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# Standard Risk Aversion and Efficient Risk Sharing

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## Abstract

This paper analyzes the risk attitude and investment behavior of a group of heterogeneous consumers who face an undesirable background risk. It is shown that standard risk aversion at the individual level does not imply standard risk aversion at the group level under efficient risk sharing. This points to a potential divergence between individual and collective investment choices in the presence of background risk. We show that if the members' absolute risk tolerance is increasing and satisfies a strong form of concavity, then the group has standard risk aversion.

*Keywords:* Standard risk aversion; Efficient risk sharing; Background risk; Portfolio choice.

*JEL classification:* D70, D81, G11.

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# 1 Introduction

Both conventional wisdom and empirical evidence suggest that people are more reluctant to invest in risky assets when they face other sources of uninsurable and undesirable “background” risk (e.g., labor income risk).<sup>1</sup> In a seminal paper, Kimball (1993) shows that an expected-utility maximizer with decreasing absolute risk aversion (DARA) and decreasing absolute prudence (DAP) will have this type of response to background risk. The combination of DARA and DAP is referred to as standard risk aversion. In the present study, we ask whether a group of diverse individuals, who share risks efficiently among themselves and make investment decisions jointly, will respond to background risk in the same way. Specifically, we want to identify the conditions under which the group’s preferences (or aggregate utility function) exhibit standard risk aversion.

It is well-known that if all members have DARA preferences, then the aggregate utility function will have the same property. However, this is not true in general for DAP, as we will show below. Thus, standard risk aversion at the individual level is not enough to ensure standard risk aversion at the group level under an efficient risk-sharing arrangement. This points to a potential divergence between individual choices and efficient collective choices. To fix ideas, consider a household of two adults, each with standard risk aversion. When acting alone, each of them would like to reduce their exposure to risky assets in the presence of an undesirable background risk, but as a family they may choose otherwise. To avoid this rather absurd prediction, it is necessary to impose some stronger restrictions on individual members’ preferences. The main contribution of this paper is to provide one such restriction. Specifically, we show that if each individual member’s absolute risk tolerance is increasing and satisfies a strong form of concavity (which is stronger than DAP) then the aggregate utility function is standard. Since standard risk aversion implies proper risk aversion and risk vulnerability, our result also ensures that the group’s preferences will have these properties.<sup>2</sup>

## 2 The Model

Consider a static model with a group made up of  $N$  individuals,  $N$  being an integer greater than one. The group has a sure amount of initial wealth  $W > 0$ , which can be invested in two types of assets: a safe asset with a riskless rate of return  $r > 0$  and a risky asset with a random rate of return  $\tilde{R}$ . Let  $\alpha$  and  $W - \alpha$  denote, respectively, the amount of risky and safe investment. The

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<sup>1</sup>See, for instance, Guiso and Paiella (2008) and Calvet and Sodini (2014) for empirical evidence on this.

<sup>2</sup>The notions of “proper risk aversion” and “risk vulnerability” are introduced by Pratt and Zeckhauser (1987) and Gollier and Pratt (1996), respectively. For a textbook treatment of these concepts, see Gollier (2001b, Chapter 9).

gross return from this portfolio is given by

$$(W - \alpha)(1 + r) + \alpha(1 + \tilde{R}) = \omega + \alpha\tilde{x},$$

where  $\omega \equiv W(1 + r) > 0$  and  $\tilde{x} \equiv \tilde{R} - r$  is the excess return from the risky asset. The random variable  $\tilde{x}$  is drawn from a compact interval  $X \subseteq \mathbb{R}$  according to some probability distribution. Apart from the risky investment, the group also faces an exogenous, uninsurable background risk  $\tilde{y}$  in final wealth. The background risk is drawn from a compact interval  $Y \subseteq \mathbb{R}$ ; it can take both positive and negative values and is statistically independent of  $\tilde{x}$ .<sup>3</sup> The probability distributions of  $\tilde{x}$  and  $\tilde{y}$  are known to all group members, so there is no disagreement in their probabilistic beliefs. The sum of investment returns and background risk is used to finance the members' consumption. The group as a whole thus faces the following budget constraint:

$$\sum_{i=1}^N \tilde{c}_i \leq \omega + \alpha\tilde{x} + \tilde{y}, \quad (1)$$

where  $\tilde{c}_i$  denotes member  $i$ 's consumption. Each member's preferences can be represented by  $E[u_i(\tilde{c}_i)]$ , for  $i \in \{1, 2, \dots, N\}$ . The utility function  $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$  is at least five times differentiable, strictly increasing, strictly concave and satisfies the Inada condition  $\lim_{c \rightarrow 0} u'_i(c) = \infty$ .

