On the extension of a preorder under translation invariance

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Abstract

This paper proves the existence, for a translation-invariant preorder on a divisible commutative group, of a complete preorder extending the preorder in question and satisfying *translation invariance*. We also prove that the extension may inherit a property of continuity. As an application, we prove the existence of a complete translation-invariant strict preorder on R which transgresses *scalar invariance* and also the existence of a complete translation-invariant preorder satisfying the social choice axioms *strong Pareto* and *..xed–step-anonymity* on a set X^{N_0}, where X is a divisible commutative group. Moreover, the two extension results are used to make *scalar invariance* appear as a consequence of *translation invariance* under a continuity requirement.

1- Introduction

(Szpilrajn 1930) extension theorem may be stated as follows. For any re‡exive and transitive binary relation (i.e. a preorder) on a given set, there exists a complete preorder which is an extension of the given preorder. Szpilrajn theorem proved of great utility in mathematical social choice theory as in some other branches of mathematics. There exists today stronger versions of Szpilrajn theorem, requiring weaker assumptions on the initial binary relation or imposing additional conditions on the relation extension. We refer to (Alcantud-Diaz 2014) for an overview on the applications and extensions of Szpilrajn theorem.

The present paper establishes the existence, for any preorder on a divisible commutative group satisfying *translation invariance*, of a complete preorder extending the given preorder and satisfying *translation invariance* (section 3, theorem 1). In (Demuynck-Lauwers 2009) the existence of an extension under the conditions *translation invariance* and *scalar invariance* is proven. However, the result proved here is stronger in the sense that it is freed from the *scalar invariance* assumption. The proof of theorem 1 follows the same diagram as the proof of Szpilrajn theorem. Starting from a preorder satisfying *translation invariance*, one adds comparisons on some pairs of alternatives in such a way that *translation invariance* remains satis..ed. Then, an argument based on Zorn's lemma makes it possible to extend the procedure to the whole space.

We also prove a second extension theorem which asserts that the former extension result (theorem 1) holds under an additional requirement of continuity (section 4, theorem 2). The proof is an adaptation of the proof of (Ja¤ray 1975) to the *translation invariance* case. It relies on the construction of a relation that

[1] I am grateful to an anonymous referee who, when reviewing another paper, guided me towards the issue of extending a preorder under translation-invariance.

is used to "clean" the extended preorder given by theorem 1 from undesirable rankings that transgress the continuity requirement.

As an application, we give two examples, the first of which shows the existence of a complete translation-invariant strict preorder on R which transgresses *scalar invariance* and the second shows the existence of a complete translation-invariant preorder satisfying the social choice axioms *strong Pareto* and *fixed–step-anonymity* on a set X^{N_0}, where X is a divisible commutative group.

Moreover, theorems 1 and 2 are used to make *scalar invariance* appear as a consequence of *translation invariance* under a continuity requirement (Corollary 2, section 5) or under a Pareto axiom (Theorem 3, section 6).

2- Preliminaries

N_0 is the set of positive integers. Q is the set of rational numbers. $(X,+)$ is a divisible commutative group. B being a binary relation on X and x,y two elements of X, xBy is denoted $x \succsim_B y$, [xBy and non(yBx)] is denoted $x \succ_B y$ and [xBy and yBx] is denoted $x \sim_B y$. The symbols $\leq, \geq, <, >$ are used for the natural order on R, except in example 1, section 4. A reflexive and transitive binary relation on X is a preorder on X. If, on top of that, for all x,y either $x \succsim_B y$ or $x \precsim_B y$, it is a complete preorder. A binary relation B_1 is said to be a subrelation to a binary relation B_2 , or B_2 an extension of B_1, if for all x,y in X,

$$x \succsim_{B_1} y \Longrightarrow x \succsim_{B_2} y$$

and

$$x \succ_{B_1} y \Longrightarrow x \succ_{B_2} y$$

Axiom *translation invariance* (TI) A preorder R satisfies *translation invariance* if:

$$\forall (x,y) \in X \times X, \forall u \in X, [x \succsim_R y \Leftrightarrow x+u \succsim_R y+u]$$

Axiom *division invariance* (DI) A preorder R satisfies division-invariance if:

$$\forall x \in X, \forall n \in N, \left(x \succsim_R y \Leftrightarrow \frac{1}{n}x \succsim_R \frac{1}{n}y \right)$$

Lemma 1 If a preorder R on X satisfies TI, then there exists a preorder $\hat{R}$ on X of which R is a subrelation and such that $\hat{R}$ satisfies TI and DI.

Proof: First, notice that under R, it is possible to sum inequalities. Indeed, by TI, if a,b,u,v are such that $a \succsim_R b$ and $u \succsim_R v$, then $a+u \succsim_R b+u$ and $b+u \succsim_R b+v$. By transitivity, $a+u \succsim_R b+v$. For each positive integer n, consider the binary relation R_n defined by

$$x \succsim_{R_n} y \text{ iff } nx \succsim_R ny$$

If x,y are such that $x \succsim_R y$, we can sum n times this inequality. Thus, $x \succsim_{R_n} y$. Likewise, it is easily seen that $x \succ_R y$ implies $x \succ_{R_n} y$. As a result,

R is a subrelation to R_n. Moreover, R_n is reflexive and transitive. It is easily checked that R_n satisfies TI.

