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Abstract

This paper analyzes the macroeconomic implications of asset price bubbles and crashes using an overlapping-generation model with endogenous labor supply. This model highlights the effects of asset price fluctuations on individuals' labor supply decision, and shows how these fluctuations can propagate to the aggregate economy through the labor-market channel. We show that asset bubbles can potentially crowd in productive investment and promote employment. This is more likely to happen when both the elasticity of intertemporal substitution for consumption and the Frisch elasticity of labor supply are large.

Keywords: Asset Bubbles, Overlapping Generations, Endogenous Labor.

JEL classification: E22, E44.

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1 Introduction

Economists have long been interested in the macroeconomic effects of asset price bubbles and crashes. In a seminal paper, Tirole (1985) shows that asset price bubbles can be sustained in an economy with overlapping generations of rational consumers. Weil (1987) extends this research by including the possibility of bubble burst. Many subsequent studies have adopted a similar OLG framework to analyze the nature and consequences of asset bubbles.[1] The models of Tirole (1985) and Weil (1987), however, have two features that are at odd with empirical evidence. First, both of them assume that individual labor supply and total employment are exogenously given. As a result, the labor market in their models are largely unrelated to and unaffected by the fluctuations in asset prices. The data, however, show that aggregate labor input tends to move closely with asset prices. In particular, the bursting of asset bubbles is often followed by a rapid deterioration in labor market conditions (see Section 2 for details). Second, both studies suggest that the formation of asset bubbles will crowd out investment in physical capital and impede economic growth, while the bursting of these bubbles will have the reverse effects. These predictions are also difficult to square with the data. For instance, private nonresidential fixed investment in the U.S. has increased significantly during the formation of the "dot-com" bubble in the 1990s and the housing bubble in the 2000s; and dropped precipitously when these bubbles collapsed. Empirical studies, such as Chirinko and Schaller (2001, 2011) and Gan (2007), provide solid evidence showing a positive effect of asset bubbles on private investment in the U.S. and in Japan. Martin and Ventura (2012) also observe that asset bubbles in these countries are often associated with robust economic growth.

In this paper, we show that these conflicts between theory and evidence can potentially be resolved by relaxing the assumption of exogenous labor supply. Specifically, we consider a two-period OLG model in which consumers can choose how much time to work, and how much to save and consume in their first period of life. There are two types of assets in this economy: physical capital and an intrinsically worthless asset. The latter is similar in nature to fiat money and unbacked government debt. Asset bubble is said to occur when this type of asset is traded across generations at a positive price. Following Weil (1987), we assume that asset bubbles may randomly crash in any time period. A crash happens when the price of the intrinsically worthless asset falls abruptly and unexpectedly to its fundamental value which is zero. Thus, unlike the deterministic model of Tirole (1985), there is a substantial downside risk associated with the intrinsically worthless asset. A key question is whether this type of risk will spawn uncertainty at the aggregate level. The

[1] Recent examples include Caballero and Krishnamurthy (2006), Farhi and Tirole (2012) and Ventura (2012) among many others. For a brief survey of rational bubble theories, see Miao (2014).

answer depends crucially on the endogeneity of labor supply, and the reason is simple. Suppose an asset bubble exists in the currrent period. Since the next-period stock of aggregate capital is predetermined in the current period, it is independent of the next-period state of the asset bubble. If labor supply is also exogenous as in Weil's (1987) model, then even if a crash happens next period it will have *no immediate impact* on aggregate output and factor prices.[2] Hence, the stochastic bubble does not generate any uncertainty at the aggregate level. This is different once we allow for an endogenous labor supply. In general, individuals' labor supply decision is contingent on the state of the asset bubble. As a result, the possibility of a crash in the future will create uncertainty in future labor input and future prices, which will in turn affect consumers' choice in the current period. This provides a simple and intuitive mechanism through which bubbles and crashes can affect the aggregate economy. The present study provides the first attempt to formulate and analyze this mechanism in a rational bubble model.[3] We find that the existence of stochastic asset bubbles can potentially crowd in productive investment, but this happens only if the bubbles can induce the consumers to work longer hours and cut back consumption when young. These effects are more likely to take place when both the elasticity of intertemporal substitution (IES) and the Frisch elasticity of labor supply are large.

Several recent studies have explored other channels through which asset bubbles can crowd in productive investment and foster economic growth in the context of OLG models. For instance, Martin and Ventura (2012) and Ventura (2012) present models in which asset bubbles can improve investment efficiency by shifting resources from less productive firms or countries to more productive ones. Caballero and Krishnamurthy (2006) and Farhi and Tirole (2012) develop models in which asset bubbles can facilitate investment by providing liquidity to financially constrained firms. For analytical convenience, these studies typically ignore the intertemporal substitution in consumption and the intratemporal substitution between consumption and labor.[4] The present study contributes to this literature by showing that these fundamental economic forces are crucial in understanding the effects of asset price bubbles and crashes.

The rest of this paper is organized as follows. Section 2 provides evidence showing that aggregate labor hours and private investment tend to move closely with asset prices during episodes of asset bubbles. Section 3 describes the setup of the model. Section 4 defines the equilibrium concepts and

[2] We assume that factor markets are competitive so that factor prices (i.e., the rental price of capital and wage rate) are determined by the marginal products of capital and labor.

[3] In an earlier study (Shi and Suen, 2014), we extend the deterministic model of Tirole (1985) to allow for an endogenous labor supply, and show that asset bubbles can potentially crowd in private investment. This study, however, does not consider the possibility of bubble crashes.

[4] In addition to an exogenous labor supply, these studies also assume that consumers (or investors) are risk neutral and only care about old-age consumption. Thus, the consumers will save all their income when young.

investigates the main properties of the model. Section 5 concludes.

2 Two Cases of Asset Bubbles in the U.S.

In this section, we use two recent cases of asset bubbles in the United States to demonstrate the pattern of comovement among asset prices, aggregate labor hours and private investment. The first case study is the "dot-com bubble" which is formed during the second half of the 1990s.[5] The second case is the housing price bubble in the 2000s.[6] Unless otherwise stated, all the data reported below are obtained from the Federal Reserve Economic Data (FRED) website.

Figure 1 shows the Dow Jones Industrial Average index during 1995-2003 and compares it to the aggregate weekly hours index in the Current Employment Survey (CES) data. Figure 2 compares the Dow Jones index to private nonresidential fixed investment (deflated by GDP deflator) over the period 1995Q1-2003Q4. These diagrams show that both aggregate labor hours and private investment have moved closely with stock prices during the "dotcom bubble" episode. Between 1995 and 2000, aggregate labor hours and real private nonresidential investment have recorded an average annual growth rate of 2.6 percent and 7.1 percent, respectively. Both figures are much higher than their long-term values.[7] Similar patterns can be observed during the housing price bubble episode. Figures 3 and 4 show the Case-Shiller 20-City Home Price Index over the period 2003-2010, and compare it to the same measures of aggregate labor hours and private investment. Between mid-2003 and mid-2006, aggregate labor hours and private investment have been growing at an average annual rate of 2.4 percent and 5.6 percent, respectively. These are again much higher than their long-term values.

3 The Model

3.1 The Environment

Time is discrete and is denoted by $t \in \{0, 1, 2, ...\}$. The economy under study is inhabited by an infinite sequence of overlapping generations. In each period, a new generation of identical consumers

[5] Both the Dow Jones index and the S&P 500 have tripled between January 1995 and January 2000; and collapsed shortly afterward. Ofek and Richardson (2002) and LeRoy (2004) provide detailed account on why this surge in stock prices cannot be explained by the growth in fundamentals (e.g., corporate earnings and dividends), and thus suggest the existence of an asset bubble.

[6] According to the Case-Shiller 20-City Home Price Index, housing prices in the U.S. have increased by 46 percent between June 2003 and June 2010. Shiller (2007) and many other studies argue that this surge represents a substantial deviation from the fundamentals (e.g., rent and construction costs).

[7] The average annual growth rate of the same labor hours index was 1.5 percent during 1963-2013. The average annual growth rate of real private nonresidential investment was 3.1 percent during 1943-2012.

is born. The size of generation t is given by $N_t = (1+n)^t$, with $n > 0$. Each consumer lives two periods, which we will refer to as the young age and the old age. In each period, each consumer has one unit of time which can be allocated between work and leisure. Retirement is mandatory in the old age, so the labor supply of old consumers is zero. Young consumers, on the other hand, can choose how much time to spend on work and how much to save and consume. There is a single commodity in this economy which can be used for consumption and capital accumulation. All prices are expressed in terms of this commodity.

Consider a consumer who is born in period $t \geq 0$. Let $c_{y,t}$, $c_{o,t+1}$ and l_t denote, respectively, his young-age consumption, old-age consumption and labor supply when young. The consumer's expected lifetime utility is given by

$$E_t \left[\frac{c_{y,t}^{1-\sigma}}{1-\sigma} - A \frac{l_t^{1+\psi}}{1+\psi} + \beta \frac{c_{o,t+1}^{1-\sigma}}{1-\sigma} \right], \tag{1}$$

where $\sigma > 0$ is the coefficient of relative risk aversion and the reciprocal of the elasticity of intertemporal substitution (EIS) for consumption, $\psi \geq 0$ is the reciprocal of the Frisch elasticity of labor supply, $\beta \in (0,1)$ is the subjective discount factor and A is a positive constant.[8] The consumer can invest in two types of assets: physical capital and an intrinsically worthless asset. The latter is called "intrinsically worthless" because it has no consumption value and cannot be used in the production of goods. The only motivation for holding this asset is to resell it at a higher price in the next period. The total supply of the intrinsically worthless asset is fixed and is denoted by $M > 0$.[9]

Let $\widetilde{p}_{t+1} \geq 0$ be the price of the intrinsically worthless asset in period $t+1$, which is unknown in period t. Since the fundamental value of this asset is zero, any strictly positive price will be referred to as an asset bubble. Following Weil (1987), we assume that $\widetilde{p}_{t+1}$ can be separated into a random component ε_{t+1} and a deterministic component p_{t+1} according to $\widetilde{p}_{t+1} \equiv \varepsilon_{t+1} p_{t+1}$. The random component, or asset price shock, is exogenous and follows a Markov chain with two possible states $\{0, 1\}$; transition probabilities

$$\Pr\{\varepsilon_{t+1} = 1 | \varepsilon_t = 1\} = q \in (0,1),$$

$$\Pr\{\varepsilon_{t+1} = 0 | \varepsilon_t = 0\} = 1;$$

[8] All young consumers will supply one unit of labor inelastically if $A = 0$. In this case, our model is identical to the production economy in Weil (1987).

[9] In period 0, all assets are owned by a group of "initial-old" consumers. The decisions of these consumers are trivial and do not play any role in the following analysis.

and initial value $\varepsilon_0 = 1$. The asset price shock is the only source of uncertainty in this economy. On the other hand, the time path of the deterministic component, $\{p_t\}_{t=0}^{\infty}$, is endogenously determined in equilibrium. At the beginning of each period t, the value of ε_t is revealed and publicly observed. Suppose $\varepsilon_t = 1$ and $p_t > 0$ so that an asset bubble exists in period t. Then, with probability q, the price of the intrinsically worthless asset will remain on the deterministic time path in the next period (i.e., $\widetilde{p}_{t+1} = p_{t+1}$); and with probability $(1-q)$, it will drop to zero. One can think of the latter scenario as the result of a sudden, unanticipated change in market sentiment which triggers a crash in the financial market. The parameter q can be interpreted as the persistence of asset bubbles. Since the probability of moving from state $\varepsilon = 1$ to state $\varepsilon = 0$ is strictly positive, every asset bubble will eventually crash (in other words, $\widetilde{p}_t$ will converge in probability to zero as t tends to infinity). The timing of the crash, however, is uncertain. Figure 5 shows the probability tree diagram for the asset price shock. The dark line in the diagram traces the time path of ε_t before the crash. We will refer to this as the *pre-crash economy* and the other parts of the diagram as the *post-crash economy.* Once the crash state is reached, $\widetilde{p}_t$ will remain zero forever. Hence, there is no incentive to hold the intrinsically worthless asset in the post-crash economy.

