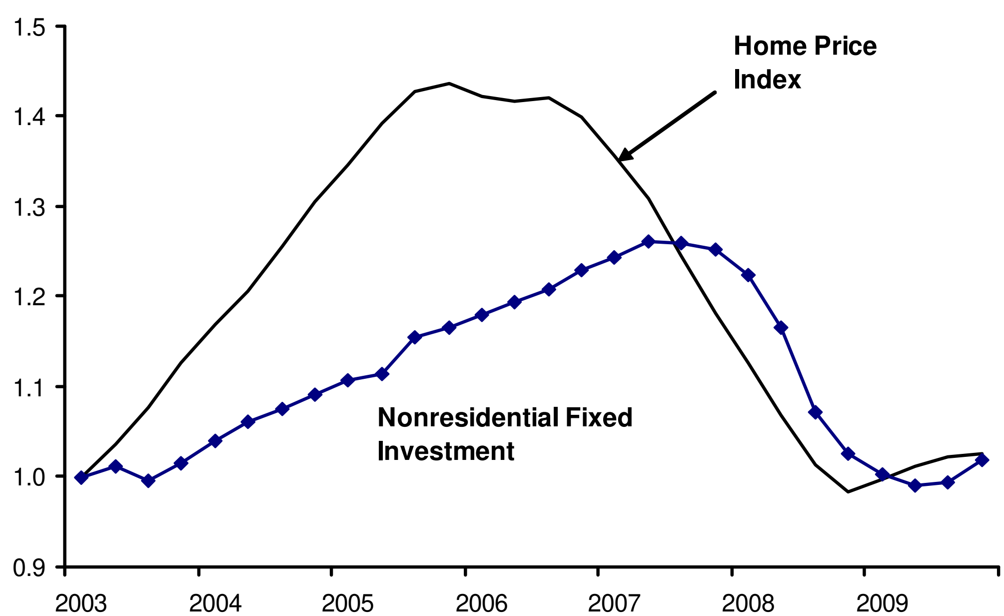

Figure 8: Private Nonresidential Fixed Investment and Home Price Index, 2003Q3 to 2010Q3.

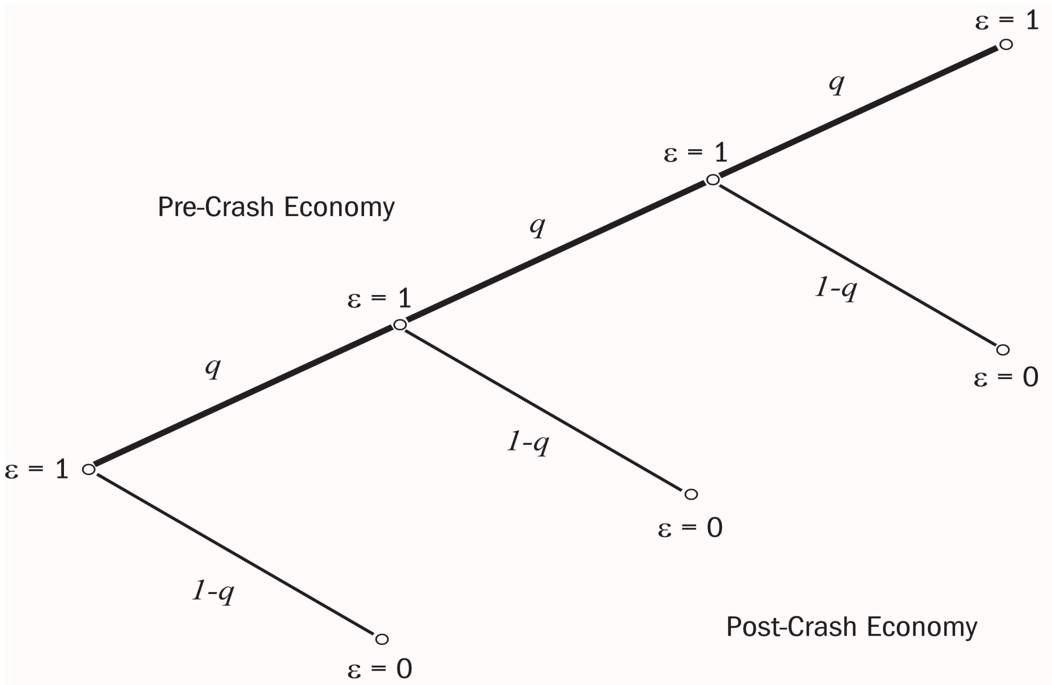

Figure 9: Probability Tree Diagram of the Asset Price Shock.