Consider the binary relation

$$\widehat{R} = \cup_{n \in \mathbb{N}'} R_n$$

defined on X by $x \succsim_{\widehat{R}} y$ iff there is n such that $x \succsim_{R_n} y$.

R is a subrelation to $\widehat{R}$. Moreover, $\widehat{R}$ is reflexive and transitive. It is a preorder. Since for each positive integer n, R_n satisfies TI, we deduce that $\widehat{R}$ satisfies TI. The lemma is proved if we show that $\widehat{R}$ satisfies DI. Let n be a positive integer, and x, y such that $x \succsim_{\widehat{R}} y$. There exists a positive integer m such that $x \succsim_{R_m} y$. Thus $mx \succsim_R my$. We can write that as $mn(\frac{1}{n}x) \succsim_R mn(\frac{1}{n}y)$. Thus $\frac{1}{n}x \succsim_{R_{mn}} \frac{1}{n}y$, what implies $\frac{1}{n}x \succsim_{\widehat{R}} \frac{1}{n}y$. $\widehat{R}$ satisfies DI. $\blacksquare$

Remark 1 (1) It is easily seen that $\widehat{R}$ is the minimal preorder satisfying TI and DI, of which R is a subrelation. (2) If R is complete, since R is a subrelation to $\widehat{R}$, we have necessarily $R = \widehat{R}$. This shows that if the preorder is complete, TI implies DI.

3- The translation-invariant extension theorem

Theorem 1 Let R be a preorder on X satisfying TI. Then there exists a complete preorder on X satisfying TI, of which R is a subrelation.

Proof: If R is a complete preorder, there is nothing to prove. Suppose that R is not complete. Consider the preorder $\widehat{R}$ built in the proof of lemma 1, and the set $\mathcal{R}$ of all preorders on X satisfying TI and DI, and of which R is a subrelation. $\mathcal{R}$ is not empty since $\widehat{R} \in \mathcal{R}$. Let (R_α) be a chain in $\mathcal{R}$, i.e. for any α, α', R_α is a subrelation to $R_{\alpha'}$ or $R_{\alpha'}$ is a subrelation to R_α. Notice that (1) the relation $\cup_\alpha (R_\alpha)$ defined on X by: $x[\cup_\alpha (R_\alpha)]y$ iff there is α such that $xR_\alpha y$, is a preorder, (2) it satisfies TI and DI, (3) R is a subrelation to $\cup_\alpha (R_\alpha)$, (4) for all α, R_α is a subrelation to $\cup_\alpha (R_\alpha)$. Hence, in the set $\mathcal{R}$, every chain admits an upper bound. According to Zorn's lemma, $\mathcal{R}$ admits at least a maximal element. Denote M such a maximal element in $\mathcal{R}$. Suppose we can prove the following claim:

Claim 1 For any non complete R' in $\mathcal{R}$ and any pair of R'-incomparable alternatives (x_0, y_0), there exists a preorder R'_1 in $\mathcal{R}$ to which R' is a subrelation and such that x_0 and y_0 are R'_1-comparable.

Then, if M were not complete, there would exist a preorder in $\mathcal{R}$ to which M is a strict subrelation. This would contradict that M is maximal in $\mathcal{R}$. Therefore, if the claim holds, M would be necessarily complete. M would be the preorder we are looking for.

What remains of the proof is devoted to establish claim 1. This is done through the following 6 steps.

If there is no non complete preorder in $\mathcal{R}$, the theorem is proved since $\mathcal{R}$ is not empty. Let R' be a non complete preorder in $\mathcal{R}$ and x_0, y_0 be two R'-incomparable elements of X.

Consider the binary relation B on X: $x \succsim_B y$ iff either $x \succsim_{R'} y$ or there is a positive rational q such that $x - y = q(x_0 - y_0)$.

We prove successively that the two clauses of the definition of B are exclusive (step 1), that the indifference relations are equal (step 2), that R' is a subrelation to B (step 3), that B is weakly acyclic (this prepares for transitivity) (step 4), that R' is a subrelation to the transitive closure of B (step 5), that the transitive closure of B satisfies TI and DI (step 6). The transitive closure of B is then the required preorder.

Step 1: the two clauses are exclusive. If there is a positive rational q such that $x - y = q(x_0 - y_0)$, then x, y are R'-incomparable. Suppose not. For instance suppose $x \succsim_{R'} y$. By TI, $x - y \succsim_{R'} 0$. By DI, for all positive integer n, $\frac{1}{n}(x - y) \succsim_{R'} 0$. Recall that it is possible to sum inequalities (see the proof of lemma 1). We sum m times the inequality $\frac{1}{n}(x - y) \succsim_{R'} 0$, m being a positive integer. We obtain $\frac{m}{n}(x - y) \succsim_{R'} 0$. Take $\frac{n}{m} = q$. It gives $x_0 - y_0 \succsim_{R'} 0$, what contradicts x_0, y_0 being incomparable. The case $y \succsim_{R'} x$ is similar.