3.2 Consumer's Problem

We now analyze the consumer's problem both before and after the crash. To distinguish between these two states of the world, all variables in the post-crash economy will be indicated by a hat ($\hat{}$). First, consider the consumer's problem in the post-crash economy, which is deterministic. Specifically, this is given by

$$\max_{\widehat{c}_{y,t},\widehat{s}_t,\widehat{l}_t,\widehat{c}_{o,t+1}} \left[\frac{\widehat{c}_{y,t}^{1-\sigma}}{1-\sigma} - A\frac{\widehat{l}_t^{1+\psi}}{1+\psi} + \beta\frac{\widehat{c}_{o,t+1}^{1-\sigma}}{1-\sigma} \right]$$

subject to the budget constraints:

$$\widehat{c}_{y,t} + \widehat{s}_t = \widehat{w}_t\widehat{l}_t, \qquad \text{and} \qquad \widehat{c}_{o,t+1} = \widehat{R}_{t+1}\widehat{s}_t,$$

where $\widehat{s}_t$ denotes savings in physical capital, $\widehat{w}_t$ is the market wage rate, and $\widehat{R}_{t+1}$ is the gross return from savings between periods t and $t+1$. The solution of this problem is given by

$$\widehat{c}_{y,t} = \left(\beta\widehat{R}_{t+1}\right)^{-\frac{1}{\sigma}}\widehat{c}_{o,t+1} = \frac{\widehat{w}_t\widehat{l}_t}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}}, \tag{2}$$

$$\widehat{l}_t = A^{-\frac{1}{\sigma+\psi}} \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \widehat{w}_t^{\frac{1-\sigma}{\sigma+\psi}}, \tag{3}$$

$$\widehat{s}_t = \Sigma \left(\widehat{R}_{t+1} \right) \widehat{w}_t \widehat{l}_t, \qquad \text{where } \Sigma \left(\widehat{R}_{t+1} \right) \equiv \frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}}. \tag{4}$$

The function $\Sigma : \mathbb{R}_+ \to [0, 1]$ defined in (4) summarizes two effects of interest rate on savings. First, holding other things constant, a higher interest rate will bring more interest income in the old age. This creates an income effect which encourages young-age consumption and discourages saving. Second, a higher interest rate will make old-age consumption cheaper relative to young-age consumption. This creates an intertemporal substitution effect which promotes saving. The latter effect dominates if and only if $\sigma < 1$. In this case, $\Sigma(\cdot)$ is a strictly increasing function. The two effects exactly cancel out when $\sigma = 1$. In this case, $\Sigma(\cdot)$ is a positive constant. The consumer's propensity to consume in the post-crash economy is given by

$$\frac{\widehat{c}_{y,t}}{\widehat{w}_t \widehat{l}_t} = 1 - \Sigma \left(\widehat{R}_{t+1} \right) = \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{-1}. \tag{5}$$

Next, consider the consumer's problem in the post-crash economy. Let m_t be the consumer's demand for the intrinsically worthless asset in period t. A young consumer now faces the following budget constraint

$$c_{y,t} + s_t + p_t m_t = w_t l_t. \tag{6}$$

Except in some special cases (which we will discuss below), the gross return from physical capital between periods t and $t+1$ will depend on the realization of ε_{t+1} and is thus uncertain in period t. Let R_{t+1} and $\widehat{R}_{t+1}$ denote, respectively, the gross return when $\varepsilon_{t+1} = 1$ and $\varepsilon_{t+1} = 0$. The consumer's old-age consumption is then given by

$$c_{o,t+1} = \begin{cases} R_{t+1} s_t + p_{t+1} m_t & \text{with probability } q, \\ \widehat{R}_{t+1} s_t & \text{with probability } 1-q. \end{cases} \tag{7}$$

Taking $\left\{ w_t, p_t, p_{t+1}, R_{t+1}, \widehat{R}_{t+1} \right\}$ as given, the consumer's problem is to choose an allocation $\{c_{y,t}, s_t, l_t, m_t, c_{o,t+1}\}$ so as to maximize his expected lifetime utility in (1), subject to the budget constraints in (6) and (7), and the non-negativity constraint: $m_t \geq 0$.[10] The Euler equation for

[10] Given a constant-relative-risk-aversion (CRRA) utility function, it is never optimal for the consumer to choose $c_{y,t} = 0$ or $c_{o,t+1} = 0$, regardless of the state of the asset bubble. Hence, the non-negativity constraint for these variables is never binding. It is also never optimal to have $s_t \leq 0$ and $l_t = 0$. Suppose the contrary that $s_t \leq 0$, then the consumer will end up having $c_{o,t+1} \leq 0$ when $\varepsilon_{t+1} = 0$, which cannot be optimal. This, together with $m_t \geq 0$,

consumption and the optimality condition for labor supply are given by

$$c_{y,t}^{-\sigma} = \beta \left[qR_{t+1} \left(R_{t+1} s_t + p_{t+1} m_t \right)^{-\sigma} + (1-q) \widehat{R}_{t+1} \left(\widehat{R}_{t+1} s_t \right)^{-\sigma} \right], \tag{8}$$

$$w_t c_{y,t}^{-\sigma} = A l_t^{\psi}. \tag{9}$$

The optimal choice of m_t is determined by

$$p_t c_{y,t}^{-\sigma} \geq \beta E_t \left[\widetilde{p}_{t+1} \left(c_{o,t+1} \right)^{-\sigma} \right] = \beta q p_{t+1} \left(R_{t+1} s_t + p_{t+1} m_t \right)^{-\sigma}, \tag{10}$$

with equality holds in the first part if $m_t > 0$. Equation (10) states it is optimal to choose $m_t = 0$ if the marginal cost of holding this asset (which is $p_t c_{y,t}^{-\sigma}$) exceeds the marginal benefit (which is $\beta E_t \left[\widetilde{p}_{t+1} \left(c_{o,t+1} \right)^{-\sigma} \right]$). This equation can be rewritten as

$$p_t \geq E_t \left[\beta \left(\frac{c_{o,t+1}}{c_{y,t}} \right)^{-\sigma} \widetilde{p}_{t+1} \right],$$

which is the standard consumption-based asset pricing equation.

We now explore the conditions under which the optimal choice of m_t is strictly positive. Consider a young consumer who initially chooses $m_t = 0$. Suppose now he is considering increasing it to $\xi/p_t > 0$, where $\xi > 0$ is infinitesimal. In order to balance his budget, the consumer will simultaneously reduce s_t by ξ. Define $\pi_{t+1} \equiv p_{t+1}/p_t$ as the gross return from the intrinsically worthless asset when $\varepsilon_{t+1} = 1$. Increasing m_t from zero to ξ/p_t will generate an expected return of $q\pi_{t+1}\xi$, which will in turn increase expected future utility by $q\pi_{t+1} \left(R_{t+1} s_t \right)^{-\sigma} \xi$. At the same time, the reduction in s_t will lower expected future utility by

$$\left[qR_{t+1} \left(R_{t+1} s_t \right)^{-\sigma} + (1-q) \widehat{R}_{t+1} \left(\widehat{R}_{t+1} s_t \right)^{-\sigma} \right] \xi. \tag{11}$$

Such an increase in m_t is desirable if and only if

$$q\pi_{t+1} \left(R_{t+1} s_t \right)^{-\sigma} \xi > \left[qR_{t+1} \left(R_{t+1} s_t \right)^{-\sigma} + (1-q) \widehat{R}_{t+1} \left(\widehat{R}_{t+1} s_t \right)^{-\sigma} \right] \xi,$$

means that consumers will never borrow. Finally, since labor income is the only source of lifetime income, it is never optimal to choose $l_t = 0$.

which can be simplified to become

$$q\pi_{t+1} > \left[q + (1-q)\left(\frac{\widehat{R}_{t+1}}{R_{t+1}}\right)^{1-\sigma}\right] R_{t+1}. \tag{12}$$

Equation (12) states that the consumer is willing to hold the intrinsically worthless asset if and only if its expected return $q\pi_{t+1}$ exceeds a certain threshold. This threshold level is determined by three factors: (i) the persistence of asset bubble q; (ii) the state-dependent returns from physical capital R_{t+1} and $\widehat{R}_{t+1}$; and (iii) the preference parameter σ. If the gross return from physical capital is *not* state-dependent, i.e., $R_{t+1} = \widehat{R}_{t+1}$, then the above condition becomes $q\pi_{t+1} > R_{t+1}$. If the utility function for consumption is logarithmic, i.e., $\sigma = 1$, then the expression in (11) can be simplified to $s_t^{-1}\xi$. In this case, both the marginal benefit and the marginal cost of increasing m_t are independent of $\widehat{R}_{t+1}$, and the condition in (12) will again be simplified to become $q\pi_{t+1} > R_{t+1}$.

Suppose the condition in (12) is valid. Then the optimal investment in the intrinsically worthless asset, denoted by $a_t \equiv p_t m_t$, is given by

$$a_t \equiv p_t m_t = \frac{p_t}{p_{t+1}}\left(\Omega_{t+1}\widehat{R}_{t+1} - R_{t+1}\right) s_t, \tag{13}$$

where

$$\Omega_{t+1} \equiv \left[\frac{q\left(\pi_{t+1} - R_{t+1}\right)}{(1-q)\,\widehat{R}_{t+1}}\right]^{\frac{1}{\sigma}}. \tag{14}$$

It is straightforward to show that $\Omega_{t+1}\widehat{R}_{t+1} > R_{t+1}$ is equivalent to (12). The consumer's propensity to consume in the pre-crash economy is given by

$$\frac{c_{y,t}}{w_t l_t} = \left\{1 + \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1}\widehat{R}_{t+1}}\left[1 + \frac{p_t}{p_{t+1}}\left(\Omega_{t+1}\widehat{R}_{t+1} - R_{t+1}\right)\right]\right\}^{-1}. \tag{15}$$

The formal derivation of (15) is shown in the Appendix.

3.3 Production

On the supply side of the economy, there is a large number of identical firms. In each period t, each firm hires labor (L_t) and physical capital (K_t) from the competitive factor markets, and produces output (Y_t) according to a Cobb-Douglas production function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}, \qquad \text{with } \alpha \in (0,1).$$

Since the production function exhibits constant returns to scale, we can focus on the problem faced by a single representative firm. We assume that physical capital is fully depreciated after one period, so that R_t coincides with the rental price of physical capital at time t. The representative firm's problem is given by

$$\max_{K_t, L_t} \left\{ K_t^{\alpha} L_t^{1-\alpha} - R_t K_t - w_t L_t \right\},$$

and the first-order conditions are

$$R_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha} \qquad \text{and} \qquad w_t = (1-\alpha) K_t^{\alpha} L_t^{-\alpha}. \tag{16}$$

Since the firm's problem is not directly affected by the asset price shock, the above equations are valid both before and after the asset bubble crashes.

4 Equilibrium

In this section, we will define and characterize an equilibrium in which the intrinsically worthless asset is valued at some point, i.e., $\widetilde{p}_t > 0$ for some t. We will refer to this type of equilibrium as a *bubbly equilibrium.* Such an equilibrium will have to take into account the stochastic timing of the crash, as well as the interactions between the pre-crash and post-crash economies. Firstly, given the timing of events, the equilibrium allocations in the pre-crash economy will determine the initial state of the post-crash economy. Secondly, when the consumers are making their decisions before the crash, say in some period t, their anticipated value of $\widehat{R}_{t+1}$ will have to be consistent with a post-crash equilibrium in the following period. Thus, the equilibrium quantities and prices in the post-crash economy will also affect the equilibrium outcomes before the crash.[11]

4.1 Post-crash Equilibrium

We begin by fully characterizing the equilibrium of the post-crash economy. Suppose the crash happens in some period $T > 0$, i.e., $\varepsilon_{T-1} = 1$ and $\varepsilon_T = 0$. Then the economy is free of asset bubbles from period T onward. Given an initial value $\widehat{K}_T > 0$, a post-crash equilibrium is made up of sequences of allocation $\left\{ \widehat{c}_{y,t}, \widehat{s}_t, \widehat{l}_t, \widehat{c}_{o,t} \right\}_{t=T}^{\infty}$, aggregate inputs $\left\{ \widehat{K}_t, \widehat{L}_t \right\}_{t=T}^{\infty}$, and prices $\left\{ \widehat{w}_t, \widehat{R}_t \right\}_{t=T}^{\infty}$ such that for all $t \geq T$, (i) the allocation $\left\{ \widehat{c}_{y,t}, \widehat{s}_t, \widehat{l}_t, \widehat{c}_{o,t+1} \right\}$ solves the consumer's problem in period

[11] The second type of interaction is absent from Weil's (1987) model.

t given $\widehat{w}_t$ and $\widehat{R}_{t+1}$; (ii) old-age consumption in period T is determined by

$$N_{T-1}\widehat{c}_{o,T} = \widehat{R}_T \widehat{K}_T;$$

(iii) the aggregate inputs $\left\{\widehat{K}_t, \widehat{L}_t\right\}$ solve the representative firm's problem in period t given $\widehat{w}_t$ and $\widehat{R}_t$; and (iv) all markets clear in every period, i.e., $\widehat{L}_t = N_t \widehat{l}_t$ and $\widehat{K}_{t+1} = N_t \widehat{s}_t$ for all t.

Define $\widehat{k}_t \equiv \widehat{K}_t / N_t$. Then the equilibrium dynamics of $\widehat{k}_t$ and $\widehat{R}_t$ are determined by[12]

$$\widehat{k}_{t+1} = \frac{1-\alpha}{\alpha(1+n)} \left[\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_t \widehat{k}_t, \tag{17}$$

$$\widehat{R}_t^{\eta} \widehat{k}_t = \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}, \tag{18}$$

where $\eta \equiv \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha}\frac{1-\sigma}{\sigma+\psi} > 0$. The initial value $\widehat{k}_T = \widehat{K}_T / N_T$ is predetermined in the pre-crash economy. Once the equilibrium time path of $\widehat{k}_t$ and $\widehat{R}_t$ are known, all other variables in the post-crash equilibrium can be uniquely determined.