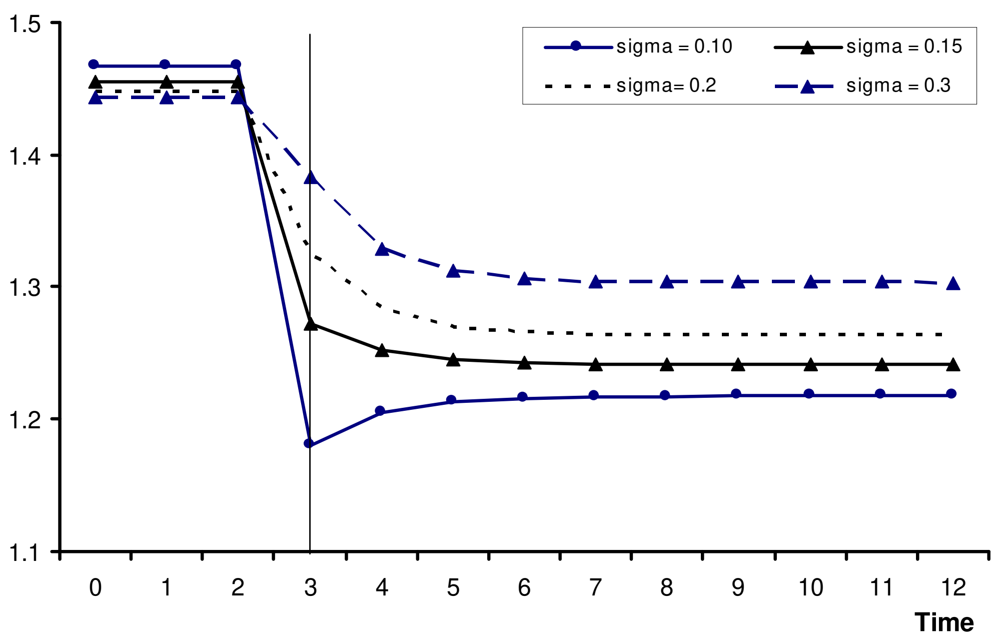

FIgure 10: Time Paths of Interest Rate under Different Values of σ.