Step 2: equivalence of indifferences. Clearly, $x \sim_{R'} y \Rightarrow x \sim_B y$. We show now that $x \sim_B y$ entails $x \sim_{R'} y$. According to the definition of B, it is enough to prove that x and y are necessarily R'-comparable. Suppose not. Then $x \succsim_B y$ implies that there is some positive rational q such that $x - y = q(x_0 - y_0)$. We have also $y \succsim_B x$. Thus, for some positive rational q', $y - x = q'(x_0 - y_0)$. We see that this gives $q'(x_0 - y_0) = -q(x_0 - y_0)$, what implies $x_0 - y_0 = 0$ because q, q' are both positive. But that contradicts x_0, y_0 being R'-incomparable.

Step 3: R' is a subrelation to B. This is a direct consequence of $x \succsim_{R'} y \Rightarrow x \succsim_B y$ (definition of B) and $x \sim_B y \Leftrightarrow x \sim_{R'} y$ (step 2).

Step 4: B is weakly-acyclic. We show that for all x, y, z in X : $x \succsim_B y$ and $y \succsim_B z \Rightarrow x \succsim_B z$ or non$(z \succsim_B x)$.

One of the four following cases is implied by $x \succsim_B y$ and $y \succsim_B z$. (1) $x \succsim_{R'} y$ and $y \succsim_{R'} z$, (2) there are q, q' such that $x - y = q(x_0 - y_0)$ and $y - z = q'(x_0 - y_0)$, (3) $x \succsim_{R'} y$ and there is q' such that $y - z = q'(x_0 - y_0)$, (4) there is q such that $x - y = q(x_0 - y_0)$ and $y \succsim_{R'} z$. Consider successively the four cases:

(1) By transitivity of R': $x \succsim_{R'} z$. Thus, $x \succsim_B z$.

(2) $x - y = q(x_0 - y_0)$ and $y - z = q'(x_0 - y_0)$ entails $x - z = (q + q')(x_0 - y_0)$. Thus $x \succsim_B z$.

(3) Suppose we had $z \succsim_B x$. We would have either $z \succsim_{R'} x$ or $z - x = q''(x_0 - y_0)$. Both possibilities contradict $x \succsim_{R'} y$ and $y - z = q'(x_0 - y_0)$. Indeed, with $x \succsim_{R'} y$, $z \succsim_{R'} x$ gives $z \succsim_{R'} y$ what contradicts $y - z = q'(x_0 - y_0)$ (step 1); whereas $y - z = q'(x_0 - y_0)$ with $z - x = q''(x_0 - y_0)$ implies $y - x = (q' + q'')(x_0 - y_0)$, what contradicts $x \succsim_{R'} y$. As a result, we have non$(z \succsim_B x)$.

(4) This case is similar to (3)

Remark 2 Let x, y, z be such that $x \succsim_B y$ and $y \succsim_B z$. Weak acyclicity entails that if one of the comparisons $x \succsim_B y$ and $y \succsim_B z$ is a strict preference, then either the comparison on (x, z) is $x \succ_B z$ or x and z are B-incomparable.

Step 5: R' is a subrelation to the transitive closure of B. Consider $\overline{B}$ the transitive closure of B defined by: $x \succsim_{\overline{B}} y$ if there is a sequence $(z_i)_{i=1}^{n}$ such that $x \succsim_B z_1, z_1 \succsim_B z_2 \ldots$ and $z_n \succsim_B y$. It is clear that $x \succsim_{R'} y$ implies $x \succsim_{\overline{B}} y$ (step 3: R' is a subrelation to B). It is enough to prove that $x \succsim_{\overline{B}} y$ implies non$(y \succ_{R'} x)$.

For a positive integer n, consider the statement Q_n: "If there is a sequence $(z_i)_{i=1}^n$ such that $x \succsim_B z_1 \succsim_B z_2 .... \succsim_B z_n \succsim_B y$, then non$(y \succ_{R'} x)$." Let's prove by induction that Q_n is true for all positive integers. Notice that when the sequence (z_i) has n terms, there is $n+1$ successive comparisons.

$n = 1$: We have $x \succsim_B z_1 \succsim_B y$. By step 4, we have $x \succsim_B y$ or non$(y \succsim_B x)$. Both possibilities contradict $y \succ_{R'} x$. So, we have non$(y \succ_{R'} x)$.

Suppose that Q_n is true and let's show that Q_{n+1} is true. Consider the sequence of $n+2$ comparisons: $x \succsim_B z_1 \succsim_B z_2 .... \succsim_B z_{n+1} \succsim_B y$.

Each one of these comparisons comes either from the clause $x \succsim_{R'} y$ or the clause $x - y = q(x_0 - y_0)$ of the definition of B. If there is two successive comparisons coming from the clause $x \succsim_{R'} y$, say $z_p \succsim_{R'} z_{p+1} \succsim_{R'} z_{p+2}$ (with $p = 0, ..., n+2$ and the convention: $z_0 = x$ and $z_{n+2} = y$), by transitivity of R' we have: $x \succsim_B ... z_p \succsim_B z_{p+2} .... \succsim_B y$ which constitutes a sequence of $n+1$ comparisons. By Q_n we have non$(y \succ_{R'} x)$. If there is two successive comparisons coming from the clause $x - y = q(x_0 - y_0)$, say $z_p \succsim_B z_{p+1} \succsim_B z_{p+2}$, then $z_p - z_{p+1} = q(x_0 - y_0)$ and $z_{p+1} - z_{p+2} = q'(x_0 - y_0)$. Thus, $z_p - z_{p+2} = (q + q')(x_0 - y_0)$ so that $z_p \succsim_B z_{p+2}$. We have again reduced the number of comparisons to $n+1$. Thus, we have also non$(y \succ_{R'} x)$. It remains to consider the cases where the comparisons are alternate. Two cases must be considered: $n+2$ even and $n+2$ odd.

