Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis

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Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis*

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Abstract

We consider two-sided matching problems where agents on one side of the market (hospitals) are required to satisfy certain distributional constraints. We show that when the preferences and constraints of the hospitals can be represented by an M$^\natural$-concave function, (i) the generalized Deferred Acceptance (DA) mechanism is strategyproof for doctors, (ii) it produces the doctor-optimal stable matching, and (iii) its time complexity is proportional to the square of the number of possible contracts. Furthermore, we provide sufficient conditions under which the generalized DA mechanism satisfies these desirable properties. These conditions are applicable to various existing works and

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enable new applications as well, thereby providing a recipe for developing desirable mechanisms in practice.
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1 Introduction

The theory of two-sided matching has been extensively developed, and it has been applied to design clearinghouse mechanisms in various markets in practice.[1] As the theory has been applied to increasingly diverse types of environments, however, researchers and practitioners have encountered various forms of distributional constraints. As these features have been precluded from consideration until recently, they pose new challenges for market designers.

The *regional maximum quotas* provide an example of distributional constraints. Under the regional maximum quotas, each agent on one side of the market (who we call a hospital) belongs to a region, and there is an upper bound on the number of agents on the other side (who we call doctors) who can be matched in each region. Regional maximum quotas exist in many markets in practice. A case in point is medical residency matching in Japan. Although the match organizers initially employed the standard Deferred Acceptance (DA) mechanism (Gale and Shapley, 1962), it was criticized as placing too many doctors in urban areas and causing doctor shortages in rural areas. To address this criticism, the Japanese government now imposes a regional maximum quota on each region of the country. Regulations that are mathematically isomorphic to regional maximum quotas are utilized in various contexts, such as Chinese graduate admission, Ukrainian college admission, and Scottish probationary teacher matching, among others (Kamada and Kojima, 2012, 2015).

Furthermore, there are many matching problems in which *minimum quotas* are imposed. School districts may need at least a certain number of students in each school in order for the school to operate, as in college admissions in Hungary (Biro, Fleiner, Irving, and Manlove, 2010). The cadet-

[1] See Roth and Sotomayor (1990) for a comprehensive survey of many results in this literature.

branch matching program organized by United States Military Academy imposes minimum quotas on the number of cadets who must be assigned to each branch (Sönmez and Switzer, 2013). Yet another type of constraints takes the form of *diversity constraints*. Public schools are often required to satisfy balance between different types of students, typically in terms of socioeconomic status (Ehlers, Hafalir, Yenmez, and Yildirim, 2014). Several mechanisms have been proposed for each of these various constraints, but previous studies have focused on tailoring mechanisms to specific settings, rather than providing a general framework.[2]

This paper develops a general framework for handling various distributional constraints in the setting of *matching with contracts* (Hatfield and Milgrom, 2005). We begin with a simple model in which, on one side of the market, there exists just one hypothetical representative agent, *the hospitals*. Although extremely simple, this model proves useful. More specifically, we offer methods to aggregate the preferences of individual hospitals and *distributional constraints* into a preference of this representative agent and, as detailed later, use this aggregation to help analyze matching with constraints.[3]

For this model with *the hospitals*, the crux of our analysis is to associate the preference of the hospitals with a mathematical concept called $M^\natural$-concavity (Murota and Shioura, 1999).[4] $M^\natural$-concavity is an adaptation of concavity to functions on discrete domains, and has been studied extensively in discrete convex analysis, a branch of discrete mathematics. We

[2]Examples of papers that accommodate specific constraints include Ehlers, Hafalir, Yenmez, and Yildirim (2014); Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2016); Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo (2016); Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014); Kamada and Kojima (2015). Needless to say, we do not claim to subsume all the results in the existing studies. For instance, Kamada and Kojima (2017a) allow for general choice functions that satisfy substitutability, while our study focuses on choice functions that satisfy $M^\natural$-concavity, which is a stronger requirement. Another notable example is the study of matching with minimum quotas by Fragiadakis and Troyan (2017). Their mechanisms are different from ours, and whether there is any way to reduce their problem to our framework, or even any matching framework with substitutability, is an open question.

