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# On the extension of a preorder under translation invariance

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## Abstract

This paper proves the existence, for a translation-invariant preorder on a divisible commutative group, of a complete preorder extending the preorder in question and satisfying *translation invariance*. We also prove that the extension may inherit a property of continuity. As an application, we prove the existence of a complete translation-invariant strict preorder on  $\mathbb{R}$  which transgresses *scalar invariance* and also the existence of a complete translation-invariant preorder satisfying the social choice axioms *strong Pareto* and *fixed-step-anonymity* on a set  $X^{\mathbb{N}_0}$ , where  $X$  is a divisible commutative group. Moreover, the two extension results are used to make *scalar invariance* appear as a consequence of *translation invariance* under a continuity requirement or under a Pareto axiom.

## 1- Introduction

(Szpilrajn 1930) extension theorem may be stated as follows. For any reflexive and transitive binary relation (i.e. a preorder) on a given set, there exists a complete preorder which is an extension of the given preorder. Szpilrajn theorem proved of great utility in mathematical social choice theory as in some other branches of mathematics. There exists today stronger versions of Szpilrajn theorem, requiring weaker assumptions on the initial binary relation or imposing additional conditions on the relation extension. We refer to (Alcantud-Diaz 2014) for an overview on the applications and extensions of Szpilrajn theorem.

The present paper establishes the existence, for any preorder on a divisible commutative group satisfying *translation invariance*, of a complete preorder extending the given preorder and satisfying *translation invariance* (section 3, theorem 1). In (Demuyneck-Lauwers 2009) the existence of an extension under the conditions *translation invariance* and *scalar invariance* is proven. However, the result proved here is stronger in the sense that it is freed from the *scalar invariance* assumption. The proof of theorem 1 follows the same diagram as the proof of Szpilrajn theorem. Starting from a preorder satisfying *translation invariance*, one adds comparisons on some pairs of alternatives in such a way that *translation invariance* remains satisfied. Then, an argument based on Zorn's lemma makes it possible to extend the procedure to the whole space.

We also prove a second extension theorem which asserts that the former extension result (theorem 1) holds under an additional requirement of continuity (section 4, theorem 2). The proof is an adaptation of the proof of (Jaffray 1975) to the *translation invariance* case. It relies on the construction of a relation that

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<sup>1</sup>I am grateful to an anonymous referee who, when reviewing another paper, guided me towards the issue of extending a preorder under translation-invariance.

is used to "clean" the extended preorder given by theorem 1 from undesirable rankings that transgress the continuity requirement.

As an application, we give two examples, the first of which shows the existence of a complete translation-invariant strict preorder on  $\mathbb{R}$  which transgresses *scalar invariance* and the second shows the existence of a complete translation-invariant preorder satisfying the social choice axioms *strong Pareto* and *fixed-step-anonymity* on a set  $X^{\mathbb{N}_0}$ , where  $X$  is a divisible commutative group.

Moreover, theorems 1 and 2 are used to make *scalar invariance* appear as a consequence of *translation invariance* under a continuity requirement (Corollary 2, section 5) or under a Pareto axiom (Theorem 3, section 6).

## 2- Preliminaries

$\mathbb{N}_0$  is the set of positive integers.  $Q$  is the set of rational numbers.  $(X, +)$  is a divisible commutative group.  $B$  being a binary relation on  $X$  and  $x, y$  two elements of  $X$ ,  $xB y$  is denoted  $x \succsim_B y$ ,  $[xB y \text{ and } \text{non}(yB x)]$  is denoted  $x \succ_B y$  and  $[xB y \text{ and } yB x]$  is denoted  $x \sim_B y$ . The symbols  $\leq, \geq, <, >$  are used for the natural order on  $\mathbb{R}$ , except in example 1, section 4. A reflexive and transitive binary relation on  $X$  is a preorder on  $X$ . If, on top of that, for all  $x, y$  either  $x \succsim_B y$  or  $x \precsim_B y$ , it is a complete preorder. A binary relation  $B_1$  is said to be a subrelation to a binary relation  $B_2$ , or  $B_2$  an extension of  $B_1$ , if for all  $x, y$  in  $X$ ,

$$x \succsim_{B_1} y \implies x \succsim_{B_2} y$$

and

$$x \succ_{B_1} y \implies x \succ_{B_2} y$$

**Axiom translation invariance (TI)** A preorder  $R$  satisfies *translation invariance* if:

$$\forall (x, y) \in X \times X, \forall u \in X, [x \succsim_R y \implies x + u \succsim_R y + u]$$

**Axiom division invariance (DI)** A preorder  $R$  satisfies *division-invariance* if:

$$\forall x \in X, \forall n \in \mathbb{N}, \left[ x \succsim_R y \implies \frac{1}{n}x \succsim_R \frac{1}{n}y \right]$$

**Lemma 1** If a preorder  $R$  on  $X$  satisfies **TI**, then there exists a preorder  $\hat{R}$  on  $X$  of which  $R$  is a subrelation and such that  $\hat{R}$  satisfies **TI** and **DI**.

**Proof:** First, notice that under  $R$ , it is possible to sum inequalities. Indeed, by **TI**, if  $a, b, u, v$  are such that  $a \succsim_R b$  and  $u \succsim_R v$ , then  $a + u \succsim_R b + u$  and  $b + u \succsim_R b + v$ . By transitivity,  $a + u \succsim_R b + v$ . For each positive integer  $n$ , consider the binary relation  $R_n$  defined by

$$x \succsim_{R_n} y \text{ iff } nx \succsim_R ny$$

If  $x, y$  are such that  $x \succsim_R y$ , we can sum  $n$  times this inequality. Thus,  $x \succsim_{R_n} y$ . Likewise, it is easily seen that  $x \succ_R y$  implies  $x \succ_{R_n} y$ . As a result,

$R$  is a subrelation to  $R_n$ . Moreover,  $R_n$  is reflexive and transitive. It is easily checked that  $R_n$  satisfies **TI**.