$n+2$ even: The sequence of comparisons either begin or ends with a comparison from R'. Suppose it begins with a comparison from R': $x \succsim_{R'} z_1 \succsim_B z_2 .... \succsim_{R'} z_{n+1} \succsim_B y$. Apply Q_n to $z_1 \succsim_B z_2 .... \succsim_{R'} z_{n+1} \succsim_B y$. It gives non$(y \succ_{R'} z_1)$. Since $x \succsim_{R'} z_1$, we cannot have $y \succ_{R'} x$. If the sequence of comparisons ends with a comparison from R', the proof is similar. So it is omitted.

$n+2$ odd: If the sequence of comparisons begins with a comparison from R', the proof is also similar. So it is omitted. If the sequence of comparisons begins with a comparison from the clause $x - y = q(x_0 - y_0)$, we have

$$x \succsim_B z_1 \succsim_{R'} z_2 .... \succsim_{R'} z_{n+1} \succsim_B y \tag{1}$$

Denote (x, z_1) by (α_1, β_1) , (z_2, z_3) by (α_2, β_2) .... $\left(z_{2(p-1)}, z_{2p-1}\right)$ by $\left(\alpha_p, \beta_p\right)$ with $p = 1, ..., \frac{n+1}{2}$ and the convention $z_0 = x$ and $z_{n+2} = y$. Since comparisons $x \succsim_B z_1, z_2 \succsim_B z_3 .... z_{n-1} \succsim_B z_n, z_{n+1} \succsim_B y$ come from the clause $x - y = q(x_0 - y_0)$, we have $\alpha_p - \beta_p = q_p (x_0 - y_0)$ for $p = 1, ..., \frac{n+3}{2}$. Moreover, according to (1), $\beta_p \succsim_{R'} \alpha_{p+1}$ for $p = 1, ..., \frac{n+1}{2}$. Thus

$$\begin{aligned}
\alpha_1 - q_1(x_0 - y_0) &\succsim_{R'} \alpha_2 \\
\alpha_2 - q_2(x_0 - y_0) &\succsim_{R'} \alpha_3 \\
&..... \\
\alpha_{(n+1)/2} - q_{(n+1)/2}(x_0 - y_0) &\succsim_{R'} \alpha_{(n+3)/2}
\end{aligned}$$

We can sum these inequalities (this is established in the proof of lemma 1).

We obtain

$$\alpha_1 + \sum_{2}^{(n-1)/2} \alpha_p - \sum_{2}^{(n-1)/2} q_p (x_0 - y_0) \succsim_{R'} \sum_{2}^{(n-1)/2} \alpha_p + \alpha_{(n+3)/2}$$

By TI we obtain

$$\alpha_1 - \sum_{1}^{(n-1)/2} q_p (x_0 - y_0) \succsim_{R'} \alpha_{(n+3)/2}$$

But $\alpha_1 = x_1$ and $\alpha_{(n+3)/2} = y$. Denote $q = \sum_1^{(n+1)/2} q_p$. Thus

$$x - q(x_0 - y_0) \succsim_{R'} y$$

By TI, $x - y \succsim_{R'} q(x_0 - y_0)$. If we had $y \succ_{R'} x$, it would give $0 \succ_{R'} x - y \succsim_{R'} q(x_0 - y_0)$. By transitivity of R' and by TI, x_0 and y_0 would be R'-comparable, which is not the case. As a result, we have non($y \succ_{R'} x$). Step 5 is proved.

Remark 3 R' is a subrelation to $\overline{B}$, but B is not.

Step 6: $\overline{B}$ satisfies TI. As R' is translation-invariant, B is clearly translation-invariant. It is easily deduced that $\overline{B}$ is also translation-invariant. Likewise, it is easily seen that $\overline{B}$ satisfies DI. Thus, $\overline{B}$ is the required preorder.□

Corollary 1 Let B be a reflexive binary relation satisfying TI. Then there exists a complete preorder satisfying TI, of which B is a subrelation, iff B is a subrelation to its transitive closure.

Proof: Necessity: the condition that B is a subrelation to its transitive closure is necessary and sufficient for the existence of a complete preorder of which B is a subrelation (Suzumura 1976, Bossert 2008). Sufficiency: denote $\overline{B}$ the transitive closure of B. It easily seen that $\overline{B}$ is a preorder satisfying TI. Apply theorem 1 to $\overline{B}$ to deduce that there exists a complete preorder satisfying TI, of which $\overline{B}$ is a subrelation. Since B is a subrelation to $\overline{B}$, it is also a subrelation to the complete preorder.□

4- Examples of application

Example 1: A translation-invariant and complete strict preorder on R with $\pi < 0 < 1$.