[3]In fact, our results readily generalize for cases with multiple, separate hospitals each of which satisfies $M^\natural$-concavity. The assumption of exactly one agent on the hospital side is made for simplicity only and, as stated above, that model proves sufficient for our purposes. A similar technique has been used by Kamada and Kojima (2017a).

[4]The letter M in $M^\natural$-concavity comes from "matroid," a mathematical structure that plays an important role in this paper. The symbol $\natural$ is read "natural."

show that if the hospitals' aggregated preference can be represented by an $M^\natural$-concave function, then the following key properties in two-sided matching hold: (i) the generalized Deferred Acceptance (DA) mechanism (Hatfield and Milgrom, 2005) is strategyproof for doctors, (ii) the resulting matching is stable (in the sense of Hatfield and Milgrom (2005)) and optimal for each doctor among all stable matchings, and (iii) the time complexity of the generalized DA mechanism is proportional to the square of the number of possible contracts.

Equipped with this general result, we study conditions under which the hospitals' preference can be represented by an $M^\natural$-concave function. We start by separating the preference of the hospitals into two parts. More specifically, we divide the preference of the hospitals into hard distributional constraints for the contracts to be feasible, and soft preferences over a family of feasible contracts. Drawing upon techniques from discrete convex analysis, we first show that if the hospitals' preference is represented by an $M^\natural$-concave function, then a family of the sets of contracts that satisfy hard distributional constraints (which we call *hospital-feasible* contracts) must constitute a matroid. Next, we show that if the hard distributional constraints constitute a matroid and the soft preferences satisfy certain easy-to-verify conditions (e.g., they can be represented as a sum of values associated with individual contracts), then the hospitals' preference can be represented by an $M^\natural$-concave function; thus the generalized DA mechanism satisfies desirable properties.

One of the main motivations of our work is to provide an easy-to-use recipe, or a toolkit, for organizing matching mechanisms under constraints. Although our general result is stated in terms of the abstract $M^\natural$-concavity condition, market designers do not need advanced knowledge of discrete convex analysis or matching theory. On the contrary, our sufficient conditions in the preceding sections suffice for most practical applications. To use our tool, all one needs to show is that the given hard distributional constraints produce a matroid (as pointed out in the preceding paragraph, requirements over soft preferences turn out to be elementary, e.g., the sum of the individual contract values). Fortunately, there exists a vast literature on matroid theory, and what kinds of constraints produce a matroid is well-understood. Therefore, it usually suffices to show that the hard distributional constraints can be mapped into existing results in matroid theory. We confirm this fact by demonstrating that most distributional constraints can be represented using our method. The list of applications includes matching markets with regional maximum quotas (Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014; Kamada and Kojima, 2015, 2017a,b), regional minimum quotas (Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo, 2016), diversity requirements in school choice (Ehlers, Hafalir, Yenmez, and Yildirim, 2014), the student-project allocation problem (Abraham, Irving, and Manlove, 2007), and the cadet-branch matching program (Sönmez and Switzer, 2013).[5]

We further demonstrate the applicability of our methodology by introducing a novel application. We examine a case in which constraints are defined based on the distance from a given ideal distribution. To our knowledge, no mechanism with desirable properties has been found in this setting before, but our general methodology enables us to find such a mechanism straightforwardly.[6] As such, we believe that this study contributes to the advance of practical market design (or "economic engineering") as emphasized in the recent literature (see Roth (2002) and Milgrom (2009) for instance), by providing tools for organizing matching clearinghouses in practice.

The rest of this paper is organized as follows. First, in the rest of this section, we discuss related literature. In Section 2, we introduce our model. In Section 3, we prove that when the hospitals' preference is represented as an $\mathrm{M}^\natural$-concave function, the above-mentioned key properties hold. In Section 4, we present several sufficient conditions under which the generalized DA satisfies these key properties. Then, in Section 5, we examine an existing application (regional maximum quotas) and a new application (distance constraints) on two-sided, many-to-one matching problems and show that the sufficient conditions described in the previous section hold in these cases. Furthermore, in Section 6, we discuss how to modify distributional constraints and soft preferences when applying our framework. Finally, Section 7 concludes this paper. Proofs are deferred to the online appendix unless noted otherwise.