For any $\sigma > 0$, the dynamical system in (17)-(18) has a unique steady state, which we will refer to as the *post-crash steady state*. This result is formally stated in Proposition 1. All proofs can be found in the Appendix.

Proposition 1 *A unique post-crash steady state exists for any* $\sigma > 0$. *The steady-state values* $\left(\widehat{R}^*, \widehat{k}^*\right)$ *are determined by*

$$\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}^*\right)^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}} = \frac{(1+n)\alpha}{1-\alpha}, \tag{19}$$

$$\widehat{k}^* = (1-\alpha)^{\frac{1-\sigma}{\sigma+\psi}} A^{-\frac{1}{\sigma+\psi}} \left[1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \left(\frac{\alpha}{\widehat{R}^*} \right)^{\eta}. \tag{20}$$

Next, we consider the stability of the post-crash steady state. If the EIS is no less than one, i.e., $\sigma \leq 1$, then this steady state is globally saddle-path stable. This means starting from any initial value $\widehat{k}_T > 0$ there exists a unique set of time paths $\left\{\widehat{k}_t, \widehat{R}_{t+1}\right\}_{t=T}^{\infty}$ that solves (17)-(18) and converges to the post-crash steady state. In addition, if $\widehat{k}_T$ is greater (or less) than the steady-state value $\widehat{k}^*$, then $\widehat{k}_t$ will decline (or increase) monotonically during the transition and $\widehat{R}_t$ will rise (or fall) monotonically towards $\widehat{R}^*$. These results are formally stated in Proposition 2.

[12]The derivation of these equations and further details of the post-crash economy can be found in an online appendix available on the author's website.

Proposition 2 *Suppose $\sigma \leq 1$. Then for any initial value $\widehat{k}_T > 0$, there exists a unique post-crash equilibrium with $\left\{\widehat{k}_t, \widehat{R}_{t+1}\right\}_{t=T}^{\infty}$ that converges monotonically to the post-crash steady state. In particular, the value of $\widehat{R}_T$ is uniquely determined by $\widehat{R}_T = \Phi\left(\widehat{k}_T\right)$, where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly decreasing function. In the transitional dynamics, $\widehat{R}_t$ and $\widehat{k}_t$ will move in opposite directions so that $\left(\widehat{k}_t - \widehat{k}^*\right)\left(\widehat{R}_t - \widehat{R}^*\right) \leq 0$ for all $t \geq T$.*

When $\sigma > 1$, the post-crash steady state can be either a sink or a saddle. A sink means that there are multiple equilibrium time paths that originate from the same initial value $\widehat{k}_T > 0$ and converge to the unique post-crash steady state. In other words, local indeterminacy may occur when $\sigma > 1$. In this study, we confine our attention to equilibrium time paths that can be uniquely determined. Hence, we focus on the case of $\sigma \leq 1$. Intuitively, $\sigma \leq 1$ means that the intertemporal substitution effect of a higher interest rate is no weaker than the income effect. This assumption is not uncommon in OLG models. For instance, Galor and Ryder (1989) show that this assumption plays an important role in establishing the existence, uniqueness and global stability of stationary equilibrium in the absence of labor-leisure choice. Fuster (1999) uses this assumption to establish the existence and uniqueness of non-stationary equilibrium in a model with uncertain lifetime and accidental bequest. More recently, Andersen and Bhattacharya (2013) adopt the same assumption to analyze the welfare implications of unfunded pensions in an OLG model with endogenous labor supply. In the rational bubble literature, Weil (1987, Section 2) focuses on equilibria in which the interest elasticity of savings is non-negative. For CRRA utility functions, this elasticity is nonnegative if and only if $\sigma \leq 1$.

4.2 Bubbly Equilibrium

We are now ready to state the complete definition of a bubbly equilibrium. Given the initial conditions, $K_0 > 0$ and $\varepsilon_0 = 1$, a bubbly equilibrium consists of two sets of allocations, prices and aggregate inputs, $\{c_{y,t}, c_{o,t}, l_t, s_t, m_t, R_t, w_t, p_t, K_t, L_t\}_{t=0}^{\infty}$ and $\left\{\widehat{c}_{y,t}, \widehat{c}_{o,t}, \widehat{l}_t, \widehat{s}_t, \widehat{R}_t, \widehat{w}_t, \widehat{K}_t, \widehat{L}_t\right\}_{t=0}^{\infty}$, that satisfy the following conditions in every period $t \geq 0$.

1. If $\varepsilon_t = 0$, then $\left\{\widehat{c}_{y,\tau}, \widehat{c}_{o,\tau}, \widehat{l}_\tau, \widehat{s}_\tau, \widehat{R}_\tau, \widehat{w}_\tau, \widehat{K}_\tau, \widehat{L}_\tau\right\}_{\tau=t}^{\infty}$ form a post-crash equilibrium.
2. If $\varepsilon_t = 1$, then
   (i) given $\left\{w_t, p_t, p_{t+1}, R_{t+1}, \widehat{R}_{t+1}\right\}$, the allocation $\{c_{y,t}, c_{o,t+1}, l_t, s_t, m_t\}$ solves the consumer's problem in period t, i.e., (6)-(10) are satisfied;
   (ii) given R_t and w_t, the aggregate inputs K_t and L_t solve the firm's problem in period t;

(iii) all markets clear in every period, i.e., $L_t = N_t l_t$, $K_{t+1} = N_t s_t$ and $N_t m_t = M$ for all t;

(iv) if $\varepsilon_{t+1} = 0$, then $\widehat{K}_{t+1} = K_{t+1}$.

The last condition states that if the crash happens in period $t+1$, then K_{t+1} will provide the initial condition for the post-crash equilibrium.

Before proceeding further, we first highlight the main difference between our model and the one in Weil (1987). In both models, the stock of aggregate capital is predetermined in the previous period. Thus, K_{t+1} is contingent on ε_t but not on ε_{t+1}. In the production economy of Weil (1987), every young consumer supplies one unit of labor inelastically regardless of the state of the asset bubble; hence $L_{t+1} = \widehat{L}_{t+1} = N_{t+1}$, for all t. Since neither K_{t+1} nor L_{t+1} depend on ε_{t+1}, a bubble crash in period $t+1$ will have no immediate impact on aggregate output and factor prices. In particular, the gross return from physical capital is never affected by the realization of the asset price shock, so that $R_{t+1} = \widehat{R}_{t+1}$ for all t. Thus, the stochastic bubble does not generate any aggregate uncertainty in Weil's model. Differently, in our model, the equilibrium quantity of L_{t+1} is endogenously determined by individuals' labor supply decisions. If the optimal choice of l_{t+1} is contingent on ε_{t+1}, then asset price fluctuations will affect the aggregate economy through the labor market. Our next proposition shows that this mechanism is operative only if $\sigma \neq 1$.

Proposition 3 *Suppose the utility function for consumption is logarithmic, i.e., $\sigma = 1$. Then the optimal labor supply is constant over time and does not depend on the state of the asset bubble. Specifically,*

$$l_t = \widehat{l}_t = \left(\frac{1+\beta}{A}\right)^{\frac{1}{1+\psi}}, \qquad \textit{for all } t \geq 0.$$

This result holds because the income and substitution effects of wage rate on labor supply cancel out each other when $\sigma = 1$. As a result, individual labor supply is independent of current consumption and current wage rate. Without the labor-market channel, the asset price shock will not generate any aggregate uncertainty. Thus, our model is effectively the same as the production economy in Weil (1987) when $\sigma = 1$.

When $\sigma < 1$, the optimal choice of l_t is not a constant in general, and it will depend on the current state of the asset price shock. The rest of this paper is devoted to analyzing the effects of bubbles and crashes in this case. To simplify the analysis, suppose the economy is initially in a *pre-crash steady state*. Specifically, a pre-crash steady state is a stationary equilibrium in the pre-crash economy with the following features: (i) the market wage rate (w^*) and the expected return from the bubbly asset ($q\pi^*$) are identical in every period; and (ii) the state-contingent returns for physical capital are identical in every period. Let R^* be the return for physical captial if the asset bubble prevails in the next period and $\widehat{R}_0^*$ be the return if it crashes. These two conditions ensure that every cohort of young consumers in the pre-crash economy faces the same economic conditions and thus make the same choices. Formally, a pre-crash steady state consists of a set of values $\left\{c_y^*, c_o^*, l^*, s^*, a^*, R^*, \widehat{R}_0^*, w^*, \pi^*, k^*\right\}$ such that the following are true in the bubbly equilibrium defined earlier: if $\varepsilon_t = 1$, then $p_{t+1}/p_t = \pi^*$, $K_t = N_t k^*$, $L_t = N_t l^*$, $p_t m_t = a^* > 0$, and $(c_{y,t}, c_{o,t}, s_t, l_t, R_t, w_t) = \left(c_y^*, c_o^*, s^*, l^*, R^*, w^*\right)$. Once the asset bubble crashes, the post-crash economy will begin with initial conditions k^* and $\widehat{R}_0^* \equiv \Phi(k^*)$ and converge to the post-crash steady state $\left(\widehat{R}^*, \widehat{k}^*\right)$.

A pre-crash steady state can be characterized as follows: Using the market-clearing condition for the intrinsically worthless asset, i.e., $N_t m_t = M$, and the stationary conditions: $p_{t+1}/p_t = \pi^*$ and $p_t m_t = p_{t+1} m_{t+1} = a^*$, we can get

$$\frac{p_{t+1}}{p_t} = \pi^* = \frac{m_t}{m_{t+1}} = \frac{N_{t+1}}{N_t} = 1 + n.$$

Thus, before the crash happens, the price of the intrinsically worthless asset is growing deterministically at rate n. Given $\widehat{R}_0^* > 0$, the values $\{R^*, w^*, l^*, k^*, a^*\}$ are uniquely determined by

$$1 + \left[1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}}\right] \left(\frac{q}{1-q}\right)^{\frac{1}{\sigma}} \left(\frac{\widehat{R}_0^*}{1+n}\right)^{1-\frac{1}{\sigma}} \left(1 - \frac{R^*}{1+n}\right)^{\frac{1}{\sigma}} = \frac{1}{\alpha} \frac{R^*}{1+n}, \tag{21}$$

$$w^* = (1-\alpha) \left(\frac{\alpha}{R^*}\right)^{\frac{\alpha}{1-\alpha}}, \tag{22}$$

$$A (l^*)^{\psi+\sigma} = \beta q \left[(1+n) w^*\right]^{1-\sigma} \left[\frac{(1-\alpha) R^*}{\alpha \Omega^* \widehat{R}_0^*}\right]^{\sigma}, \tag{23}$$

$$k^* = l^* \left(\frac{\alpha}{R^*}\right)^{\frac{1}{1-\alpha}}, \tag{24}$$

$$a^* = \left(\Omega^* \widehat{R}_0^* - R^*\right) k^*, \tag{25}$$

where

$$\Omega^* \equiv \left[\frac{q(1+n-R^*)}{(1-q)\widehat{R}_0^*}\right]^{\frac{1}{\sigma}}.$$

A detailed derivation of these equations can be found in the Appendix. Once these values are known, the remaining variables $\left\{c_y^*, c_o^*, s^*\right\}$ can be uniquely determined from the consumer's budget constraints. Equations (21)-(24) implicitly define a one-to-one mapping between $\widehat{R}_0^*$ and k^*, which we will denote by $k^* = \Gamma\left(\widehat{R}_0^*\right)$. This, together with the mapping $\widehat{R}_0^* = \Phi(k^*)$ mentioned in Proposition 2, can be used to determine the value of $\widehat{R}_0^*$ and k^*.

4.3 Expansionary Effect of Asset Bubbles

We now turn to the main subject of this paper, which is the potential expansionary effect of asset bubbles. Specifically, we want to identify the conditions under which the pre-crash steady state has a higher level of labor supply and capital-labor ratio than the post-crash steady state, i.e., $l^* > \widehat{l}^*$ and $k^* > \widehat{k}^*$.[13] We begin by stating an intermediate result.