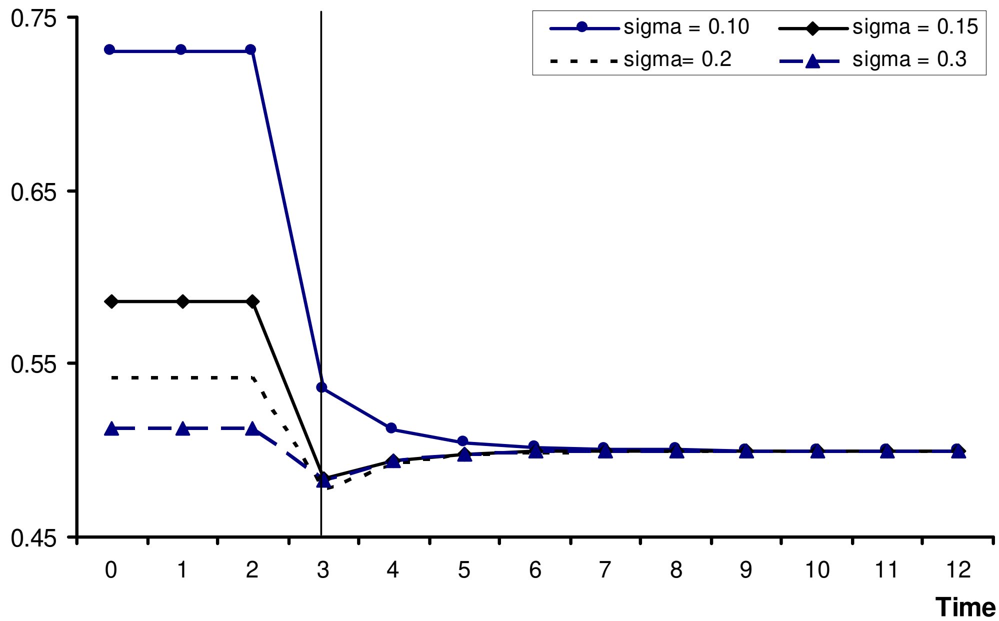

Figure 11: Time Paths of Labor Supply under Different Values of σ.

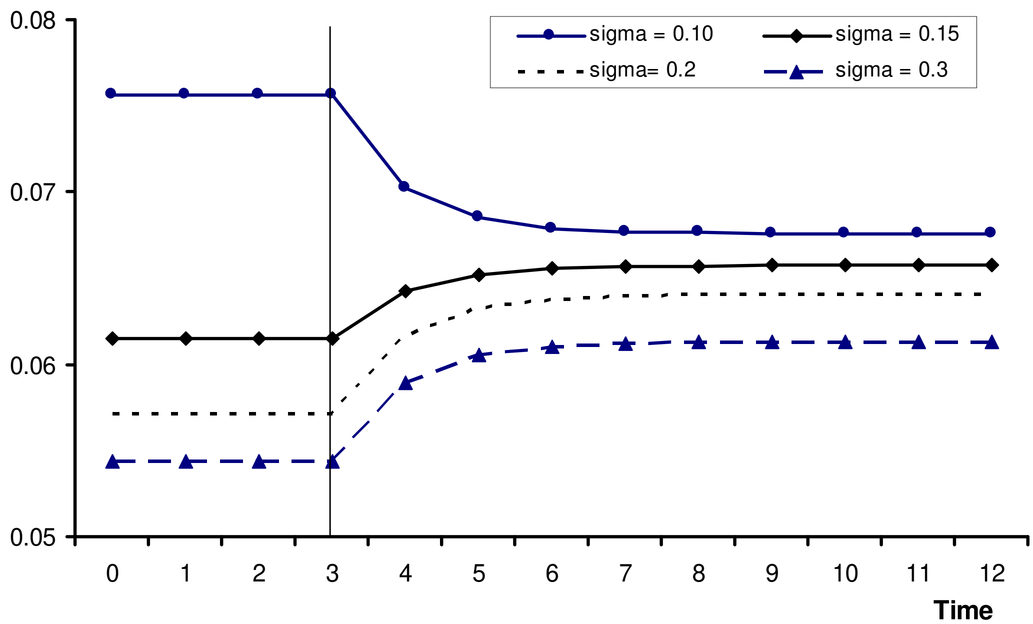

Figure 12: Time Paths of Capital under Different Values of σ.