Related literature

Although matching with constraints is a fairly new research topic, questions related to this issue have been studied in the literature in various specific contexts. In the U.K. medical match in the 20th century, some hospitals pre-

[5]Most of these results are presented in the online appendix.

[6]In the online appendix, we examine another novel application. Specifically, we study a setting where hospitals are partitioned into regions, and each region is associated with a regional maximum quota as well as a subset of doctors who have priority to be placed in that region over other doctors.

ferred to hire at most one female doctor (Roth, 1991). In school choice, many schools are subject to diversity constraints in terms of socioeconomic status and academic performance (Abdulkadiroğlu, 2005; Abdulkadiroğlu and Sönmez, 2003; Echenique and Yenmez, 2015; Ehlers, Hafalir, Yenmez, and Yildirim, 2014; Hafalir, Yenmez, and Yildirim, 2013; Kojima, 2012). Constraints placed over *sets* of agents have been studied in the context of student-project allocations (Abraham, Irving, and Manlove, 2007), college admission (Biro, Fleiner, Irving, and Manlove, 2010), and medical residency matching (Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo, 2016; Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014; Kamada and Kojima, 2015, 2017a,b). Our marginal contribution over these existing studies is to present a unified framework and analyze these specific markets as well as others using a single technique: as will be seen below, our theory can be applied to a wide variety of existing applications as well as new ones (Section 5).

Our paper is at the intersection of discrete mathematics and economics. In the former research field, there is a vast literature on discrete optimization. Its insight has been used in a broad range of applications such as scheduling, facility location, and structural analysis of engineering systems among others: see Murota (2000) or Schrijver (2003) or Korte and Vygen (2012), for instance. Recent advances in discrete convex analysis have found applications in exchange economies with indivisible goods (Murota, 2003; Murota and Tamura, 2003; Sun and Yang, 2006), systems analysis (Murota, 2003), inventory management (Huh and Janakiraman, 2010; Zipkin, 2008) and auction (Murota, Shioura, and Yang, 2013). As this long, and yet partial, list suggests, techniques from this literature can be applied to a wide variety of problems. We add matching problems to this list. As suggested by our analysis, results from discrete convex analysis may provide useful tools for studying matching specifically as well as economics in general.

This paper is not the first to apply discrete convex analysis to matching problems.[7] Fujishige and Tamura (2006, 2007) and Murota and Yokoi (2015) apply discrete convex analysis to study matching problems, and some of our analysis draws upon their results.[8] [9] Our marginal contributions are twofold.

[7]See the survey paper by Murota (2016) on applications of discrete convex analysis to economics.

[8]More specifically, these works deal with a many-to-many matching problem, in which a doctor/worker can work at multiple hospitals/firms. Fujishige and Tamura (2006, 2007) also consider continuous transfer and quasilinear payoff functions.

[9]See also an earlier contribution by Fleiner (2001) who applies matroid theory to match-

First, while incentive issues are not a central topic in these existing studies, they are one of the main issues in our analysis, i.e., we apply our technique to show that the generalized DA mechanism is strategyproof for doctors. Such a strategic question is a natural issue in economics, but it is rarely studied in the optimization literature. In this sense, we provide new economic questions to the discrete optimization literature. Second, we are the first to establish that various constraints found in practice can be addressed by the technique of discrete convex analysis.