Proposition 4 *Suppose $\sigma < 1$. Then the existence of asset bubble is associated with a higher level of steady-state interest rate, i.e., $R^* > \widehat{R}^*$.*

The above result can be attributed to two factors. Firstly, since aggregate uncertainty exists before the crash happens, consumers will demand a higher return from savings in the pre-crash steady state. Secondly, even in the absence of uncertainty, the existence of asset bubble tends to lower the capital-labor ratio and drives up the steady-state interest rate.[14]

Using (24), which is valid in both the pre-crash and post-crash economies, we can get

$$k^* = l^* \left(\frac{\alpha}{R^*}\right)^{\frac{1}{1-\alpha}} > \widehat{l}^* \left(\frac{\alpha}{\widehat{R}^*}\right)^{\frac{1}{1-\alpha}} = \widehat{k}^* \qquad \Leftrightarrow \qquad \frac{l^*}{\widehat{l}^*} > \left(\frac{R^*}{\widehat{R}^*}\right)^{\frac{1}{1-\alpha}} > 1. \tag{26}$$

This shows that asset bubbles can potentially crowd in productive investment, but this happens only if there is a sufficiently large expansion in labor supply among the young consumers. In both economies, optimal labor supply is determined by equation (9), which can be restated as

$$Al_t^{\psi+\sigma} = w_t^{1-\sigma} \left(\frac{c_{y,t}}{w_t l_t}\right)^{-\sigma}. \tag{27}$$

Equation (27) shows that individual labor supply is jointly determined by the current wage rate and the propensity to consume when young. Holding the propensity to consume constant, individual labor supply is an increasing function in wage rate when $\sigma < 1$. Since $R^* > \widehat{R}^*$ implies $w^* < \widehat{w}^*$, this wage-rate effect alone will lower the supply of labor in the pre-crash steady state. Thus, $l^* > \widehat{l}^*$

[13] Note that $k^* > \widehat{k}^*$ also means that the post-crash economy will start with a higher capital-labor ratio than its steady-state value. Thus, by the results in Proposition 2, $\widehat{k}_t$ is strictly decreasing towards $\widehat{k}^*$ in the transition dynamics.

[14] See Shi and Suen (2014) Proposition 2 for a proof of this statement. Other rational bubble models, such as Tirole (1985), Weil (1987), Olivier (2000), and Farhi and Tirole (2012), also predict a higher long-run interest rate in the presence of asset bubble.

is possible *only if* the consumers have a lower propensity to consume before the crash, i.e.,

$$\frac{\widehat{c}_y^*}{\widehat{w}^*\widehat{l}^*} > \frac{c_y^*}{w^* l^*}.$$

Using (5) and (15), one can express these propensities in terms of $\widehat{R}^*$, R^*, $\widehat{R}^*$ and $\pi^* = 1 + n$.

To summarize, asset bubbles can potentially crowd in productive investment in our model, but this happens only if these bubbles can induce the young consumers to consume less and work more. This is more likely to happen when the EIS for consumption (i.e., $1/\sigma$) and the Frisch elasticity of labor supply (i.e., $1/\psi$) are large. The exact conditions for $l^* > \widehat{l}^*$ and $k^* > \widehat{k}^*$, expressed in terms of R^*, $\widehat{R}_0^*$ and $\widehat{R}^*$, are shown in Proposition 5.

Proposition 5 *Suppose* $\sigma < 1$. *Then* $l^* > \widehat{l}^*$ *if and only if*

$$\left[\frac{q(1+n)}{\widehat{R}^*}\right]^{\frac{1}{\sigma}} \left(\frac{R^*}{\widehat{R}^*}\right)^{-\frac{\alpha(1-\sigma)}{(1-\alpha)\sigma}} > \frac{\Omega^* \widehat{R}_0^*}{R^*},$$

and the asset bubble can crowd in productive investment, i.e., $k^* > \widehat{k}^*$, *if and only if*

$$\left[\frac{q(1+n)}{\widehat{R}^*}\right]^{\frac{1}{\sigma}} \left(\frac{R^*}{\widehat{R}^*}\right)^{-\left[1+\frac{\psi+\sigma}{(1-\alpha)\sigma}\right]} > \frac{\Omega^* \widehat{R}_0^*}{R^*}.$$

4.4 Numerical Examples

In this section we use some numerical examples to illustrate the effects of an asset bubble crash in our model. We mention at the outset that these examples are only intended to demonstrate the working of the model and the theoretical results in the previous section. Thus, some of the parameter values are specifically chosen so that asset bubbles can crowd in productive investment.

Suppose one model period takes 30 years. Set the annual subjective discount factor to 0.9950 and the annual employment growth rate to 1.6 percent.[15] These values imply $\beta = (0.9950)^{30} = 0.8604$ and $n = (1.0160)^{30} - 1 = 0.6099$. In addition, we set $q = 0.90$, $\alpha = 0.30$ and $\psi = 0$. Our choice of q and n implies that the expected return from the intrinsically worthless asset is $q(1+n) = 1.4490$. We consider four different values of σ between 0.10 and 0.30. For each value of σ, the parameter A is chosen so that $\widehat{l}^*$ is 0.50.[16] Using these parameter values, we solve for the equilibrium time paths under the following scenario: Suppose the economy starts from a pre-crash steady state at

[15] The latter is consistent with the average annual growth rate of U.S. employment over the period 1953-2008.

[16] Under the assumption of indivisible labor ($\psi = 0$), the variable l_t is more suitably interpreted as the labor force participation rate at time t. Thus, we choose a target value of $\widehat{l}^*$ based on the average labor force participation rate in the United States during the postwar period, which is about 0.50.

Table 1

Pre-crash and Post-crash Steady States

	$\sigma = 0.10$		$\sigma = 0.15$		$\sigma = 0.20$		$\sigma = 0.30$	
	Post-crash	Pre-crash	Post-crash	Pre-crash	Post-crash	Pre-crash	Post-crash	Pre-crash
R	1.2176	1.4671	1.2416	1.4548	1.2637	1.4485	1.3036	1.4434
c_y	0.0832	0.0374	0.0846	0.0538	0.0858	0.0640	0.0878	0.0758
l	0.5000	0.7306	0.5000	0.5862	0.5000	0.5416	0.5000	0.5132
k	0.0676	0.0757	0.0657	0.0614	0.0641	0.0571	0.0613	0.0544
y	0.2743	0.3701	0.2720	0.2980	0.2700	0.2758	0.2664	0.2617
a	0	0.0998	0	0.0559	0	0.0371	0	0.0198

Note: The notation y denotes per-worker output, i.e., $y = k^{\alpha} l^{1-\alpha}$.

time $t = 0$, and suppose the bubble bursts unexpectedly at time $t = 3$.[17] The economy then converges to the unique post-crash steady state. The transition dynamics in the post-crash economy is computed using backward shooting method.

Table 1 shows the key variables in the pre-crash and post-crash steady states under different values of σ. The first row reports the value of $\widehat{R}^*$ and R^*. In all four cases, the return from physical capital is higher in the pre-crash steady state than in the post-crash steady state, which is consistent with the prediction of Proposition 4. In all the reported cases, we have $l^* > \widehat{l}^*$ which means labor supply is higher before the crash. In particular, the gap between l^* and $\widehat{l}^*$ widens as σ decreases. This captures an increasingly stronger intertemporal substitution effect which induce the young consumers to consume less and work more. When $\sigma = 0.1$, the difference between l^* and $\widehat{l}^*$ is sufficiently large so that asset bubble can also crowd in productive investment (i.e., $k^* > \widehat{k}^*$).

Figures 6-8 show the time path of interest rate (R), labor supply (l) and capital-labor ratio (k) before and after the crash. In all four cases, the crash induces an immediate reduction in interest rate and labor supply. During the transition in the post-crash economy, $\widehat{R}_t$ and $\widehat{k}_t$ move in opposite directions as predicted by Proposition 2. In the more interesting case where asset bubble crowds in physical capital (i.e., $\sigma = 0.1$), labor supply and productive investment fall markedly at the time of the crash and continue to decline afterward.

[17] In other words, we consider a particular sequence of asset price shocks in which $\varepsilon_t = 1$ for $t \in \{0, 1, 2\}$ and $\varepsilon_t = 0$ for $t \geq 3$. As explained earlier, the non-stationary bubbleless equilibrium will always begin with the same initial values k^* and $\widehat{R}_0^*$ regardless of the timing of the crash. Thus, the exact time period when the crash happens is immaterial.

5 Concluding Remarks

This paper contributes to the stochastic bubble literature by demonstrating the importance of endogenous labor supply and intertemporal substitution in understanding the effects of asset price bubbles and crashes. In particular, we show that stochastic bubbles can crowd in productive investment and promote aggregate employment when the intertemporal substitution effect is sufficiently strong. We remark that the present study is mainly theoretical in nature and more effort is needed in order to generate realistic quantitative results. In particular, expanding the consumer's planning horizon (and thus reducing the length of each model period) is crucial for matching the model to the data. Introducing other model features, such as financial market imperfections and heterogeneity in firm productivity as in Martin and Ventura (2012) and Farhi and Tirole (2012), may also help expand the range of parameter values under which asset bubbles can crowd in productive investment. We leave these possibilities for future research.

Appendix

Derivation of Equation (15)

Consider the consumer's problem in the pre-crash economy. The first-order conditions for an interior solution of (s_t, m_t, l_t) are given by

$$(w_t l_t - s_t - p_t m_t)^{-\sigma} = \beta \left[q R_{t+1} \left(R_{t+1} s_t + p_{t+1} m_t \right)^{-\sigma} + (1-q) \widehat{R}_{t+1} \left(\widehat{R}_{t+1} s_t \right)^{-\sigma} \right], \tag{28}$$

$$(w_t l_t - s_t - p_t m_t)^{-\sigma} = \beta q \left(\frac{p_{t+1}}{p_t} \right) \left(R_{t+1} s_t + p_{t+1} m_t \right)^{-\sigma}, \tag{29}$$

$$A l_t^{\psi} = w_t \left(w_t l_t - s_t - p_t m_t \right)^{-\sigma}. \tag{30}$$

Define $\pi_{t+1} \equiv p_{t+1}/p_t$. Combining (28) and (29), and rearranging terms gives

$$R_{t+1} s_t + p_{t+1} m_t = \underbrace{\left[\frac{q \left(\pi_{t+1} - R_{t+1} \right)}{(1-q) \widehat{R}_{t+1}} \right]^{\frac{1}{\sigma}}}_{\Omega_{t+1}} \left(\widehat{R}_{t+1} s_t \right), \tag{31}$$

which implies

$$m_t = \frac{1}{p_{t+1}} \left(\Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) s_t,$$

$$s_t + p_t m_t = \left[1 + \frac{p_t}{p_{t+1}} \left(\Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) \right] s_t. \tag{32}$$

Using (29), (31) and (32), we can get

$$s_t = \left\{ \frac{\left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + \left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}} \left[1 + \frac{p_t}{p_{t+1}} \left(\Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) \right]} \right\} w_t l_t, \tag{33}$$

$$c_{y,t} = w_t l_t - (s_t + p_t m_t) = \left\{ \frac{\Omega_{t+1} \widehat{R}_{t+1}}{\Omega_{t+1} \widehat{R}_{t+1} + \left(\beta q \pi_{t+1} \right)^{\frac{1}{\sigma}} \left[1 + \frac{p_t}{p_{t+1}} \left(\Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) \right]} \right\} w_t l_t. \tag{34}$$

Equation (15) can be obtained by simplifying the last equation.

Proof of Proposition 1

In any post-crash steady state, we have $\widehat{k}_{t+1} = \widehat{k}_t = \widehat{k}^*$ and $\widehat{R}_{t+1} = \widehat{R}_t = \widehat{R}^*$ for all t. Substituting these into (4) and rearranging terms gives

$$\Theta\left(\widehat{R}^*\right) \equiv \frac{\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}-1}} = \frac{(1+n)\,\alpha}{1-\alpha}. \tag{35}$$

Substituting the same steady state conditions into (18) and rearranging terms gives (20). Note that the function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ defined in (35) is continuously differentiable and satisfies $\Theta(0) = 0$. Straightforward differentiation gives

$$\Theta'\left(\widehat{R}\right) = \frac{\beta^{\frac{1}{\sigma}}\widehat{R}^{\frac{1}{\sigma}-1}\left(\frac{1}{\sigma}+\beta^{\frac{1}{\sigma}}\widehat{R}^{\frac{1}{\sigma}-1}\right)}{\left(1+\beta^{\frac{1}{\sigma}}\widehat{R}^{\frac{1}{\sigma}-1}\right)^2} > 0, \qquad \text{for any } \sigma > 0.$$

Hence, there exists a unique value of $\widehat{R}^* > 0$ that solves (35). Using (20), one can obtain a unique value of $\widehat{k}^* > 0$. This proves Proposition 1.

Proof of Proposition 2

First, consider the case when $\sigma = 1$. Equations (17) and (18) now become

$$\widehat{k}_{t+1} = \frac{1-\alpha}{\alpha(1+n)}\left(\frac{\beta}{1+\beta}\right)\widehat{R}_t\widehat{k}_t, \qquad \text{and} \qquad \widehat{R}_t^{\frac{1}{1-\alpha}}\widehat{k}_t = \alpha^{\frac{1}{1-\alpha}}\left(\frac{1+\beta}{A}\right)^{\frac{1}{1+\psi}}. \tag{36}$$

Combining the two gives

$$\widehat{k}_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\left(\frac{1+\beta}{A}\right)^{\frac{1-\alpha}{1+\psi}}\widehat{k}_t^{\alpha}.$$

Since $\alpha \in (0,1)$, there exists a unique non-trivial steady state $\widehat{k}^* > 0$ which is globally stable. The second equation in (36) can be rewritten as

$$\widehat{R}_t = \alpha\left(\frac{1+\beta}{A}\right)^{\frac{1-\alpha}{1+\psi}}\left(\widehat{k}_t\right)^{\alpha-1} \equiv \Phi\left(\widehat{k}_t\right),$$

where $\Phi(\cdot)$ is a strictly decreasing function.