Consider the binary relation

$$\widehat{R} = \cup_{n \in \mathbb{N}_0} R_n$$

defined on  $X$  by  $x \lesssim_{\widehat{R}} y$  iff there is  $n$  such that  $x \lesssim_{R_n} y$ .

$R$  is a subrelation to  $\widehat{R}$ . Moreover,  $\widehat{R}$  is reflexive and transitive. It is a preorder. Since for each positive integer  $n$ ,  $R_n$  satisfies **TI**, we deduce that  $\widehat{R}$  satisfies **TI**. The lemma is proved if we show that  $\widehat{R}$  satisfies **DI**. Let  $n$  be a positive integer, and  $x, y$  such that  $x \lesssim_{\widehat{R}} y$ . There exists a positive integer  $m$  such that  $x \lesssim_{R_m} y$ . Thus  $mx \lesssim_R my$ . We can write that as  $mn(\frac{1}{n}x) \lesssim_R mn(\frac{1}{n}y)$ . Thus  $\frac{1}{n}x \lesssim_{R_{mn}} \frac{1}{n}y$ , what implies  $\frac{1}{n}x \lesssim_{\widehat{R}} \frac{1}{n}y$ .  $\widehat{R}$  satisfies **DI**. ■

**Remark 1** (1) It is easily seen that  $\widehat{R}$  is the minimal preorder satisfying **TI** and **DI**, of which  $R$  is a subrelation. (2) If  $R$  is complete, since  $R$  is a subrelation to  $\widehat{R}$ , we have necessarily  $R = \widehat{R}$ . This shows that if the preorder is complete, **TI** implies **DI**.

### 3- The translation-invariant extension theorem

**Theorem 1** Let  $R$  be a preorder on  $X$  satisfying **TI**. Then there exists a complete preorder on  $X$  satisfying **TI**, of which  $R$  is a subrelation.

**Proof:** If  $R$  is a complete preorder, there is nothing to prove. Suppose that  $R$  is not complete. Consider the preorder  $\widehat{R}$  built in the proof of lemma 1, and the set  $\mathfrak{R}$  of all preorders on  $X$  satisfying **TI** and **DI**, and of which  $R$  is a subrelation.  $\mathfrak{R}$  is not empty since  $\widehat{R} \in \mathfrak{R}$ . Let  $(R_\alpha)$  be a chain in  $\mathfrak{R}$ , i.e. for any  $\alpha, \alpha'$ ,  $R_\alpha$  is a subrelation to  $R_{\alpha'}$  or  $R_{\alpha'}$  is a subrelation to  $R_\alpha$ . Notice that (1) the relation  $\cup_\alpha (R_\alpha)$  defined on  $X$  by:  $x [\cup_\alpha (R_\alpha)] y$  iff there is  $\alpha$  such that  $x R_\alpha y$ , is a preorder, (2) it satisfies **TI** and **DI**, (3)  $R$  is a subrelation to  $\cup_\alpha (R_\alpha)$ , (4) for all  $\alpha$ ,  $R_\alpha$  is a subrelation to  $\cup_\alpha (R_\alpha)$ . Hence, in the set  $\mathfrak{R}$ , every chain admits an upper bound. According to Zorn's lemma,  $\mathfrak{R}$  admits at least a maximal element. Denote  $M$  such a maximal element in  $\mathfrak{R}$ . Suppose we can prove the following claim:

**Claim 1** For any non complete  $R'$  in  $\mathfrak{R}$  and any pair of  $R'$ -incomparable alternatives  $(x_0, y_0)$ , there exists a preorder  $R'_1$  in  $\mathfrak{R}$  to which  $R'$  is a subrelation and such that  $x_0$  and  $y_0$  are  $R'_1$ -comparable.

Then, if  $M$  were not complete, there would exist a preorder in  $\mathfrak{R}$  to which  $M$  is a strict subrelation. This would contradict that  $M$  is maximal in  $\mathfrak{R}$ . Therefore, if the claim holds,  $M$  would be necessarily complete.  $M$  would be the preorder we are looking for.

What remains of the proof is devoted to establish claim 1. This is done through the following 6 steps.

If there is no non complete preorder in  $\mathfrak{R}$ , the theorem is proved since  $\mathfrak{R}$  is not empty. Let  $R'$  be a non complete preorder in  $\mathfrak{R}$  and  $x_0, y_0$  be two  $R'$ -incomparable elements of  $X$ .

Consider the binary relation  $B$  on  $X$ :  $x \lesssim_B y$  iff either  $x \lesssim_{R'} y$  or there is a positive rational  $q$  such that  $x - y = q(x_0 - y_0)$ .

We prove successively that the two clauses of the definition of  $B$  are exclusive (step 1), that the indifference relations are equal (step 2), that  $R'$  is a subrelation to  $B$  (step 3), that  $B$  is weakly acyclic (this prepares for transitivity) (step 4), that  $R'$  is a subrelation to the transitive closure of  $B$  (step 5), that the transitive closure of  $B$  satisfies **TI** and **DI** (step 6). The transitive closure of  $B$  is then the required preorder.

**Step 1: the two clauses are exclusive.** If there is a positive rational  $q$  such that  $x - y = q(x_0 - y_0)$ , then  $x, y$  are  $R'$ -incomparable. Suppose not. For instance suppose  $x \succ_{R'} y$ . By **TI**,  $x - y \succ_{R'} 0$ . By **DI**, for all positive integer  $n$ ,  $\frac{1}{n}(x - y) \succ_{R'} 0$ . Recall that it is possible to sum inequalities (see the proof of lemma 1). We sum  $m$  times the inequality  $\frac{1}{n}(x - y) \succ_{R'} 0$ ,  $m$  being a positive integer. We obtain  $\frac{m}{n}(x - y) \succ_{R'} 0$ . Take  $\frac{n}{m} = q$ . It gives  $x_0 - y_0 \succ_{R'} 0$ , what contradicts  $x_0, y_0$  being incomparable. The case  $y \succ_{R'} x$  is similar.