This paper uses the framework of matching with contracts due to Hatfield and Milgrom (2005).[10] They identify a set of conditions for key results in matching with contracts. More specifically, if the choice function of every hospital satisfies substitutability, the law of aggregate demand, and the irrelevance of rejected contracts, then a generalized DA mechanism finds a stable allocation, and the mechanism is strategyproof for doctors.[11] Hatfield and Kojima (2009, 2010) further show that the generalized DA mechanism is group strategyproof for doctors.[12] Our analysis draws upon those studies, but it makes at least three marginal contributions over them. First, we provide three sufficient conditions for the key results to hold, where the requirements over soft preferences are elementary. Thus, basically all one needs to show is that the given hard distributional constraints produce a matroid. Second, there exists a vast literature on matroid theory, and what kinds of constraints produce a matroid is well-understood. Therefore, it usually suffices to show that the hard distributional constraints can be mapped into existing results in matroid theory. Lastly, the time complexity of the generalized DA mecha-

ing. His analysis is a special case of a more recent contribution by Fujishige and Tamura (2007).

[10]Fleiner (2003) obtains some of the results including the existence of a stable allocation in a framework that is more general than the model of Hatfield and Milgrom (2005). On the other hand, he does not show results regarding incentives, which are important for our purposes.

[11]Hatfield and Milgrom (2005) implicitly assume the irrelevance of rejected contracts throughout their analysis. Aygün and Sönmez (2013) point this out and show that this condition is important for the conclusions of Hatfield and Milgrom (2005), while showing that substitutability and the law of aggregate demand imply the irrelevance of rejected contracts.

[12]Other contributions in matching with contracts include Hatfield and Kojima (2008), Hatfield and Kominers (2009, 2012), Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Echenique (2012), Sönmez (2013), Sönmez and Switzer (2013), and Kominers and Sönmez (2016).

nism is polynomial under M$^\natural$-concavity while this property is not guaranteed under general substitutable preferences. This property is very important in practical application.

As stated above, one of the main goals of our paper is to identify a class of payoff functions that is general enough to represent various distributional constraints and preferences, while being tractable enough so that desirable normative properties can be established. Although this research program is still in its infancy, there are notable contributions. Hatfield and Milgrom (2005) set the agenda by introducing a family of payoff functions called endowed assignment valuations. A variant of this class of functions is proposed by Milgrom (2009) and further studied by Budish, Che, Kojima, and Milgrom (2013), while Ostrovsky and Paes Leme (2015) propose a new class of payoff functions called matroid-based valuations. We contribute to this line of research in several ways. First, we identify a superclass of payoff functions, M$^\natural$-concave functions, as the key to our approach.[13] Second, in addition to various sufficient conditions, we identify a *necessary* condition for a payoff function to allow the use of our theory in terms of the matroid structure.

This paper is part of the literature on practical market design, both in terms of content and in terms of approach. As advocated by Roth (2002), recent market design theory has focused on solving practical problems by providing detailed and concrete solutions.[14] Real problems often share common basic features, but differ substantially in details. For instance, different school districts share some common goals such as efficiency, stability (fairness) and incentive compatibility, but can differ in some details such as diversity constraints, structure of school priorities, and authoritative power of individual schools (Abdulkadiroğlu, Pathak, and Roth, 2005, 2009; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005, 2006). In this respect, our contributions are twofold. First, our framework provides mechanisms that can be applied to a variety of existing problems as discussed earlier. Second,

[13]Ostrovsky and Paes Leme (2015) demonstrate that the class of endowed assignment valuations is a strict subset of the matroid-based valuations. The matroid-based valuations is a subset of the class of valuations that satisfy the gross substitutes condition. The gross substitutes condition is equivalent to M$^\natural$-concavity (Fujishige and Yang, 2003). As Ostrovsky and Paes Leme (2015) mention, it is still an open question whether the matroid-based valuations is equivalent to the class of valuations that satisfy the gross substitutes condition.

[14]Auction market design emphasizes the importance of addressing practical problems as well (see Milgrom (2000, 2004) for instance).

we develop a theory of matching under constraints that could be applied to new problems that have not been found yet but may be found in the future.

Finally, this paper is part of the literature on matching and market design. The field is too large to even casually summarize here. Instead, we refer interested readers to surveys by Roth and Sotomayor (1990), Roth (2008), Sönmez and Ünver (2011), Abdulkadiroğlu and Sönmez (2013), Pathak (2016), and Kojima (2016).