In the present study, we focus on efficient decisions made by the group. Specifically, this means the members of the group collectively decide on a level of risky investment ( $\alpha$ ) and an allocation of consumption ( $\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_N$ ) so as to maximize a weighted average of their expected utility, i.e.,

$$\sum_{i=1}^N \lambda_i E[u_i(\tilde{c}_i)],$$

where  $\lambda_i > 0$  is the Pareto weight for member  $i$ ; subject to (1) and  $\tilde{c}_i \geq 0$  for all  $i$ . This problem can be divided into two parts: First, conditional on the choice of  $\alpha$  and the realization of  $(\tilde{x}, \tilde{y})$ , the group solves a resources allocation problem:

$$\hat{u}(z) \equiv \max_{\{\tilde{c}_1, \dots, \tilde{c}_N\}} \sum_{i=1}^N \lambda_i u_i(\tilde{c}_i), \quad (2)$$

subject to

$$\sum_{i=1}^N \tilde{c}_i \leq z \equiv \omega + \alpha\tilde{x} + \tilde{y}, \quad \text{and} \quad \tilde{c}_i \geq 0 \quad \text{for all } i.$$

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<sup>3</sup>One way to interpret this background risk is as a random net income. A positive value of  $\tilde{y}$  then represents a windfall, while a negative value can be the result of a large, unanticipated expense.

For any  $z > 0$ , the constraint set of the above problem is compact. This, together with a continuous and strictly concave objective function, ensures the existence of a unique solution. The Inada condition ensures that the optimal choice of each  $\tilde{c}_i$  is strictly positive. By the maximum theorem, the aggregate utility function  $\hat{u}(\cdot)$  is continuous and the optimal choice of each  $\tilde{c}_i$  can be determined by a single-valued continuous function  $\kappa_i(z)$ , known as the sharing rule. By the implicit function theorem, if each  $u_i(\cdot)$  is  $(m + 1)$  times differentiable, then both  $\kappa_i(\cdot)$  and  $\hat{u}(\cdot)$  are  $m$  times differentiable.<sup>4</sup> Thus, under our stated assumptions, both  $\kappa_i(\cdot)$  and  $\hat{u}(\cdot)$  are at least four times differentiable. In addition,  $\hat{u}(\cdot)$  is strictly increasing and strictly concave.

The second part of the collective decision problem is a static portfolio choice problem:

$$\max_{\alpha} E [\hat{u}(\omega + \alpha\tilde{x} + \tilde{y})]. \quad (3)$$

Note that some restrictions on the choice of  $\alpha$  are implicitly implied by the Inada condition. Since the optimal choice of all  $\tilde{c}_i$  must be strictly positive, the group must choose  $\alpha$  so that  $z \equiv \omega + \alpha\tilde{x} + \tilde{y}$  is strictly positive *for all possible realizations of*  $(\tilde{x}, \tilde{y})$ . Depending on the boundary values of  $X$  and  $Y$ , this specification can allow for short-selling of the risky asset (i.e.,  $\alpha < 0$ ) or short-selling of the safe asset (i.e.,  $\alpha > W$ ). Since the objective function in (3) is continuous and strictly concave in  $\alpha$ , a unique solution (denoted by  $\alpha^*$ ) exists.

### 3 Standard Risk Aversion of $\hat{u}$

For each member  $i \in \{1, 2, \dots, N\}$ , define  $A_i(c) \equiv -u_i''(c)/u_i'(c)$  as the Arrow-Pratt measure of absolute risk aversion and  $P_i(c) \equiv -u_i'''(c)/u_i''(c)$  as the measure of absolute prudence. The reciprocal of  $A_i(c)$ , denoted by  $T_i(c)$ , is the measure of absolute risk tolerance. The first derivative of  $T_i(c)$  is often referred to as absolute cautiousness [see, for instance, Wilson (1968) and Hara *et al.* (2007)]. Since  $\hat{u}(\cdot)$  is at least four times differentiable, we can define the corresponding measures,  $\hat{A}(z)$ ,  $\hat{T}(z)$  and  $\hat{P}(z)$ , for the aggregate utility function. Wilson (1968) shows that there is a close connection between  $T_i(c)$ ,  $\hat{T}(z)$  and the sharing rule  $\kappa_i(z)$ . Specifically,

$$\kappa_i'(z) = \frac{T_i[\kappa_i(z)]}{\hat{T}(z)} > 0 \quad \text{for all } i, \text{ and} \quad (4)$$