**Step 2: equivalence of indifferences.** Clearly,  $x \sim_{R'} y \Rightarrow x \sim_B y$ . We show now that  $x \sim_B y$  entails  $x \sim_{R'} y$ . According to the definition of  $B$ , it is enough to prove that  $x$  and  $y$  are necessarily  $R'$ -comparable. Suppose not. Then  $x \succ_B y$  implies that there is some positive rational  $q$  such that  $x - y = q(x_0 - y_0)$ . We have also  $y \succ_B x$ . Thus, for some positive rational  $q'$ ,  $y - x = q'(x_0 - y_0)$ . We see that this gives  $q'(x_0 - y_0) = -q(x_0 - y_0)$ , what implies  $x_0 - y_0 = 0$  because  $q, q'$  are both positive. But that contradicts  $x_0, y_0$  being  $R'$ -incomparable.

**Step 3:  $R'$  is a subrelation to  $B$ .** This is a direct consequence of  $x \succ_{R'} y \Rightarrow x \succ_B y$  (definition of  $B$ ) and  $x \sim_B y \Leftrightarrow x \sim_{R'} y$  (step 2).

**Step 4:  $B$  is weakly-acyclic.** We show that for all  $x, y, z$  in  $X$ :  $x \succ_B y$  and  $y \succ_B z \Rightarrow x \succ_B z$  or  $\text{non}(z \succ_B x)$ .

One of the four following cases is implied by  $x \succ_B y$  and  $y \succ_B z$ . (1)  $x \succ_{R'} y$  and  $y \succ_{R'} z$ , (2) there are  $q, q'$  such that  $x - y = q(x_0 - y_0)$  and  $y - z = q'(x_0 - y_0)$ , (3)  $x \succ_{R'} y$  and there is  $q'$  such that  $y - z = q'(x_0 - y_0)$ , (4) there is  $q$  such that  $x - y = q(x_0 - y_0)$  and  $y \succ_{R'} z$ . Consider successively the four cases:

(1) By transitivity of  $R'$ :  $x \succ_{R'} z$ . Thus,  $x \succ_B z$ .

(2)  $x - y = q(x_0 - y_0)$  and  $y - z = q'(x_0 - y_0)$  entails  $x - z = (q + q')(x_0 - y_0)$ . Thus  $x \succ_B z$ .

(3) Suppose we had  $z \succ_B x$ . We would have either  $z \succ_{R'} x$  or  $z - x = q''(x_0 - y_0)$ . Both possibilities contradict  $x \succ_{R'} y$  and  $y - z = q'(x_0 - y_0)$ . Indeed, with  $x \succ_{R'} y$ ,  $z \succ_{R'} x$  gives  $z \succ_{R'} y$  what contradicts  $y - z = q'(x_0 - y_0)$  (step 1); whereas  $y - z = q'(x_0 - y_0)$  with  $z - x = q''(x_0 - y_0)$  implies  $y - x = (q' + q'')(x_0 - y_0)$ , what contradicts  $x \succ_{R'} y$ . As a result, we have  $\text{non}(z \succ_B x)$ .

(4) This case is similar to (3)

**Remark 2** Let  $x, y, z$  be such that  $x \succ_B y$  and  $y \succ_B z$ . Weak acyclicity entails that if one of the comparisons  $x \succ_B y$  and  $y \succ_B z$  is a strict preference, then either the comparison on  $(x, z)$  is  $x \succ_B z$  or  $x$  and  $z$  are  $B$ -incomparable.

**Step 5:  $R'$  is a subrelation to the transitive closure of  $B$ .** Consider  $\overline{B}$  the transitive closure of  $B$  defined by:  $x \succ_{\overline{B}} y$  if there is a sequence  $(z_i)_{i=1}^n$  such that  $x \succ_B z_1, z_1 \succ_B z_2, \dots$  and  $z_n \succ_B y$ . It is clear that  $x \succ_{R'} y$  implies  $x \succ_{\overline{B}} y$  (step 3:  $R'$  is a subrelation to  $B$ ). It is enough to prove that  $x \succ_{\overline{B}} y$  implies  $\text{non}(y \succ_{R'} x)$ .

For a positive integer  $n$ , consider the statement  $Q_n$ : "If there is a sequence  $(z_i)_{i=1}^n$  such that  $x \lesssim_B z_1 \lesssim_B z_2 \dots \lesssim_B z_n \lesssim_B y$ , then  $\text{non}(y \succ_{R'} x)$ ." Let's prove by induction that  $Q_n$  is true for all positive integers. Notice that when the sequence  $(z_i)$  has  $n$  terms, there is  $n + 1$  successive comparisons.

$n = 1$  : We have  $x \lesssim_B z_1 \lesssim_B y$ . By step 4, we have  $x \lesssim_B y$  or  $\text{non}(y \succ_{R'} x)$ . Both possibilities contradict  $y \succ_{R'} x$ . So, we have  $\text{non}(y \succ_{R'} x)$ .

Suppose that  $Q_n$  is true and let's show that  $Q_{n+1}$  is true. Consider the sequence of  $n + 2$  comparisons:  $x \lesssim_B z_1 \lesssim_B z_2 \dots \lesssim_B z_{n+1} \lesssim_B y$ .