2 Model

A market is a tuple $(D, H, X, (\succ_d)_{d\in D}, f)$. D is a finite set of doctors and H is a finite set of hospitals. X is a finite set of contracts. Each contract $x \in X$ is bilateral, in the sense that x is associated with exactly one doctor $x_D \in D$ and exactly one hospital $x_H \in H$. Each contract can also contain some terms of contracts such as working time and wages. Each $\succ_d$ represents the strict preference of each doctor d over acceptable contracts within $X_d = \{x \in X \mid x_D = d\}$.[15] We assume each contract $x \in X$ is acceptable for x_H: if a hospital considers a contract unacceptable, it is not included in X. For notational simplicity, for $X' \subseteq X$ and $x \in X$, we write $X' + x$ and $X' - x$ to represent $X' \cup \{x\}$ and $X' \setminus \{x\}$, respectively. Also, when $x = \emptyset$, $X' + x$ means nothing is added to X', and $X' - x$ means nothing is removed from X'.

We assume some distributional constraints are enforced on feasible contracts. We assume such distributional constraints and hospital preferences are aggregated into a preference of a representative agent, which we call "the hospitals" (Section 5 and Appendix C illustrate in detail how such aggregations can be done in various applications). The preference of the hospitals is represented by a payoff function $f : 2^X \to \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ is the set of all real numbers. For two sets of contracts $X', X'' \subseteq X$, the hospitals strictly prefer X' over X'' if and only if $f(X') > f(X'')$ holds. If $X' \subseteq X$ violates some distributional constraint, then $f(X') = -\infty$. We assume f is normalized by $f(\emptyset) = 0$.[16] Also, we assume f is unique-selecting, i.e., for all

[15] More precisely, we assume that for each doctor d, the set of acceptable contracts for d is given as a subset of X_d, and $\succ_d$ represents a strict preferences over that set. Clearly, it is equivalent to a (more standard) model in which we let $\succ_d$ be a strict preference over $X_d \cup \{\emptyset\}$ where $\emptyset$ is the outside option, and say that a contract x is acceptable if $x \succ_d \emptyset$.

[16] As described later, this assumption is slightly stronger than mere normalization, be-

$X' \subseteq X$, $|\arg\max_{X'' \subseteq X'} f(X'')| = 1$ holds.[17]

Now, we introduce several concepts used in this paper.

Definition 1 (feasibility)**.** For a subset of contracts $X' \subseteq X$, we say X' is **hospital-feasible** if $f(X') \neq -\infty$. We say X' is **doctor-feasible** if for all $d \in D$, either (i) $X'_d = \{x\}$ and x is acceptable for d, or (ii) $X'_d = \emptyset$ holds, where $X'_d = \{x \in X' \mid x_D = d\}$. We say X' is **feasible** if it is doctor- and hospital-feasible. We call a feasible set of contracts a **matching**.

With a slight abuse of notation, for two sets of contracts X' and X'', we denote $X'_d \succ_d X''_d$ if either (i) $X'_d = \{x'\}$, $X''_d = \{x''\}$, and $x' \succ_d x''$ for some $x', x'' \in X_d$ that are acceptable for d, or (ii) $X'_d = \{x'\}$ for some $x' \in X_d$ that is acceptable for d and $X''_d = \emptyset$. Furthermore, we denote $X'_d \succeq_d X''_d$ if either $X'_d \succ_d X''_d$ or $X'_d = X''_d$. Also, we use notations like $x \succ_d X'_d$ or $X'_d \succ_d x$, where x is a contract and X' is a matching. Furthermore, for $X'_d \subseteq X_d$, we say X'_d is acceptable for d if either (i) $X'_d = \{x\}$ and x is acceptable for d, or (ii) $X'_d = \emptyset$ holds.

For each doctor d, its **choice function** Ch_d specifies her most preferred contract within $X' \subseteq X$, i.e., $Ch_d(X') = \{x\}$, where x is the most preferred acceptable contract in X'_d if one exists, and $Ch_d(X') = \emptyset$ if no such contract exists. Then, the choice function of all doctors Ch_D is defined as $Ch_D(X') := \bigcup_{d \in D} Ch_d(X')$.