Notice that only in this example, the symbols $\leq, \geq, <, >$ are used for something else than the natural order on R. Consider the following binary relation $\precsim$ on R :

$$x \precsim y \text{ if there is two nonnegative rationals } q, q' \text{ such that } x - y = -q + q'\pi$$

$\precsim$ is reflexive, transitive and satisfies TI. Moreover, $\precsim$ is a strict preorder, which means that $x \precsim y$ and $y \precsim x$ implies $x = y$. Indeed $x - y = -q + q'\pi$ and $y - x = -q_1 + q_1'\pi$ yields $0 = (x - y) + (y - x) = -(q + q_1) + (q' + q_1')\pi$.

Thus $(q + q_1) = (q' + q_1')\pi$. We must have $q' + q_1' = 0$ otherwise π would be rational. Thus we have also $q + q_1 = 0$. Since q, q_1, q', q_1' are nonnegative, we have $q = q_1 = q' = q_1' = 0$ and $x = y$.

Theorem 1 asserts the existence of a translation-invariant and complete preorder, say $\leq$, of which $\preceq$ is a subrelation. $\leq$ is strict like $\preceq$. Observe that $\leq$ respects the natural order of rationals. But it does not coincide with the natural order of reals. Moreover it does not satisfy *scalar invariance* since if you multiply $0 < 1$ by π the inequality is reversed. Finally, $\leq$ is not continuous. Consider a positive sequence of rational (q_n) such that $\lim q_n = \frac{1}{\pi}$. TI allows to multiply an inequality by a positive rational. Multiplying $\pi < 0$ by q_n yields $q_n\pi < q_n.0 = 0$ for all n. But $\lim q_n\pi = 1 > 0$. A question then arises: can *scalar invariance* still be transgressed under TI and continuity? An answer is provided in section 5.

Example 2: Existence of a translation-invariant, strong-Pareto, fixed–step-anonymous and complete preorder on $X^{\mathbb{N}_0}$, where X is a divisible commutative group equipped with a complete preorder R satisfying TI.

It is possible to demonstrate the existence of such a preorder using the ultrafilter technique, as in (Fleurbaey-Michel 2003, Lauwers 2009). We demonstrate here this existence without using ultrafilters, which are highly nonconstructive objects. Although our theorem 1 also makes use of the axiom of choice, one may consider that our method is nevertheless more constructive in the sense that it indicates the concrete steps of adding comparisons.

Let $Y = X^{\mathbb{N}_0}$, let R' be a preorder on Y. We first give the following definitions:

Fixed-step permutation: (Fleurbaey-Michel 2003) σ is a fixed-step permutation if there exist $k \in \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$, $\sigma(\{1, ..., kn\}) = \{1, ..., kn\}$.

Axiom *fixed-step-anonymity*: Denote $\sigma(x)$ the sequence obtained by permuting the components of $x \in Y$ according to the permutation σ. R' is fixed-step-anonymous if for all $x \in Y$ and fixed-step permutation σ, we have $x \sim_{R'} \sigma(x)$.

Axiom *strong Pareto*: R' is *strong Pareto* if, for all $x, y \in Y$ such that $\forall i \in \mathbb{N}_0$ $x_i \succsim_R y_i$ and $x_j \succ_R y_j$ for some j, we have $x \succ_{R'} y$ (x_i, y_i denote the i^{th} component of resp. x, y).

Pareto axioms capture the idea that an increase of the components of a vector must increase the ranking of the vector. Anonymity axioms express a requirement of symmetry in the treatment of individuals or dates.

The fixed-step catching-up SC. For all $x, y \in \mathbb{R}^{\mathbb{N}_0}$, $x \succsim_{SC} y$ iff there exist $k, m \in \mathbb{N}_0$ such that, for all $n \in \mathbb{N}_0$ with $n > m$, we have

$$\sum_{i=1}^{kn} x_i \geq \sum_{i=1}^{kn} y_i$$

SC is a fixed–step-anonymous preorder (Fleurbaey-Michel 2003).

Proposition 1: There exists a translation-invariant, strong-Pareto, fixed-step-anonymous and complete preorder on $\mathbb{R}^{\mathbb{N}_0}$.

Proof: Apply theorem 1 to SC. There exists a translation-invariant and complete preorder R^0 on Y of which SC is a subrelation. SC being a subrelation to R^0 entails that R^0 satisfies *strong Pareto* and *fixed–step-anonymity*. R^0 is the required preorder.$\blacksquare$

5- *Scalar invariance* as a consequence of TI and a continuity requirement

For a given nontrivial preorder R on a divisible commutative group X, $\tau_+(R)$ is the associated upper-order-topology, i.e. the topology generated by the base of open intervals: $\beta_+(R) = \{\{x \in X : x \succ_R a\}, a \in X\}$.

Theorem 2: Let R be a preorder on X satisfying TI. Then there exists a complete preorder R^0 on X satisfying TI, of which R is a subrelation, and such that $\tau_+(R^0) \subset \tau_+(R)$.

Proof: The following proof is an adaptation of the proof of (Jaffray 1975) to a translation-invariant preorder. We start from a translation-invariant complete preorder which extends R, whose existence is guaranteed by theorem 1. We then apply a clause[2] to "clean up" rankings that do not respect the upper-order-topology. It turns out that this clause is also translation-invariant, which makes it possible to build the desired preorder.