Each one of these comparisons comes either from the clause  $x \lesssim_{R'} y$  or the clause  $x - y = q(x_0 - y_0)$  of the definition of  $B$ . If there is two successive comparisons coming from the clause  $x \lesssim_{R'} y$ , say  $z_p \lesssim_{R'} z_{p+1} \lesssim_{R'} z_{p+2}$  (with  $p = 0, \dots, n + 2$  and the convention:  $z_0 = x$  and  $z_{n+2} = y$ ), by transitivity of  $R'$  we have:  $x \lesssim_B \dots z_p \lesssim_B z_{p+2} \dots \lesssim_B y$  which constitutes a sequence of  $n + 1$  comparisons. By  $Q_n$  we have  $\text{non}(y \succ_{R'} x)$ . If there is two successive comparisons coming from the clause  $x - y = q(x_0 - y_0)$ , say  $z_p \lesssim_B z_{p+1} \lesssim_B z_{p+2}$ , then  $z_p - z_{p+1} = q(x_0 - y_0)$  and  $z_{p+1} - z_{p+2} = q'(x_0 - y_0)$ . Thus,  $z_p - z_{p+2} = (q + q')(x_0 - y_0)$  so that  $z_p \lesssim_B z_{p+2}$ . We have again reduced the number of comparisons to  $n + 1$ . Thus, we have also  $\text{non}(y \succ_{R'} x)$ . It remains to consider the cases where the comparisons are alternate. Two cases must be considered:  $n + 2$  even and  $n + 2$  odd.

$n + 2$  even: The sequence of comparisons either begin or ends with a comparison from  $R'$ . Suppose it begins with a comparison from  $R'$ :  $x \lesssim_{R'} z_1 \lesssim_B z_2 \dots \lesssim_{R'} z_{n+1} \lesssim_B y$ . Apply  $Q_n$  to  $z_1 \lesssim_B z_2 \dots \lesssim_{R'} z_{n+1} \lesssim_B y$ . It gives  $\text{non}(y \succ_{R'} z_1)$ . Since  $x \lesssim_{R'} z_1$ , we cannot have  $y \succ_{R'} x$ . If the sequence of comparisons ends with a comparison from  $R'$ , the proof is similar. So it is omitted.

$n + 2$  odd: If the sequence of comparisons begins with a comparison from  $R'$ , the proof is also similar. So it is omitted. If the sequence of comparisons begins with a comparison from the clause  $x - y = q(x_0 - y_0)$ , we have

$$x \lesssim_B z_1 \lesssim_{R'} z_2 \dots \lesssim_{R'} z_{n+1} \lesssim_B y \quad (1)$$

Denote  $(x, z_1)$  by  $(\alpha_1, \beta_1)$ ,  $(z_2, z_3)$  by  $(\alpha_2, \beta_2) \dots (z_{2(p-1)}, z_{2p-1})$  by  $(\alpha_p, \beta_p)$  with  $p = 1, \dots, \frac{n+1}{2}$  and the convention  $z_0 = x$  and  $z_{n+2} = y$ . Since comparisons  $x \lesssim_B z_1, z_2 \lesssim_B z_3 \dots z_{n-1} \lesssim_B z_n, z_{n+1} \lesssim_B y$  come from the clause  $x - y = q(x_0 - y_0)$ , we have  $\alpha_p - \beta_p = q_p(x_0 - y_0)$  for  $p = 1, \dots, \frac{n+3}{2}$ . Moreover, according to (1),  $\beta_p \lesssim_{R'} \alpha_{p+1}$  for  $p = 1, \dots, \frac{n+1}{2}$ . Thus

$$\begin{aligned} \alpha_1 - q_1(x_0 - y_0) &\lesssim_{R'} \alpha_2 \\ \alpha_2 - q_2(x_0 - y_0) &\lesssim_{R'} \alpha_3 \\ &\dots \\ \alpha_{(n+1)/2} - q_{(n+1)/2}(x_0 - y_0) &\lesssim_{R'} \alpha_{(n+3)/2} \end{aligned}$$

We can sum these inequalities (this is established in the proof of lemma 1).

We obtain

$$\alpha_1 + \sum_2^{(n+1)/2} \alpha_p - \sum_2^{(n+1)/2} q_p(x_0 - y_0) \succsim_{R'} \sum_2^{(n+1)/2} \alpha_p + \alpha_{(n+3)/2}$$

By **TI** we obtain

$$\alpha_1 - \sum_1^{(n+1)/2} q_p(x_0 - y_0) \succsim_{R'} \alpha_{(n+3)/2}$$

But  $\alpha_1 = x_1$  and  $\alpha_{(n+3)/2} = y$ . Denote  $q = \sum_1^{(n+1)/2} q_p$ . Thus

$$x - q(x_0 - y_0) \succsim_{R'} y$$

By **TI**,  $x - y \succsim_{R'} q(x_0 - y_0)$ . If we had  $y \succ_{R'} x$ , it would give  $0 \succ_{R'} x - y \succsim_{R'} q(x_0 - y_0)$ . By transitivity of  $R'$  and by **TI**,  $x_0$  and  $y_0$  would be  $R'$ -comparable, which is not the case. As a result, we have  $\text{non}(y \succ_{R'} x)$ . Step 5 is proved.

**Remark 3**  $R'$  is a subrelation to  $\overline{B}$ , but  $B$  is not.

**Step 6:**  $\overline{B}$  satisfies **TI**. As  $R'$  is translation-invariant,  $B$  is clearly translation-invariant. It is easily deduced that  $\overline{B}$  is also translation-invariant. Likewise, it is easily seen that  $\overline{B}$  satisfies **DI**. Thus,  $\overline{B}$  is the required preorder. ■

**Corollary 1** Let  $B$  be a reflexive binary relation satisfying **TI**. Then there exists a complete preorder satisfying **TI**, of which  $B$  is a subrelation, iff  $B$  is a subrelation to its transitive closure.