Next, consider the case when $\sigma < 1$. To prove that the post-crash steady state is globally saddle-path stable, we will use the same "phase diagram" approach as in Tirole (1985) and Weil (1987).

To start, define a function $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}_+$ according to

$$\mathcal{F}(R) = \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left(1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1} \right)^{\frac{\sigma}{\sigma+\psi}} R^{-\eta}. \tag{37}$$

Note that the unique post-crash steady state must satisfy $\widehat{k}^* = \mathcal{F}\left(\widehat{R}^*\right)$. Taking the logarithm of both sides of (37) and differentiating the resultant expression with respect to R gives

$$\frac{R\mathcal{F}'(R)}{\mathcal{F}(R)} = \frac{1-\sigma}{\sigma+\psi} \left(\frac{\beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}}{1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}} - \widetilde{\eta} \right) = \frac{1-\sigma}{\sigma+\psi} \left[\Sigma(R) - \widetilde{\eta} \right],$$

where $\widetilde{\eta} \equiv (\sigma+\psi)\eta/(1-\sigma)$ and $\Sigma(\cdot)$ is the function defined in (4). There are two possible scenarios: (i) $\widetilde{\eta} \geq 1$ and (ii) $\widetilde{\eta} < 1$. Since $\Sigma(\cdot)$ is strictly increasing and bounded above by one, in the first scenario we have $\mathcal{F}'(R) < 0$ for all $R \geq 0$, $\lim_{R\to 0} \mathcal{F}(R) = +\infty$ and $\lim_{R\to\infty} \mathcal{F}(R) = 0$. In the second scenario, $\mathcal{F}(\cdot)$ is a U-shaped function. Figures B1 and B2 provide a graphical illustration of these two scenarios. In both diagrams, the function $\mathcal{F}(\cdot)$ and the vertical line representing $R = \widehat{R}^*$ divide the (R, k)-space into four quadrants:

$$Q_1 \equiv \left\{ (R,k) : k \leq \mathcal{F}(R),\ R \leq \widehat{R}^*, \text{ and } (R,k) \neq \left(\widehat{R}^*, \widehat{k}^*\right) \right\},$$

$$Q_2 \equiv \left\{ (R,k) : k > \mathcal{F}(R) \text{ and } R < \widehat{R}^* \right\},$$

$$Q_3 \equiv \left\{ (R,k) : k \geq \mathcal{F}(R),\ R \geq \widehat{R}^*, \text{ and } (R,k) \neq \left(\widehat{R}^*, \widehat{k}^*\right) \right\},$$

$$Q_4 \equiv \left\{ (R,k) : k < \mathcal{F}(R) \text{ and } R > \widehat{R}^* \right\}.$$

The rest of the proof is divided into a number of intermediate steps. These steps are valid both when $\widetilde{\eta} \geq 1$ and when $\widetilde{\eta} < 1$.

Step 1 For any initial value $\left(\widehat{R}_T, \widehat{k}_T\right) > 0$, there exists a unique sequence $\left\{\widehat{R}_{T+1}, \widehat{k}_{T+1}, \widehat{R}_{T+2}, \widehat{k}_{T+2}, ...\right\}$ that solves the dynamical system in (17)-(18). Whether this is part of a non-stationary post-crash equilibrium depends on the location of $\left(\widehat{R}_T, \widehat{k}_T\right)$ on the (R, k)-space. A solution $\left\{\widehat{R}_{T+1}, \widehat{k}_{T+1}, \widehat{R}_{T+2}, \widehat{k}_{T+2}, ...\right\}$ is said to originate from Q_n if $\left(\widehat{R}_T, \widehat{k}_T\right) \in Q_n$, for $n \in \{1, 2, 3, 4\}$. In the first step of the proof, it is shown that any solution that originates from Q_1 or Q_3 cannot be part of a post-crash equilibrium.

Suppose $\left(\widehat{R}_t, \widehat{k}_t\right)$ is in Q_1 for some $t \geq T$. This means either (i) $\widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right)$ and $\widehat{R}_t \leq \widehat{R}^*$, or (ii) $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$ and $\widehat{R}_t < \widehat{R}^*$. First consider the case when $\widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right)$ and $\widehat{R}_t \leq \widehat{R}^*$. Using (18), we can obtain

$$\begin{aligned}\widehat{R}_t^{\eta}\widehat{k}_t &= \alpha^{\eta}\left[\frac{(1-\alpha)^{1-\sigma}}{A}\right]^{\frac{1}{\sigma+\psi}}\left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t+1}\right)^{\frac{1}{\sigma}-1}\right]^{\frac{\sigma}{\sigma+\psi}} \\ &< \alpha^{\eta}\left[\frac{(1-\alpha)^{1-\sigma}}{A}\right]^{\frac{1}{\sigma+\psi}}\left[1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}_{t}\right)^{\frac{1}{\sigma}-1}\right]^{\frac{\sigma}{\sigma+\psi}},\end{aligned}$$

which implies $\widehat{R}_{t+1} < \widehat{R}_t \leq \widehat{R}^*$. Recall that the function $\Sigma(\cdot)$ defined in (4) is strictly increasing when $\sigma < 1$. Then it follows from (17) that

$$\begin{aligned}\widehat{k}_{t+1} &= \frac{1-\alpha}{\alpha(1+n)}\Sigma\left(\widehat{R}_{t+1}\right)\widehat{R}_t\widehat{k}_t \\ &< \frac{1-\alpha}{\alpha(1+n)}\Sigma\left(\widehat{R}^*\right)\widehat{R}_t\widehat{k}_t \leq \frac{1-\alpha}{\alpha(1+n)}\Sigma\left(\widehat{R}^*\right)\widehat{R}^*\widehat{k}_t = \widehat{k}_t.\end{aligned}$$

The last equality follows from equation (19). This result implies $\widehat{k}_{t+1} < \widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right) < \mathcal{F}\left(\widehat{R}_{t+1}\right)$. Next, consider the case when $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$ and $\widehat{R}_t < \widehat{R}^*$. Equation (18) and $\widehat{k}_t = \mathcal{F}\left(\widehat{R}_t\right)$ together imply $\widehat{R}_{t+1} = \widehat{R}_t < \widehat{R}^*$. This, together with (17), implies $\widehat{k}_{t+1} < \widehat{k}_t < \mathcal{F}\left(\widehat{R}_t\right) = \mathcal{F}\left(\widehat{R}_{t+1}\right)$. This proves the following: Any solution that originates from Q_1 is a strictly decreasing sequence and is confined in Q_1, i.e., $\left(\widehat{R}_t, \widehat{k}_t\right) \in Q_1$ for all $t \geq T$. Since both $\widehat{k}_t$ and $\widehat{R}_t$ are strictly decreasing over time, in the long run we will have either $\widehat{k}_t = 0$ or $\widehat{R}_t = 0$, which cannot happen in equilibrium.

Using a similar argument, we can show that any solution that originates from Q_3 is a strictly increasing sequence and is confined in Q_3. Using the young consumer's budget constraint and the capital market clearing condition, we can obtain the following condition

$$\widehat{s}_t = \frac{\widehat{k}_{t+1}}{1+n} < \widehat{w}_t\widehat{l}_t \leq \widehat{w}_t = (1-\alpha)\left(\frac{\alpha}{\widehat{R}_t}\right)^{\frac{\alpha}{1-\alpha}}.$$

Obviously, this will be violated at some point if both $\widehat{k}_t$ and $\widehat{R}_t$ are strictly increasing over time. Hence, any solution that originates from Q_3 cannot be part of a post-crash equilibrium.

Step 2 We now show that any solution that originates from Q_2 will never enter Q_4, i.e., $\left(\widehat{R}_T, \widehat{k}_T\right) \in Q_2$ implies $\left(\widehat{R}_t, \widehat{k}_t\right) \notin Q_4$, for all $t > T$; likewise, any solution that originates from Q_4 will never enter Q_2.

Suppose $\left(\widehat{R}_t, \widehat{k}_t\right)$ is in Q_2 for some $t \geq T$. Then we have

$$\begin{aligned} \widehat{R}_t^{\eta} \widehat{k}_t &= \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}} \\ &> \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}, \end{aligned}$$

which implies $\widehat{R}_{t+1} > \widehat{R}_t$. Suppose the contrary that $\left(\widehat{R}_{t+1}, \widehat{k}_{t+1}\right)$ is in Q_4, so that $\widehat{R}_{t+1} > \widehat{R}^* > \widehat{R}_t$ and $\widehat{k}_{t+1} < \mathcal{F}\left(\widehat{R}_{t+1}\right)$. Then, using (17) we can get

$$\begin{aligned} \widehat{R}_{t+1} \widehat{k}_{t+1} &= \frac{1-\alpha}{\alpha(1+n)} \left[\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_t \widehat{k}_t \\ &> \frac{1-\alpha}{\alpha(1+n)} \left[\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}^* \right)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}^* \right)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_t \widehat{k}_t = \widehat{R}_t \widehat{k}_t. \end{aligned} \tag{38}$$

The second line uses the fact that $\Sigma(\cdot)$ is strictly increasing and $\widehat{R}_{t+1} > \widehat{R}^*$. The last equality follows from the steady-state condition in (19). Since $\eta > 1$, we also have $\widehat{R}_{t+1}^{\eta-1} > \widehat{R}_t^{\eta-1}$. This, together with (18) and (38), implies

$$\widehat{R}_{t+1}^{\eta} \widehat{k}_{t+1} > \widehat{R}_t^{\eta} \widehat{k}_t = \alpha^{\eta} \left[\frac{(1-\alpha)^{1-\sigma}}{A} \right]^{\frac{1}{\sigma+\psi}} \left[1 + \beta^{\frac{1}{\sigma}} \left(\widehat{R}_{t+1} \right)^{\frac{1}{\sigma}-1} \right]^{\frac{\sigma}{\sigma+\psi}}$$

$$\Rightarrow \widehat{k}_{t+1} > \mathcal{F}\left(\widehat{R}_{t+1}\right),$$

which gives rise to a contradiction. Hence, any solution that originates from Q_2 will never enter Q_4. Using similar arguments, we can show that any solution that originates from Q_4 will never enter Q_2.

Step 3 Consider a solution that originates from Q_2. As shown in Step 2, $\left(\widehat{R}_T, \widehat{k}_T\right) \in Q_2$ implies $\widehat{R}_{T+1} > \widehat{R}_T$. If $\widehat{R}_{T+1} \geq \widehat{R}^*$, then the economy is in Q_3 at time $T+1$ and by the results in Step 1, we know that $\widehat{R}_t$ will diverge to infinity in the long run. If $\widehat{R}_{T+1} < \widehat{R}^*$, then using (17) we can obtain

$$\begin{aligned} \widehat{k}_{T+1} &= \frac{1-\alpha}{\alpha(1+n)} \left[\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{T+1} \right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}_{T+1} \right)^{\frac{1}{\sigma}-1}} \right] \widehat{R}_T \widehat{k}_T \\ &< \frac{1-\alpha}{\alpha(1+n)} \left[\frac{\beta^{\frac{1}{\sigma}} \left(\widehat{R}^* \right)^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}^* \right)^{\frac{1}{\sigma}-1}} \right] \widehat{k}_T = \widehat{k}_T. \end{aligned}$$

There are two possible scenarios: First, if $\widehat{R}_{T+1} < \widehat{R}^*$ and $\widehat{k}_{T+1} \leq \mathcal{F}\left(\widehat{R}_{T+1}\right)$, then the economy is in Q_1 at time $T+1$. By the results in Step 1, we know that all subsequent values of $\widehat{R}_t$ will be strictly less than $\widehat{R}^*$. Second, if $\widehat{R}_{T+1} < \widehat{R}^*$ and $\mathcal{F}\left(\widehat{R}_{T+1}\right) < \widehat{k}_{T+1}$, then that means the economy remains in Q_2 at time $T+1$. In addition, we have $\widehat{R}_{T+1} > \widehat{R}_T$ and $\widehat{k}_T > \widehat{k}_{T+1}$ which means the economy is now getting closer to the steady state $\left(\widehat{R}^*, \widehat{k}^*\right)$. Thus, any solution that originates from Q_2 has three possible fates: (i) It will enter Q_3 at some point and $\widehat{R}_t$ will then diverge to infinity. (ii) It will enter Q_1 at some point and $\widehat{R}_t$ will be strictly less than $\widehat{R}^*$ afterward. (iii) It will converge to the post-crash steady state. For reasons explained above, the first two types of solutions cannot be part of an equilibrium. Hence, a solution originating from Q_2 is an equilibrium path only if it converges to the steady state $\left(\widehat{R}^*, \widehat{k}^*\right)$. The above argument also shows that, along the convergent path, $\widehat{k}_t$ is decreasing towards $\widehat{k}^*$ while $\widehat{R}_t$ is increasing towards $\widehat{R}^*$.

Using a similar argument, we can show that any solution originating from Q_4 is an equilibrium path only if it converges to the steady state $\left(\widehat{R}^*, \widehat{k}^*\right)$, and that along the convergent path, $\widehat{k}_t$ is increasing towards $\widehat{k}^*$ while $\widehat{R}_t$ is decreasing towards $\widehat{R}^*$.