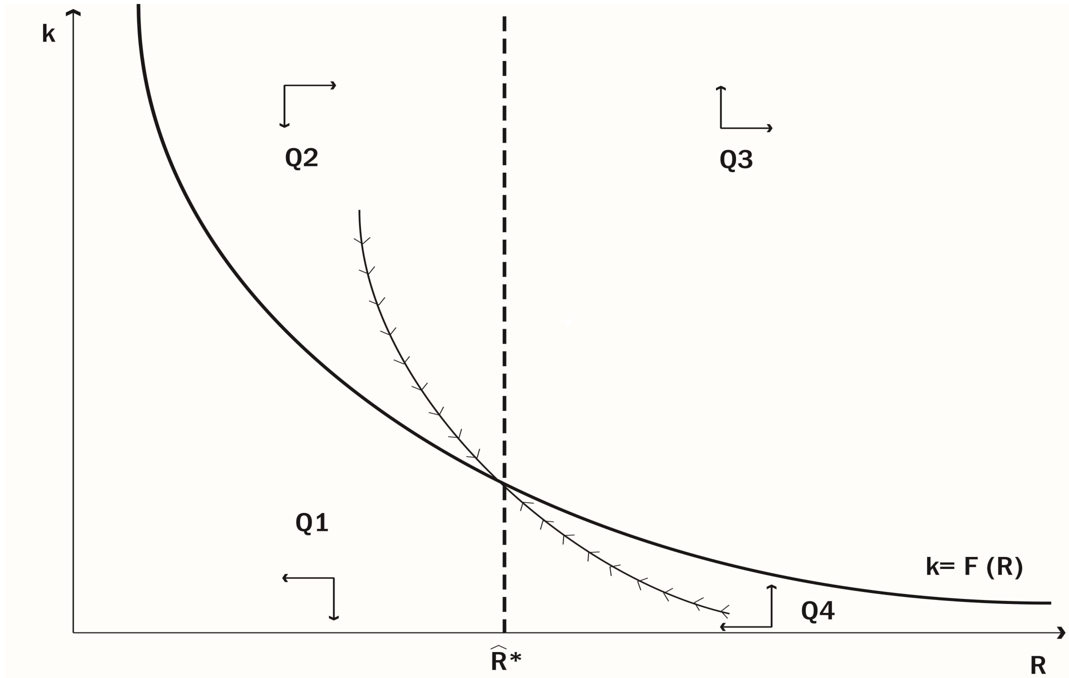

Figure B1: Phase Diagram for the case when $\widetilde{\eta} \geq 1$.

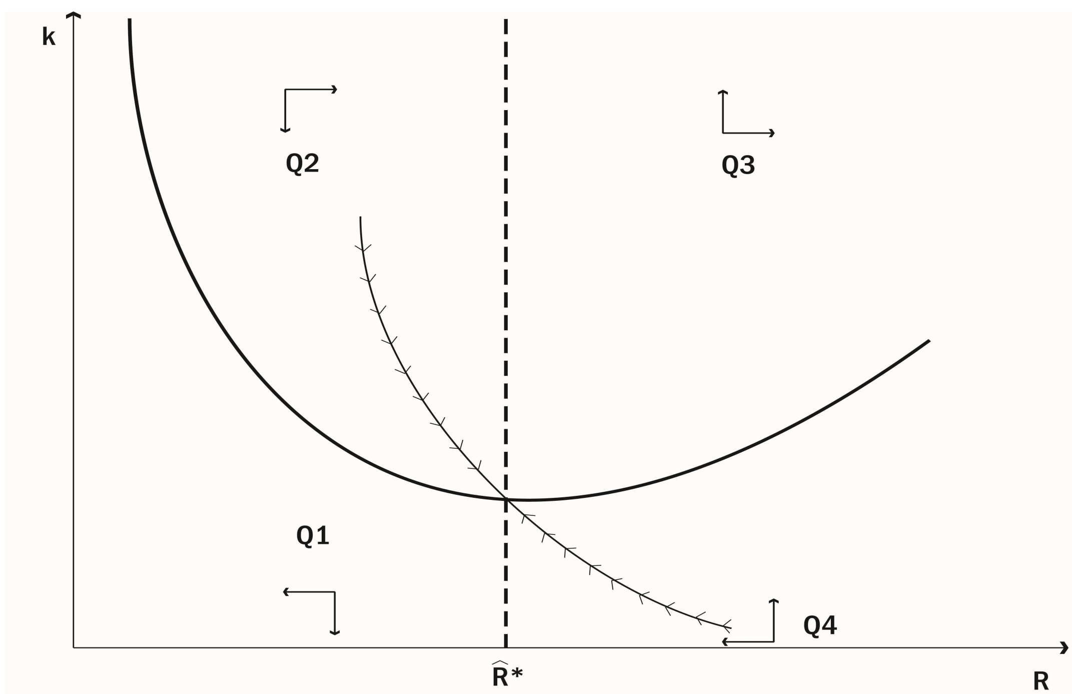

Figure B2: Phase Diagram for the case when $\widetilde{\eta} < 1$.

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