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<sup>4</sup>A formal proof of this statement is available from the author upon request. The same result is also mentioned in Hara (2006).

$$\widehat{T}(z) = \sum_{i=1}^N T_i[\kappa_i(z)]. \quad (5)$$

Differentiating both sides of (5) with respect to  $z$  gives

$$\widehat{T}'(z) = \sum_{i=1}^N \kappa_i'(z) T_i'[\kappa_i(z)]. \quad (6)$$

Since  $\sum_{i=1}^N [\kappa_i'(z)] = 1$ , the absolute cautiousness of  $\widehat{u}(\cdot)$  can be viewed as a weighted average of the individuals' absolute cautiousness (evaluated under the sharing rule).

We now consider the effect of background risk on the group's investment decision. Note that the portfolio choice problem in (3) is no different from the one faced by a *single* decision-maker (normative representative agent) with utility function  $\widehat{u}(\cdot)$ . Thus, according to the variant of Proposition 6 in Kimball (1993, p.610), any independent background risk  $\widetilde{y}$  that raises the representative agent's expected marginal utility under the optimal choice  $\alpha^*$ , i.e.,

$$E[\widehat{u}'(\omega + \alpha^* \widetilde{x} + \widetilde{y})] \geq E[\widehat{u}'(\omega + \alpha^* \widetilde{x})],$$

will lower the absolute value of  $\alpha^*$  if and only if  $\widehat{u}(\cdot)$  exhibits standard risk aversion, i.e., when both  $\widehat{A}(\cdot)$  and  $\widehat{P}(\cdot)$  are decreasing functions.

The conditions for a decreasing  $\widehat{A}(\cdot)$  are well-known in existing literature. From (4) and (5), it is obvious that if  $T_i(\cdot)$  is an increasing function (or equivalently,  $A_i(\cdot)$  is a decreasing function) for all  $i$ , then  $\widehat{A}(\cdot)$  must be decreasing. The relation between  $P_i(\cdot)$  and  $\widehat{P}(\cdot)$ , on the other hand, is less explored. Our first result is intended to fill this gap. Unless otherwise stated, all proofs can be found in the Appendix.

**Lemma 1** *The representative agent's absolute prudence is given by*

$$\widehat{P}(z) \equiv \sum_{i=1}^N [\kappa_i'(z)]^2 P_i[\kappa_i(z)], \quad (7)$$

with first derivative

$$\widehat{P}'(z) = \sum_{i=1}^N [\kappa_i'(z)]^3 P_i'[\kappa_i(z)] + \frac{2}{[\widehat{T}(z)]^2} \sum_{i=1}^N \kappa_i'(z) \left\{ T_i'[\kappa_i(z)] - \widehat{T}'(z) \right\}^2. \quad (8)$$

**Proof of Lemma 1** Differentiating  $T_i(c) \equiv -u_i'(c)/u_i''(c)$  with respect to  $c$  gives  $T_i'(c) = -1 + T_i(c)P_i(c)$  for all  $c > 0$ . The counterpart for  $\widehat{u}(\cdot)$  is  $\widehat{T}'(z) = -1 + \widehat{T}(z)\widehat{P}(z)$  for all  $z > 0$ .