Step 4 We now establish the uniqueness of saddle path. Fix $\widehat{k}_T > 0$. Suppose the contrary that there exists two saddle paths, denoted by $\left\{\widehat{R}'_t, \widehat{k}'_t\right\}_{t=T}^{\infty}$ and $\left\{\widehat{R}''_t, \widehat{k}''_t\right\}_{t=T}^{\infty}$, with $\widehat{k}'_T = \widehat{k}''_T = \widehat{k}_T$ and $\widehat{R}'_T > \widehat{R}''_T > 0$. By the results in Step 3, we know that $\lim_{t\to\infty} \widehat{R}'_t = \lim_{t\to\infty} \widehat{R}''_t = \widehat{R}^*$. Substituting $\widehat{k}'_T = \widehat{k}''_T$ and $\widehat{R}'_T > \widehat{R}''_T$ into (18) gives

$$\left(\frac{\widehat{R}'_T}{\widehat{R}''_T} \right)^{\eta} = \left[\frac{1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}'_{T+1} \right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}} \left(\widehat{R}''_{T+1} \right)^{\frac{1}{\sigma}-1}} \right]^{\frac{\sigma}{\sigma+\psi}} > 1,$$

which implies $\widehat{R}'_{T+1} > \widehat{R}''_{T+1} > 0$. Using (17), we can get

$$\frac{\widehat{k}'_{T+1}}{\widehat{k}''_{T+1}} = \frac{\Sigma\left(\widehat{R}'_{T+1}\right)}{\Sigma\left(\widehat{R}''_{T+1}\right)} \frac{\widehat{R}'_T}{\widehat{R}''_T} > 1.$$

Using (18) again, but now for $t = T+1$, gives

$$\left(\frac{\widehat{R}'_{T+1}}{\widehat{R}''_{T+1}}\right)^{\eta} \left(\frac{\widehat{k}'_{T+1}}{\widehat{k}''_{T+1}}\right) = \left[\frac{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}'_{T+2}\right)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}}\left(\widehat{R}''_{T+2}\right)^{\frac{1}{\sigma}-1}}\right]^{\frac{\sigma}{\sigma+\psi}} > 1,$$

which implies $\widehat{R}'_{T+2} > \widehat{R}''_{T+2}$. By an induction argument, we can show that $\widehat{R}'_{T+j} > \widehat{R}''_{T+j}$ implies $\widehat{k}'_{T+j} > \widehat{k}''_{T+j}$, and $\widehat{R}'_{T+j+1} > \widehat{R}''_{T+j+1}$, for all $j \geq 1$. The last result contradicts $\lim_{t\to\infty} \widehat{R}'_t = \lim_{t\to\infty} \widehat{R}''_t = \widehat{R}^*$. Hence, we can rule out the possibility of multiple saddle paths.

In sum, we have shown that any equilibrium path that originates from a given value of $\widehat{k}_T > 0$ must be unique and converge to the post-crash steady state. Hence, the dynamical system in (17)-(18) is globally saddle-path stable. The one-to-one relationship between $\widehat{R}_T$ and $\widehat{k}_T$ can be captured by a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. Since the saddle path is downward sloping in the (R, k)-space, $\Phi(\cdot)$ must be strictly decreasing. This completes the proof of Proposition 2.

Proof of Proposition 3

In the post-crash economy, optimal labor supply is determined by (3). Setting $\sigma = 1$ gives $\widehat{l}_t = \left(\frac{1+\beta}{A}\right)^{\frac{1}{1+\psi}}$ for all t. Next, consider the pre-crash economy. Substituting (34) into (30) and rearranging terms give

$$Al_t^{\psi+\sigma} = (w_t)^{1-\sigma} \left\{\frac{\Omega_{t+1}\widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}}\left[1 + \frac{p_t}{p_{t+1}}\left(\Omega_{t+1}\widehat{R}_{t+1} - R_{t+1}\right)\right]}{\Omega_{t+1}\widehat{R}_{t+1}}\right\}^{\sigma}, \tag{39}$$

where Ω_{t+1} is defined in (14). When $\sigma = 1$, the right-hand side of the above equation becomes

$$\begin{aligned} & 1 + \left(\Omega_{t+1}\widehat{R}_{t+1}\right)^{-1} (\beta q \pi_{t+1}) \left[1 + \frac{p_t}{p_{t+1}}\left(\Omega_{t+1}\widehat{R}_{t+1} - R_{t+1}\right)\right] \\ = \; & 1 + \frac{\beta(1-q)}{\pi_{t+1} - R_{t+1}} \left[\pi_{t+1} - R_{t+1} + \frac{q(\pi_{t+1} - R_{t+1})}{1-q}\right] = 1 + \beta. \end{aligned}$$

Hence, we have $Al_t^{\psi+1} = 1 + \beta$ for all t. This completes the proof of Proposition 3.

Derivation of Equations (21)-(25)

Recall that the optimal choice of s_t in the pre-crash economy is determined by equation (33). Using this and $\alpha w_t l_t = (1-\alpha) R_t k_t$, we can write the market-clearing condition for physical capital as

$$(1+n) k_{t+1} = \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1} + (\beta q \pi_{t+1})^{\frac{1}{\sigma}} \left[1 + \frac{p_t}{p_{t+1}} \left(\Omega_{t+1} \widehat{R}_{t+1} - R_{t+1} \right) \right]} \right\} \left(\frac{1-\alpha}{\alpha} \right) R_t k_t. \tag{40}$$

Combining (39) and (40) gives

$$A l_t^{\psi+\sigma} = (w_t)^{1-\sigma} \left\{ \frac{(\beta q \pi_{t+1})^{\frac{1}{\sigma}}}{\Omega_{t+1} \widehat{R}_{t+1}} \left[\frac{1-\alpha}{\alpha (1+n)} \right] \frac{R_t k_t}{k_{t+1}} \right\}^{\sigma}. \tag{41}$$

Upon setting $k_{t+1} = k_t = k^*$, $R_t = R_{t+1} = R^*$, $\widehat{R}_{t+1} = \widehat{R}_0^*$ and $\pi_{t+1} = 1+n$, equation (40) becomes

$$1+n = \left\{ \frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^* + [\beta q (1+n)]^{\frac{1}{\sigma}} \left[1 + \frac{1}{1+n} \left(\Omega^* \widehat{R}_0^* - R^* \right) \right]} \right\} \left(\frac{1-\alpha}{\alpha} \right) R^*. \tag{42}$$

$$\Rightarrow 1 + \left[1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \right] \left(\frac{\Omega^* \widehat{R}_0^*}{1+n} \right) = \frac{1}{\alpha} \frac{R^*}{1+n}.$$

Equation (21) can be obtained by rearranging the terms in the above equation. Similarly, after substituting the stationarity conditions into (41), we can obtain

$$A (l^*)^{\psi+\sigma} = (w^*)^{1-\sigma} \left\{ \frac{[\beta q (1+n)]^{\frac{1}{\sigma}}}{\Omega^* \widehat{R}_0^*} \left(\frac{1-\alpha}{\alpha} \right) \frac{R^*}{1+n} \right\}^{\sigma}.$$

Equation (23) follows immediately from this equation. Equations (22) and (24) can be obtained from (16). Finally, equation (25) can be obtained from (13).

Define $\theta^* \equiv R^*/(1+n)$. Then we can rewrite (21) as

$$\Psi(\theta^*) \equiv 1 + \left[1 + (\beta q)^{-\frac{1}{\sigma}} (1+n)^{1-\frac{1}{\sigma}} \right] \left(\frac{q}{1-q} \right)^{\frac{1}{\sigma}} \left(\frac{\widehat{R}_0^*}{1+n} \right)^{1-\frac{1}{\sigma}} (1-\theta^*)^{\frac{1}{\sigma}} = \frac{\theta^*}{\alpha}. \tag{43}$$

For any $\widehat{R}_0^* > 0$ and $\sigma > 0$, $\Psi : [0,1] \to \mathbb{R}_+$ is a strictly decreasing function that satisfies $\Psi(0) > 0$ and $\Psi(1) = 1 < 1/\alpha$. Meanwhile, the right-hand side of the above equation is a straight line that passes through the origin and $1/\alpha$ (when $\theta^* = 1$). Thus, for any $\widehat{R}_0^* > 0$ and $\sigma > 0$, there exists a unique $\theta^* \in (0,1)$ that solves (43). Once $R^* \equiv (1+n)\theta^*$ is determined, the value of $\{k^*, w^*, l^*, a^*\}$ can be uniquely determined using (22)-(25).

Proof of Proposition 4

The proof of this result is divided into two parts: First, by comparing the optimal labor supply and capital market clearing condition in the pre-crash and post-crash steady states, we show that $\sigma < 1$, $a^* > 0$ and $R^* \leq \widehat{R}^*$ together imply $k^* \geq \widehat{k}^*$. Thus, by the results in Proposition 2, we will have $\widehat{R}_0^* \leq \widehat{R}^*$. Second, we show that the same conditions $\sigma < 1$, $a^* > 0$ and $R^* \leq \widehat{R}^*$ also imply $\widehat{R}_0^* > \widehat{R}^*$. Hence, there is a contradiction and it must be the case that $R^* > \widehat{R}^*$ when $\sigma < 1$.

We begin by establishing some useful intermediate results. First, in any pre-crash steady state, $a^* > 0$ if and only $\Omega^* \widehat{R}_0^* > R^*$. Using the definition of Ω^*, we can rewrite this condition as

$$\frac{q\,(1+n-R^*)}{R^*} > (1-q)\left(\frac{R^*}{\widehat{R}_0^*}\right)^{\sigma-1}. \tag{44}$$

Next, we compare the optimal labor supply in the two steady states. In the post-crash steady state,

$$A\left(\widehat{l}^*\right)^{\psi+\sigma} = \left[(1-\alpha)\left(\frac{\alpha}{\widehat{R}^*}\right)^{\frac{\alpha}{1-\alpha}}\right]^{1-\sigma} \beta^{\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{\frac{1}{\sigma}}\left(\frac{1-\alpha}{\alpha}\right)\frac{1}{1+n}.$$

The counterpart of this in the pre-crash steady state is equation (23). Taken together, they imply

$$\left(\frac{l^*}{\widehat{l}^*}\right)^{\psi+\sigma} = \left(\frac{R^*}{\widehat{R}^*}\right)^{-\frac{\alpha(1-\sigma)}{1-\alpha}}\left[\frac{q\,(1+n)}{\widehat{R}^*}\right]\left(\frac{R^*}{\Omega^*\widehat{R}_0^*}\right)^{\sigma}. \tag{45}$$

Using (19), we can write

$$\frac{1-\alpha}{\alpha}\frac{\widehat{R}^*}{1+n} = 1 + \beta^{-\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{1-\frac{1}{\sigma}}.$$

A similar equation for the pre-crash economy can be obtained from (42), i.e.,

$$\frac{1-\alpha}{\alpha}\frac{R^*}{1+n} = 1 + \frac{\Omega^*\widehat{R}_0^* - R^*}{1+n} + \left[\beta q\,(1+n)\right]^{-\frac{1}{\sigma}}\Omega^*\widehat{R}_0^*.$$

Combining the two gives

$$\frac{1-\alpha}{\alpha\,(1+n)}\left(R^* - \widehat{R}^*\right) = \frac{\Omega^*\widehat{R}_0^* - R^*}{1+n} + \left[\beta q\,(1+n)\right]^{-\frac{1}{\sigma}}\Omega^*\widehat{R}_0^* - \beta^{-\frac{1}{\sigma}}\left(\widehat{R}^*\right)^{1-\frac{1}{\sigma}}.$$

Hence, $\Omega^*\widehat{R}_0^* > R^*$ and $R^* \leq \widehat{R}^*$ together imply

$$\left[q\,(1+n)\right]^{-\frac{1}{\sigma}}\Omega^*\widehat{R}_0^* \leq \left(\widehat{R}^*\right)^{1-\frac{1}{\sigma}}$$

$$\Rightarrow 1 \leq \frac{q(1+n)}{\widehat{R}^*}\left(\frac{\widehat{R}^*}{\Omega^* \widehat{R}_0^*}\right)^{\sigma}. \tag{46}$$

Equations (45)-(46) and $R^* \leq \widehat{R}^*$ then imply $l^* \geq \widehat{l}^*$. Using (24), we can get

$$k^* = l^* \left(\frac{\alpha}{R^*}\right)^{\frac{1}{1-\alpha}} \geq \widehat{l}^* \left(\frac{\alpha}{\widehat{R}^*}\right)^{\frac{1}{1-\alpha}} = \widehat{k}^*.$$

This establishes the first part of the proof. On the other hand, using the definition of Ω^*, we can rewrite (46) as

$$\frac{(1+n-R^*)}{1+n} \leq (1-q)\left(\frac{\widehat{R}^*}{\widehat{R}_0^*}\right)^{\sigma-1}. \tag{47}$$

If $\sigma < 1$ and $R^* \leq \widehat{R}^*$ are true, then we can combine (44) and (47) to get

$$\frac{(1+n-R^*)}{1+n} \leq (1-q)\left(\frac{\widehat{R}^*}{\widehat{R}_0^*}\right)^{\sigma-1} \leq (1-q)\left(\frac{R^*}{\widehat{R}_0^*}\right)^{\sigma-1} < \frac{q(1+n-R^*)}{R^*},$$

which implies $R^* < q(1+n)$, or equivalently,

$$\frac{(1+n-R^*)}{1+n} > (1-q).$$

Equation (47) then implies $\widehat{R}_0^* > \widehat{R}^*$ which is inconsistent with the fact that $k^* \geq \widehat{k}^*$. Hence, it must be the case that $R^* > \widehat{R}^*$. This completes the proof of Proposition 4.