**Proof:** Necessity: the condition that  $B$  is a subrelation to its transitive closure is necessary and sufficient for the existence of a complete preorder of which  $B$  is a subrelation (Suzumura 1976, Bossert 2008). Sufficiency: denote  $\overline{B}$  the transitive closure of  $B$ . It easily seen that  $\overline{B}$  is a preorder satisfying **TI**. Apply theorem 1 to  $\overline{B}$  to deduce that there exists a complete preorder satisfying **TI**, of which  $\overline{B}$  is a subrelation. Since  $B$  is a subrelation to  $\overline{B}$ , it is also a subrelation to the complete preorder. ■

## 4- Examples of application

**Example 1:** A translation-invariant and complete strict preorder on  $\mathbb{R}$  with  $\pi < 0 < 1$ .

Notice that only in this example, the symbols  $\leq, \geq, <, >$  are used for something else than the natural order on  $\mathbb{R}$ . Consider the following binary relation  $\precsim$  on  $\mathbb{R}$ :

$$x \precsim y \text{ if there is two nonnegative rationals } q, q' \text{ such that } x - y = -q + q'\pi$$

$\precsim$  is reflexive, transitive and satisfies **TI**. Moreover,  $\precsim$  is a strict preorder, which means that  $x \precsim y$  and  $y \precsim x$  implies  $x = y$ . Indeed  $x - y = -q + q'\pi$  and  $y - x = -q_1 + q'_1\pi$  yields  $0 = (x - y) + (y - x) = -(q + q_1) + (q' + q'_1)\pi$ .

Thus  $(q + q_1) = (q' + q'_1)\pi$ . We must have  $q' + q'_1 = 0$  otherwise  $\pi$  would be rational. Thus we have also  $q + q_1 = 0$ . Since  $q, q_1, q', q'_1$  are nonnegative, we have  $q = q_1 = q' = q'_1 = 0$  and  $x = y$ .

Theorem 1 asserts the existence of a translation-invariant and complete pre-order, say  $\leq$ , of which  $\lesssim$  is a subrelation.  $\leq$  is strict like  $\lesssim$ . Observe that  $\leq$  respects the natural order of rationals. But it does not coincide with the natural order of reals. Moreover it does not satisfy *scalar invariance* since if you multiply  $0 < 1$  by  $\pi$  the inequality is reversed. Finally,  $\leq$  is not continuous. Consider a positive sequence of rational  $(q_n)$  such that  $\lim q_n = \frac{1}{\pi}$ . **TI** allows to multiply an inequality by a positive rational. Multiplying  $\pi < 0$  by  $q_n$  yields  $q_n\pi < q_n \cdot 0 = 0$  for all  $n$ . But  $\lim q_n\pi = 1 > 0$ . A question then arises: can *scalar invariance* still be transgressed under **TI** and continuity? An answer is provided in section 5.

**Example 2:** Existence of a translation-invariant, strong-Pareto, fixed-step-anonymous and complete preorder on  $X^{\mathbb{N}_0}$ , where  $X$  is a divisible commutative group equipped with a complete preorder  $R$  satisfying **TI**.

It is possible to demonstrate the existence of such a preorder using the ultrafilter technique, as in (Fleurbaey-Michel 2003, Lauwers 2009). We demonstrate here this existence without using ultrafilters, which are highly nonconstructive objects. Although our theorem 1 also makes use of the axiom of choice, one may consider that our method is nevertheless more constructive in the sense that it indicates the concrete steps of adding comparisons.

Let  $Y = X^{\mathbb{N}_0}$ , let  $R'$  be a preorder on  $Y$ . We first give the following definitions:

**Fixed-step permutation:** (Fleurbaey-Michel 2003)  $\sigma$  is a fixed-step permutation if there exist  $k \in \mathbb{N}_0$  such that for all  $n \in \mathbb{N}_0$ ,  $\sigma(\{1, \dots, kn\}) = \{1, \dots, kn\}$ .

**Axiom fixed-step-anonymity:** Denote  $\sigma(x)$  the sequence obtained by permuting the components of  $x \in Y$  according to the permutation  $\sigma$ .  $R'$  is fixed-step-anonymous if for all  $x \in Y$  and fixed-step permutation  $\sigma$ , we have  $x \sim_{R'} \sigma(x)$ .

**Axiom strong Pareto:**  $R'$  is strong Pareto if, for all  $x, y \in Y$  such that  $\forall i \in \mathbb{N}_0$   $x_i \lesssim_R y_i$  and  $x_j \succ_R y_j$  for some  $j$ , we have  $x \succ_{R'} y$  ( $x_i, y_i$  denote the  $i^{th}$  component of resp.  $x, y$ ).

Pareto axioms capture the idea that an increase of the components of a vector must increase the ranking of the vector. Anonymity axioms express a requirement of symmetry in the treatment of individuals or dates.

**The fixed-step catching-up SC.** For all  $x, y \in \mathbb{R}^{\mathbb{N}_0}$ ,  $x \lesssim_{SC} y$  iff there exist  $k, m \in \mathbb{N}_0$  such that, for all  $n \in \mathbb{N}_0$  with  $n > m$ , we have

$$\sum_{i=1}^{kn} x_i \geq \sum_{i=1}^{kn} y_i$$

$SC$  is a fixed-step-anonymous preorder (Fleurbaey-Michel 2003).

**Proposition 1:** There exists a translation-invariant, strong-Pareto, fixed-step-anonymous and complete preorder on  $\mathbb{R}^{\mathbb{N}_0}$ .

**Proof:** Apply theorem 1 to  $SC$ . There exists a translation-invariant and complete preorder  $R'$  on  $Y$  of which  $SC$  is a subrelation.  $SC$  being a subrelation to  $R'$  entails that  $R'$  satisfies *strong Pareto* and *fixed-step-anonymity*.  $R'$  is the required preorder. ■

## 5- *Scalar invariance* as a consequence of **TI** and a continuity requirement

For a given nontrivial preorder  $R$  on a divisible commutative group  $X$ ,  $\tau_+(R)$  is the associated upper-order-topology, i.e. the topology generated by the base of open intervals:  $\beta_+(R) = \{\{x \in X : x \prec_R a\}, a \in X\}$ .

**Theorem 2:** Let  $R$  be a preorder on  $X$  satisfying **TI**. Then there exists a complete preorder  $R'$  on  $X$  satisfying **TI**, of which  $R$  is a subrelation, and such that  $\tau_+(R') \subset \tau_+(R)$ .

**Proof:** The following proof is an adaptation of the proof of (Jaffray 1975) to a translation-invariant preorder. We start from a translation-invariant complete preorder which extends  $R$ , whose existence is guaranteed by theorem 1. We then apply a clause<sup>2</sup> to "clean up" rankings that do not respect the upper-order-topology. It turns out that this clause is also translation-invariant, which makes it possible to build the desired preorder.