Let R_1 be a complete preorder extending R and satisfying TI. Let $x, y \in X$. Consider the following clause :

$C(x,y)$: "There exists $B \in \beta_+(R)$ containing x such that, for all $B^0 \in \beta_+(R)$ containing y, we can find $x^0 \in B^0$ such that for all $z \in B$, we have $z \succ_{R_1} x^0$"

Because R_1 satisfies TI, it is easily seen that if $C(x,y)$ is true, $C(x+h, y+h)$ is true for all h in X. Moreover, if $C(x,y)$ is true, it is clear that we cannot have $C(y,x)$ true. Thus, we can define a asymmetric relation R_2 checking TI as follows: $x \succ_{R_2} y$ iff $C(x,y)$ is true.

We prove now that R_2 is negatively transitive, i.e.

$$\text{not}(x \succ_{R_2} y) \text{ and not}(y \succ_{R_2} z) \text{ implies not}(x \succ_{R_2} z)$$

We have:

Not$(x \succ_{R_2} y) \iff$ for all $B_1 \in \beta_+(R)$ containing x, there exists $B_1^0 \in \beta_+(R)$ containing y such that [for all x_1^0 in B_1^0, there exists x_1^{00} in B_1 such that $x_1^{00} \succsim_{R_1} x_1^0$].

Not$(y \succ_{R_2} z) \iff$ for all $B_2 \in \beta_+(R)$ containing y, there exists $B_2^0 \in \beta_+(R)$ containing z such that [for all x_2^0 in B_2^0, there exists x_2^{00} in B_2 such that $x_2^{00} \succsim_{R_1} x_2^0$].

Let B_1 be in $\beta_+(R)$ containing x and B_1^0 be the interval which existence is asserted by the clause "not$(x \succ_{R_2} y)$". Take B_1^0 as the interval B_2 of the

[2] This clause combines the two clauses proposed by (Jaffray 1975) in the proof of his theorem 1, the first of which defines a preorder on $\beta_+(R)$ and the second a preorder on X.

clause "not($y \prec_{R_3} z$)". Thus, there exists $B_2' \in \beta_+(R)$ containing z such that [for all x_2'in B_2', there exists x_2'' in B_1' such that $x_2'' \succsim_{R_1} x_2'$]. Now apply the clause "not($x \succ_{R_2} y$)" for x_2'' instead of x_1' and deduce that there exists x_1'' in B_1 such that $x_1'' \succsim_{R_1} x_2''$. By transitivity of R_2, $x_1'' \succsim_{R_1} x_2''$ and $x_2'' \succsim_{R_1} x_2'$ gives $x_1'' \succsim_{R_1} x_2'$.

Summing up: for some B_1 in $\beta_+(R)$ containing x, we have found $B_2' \in \beta_+(R)$ containing z such that [for all x_2'in B_2' there exists x_1'' in B_1 such that $x_1'' \succsim_{R_1} x_1'$]. This is exactly the clause not($x \prec_{R_2} z$).

Since asymmetry and negative transitivity imply transitivity, R_2 is transitive.

Now let R' be the following binary relation:

$$x \precsim_{R'} y \text{ iff } [(x \prec_{R_2} y) \text{ or not}(x \succ_{R_2} y)]$$

The transitivity and negative transitivity of R_2 implies the transitivity of R'. Moreover, R' is complete and satisfies TI.

We show now that R is a subrelation to R'. Indeed, let x, y be such that $x \prec_R y$. In the clause $C(x, y)$, take $B = \{z \in X : z \prec_R y\}$. We have $x \in B$ and for all B' containing y,we have $z \prec_{R_1} y$ for all $z \in B$. Hence the clause $C(x, y)$ is true and $x \prec_{R_2} y$. Consequently, $x \prec_{R'} y$. If x, y are such that $x \sim_R y$, the clause $C(x, y)$ cannot be satisfied. To see it, it suffices to notice that an interval containing x necessarily contains y and vice versa. If we take $B' = B$ in the clause $C(x, y)$, there is no x' in B such that for all $z \in B$, we have $z \prec_{R_1} x'$. Thus we have not($x \prec_{R_2} y$). In the same way, we have not($y \prec_{R_2} x$).Consequently, $x \sim_{R'} y$.

It remains to show that $\tau_+(R') \subset \tau_+(R)$. Let $y \in X$. We show that any subset in $\beta_+(R')$, the base of open intervals generating $\tau_+(R')$, is open with respect to $\tau_+(R)$. Let $x \in B = \{z \in X : z \prec_{R'} y\}$. By the definition of R', there is B_x in $\beta_+(R)$, containing x, such that for all $B_y \in \beta_+(R)$ containing y,we can find $x' \in B_y$ such that for all $z \in B_x$, we have $z \prec_{R_1} x'$. We can see that this implies that for all $z \in B_x$, we have $z \prec_{R'} y$. Hence $B_x \subset B$. Recap: for all x in B, we found B_x in $\beta_+(R)$ containing x such that $B_x \subset B$.As a result, B is a union of open sets of $\tau_+(R)$. It is thus an open set of $\tau_+(R)$.■

Remark 5: Theorem 2 holds if we replace $\tau_+(R)$ and $\tau_+(R')$ respectively by $\tau_-(R)$ and $\tau_-(R')$ the lower-order-topologies.