Proof of Proposition 5

The necessary and sufficient condition for $l^* > \widehat{l}^*$ follows immediately from (45). Using (24), we can get

$$\frac{k^*}{\widehat{k}^*} = \frac{l^*}{\widehat{l}^*}\left(\frac{\widehat{R}^*}{R^*}\right)^{\frac{1}{1-\alpha}}.$$

The necessary and sufficient condition for $k^* > \widehat{k}^*$ is obtained by combining this and (45).

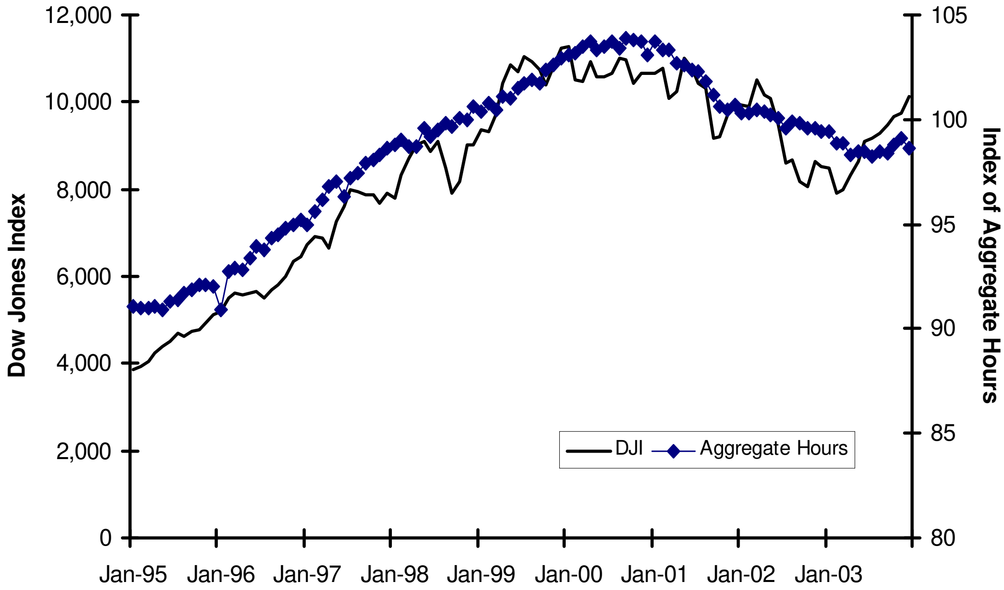

Figure 1: Aggregate Hours and Dow Jones Index, 1995-2003.

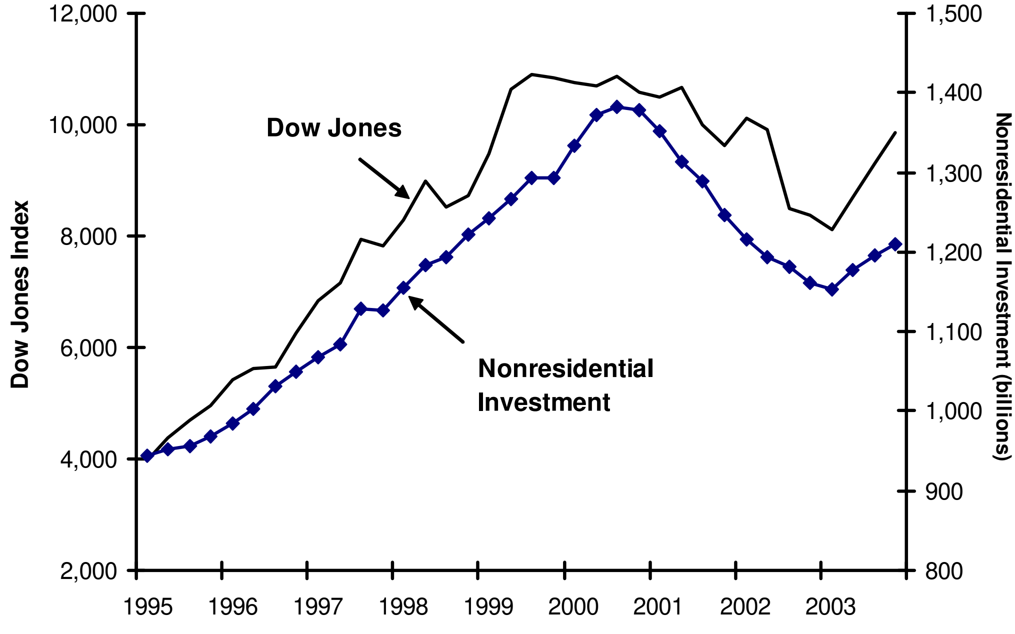

Figure 2: Private Nonresidential Fixed Investment and Dow Jones Index, 1995Q1 to 2003Q4.

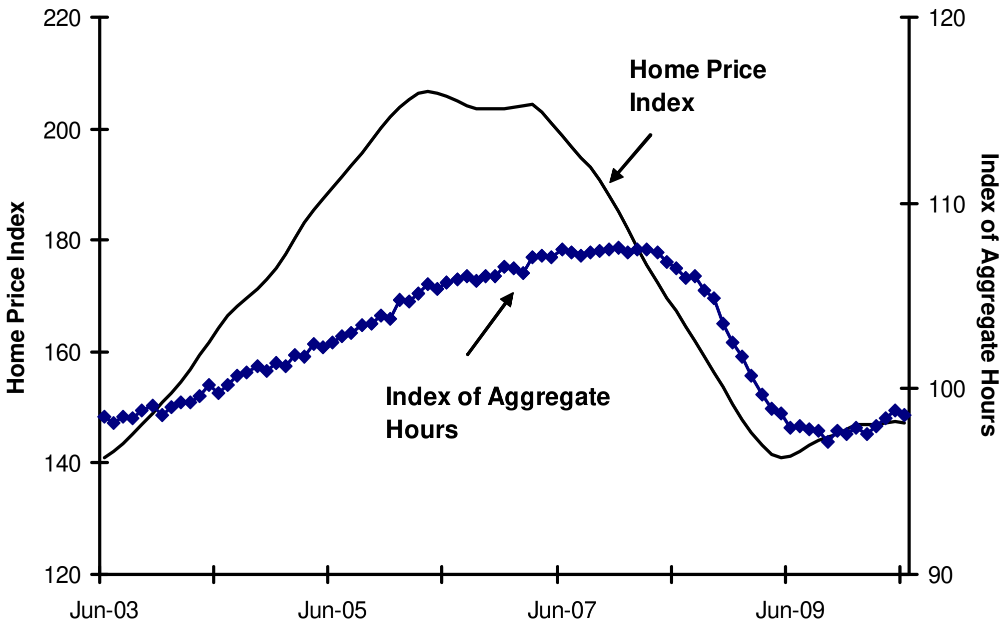

Figure 3: Aggregate Hours and Home Price Index, June 2003 to June 2010.

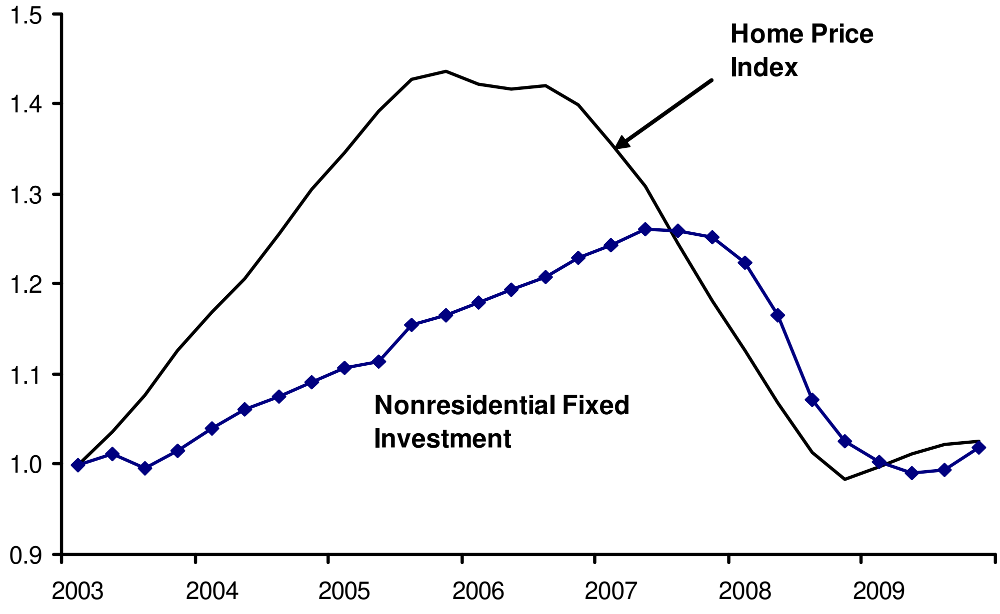

Figure 4: Private Nonresidential Fixed Investment and Home Price Index, 2003Q3 to 2010Q3.

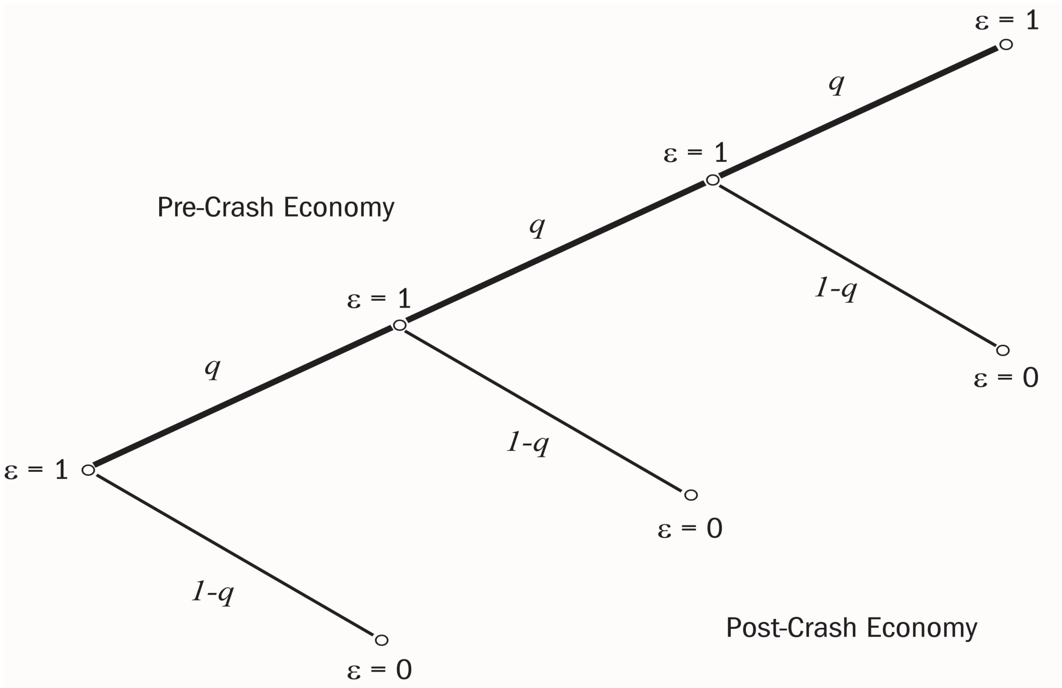

Figure 5: Probability Tree Diagram of the Asset Price Shock.