Let  $R_1$  be a complete preorder extending  $R$  and satisfying **TI**. Let  $x, y \in X$ . Consider the following clause :

$C(x, y)$ : "There exists  $B \in \beta_+(R)$  containing  $x$  such that, for all  $B' \in \beta_+(R)$  containing  $y$ , we can find  $x' \in B'$  such that for all  $z \in B$ , we have  $z \prec_{R_1} x'$ "

Because  $R_1$  satisfies **TI**, it is easily seen that if  $C(x, y)$  is true,  $C(x+h, y+h)$  is true for all  $h$  in  $X$ . Moreover, if  $C(x, y)$  is true, it is clear that we cannot have  $C(y, x)$  true. Thus, we can define a asymmetric relation  $R_2$  checking **TI** as follows:  $x \prec_{R_2} y$  iff  $C(x, y)$  is true.

We prove now that  $R_2$  is negatively transitive, i.e.

$$\text{not}(x \prec_{R_2} y) \text{ and } \text{not}(y \prec_{R_2} z) \text{ implies } \text{not}(x \prec_{R_2} z)$$

We have:

$\text{Not}(x \prec_{R_2} y) \iff$  for all  $B_1 \in \beta_+(R)$  containing  $x$ , there exists  $B'_1 \in \beta_+(R)$  containing  $y$  such that [for all  $x'_1$  in  $B'_1$ , there exists  $x''_1$  in  $B_1$  such that  $x''_1 \succ_{R_1} x'_1$ ].

$\text{Not}(y \prec_{R_2} z) \iff$  for all  $B_2 \in \beta_+(R)$  containing  $y$ , there exists  $B'_2 \in \beta_+(R)$  containing  $z$  such that [for all  $x'_2$  in  $B'_2$ , there exists  $x''_2$  in  $B_2$  such that  $x''_2 \succ_{R_1} x'_2$ ].

Let  $B_1$  be in  $\beta_+(R)$  containing  $x$  and  $B'_1$  be the interval which existence is asserted by the clause " $\text{not}(x \prec_{R_2} y)$ ". Take  $B'_1$  as the interval  $B_2$  of the

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<sup>2</sup>This clause combines the two clauses proposed by (Jaffray 1975) in the proof of his theorem 1, the first of which defines a preorder on  $\beta_+(R)$  and the second a preorder on  $X$ .

clause " $\text{not}(y \prec_{R_3} z)$ ". Thus, there exists  $B'_2 \in \beta_+(R)$  containing  $z$  such that [for all  $x'_2$  in  $B'_2$ , there exists  $x''_2$  in  $B'_1$  such that  $x''_2 \succ_{R_1} x'_2$ ]. Now apply the clause " $\text{not}(x \prec_{R_2} y)$ " for  $x''_2$  instead of  $x'_1$  and deduce that there exists  $x'_1$  in  $B_1$  such that  $x'_1 \succ_{R_1} x''_2$ . By transitivity of  $R_2$ ,  $x'_1 \succ_{R_1} x''_2$  and  $x''_2 \succ_{R_1} x'_2$  gives  $x'_1 \succ_{R_1} x'_2$ .

Summing up: for some  $B_1$  in  $\beta_+(R)$  containing  $x$ , we have found  $B'_2 \in \beta_+(R)$  containing  $z$  such that [for all  $x'_2$  in  $B'_2$  there exists  $x''_1$  in  $B_1$  such that  $x''_1 \succ_{R_1} x'_1$ ]. This is exactly the clause  $\text{not}(x \prec_{R_2} z)$ .

Since asymmetry and negative transitivity imply transitivity,  $R_2$  is transitive.

Now let  $R'$  be the following binary relation:

$$x \prec_{R'} y \text{ iff } [(x \prec_{R_2} y) \text{ or } \text{not}(x \succ_{R_2} y)]$$

The transitivity and negative transitivity of  $R_2$  implies the transitivity of  $R'$ . Moreover,  $R'$  is complete and satisfies **TI**.

We show now that  $R$  is a subrelation to  $R'$ . Indeed, let  $x, y$  be such that  $x \prec_R y$ . In the clause  $C(x, y)$ , take  $B = \{z \in X : z \prec_R y\}$ . We have  $x \in B$  and for all  $B'$  containing  $y$ , we have  $z \prec_{R_1} y$  for all  $z \in B$ . Hence the clause  $C(x, y)$  is true and  $x \prec_{R_2} y$ . Consequently,  $x \prec_{R'} y$ . If  $x, y$  are such that  $x \sim_R y$ , the clause  $C(x, y)$  cannot be satisfied. To see it, it suffices to notice that an interval containing  $x$  necessarily contains  $y$  and vice versa. If we take  $B' = B$  in the clause  $C(x, y)$ , there is no  $x'$  in  $B$  such that for all  $z \in B$ , we have  $z \prec_{R_1} x'$ . Thus we have  $\text{not}(x \prec_{R_2} y)$ . In the same way, we have  $\text{not}(y \prec_{R_2} x)$ . Consequently,  $x \sim_{R'} y$ .

It remains to show that  $\tau_+(R') \subset \tau_+(R)$ . Let  $y \in X$ . We show that any subset in  $\beta_+(R')$ , the base of open intervals generating  $\tau_+(R')$ , is open with respect to  $\tau_+(R)$ . Let  $x \in B = \{z \in X : z \prec_{R'} y\}$ . By the definition of  $R'$ , there is  $B_x$  in  $\beta_+(R)$ , containing  $x$ , such that for all  $B_y \in \beta_+(R)$  containing  $y$ , we can find  $x' \in B_y$  such that for all  $z \in B_x$ , we have  $z \prec_{R_1} x'$ . We can see that this implies that for all  $z \in B_x$ , we have  $z \prec_{R'} y$ . Hence  $B_x \subset B$ . Recap: for all  $x$  in  $B$ , we found  $B_x$  in  $\beta_+(R)$  containing  $x$  such that  $B_x \subset B$ . As a result,  $B$  is a union of open sets of  $\tau_+(R)$ . It is thus an open set of  $\tau_+(R)$ . ■

**Remark 5:** Theorem 2 holds if we replace  $\tau_+(R)$  and  $\tau_+(R')$  respectively by  $\tau_-(R)$  and  $\tau_-(R')$  the lower-order-topologies.