Remark 6: The inclusion $\tau_+(R') \subset \tau_+(R)$ entails the upper semicontinuity of the extension with respect to any topology on X stronger than $\tau_+(R)$. Upper semicontinuity is used here in the sense that lower sections $\{x \in X : x \prec_R a\}$ are open. But it is not necessary for the topology on X to be stronger than $\tau_+(R)$ to have the upper semicontinuity of the extension. For more information on this issue, see (Jaffray 1975), section 5.

Axiom *scalar invariance*: For all nonnegative real α and vectors x, y in a real vector space equipped with a preorder R, Y, $x \succsim_R y \Rightarrow \alpha x \succsim_R \alpha y$.

Corollary 2: Let Y be a real normed vector space. Denote t the topology induced by the norm of Y. Let R be a preorder on Y satisfying TI and $\tau_+(R) \subset t$. Let R' be one of the complete preorders which existence is asserted by theorem

2, i.e. a complete preorder of which R is a subrelation, satisfying TI and such that $\tau_+(R') \subset \tau_+(R)$. Then R' satisfies *scalar invariance*.

Proof: We have $\tau_+(R') \subset t$. Let α be a nonnegative real and x, y two vectors in Y such that $x \succsim_R y$. Using TI and DI we get $q(x - y) \succsim_{R'} 0$ for any nonnegative rational number q. Let (q_n) be a nonnegative sequence of rationals converging to α. The sequence $q_n(x - y)$ converges to $\alpha(x - y)$. On the other hand, $q_n(x - y) \in C_+ = \{z \in Y : z \succsim_{R'} 0\}$ and C_+ is closed since $\tau_+(R') \subset t$. Thus, the limit of the sequence $(q_n(x - y))$, which is $\alpha(x - y)$, belongs to C_+. As a result $\alpha(x - y) \succsim_{R'} 0$. What yields, by TI, $\alpha x \succsim_{R'} \alpha y$.■

An immediate consequence of corollary 2 is the following:

Corollary 3: Let R be a complete preorder on Y, a real normed vector space, satisfying TI and $\tau_+(R) \subset t$, where t is the topology induced by the norm of Y. Then R satisfies *scalar invariance*.

Remark 7: $\tau_+(R) \subset t$ is a continuity requirement. Under that continuity requirement and TI, *scalar invariance* is, in a sense, satisfied since every complete preorder extending the original preorder and satisfying the same axiom of continuity and TI must satisfy *scalar invariance*.

Remark 8: (Demuynck-Lauwers 2009) showed that a given preorder satisfying TI and *scalar invariance* can be extended into a complete preorder satisfying TI and *scalar invariance*. Corollary 2 shows that if, in addition, the initial preorder satisfies upper semicontinuity, then it admits an extension which also satisfies upper semicontinuity in addition to the axioms TI and *scalar invariance*.

Remark 9: While Corollary 3 presents *scalar invariance* as a consequence of TI and a condition of continuity, (Weibull 1985) theorem A has shown that under conditions TI, *scalar invariance* and a continuity requirement called *scalar continuity*, a complete preorder verifies a stronger condition of continuity that results in representability, i.e. the existence of a real-valued order-preserving continuous function. For more information on *scalar continuity* and its properties in the context of a monotone order, see (Mitra-Ozbek 2013).

6- *Scalar invariance* as a consequence of TI and a weak Pareto axiom

We are now in the space $l_\infty^r = \{(x_1, x_2, ...) : x_i \in \mathbb{R} \text{ and } \sup |x_i| e^{-ri} < +\infty\}$, where r is a nonnegative real. This space is suitable for studying economic decisions in discrete time, infinite horizon and exponentially growing economy. If $r = 0$, the economy remains bounded.

Axiom *super weak Pareto*: if $\inf(x_i - y_i)e^{-ri} > 0$ then $x \succ_R y$.

The following lemma is a slight strengthening of theorem 4 of (Mabrouk 2011). It will be used to prove theorem 3.

Lemma 3: (wrong) If a complete preorder R' on l_∞^r satisfies *super weak Pareto* and TI, then for every $u \in l_\infty^r$ such that $u \succ_{R'} 0$ there exists a non-zero, continuous, positive (in the sense that if $x_i \geq 0$ for all i then $\varphi(x) \geq 0$) linear functional φ_u on l_∞^r such that $\varphi_u(x) > \varphi_u(y) \Rightarrow x \succ_{R'} y$ and $\varphi_u(u) > 0$.

Proof: We refer to the proof of theorem 4 in (Mabrouk 2011). The notations there are the same, except for the axiom TI instead of which a weaker axiom called "weak inv $(a_i + x_i)$" was used in (Mabrouk 2011) [3]. For the convenience of the reader, we recall some definitions and results: $l_{1\,++}^{r\pm} = \{x \in l_1^r : \inf x_i e_{\circledR}^{ri} > 0\}$, $S = \{s \in l_1^r \,/\, s \succsim_{R^0} 0\}$ and $Q = \{q \in l_1^r : q = s + p, s \in S, p \in l_{1\,++}^{r\pm}\}$. In the proof of theorem 4 of (Mabrouk 2011), Q is proved to be open and convex and to have the following properties: (i) $0 \notin Q$ (ii) $\mu q \in Q$ whenever $q \in Q$ and μ is a positive real.