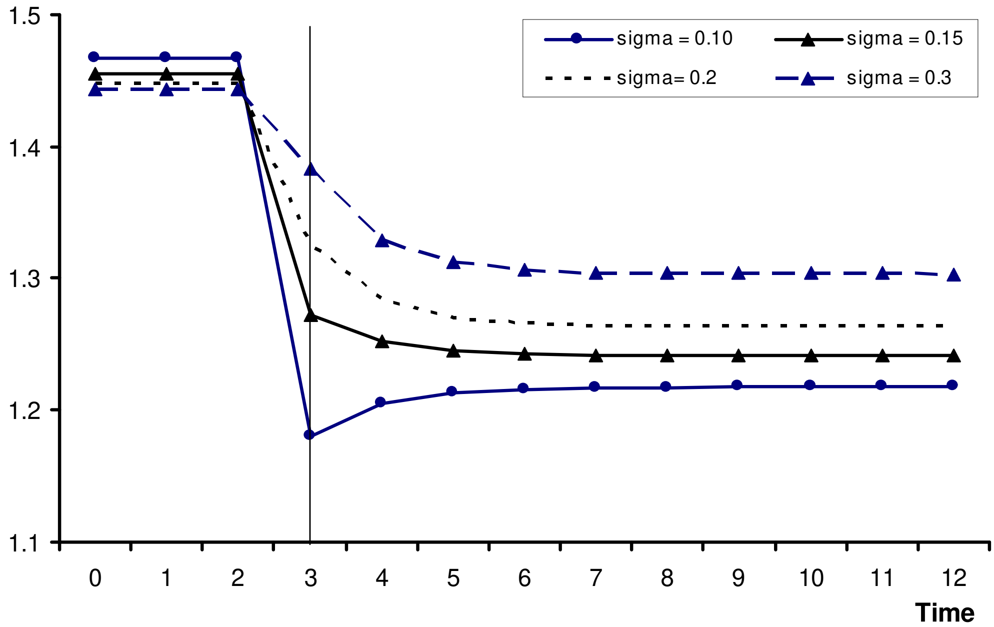

FIgure 6: Time Paths of Interest Rate under Different Values of σ.

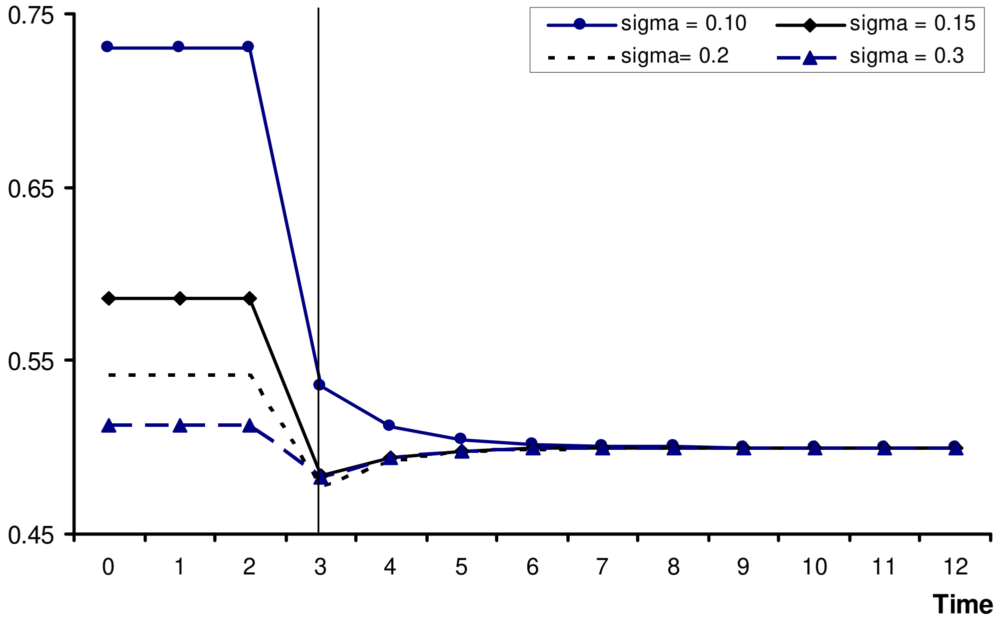

Figure 7: Time Paths of Labor Supply under Different Values of σ.

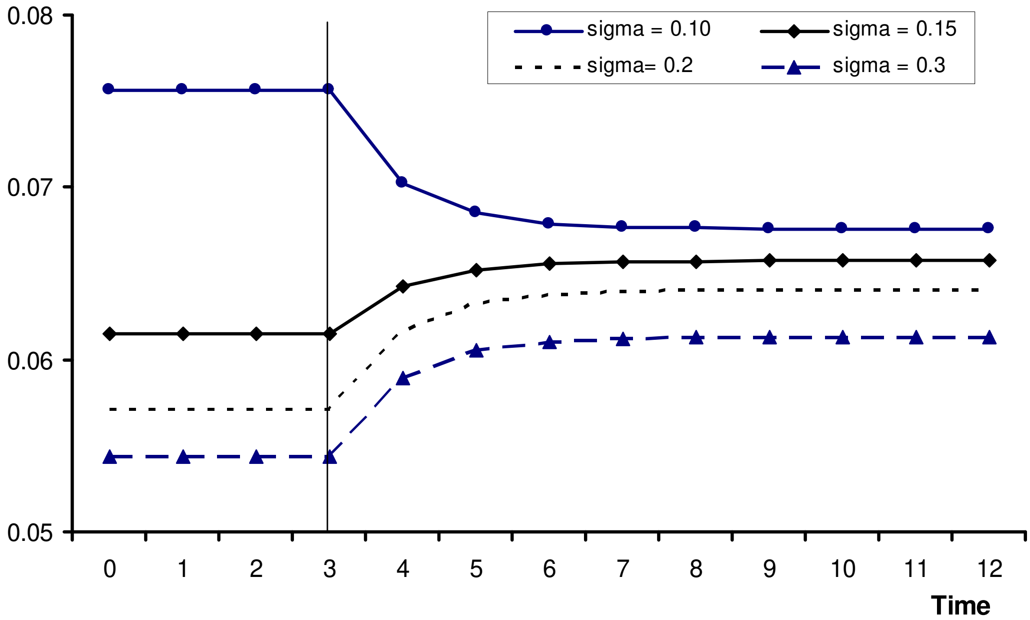

Figure 8: Time Paths of Capital under Different Values of σ.

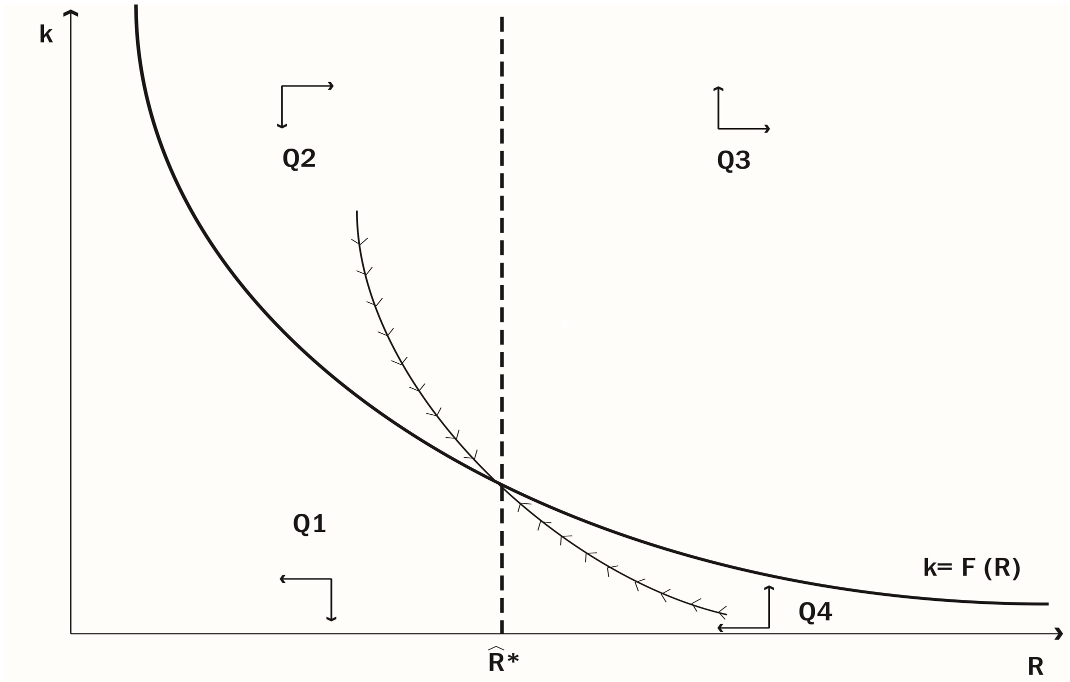

Figure B1: Phase Diagram for the case when $\widetilde{\eta} \geq 1$.

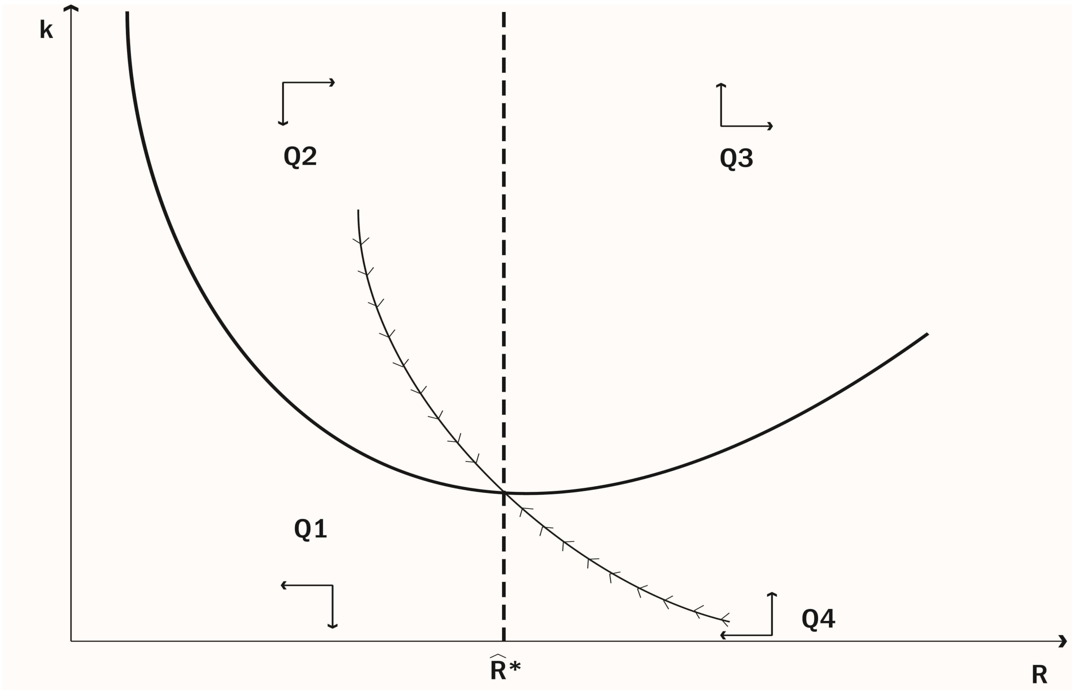

Figure B2: Phase Diagram for the case when $\widetilde{\eta} < 1$.

References

[1] Andersen, Torben M. and Joydeep Bhattacharya (2013) Unfunded pensions and endogenous labor supply. *Macroeconomic Dynamics* 17, 971-997.

[2] Caballero, Ricardo J. and Arvind Krishnamurthy (2006) Bubbles and capital flow volatility: causes and risk management. *Journal of Economic Theory* 53, 35-53.

[3] Chirinko, Robert S. and Huntley Schaller (2001) Business fixed investment and "bubbles": the Japanese case. *American Economic Review* 91, 663-680.

[4] Chirinko, Robert S. and Huntley Schaller (2011) Fundamentals, misvaluation, and business investment. *Journal of Money, Credit and Banking* 43, 1423-1442.

[5] Farhi, Emmanuel and Jean Tirole (2012) Bubbly liquidity. *Review of Economic Studies* 79, 678-706.

[6] Fuster, Luisa (1999) Effects of uncertain lifetime and annuity insurance on capital accumulation and growth. *Economic Theory* 13, 429-445.

[7] Galor, Oded and Harl E. Ryder (1989) Existence, uniqueness and stability of equilibrium in an overlapping-generations model with productive capital. *Journal of Economic Theory* 49, 360-375.

[8] Gan, Jie (2007) The real effects of asset market bubbles: loan- and firm-level evidence of a lending channel. *Review of Financial Studies* 20, 1941-1973.

[9] LeRoy, Stephen F. (2004) Rational exuberance.*Journal of Economic Literature* 42, 783-804.

[10] Martin, Alberto and Jaume Ventura (2012) Economic growth with bubbles. *American Economic Review* 102, 3033-3058.

[11] Miao, Jianjun (2014) Introduction to economic theory of bubbles. *Journal of Mathematical Economics* 53, 130-136.

[12] Ofek, Eli and Matthew Richardson (2002) The valuation and market rationality of internet stock prices. *Oxford Review of Economic Policy* 18, 265-287.

[13] Olivier, Jacques (2000) Growth-enhancing bubbles. *International Economic Review* 41, 133-151.

[14] Shi, Lisi and Richard M.H. Suen (2014) Asset bubbles in an overlapping generations model with endogenous labor supply. *Economics Letters* 123, 164-167.

[15] Shiller, Robert J. (2007) Understanding recent trends in house prices and home ownership. NBER working paper 13553.

[16] Tirole, Jean (1985) Asset bubbles and overlapping generations. *Econometrica* 53, 1499-1528.

[17] Ventura, Jaume (2012) Bubbles and capital flows. *Journal of Economic Theory* 147, 738-758.

[18] Weil, Philippe (1987) Confidence and the real value of money in an overlapping generations economy. *Quarterly Journal of Economics* 102, 1-22.