**Remark 6:** The inclusion  $\tau_+(R') \subset \tau_+(R)$  entails the upper semicontinuity of the extension with respect to any topology on  $X$  stronger than  $\tau_+(R)$ . Upper semicontinuity is used here in the sense that lower sections  $\{x \in X : x \prec_R a\}$  are open. But it is not necessary for the topology on  $X$  to be stronger than  $\tau_+(R)$  to have the upper semicontinuity of the extension. For more information on this issue, see (Jaffray 1975), section 5.

**Axiom scalar invariance:** For all nonnegative real  $\alpha$  and vectors  $x, y$  in a real vector space equipped with a preorder  $R$ ,  $Y$ ,  $x \succ_R y \implies \alpha x \succ_R \alpha y$ .

**Corollary 2:** Let  $Y$  be a real normed vector space. Denote  $t$  the topology induced by the norm of  $Y$ . Let  $R$  be a preorder on  $Y$  satisfying **TI** and  $\tau_+(R) \subset t$ . Let  $R'$  be one of the complete preorders whose existence is asserted by theorem

2, i.e. a complete preorder of which  $R$  is a subrelation, satisfying **TI** and such that  $\tau_+(R') \subset \tau_+(R)$ . Then  $R'$  satisfies *scalar invariance*.

**Proof:** We have  $\tau_+(R') \subset t$ . Let  $\alpha$  be a nonnegative real and  $x, y$  two vectors in  $Y$  such that  $x \succsim_R y$ . Using **TI** and **DI** we get  $q(x - y) \succsim_{R'} 0$  for any nonnegative rational number  $q$ . Let  $(q_n)$  be a nonnegative sequence of rationals converging to  $\alpha$ . The sequence  $q_n(x - y)$  converges to  $\alpha(x - y)$ . On the other hand,  $q_n(x - y) \in C_+ = \{z \in Y : z \succsim_{R'} 0\}$  and  $C_+$  is closed since  $\tau_+(R') \subset t$ . Thus, the limit of the sequence  $(q_n(x - y))$ , which is  $\alpha(x - y)$ , belongs to  $C_+$ . As a result  $\alpha(x - y) \succsim_{R'} 0$ . What yields, by **TI**,  $\alpha x \succsim_{R'} \alpha y$ . ■

An immediate consequence of corollary 2 is the following:

**Corollary 3:** Let  $R$  be a complete preorder on  $Y$ , a real normed vector space, satisfying **TI** and  $\tau_+(R) \subset t$ , where  $t$  is the topology induced by the norm of  $Y$ . Then  $R$  satisfies *scalar invariance*.

**Remark 7:**  $\tau_+(R) \subset t$  is a continuity requirement. Under that continuity requirement and **TI**, *scalar invariance* is, in a sense, satisfied since every complete preorder extending the original preorder and satisfying the same axiom of continuity and **TI** must satisfy *scalar invariance*.

**Remark 8:** (Demuynck-Lauwers 2009) showed that a given preorder satisfying **TI** and *scalar invariance* can be extended into a complete preorder satisfying **TI** and *scalar invariance*. Corollary 2 shows that if, in addition, the initial preorder satisfies upper semicontinuity, then it admits an extension which also satisfies upper semicontinuity in addition to the axioms **TI** and *scalar invariance*.

**Remark 9:** While Corollary 3 presents *scalar invariance* as a consequence of **TI** and a condition of continuity, (Weibull 1985) theorem A has shown that under conditions **TI**, *scalar invariance* and a continuity requirement called *scalar continuity*, a complete preorder verifies a stronger condition of continuity that results in representability, i.e. the existence of a real-valued order-preserving continuous function. For more information on *scalar continuity* and its properties in the context of a monotone order, see (Mitra-Ozbek 2013).

## 6- *Scalar invariance* as a consequence of **TI** and a weak Pareto axiom

We are now in the space  $l_\infty^r = \{(x_1, x_2, \dots) : x_i \in \mathbb{R} \text{ and } \sup |x_i| e^{-ri} < +\infty\}$ , where  $r$  is a nonnegative real. This space is suitable for studying economic decisions in discrete time, infinite horizon and exponentially growing economy. If  $r = 0$ , the economy remains bounded.

**Axiom *super weak Pareto*:** if  $\inf(x_i - y_i)e^{-ri} > 0$  then  $x \succ_R y$ .

The following lemma is a slight strengthening of theorem 4 of (Mabrouk 2011). It will be used to prove theorem 3.

**Lemma 3:** (wrong) If a complete preorder  $R'$  on  $l_\infty^r$  satisfies *super weak Pareto* and **TI**, then for every  $u \in l_\infty^r$  such that  $u \succ_{R'} 0$  there exists a non-zero, continuous, positive (in the sense that if  $x_i \geq 0$  for all  $i$  then  $\varphi(x) \geq 0$ ) linear functional  $\varphi_u$  on  $l_\infty^r$  such that  $\varphi_u(x) > \varphi_u(y) \Rightarrow x \succ_{R'} y$  and  $\varphi_u(u) > 0$ .