Now let u be in l_1^r such that $u \succ_{R^0} 0$. The idea is to consider the convex hull Q^0 of the set Q and vector u instead of the set Q. We have $Q^0 = \{q^0 \in l_1^r : \exists(\lambda, q) \in [0,1] \times Q, q^0 = \lambda q + (1 - \lambda)u\}$. problem: Q^0 is not open!!... We show that $0 \notin Q^0$. Suppose not. There would exist λ in $[0,1]$ with $\lambda q + (1 - \lambda)u = 0$. Since $0 \notin Q$ and $u \succ_{R^0} 0$, we have $\lambda \neq 0$ and $\lambda \neq 1$. Thus we would have $\frac{\lambda}{1-\lambda}q + u = 0$. But $\frac{\lambda}{1-\lambda}q \in Q$. Thus $\frac{\lambda}{1-\lambda}q \succ_{R^0} 0$. Since $u \succ_{R^0} 0$, by TI we would have $\frac{\lambda}{1-\lambda}q + u \succ_{R^0} 0$. A contradiction. Since $0 \notin Q^0$, thanks to Hahn–Banach theorem, there exist a non-zero continuous linear functional φ_u supporting Q^0. This is written: for all q^0 in Q^0, $\varphi_u(q^0) > 0$. In particular, $\varphi_u(u) > 0$. One shows, literally as in the proof of theorem 4 of (Mabrouk 2011), that for all x, y in l_1^r, $\varphi_u(x) > \varphi_u(y) \Rightarrow x \succ_{R^0} y$ and that φ_u is positive.■

Theorem 3: Let R be a preorder on l_1^r satisfying TI and *super weak Pareto*. Let R^0 be one of the complete preorders which existence is asserted by theorem 1, i.e. a complete preorder of which R is a subrelation and satisfying TI. Then R^0 satisfies *scalar invariance*.

Proof: Since R is a subrelation to R^0, R^0 also satisfies *super weak Pareto*. Let $x, y \in l_1^r$ such that $x \succ_{R^0} y$. Denote $u = x - y$. We have $u \succ_{R^0} 0$. Apply lemma 3. There exists a non-zero, continuous, positive linear functional φ_u on l_1^r such that $\forall x^0, y^0 \in l_1^r, \varphi_u(x^0) > \varphi_u(y^0) \Rightarrow x^0 \succ_{R^0} y^0$ and $\varphi_u(u) > 0$.Let α be a positive real. Multiplying this inequality by α, one gets $\alpha\varphi_u(u) = \varphi_u(\alpha u) > 0$. Replace u by $x - y$. Then $\varphi_u(\alpha u) = \varphi_u(\alpha(x - y)) = \varphi_u(\alpha x - \alpha y) = \varphi_u(\alpha x) - \varphi_u(\alpha y) > 0$. Hence, $\varphi_u(\alpha x) > \varphi_u(\alpha y)$ and $\alpha x \succ_{R^0} \alpha y$. We have shown that for all positive real α and $x, y \in l_1^r, x \succ_{R^0} y \Rightarrow \alpha x \succ_{R^0} \alpha y$. Moreover, $x \sim_{R^0} y$ implies $\alpha x \sim_{R^0} \alpha y$ (since if we had for example $\alpha x \succ_{R^0} \alpha y$, we could multiply this last inequality by $\frac{1}{\alpha}$ and get $x \succ_{R^0} y$, a contradiction). This proves *scalar invariance*.■

Theorem 3 indicates that *scalar invariance* is satisfied under TI and *super weak Pareto* in the same sense as in remark 7. TI together with *scalar invariance* is called *strong invariance* in the terminology of (Mitra-Ozbek 2013)[4]. If we accept this justification of *scalar invariance* by TI, we are led to admit that, under *super weak Pareto*, the axiom , *strong invariance* is in a way a consequence of the axiom TI.

An immediate consequence of theorem 3 is the following:

Corollary 4: Every complete preorder R^0 satisfying TI and *super weak Pareto*, satisfies *scalar invariance*.

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[3] The definition of "weak inv $(a_i + x_i)$" is: $\forall x, y, u \in X, [x \succ_R y \Rightarrow x + u \succsim_R y + u]$. Of course, lemma 2 holds with weak inv $(a_i + x_i)$ instead of TI.

[4] In the terminology of (D'Aspremont-Gevers 2002), it is called *invariance with respect to common rescaling and individual change of origin*.

Remark 10: Since theorem 1 and lemma 3 hold in ..nite dimension, it is also the case for theorem 3 and corollary 4. Consequently, when the preorder is complete and super-weak Pareto, *strong invariance* is equivalent to TI. Hence, theorem 18 of (D'Aspremont-Gevers 2002) or example 2 of (Mitra-Ozbek 2013) asserting the linear representability of a complete preorder respecting TI, *scalar invariance, weak Pareto* and another axiom, hold without imposing *scalar invariance*.

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