Substituting these questions into (6), and using  $\sum_{i=1}^N [\kappa'_i(z)] = 1$  gives

$$\widehat{T}(z) \widehat{P}(z) = \sum_{i=1}^N \kappa'_i(z) T_i[\kappa_i(z)] P_i[\kappa_i(z)].$$

Equation (7) follows immediately by rearranging terms and applying (4). Next, differentiating (7) with respect to  $z$  gives

$$\widehat{P}'(z) = \sum_{i=1}^N [\kappa'_i(z)]^3 P'_i[\kappa_i(z)] + 2 \sum_{i=1}^N \kappa'_i(z) \kappa''_i(z) P_i[\kappa_i(z)].$$

Differentiating (4) with respect to  $z$  gives

$$\kappa''_i(z) = \frac{\kappa'_i(z)}{\widehat{T}(z)} \left\{ T'_i[\kappa_i(z)] - \widehat{T}'(z) \right\}.$$

Equation (8) can be obtained by combining the last two equations. This completes the proof. ■

Equation (8) shows that the first derivative of  $\widehat{P}(\cdot)$  can be decomposed into two parts: The first part captures the effects of  $P'_i(\cdot)$  on  $\widehat{P}'(\cdot)$ . In particular, this term is negative if all group members have decreasing absolute prudence. The second term captures the effects due to the heterogeneity in absolute cautiousness across group members. Since  $\widehat{T}'(z)$  is the weighted average of  $\{T'_i[\kappa_i(z)]\}_{i=1}^N$  under the set of weights  $\{\kappa'_i(z)\}_{i=1}^N$ , the expression  $\sum_{i=1}^N \kappa'_i(z) \left\{ T'_i[\kappa_i(z)] - \widehat{T}'(z) \right\}^2$  is the variance of absolute cautiousness among the group members, which is always positive. This positive term suggests that efficient risk sharing has a tendency to raise the slope of  $\widehat{P}(\cdot)$ . Thus, even if all members have DAP preferences, the representative agent may not have the same attribute. This proves that standard risk aversion at the individual level does not imply standard risk aversion at the group level under an efficient risk-sharing arrangement.

This is not the first study that points to a potential discordance between individual and collective preferences under this type of arrangement. Hara *et al.* (2007) examine the second derivative of  $\widehat{T}(z)$  and the first derivative of  $\widehat{T}(z)/z$  in a similar model but without background risk.<sup>5</sup> They find that efficient risk sharing has a tendency to make  $\widehat{T}(z)$  a convex function and increase the slope of  $\widehat{T}(z)/z$ . Thus, even if all group members have concave absolute risk tolerance or increasing relative risk aversion (which is equivalent to a decreasing  $T_i(c)/c$ ), the representative agent may not have these characteristics. The concavity of  $\widehat{T}(\cdot)$  is of particular interest here due to the following

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<sup>5</sup>The function  $\widehat{T}(z)/z$  is the reciprocal of the relative risk aversion for the representative agent. Hara *et al.* (2007) refer to this as relative risk tolerance.

observation.<sup>6</sup>

**Lemma 2** *If  $\widehat{T}(\cdot)$  is increasing concave, then  $\widehat{P}(\cdot)$  is decreasing and  $\widehat{u}(\cdot)$  is standard.*

Lemma 2 suggests one way to establish the standardness of  $\widehat{u}(\cdot)$ . The next question is under what conditions will  $\widehat{T}(\cdot)$  be a concave function. Hara *et al.* (2007) have already shown that it is not enough to have a concave  $T_i(\cdot)$  for all  $i$ . This prompts us to consider a stronger form of concavity, which is the notion of “ $\rho$ -concavity” as discussed in Caplin and Nalebuff (1991).

For any  $\rho \in [-\infty, \infty]$ , a nonnegative function  $g(\cdot)$  is called  $\rho$ -concave if the transformed function  $\widetilde{g}(x) \equiv [g(x)]^\rho / \rho$  is concave. Since  $g(\cdot)$  and  $\widetilde{g}(\cdot)$  are equivalent when  $\rho = 1$ , the usual notion of concavity corresponds to the case of  $\rho = 1$ . Quasi-concavity and logconcavity of  $g(\cdot)$  correspond, respectively, to the cases of  $\rho = -\infty$  and  $\rho = 0$ . In general, if  $g(\cdot)$  is  $\rho_1$ -concave, then it is also  $\rho_2$ -concave for all  $\rho_2 \leq \rho_1$ . If both  $g(\cdot)$  and  $\widetilde{g}(\cdot)$  are twice differentiable, then  $g(\cdot)$  is  $\rho$ -concave if and only if

$$g(x)g''(x) \leq (1 - \rho) [g'(x)]^2, \quad \text{for all } x.$$

The main result of this paper is to show that if each group member’s absolute risk tolerance is  $\rho$ -concave, for some  $\rho \geq 2$ , then the representative agent’s absolute risk tolerance is a concave function. This result holds regardless of whether  $T_i(\cdot)$  is monotonic. It follows that if each  $T_i(\cdot)$  is increasing and  $\rho$ -concave, for some  $\rho \geq 2$ , then  $\widehat{T}(\cdot)$  is increasing concave and  $\widehat{u}(\cdot)$  is standard.<sup>7</sup>