**Proof:** We refer to the proof of theorem 4 in (Mabrouk 2011). The notations there are the same, except for the axiom **TI** instead of which a weaker axiom

called "weak inv  $(a_i + x_i)$ " was used in (Mabrouk 2011)<sup>3</sup>. For the convenience of the reader, we recall some definitions and results:  $l_{\infty++}^{\circ} = \{x \in l_{\infty}^r : \inf x_i e^{-ri} > 0\}$ ,  $S = \{s \in l_{\infty}^r / s \succsim_{R'} 0\}$  and  $Q = \{q \in l_{\infty}^r : q = s + p, s \in S, p \in l_{\infty++}^{\circ}\}$ . In the proof of theorem 4 of (Mabrouk 2011),  $Q$  is proved to be open and convex and to have the following properties: (i)  $0 \notin Q$  (ii)  $\mu q \in Q$  whenever  $q \in Q$  and  $\mu$  is a positive real.

Now let  $u$  be in  $l_{\infty}^r$  such that  $u \succ_{R'} 0$ . The idea is to consider the convex hull  $Q'$  of the set  $Q$  and vector  $u$  instead of the set  $Q$ . We have  $Q' = \{q' \in l_{\infty}^r : \exists(\lambda, q) \in [0, 1] \times Q, q' = \lambda q + (1 - \lambda)u\}$ . problem:  $Q'$  is not open!!... We show that  $0 \notin Q'$ . Suppose not. There would exist  $\lambda$  in  $[0, 1]$  with  $\lambda q + (1 - \lambda)u = 0$ . Since  $0 \notin Q$  and  $u \succ_{R'} 0$ , we have  $\lambda \neq 0$  and  $\lambda \neq 1$ . Thus we would have  $\frac{\lambda}{1-\lambda}q + u = 0$ . But  $\frac{\lambda}{1-\lambda}q \in Q$ . Thus  $\frac{\lambda}{1-\lambda}q \succ_{R'} 0$ . Since  $u \succ_{R'} 0$ , by **TI** we would have  $\frac{\lambda}{1-\lambda}q + u \succ_{R'} 0$ . A contradiction. Since  $0 \notin Q'$ , thanks to Hahn–Banach theorem, there exist a non-zero continuous linear functional  $\varphi_u$  supporting  $Q'$ . This is written: for all  $q'$  in  $Q'$ ,  $\varphi_u(q') > 0$ . In particular,  $\varphi_u(u) > 0$ . One shows, literally as in the proof of theorem 4 of (Mabrouk 2011), that for all  $x, y$  in  $l_{\infty}^r$ ,  $\varphi_u(x) > \varphi_u(y) \Rightarrow x \succ_{R'} y$  and that  $\varphi_u$  is positive.■

**Theorem 3:** Let  $R$  be a preorder on  $l_{\infty}^r$  satisfying **TI** and *super weak Pareto*. Let  $R'$  be one of the complete preorders which existence is asserted by theorem 1, i.e. a complete preorder of which  $R$  is a subrelation and satisfying **TI**. Then  $R'$  satisfies *scalar invariance*.

**Proof:** Since  $R$  is a subrelation to  $R'$ ,  $R'$  also satisfies *super weak Pareto*. Let  $x, y \in l_{\infty}^r$  such that  $x \succ_{R'} y$ . Denote  $u = x - y$ . We have  $u \succ_{R'} 0$ . Apply lemma 3. There exists a non-zero, continuous, positive linear functional  $\varphi_u$  on  $l_{\infty}^r$  such that  $\forall x', y' \in l_{\infty}^r, \varphi_u(x') > \varphi_u(y') \Rightarrow x' \succ_{R'} y'$  and  $\varphi_u(u) > 0$ . Let  $\alpha$  be a positive real. Multiplying this inequality by  $\alpha$ , one gets  $\alpha\varphi_u(u) = \varphi_u(\alpha u) > 0$ . Replace  $u$  by  $x - y$ . Then  $\varphi_u(\alpha u) = \varphi_u(\alpha(x - y)) = \varphi_u(\alpha x - \alpha y) = \varphi_u(\alpha x) - \varphi_u(\alpha y) > 0$ . Hence,  $\varphi_u(\alpha x) > \varphi_u(\alpha y)$  and  $\alpha x \succ_{R'} \alpha y$ . We have shown that for all positive real  $\alpha$  and  $x, y \in l_{\infty}^r, x \succ_{R'} y \Rightarrow \alpha x \succ_{R'} \alpha y$ . Moreover,  $x \sim_{R'} y$  implies  $\alpha x \sim_{R'} \alpha y$  (since if we had for example  $\alpha x \succ_{R'} \alpha y$ , we could multiply this last inequality by  $\frac{1}{\alpha}$  and get  $x \succ_{R'} y$ , a contradiction). This proves *scalar invariance*.■

Theorem 3 indicates that *scalar invariance* is satisfied under **TI** and *super weak Pareto* in the same sense as in remark 7. **TI** together with *scalar invariance* is called *strong invariance* in the terminology of (Mitra-Ozbek 2013)<sup>4</sup>. If we accept this justification of *scalar invariance* by **TI**, we are led to admit that, under *super weak Pareto*, the axiom, *strong invariance* is in a way a consequence of the axiom **TI**.

An immediate consequence of theorem 3 is the following:

**Corollary 4:** Every complete preorder  $R'$  satisfying **TI** and *super weak Pareto*, satisfies *scalar invariance*.

<sup>3</sup>The definition of "weak inv  $(a_i + x_i)$ " is:  $\forall x, y, u \in X, [x \succ_R y \Rightarrow x + u \succsim_R y + u]$ . Of course, lemma 2 holds with weak inv  $(a_i + x_i)$  instead of **TI**.

<sup>4</sup>In the terminology of (D'Aspremont-Gevers 2002), it is called *invariance with respect to common rescaling and individual change of origin*.

**Remark 10:** Since theorem 1 and lemma 3 hold in finite dimension, it is also the case for theorem 3 and corollary 4. Consequently, when the preorder is complete and super-weak Pareto, *strong invariance* is equivalent to **TI**. Hence, theorem 18 of (D'Aspremont-Gevers 2002) or example 2 of (Mitra-Ozbek 2013) asserting the linear representability of a complete preorder respecting **TI**, *scalar invariance*, *weak Pareto* and another axiom, hold without imposing *scalar invariance*.

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