**Theorem 3** *Suppose for each  $i \in \{1, 2, \dots, N\}$ ,  $T_i(\cdot)$  is  $\rho$ -concave, for some  $\rho \geq 2$ , then  $\widehat{T}(\cdot)$  is a concave function. If, in addition, each  $T_i(\cdot)$  is increasing, then  $\widehat{u}(\cdot)$  is standard.*

**Proof of Theorem 3** As shown in Theorem 4 of Hara *et al.* (2007), the second derivative of  $\widehat{T}(z)$  can be expressed as

$$\widehat{T}''(z) = \sum_{i=1}^N [\kappa_i'(z)]^2 T_i''[\kappa_i(z)] + \frac{1}{\widehat{T}(z)} \sum_{i=1}^N \kappa_i'(z) T_i'[\kappa_i(z)] \left\{ T_i'[\kappa_i(z)] - \widehat{T}'(z) \right\}.$$

Using (4) and (6), we can rewrite this as

$$\widehat{T}''(z) = \frac{1}{\widehat{T}(z)} \sum_{i=1}^N [\kappa_i'(z)] \left\{ T_i''[\kappa_i(z)] T_i[\kappa_i(z)] + (T_i'[\kappa_i(z)])^2 \right\} - \frac{[\widehat{T}'(z)]^2}{\widehat{T}(z)}.$$

<sup>6</sup>This result has appeared in Gollier (2001b, p.166). Its proof follows immediately by noting that  $\widehat{P}(z) > 0$ ,  $\widehat{T}(z) > 0$  and  $\widehat{T}''(z) = \widehat{T}'(z) \widehat{P}(z) + \widehat{T}(z) \widehat{P}'(z)$  for all  $z > 0$ .

<sup>7</sup>If  $T_i(\cdot)$  is increasing and  $\rho$ -concave for some  $\rho \geq 2$ , then it is increasing and concave in the usual sense. Thus, by Lemma 2,  $u_i(\cdot)$  has standard risk aversion.

Thus, it suffice to show that  $T_i''(c)T_i(c) + [T_i'(c)]^2 \leq 0$  for all  $c \geq 0$  and for all  $i$ . If  $T_i(\cdot)$  is  $\rho$ -concave for some  $\rho \geq 2$ , then we have  $T_i''(c)T_i(c) \leq (1 - \rho) [T_i'(c)]^2$ , which implies

$$T_i''(c)T_i(c) + [T_i'(c)]^2 \leq (2 - \rho) [T_i'(c)]^2 \leq 0.$$

This completes the proof. ■

In the economics literature, the assumption of  $\rho$ -concavity is typically imposed on the density function of some distributions.<sup>8</sup> To the best of our knowledge, this is the first study that applies this type of concavity to characterize risk preferences. Suppose individuals' absolute risk tolerance takes a power form as in Gollier (2001a, p.189), i.e.,  $T_i(c) = \alpha_i c^{\phi_i}$ , for some constants  $\alpha_i > 0$  and  $\phi_i \geq 0$ . Then  $T_i(c)$  is  $\rho$ -concave for some  $\rho \geq 2$  if and only if  $\phi_i \leq 0.5$ .

Theorem 3 has a number of implications regarding the representative agent's risk preferences. First, if  $\hat{u}(\cdot)$  has standard risk aversion then it also exhibits proper risk aversion and risk vulnerability as defined in Pratt and Zeckhauser (1987) and Gollier and Pratt (1996). Second, a decreasing  $\hat{P}(\cdot)$  also implies that the fourth derivative of  $\hat{u}(\cdot)$  is negative. Apps *et al.* (2014) show that this property is not true in general even if  $u_i''''(\cdot) < 0$  for all  $i$ .

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<sup>8</sup>For instance, Caplin and Nalebuff (1991) impose this assumption on the distribution of voters' characteristics in a voting model; Ewerhart (2013) applies this on the distribution of bidders' characteristics in auction models.

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