For the hospitals, their choice function Ch_H is defined by $Ch_H(X') = \arg\max_{X'' \subseteq X'} f(X'')$ for each $X' \subseteq X$. Since we assume payoff function f is unique-selecting, Ch_H is uniquely determined by f.

Note that if X' contains multiple contracts that are related to the same doctor d, Ch_H is allowed to choose them simultaneously.

Definition 2 (stability (Hatfield and Milgrom, 2005))**.** We say a matching X' is **stable** if $X' = Ch_H(X') = Ch_D(X')$ and there exists no $x \in X \setminus X'$ such that $x \in Ch_H(X' + x)$ and $x \in Ch_D(X' + x)$.[18]

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cause it implies that $\emptyset$ is hospital-feasible.

[17]Observe that strict preference of the hospitals, a standard assumption in matching theory, implies f is unique-selecting, but the converse does not hold. Also we note that if f is not unique-selecting, we can obtain a unique-selecting function by modifying f very slightly. Let us define a total order relation on X, and for each $x \in X$, let $\mathrm{rank}(x)$ represent the position of x within X according to this relation, i.e., $\mathrm{rank}(x) = i$ if x is ranked i-th. Also, let $v(x)$ denote $\epsilon \cdot 2^{-\mathrm{rank}(x)}$, where ϵ is a sufficiently small positive number. Then, $f(X') + \sum_{x \in X'} v(x)$ is unique-selecting and $\arg\max[f(X') + \sum_{x \in X'} v(x)] \subseteq \arg\max f(X')$.

[18]Hatfield and Milgrom (2005) as well as many others define stability in such a way that

We sometimes refer to the stability concept in Definition 2 as Hatfield-Milgrom (HM)-stability when we discuss the relation with other stability concepts.

Let $\mathcal{X}$ be the set of all stable matchings. We say $X' \in \mathcal{X}$ is the **doctor-optimal stable matching** if $X'_d \succeq_d X''_d$ for all $X'' \in \mathcal{X}$ and $d \in D$.[19]

A mechanism φ is a function that takes a profile of preferences of doctors $\succ_D$ as an input and returns a matching $X' \subseteq X$. Let $\succ_{D\setminus\{d\}}$ denote a profile of preferences of doctors except d, and $(\succ_d, \succ_{D\setminus\{d\}})$ denote a profile of preferences of all doctors, where d's preference is $\succ_d$ and the profile of preferences of other doctors is $\succ_{D\setminus\{d\}}$. We say φ is **strategyproof for doctors** if $\varphi_d((\succ_d, \succ_{D\setminus\{d\}})) \succeq_d \varphi_d((\succ'_d, \succ_{D\setminus\{d\}}))$ holds for all d, $\succ_d$, $\succ'_d$, and $\succ_{D\setminus\{d\}}$.

Let us introduce three properties of the choice function of the hospitals. If Ch_H satisfies these conditions, the generalized DA satisfies several desirable properties (Aygün and Sönmez, 2013; Hatfield and Milgrom, 2005).

Irrelevance of rejected contracts: for any $X' \subseteq X$ and any $x \in X \setminus X'$, $Ch_H(X') = Ch_H(X' + x)$ holds whenever $x \notin Ch_H(X' + x)$.

Substitutability (the substitutes condition): for any $X', X'' \subseteq X$ with $X' \subseteq X''$, $Re_H(X') \subseteq Re_H(X'')$ holds, where $Re_H(Y) = (Y \setminus Ch_H(Y))$.

Law of aggregate demand: for any $X', X'' \subseteq X$ with $X' \subseteq X''$, $|Ch_H(X')| \leq |Ch_H(X'')|$.

3 M$^\natural$-concavity and the generalized DA mechanism

This section introduces the concept of M$^\natural$-concavity, which imposes a restriction on the way that the hospitals evaluate sets of contracts. Then we show that if the preference of the hospitals is represented as an M$^\natural$-concave function, then a number of key conclusions in matching theory hold.

a block by a coalition that includes multiple doctors is allowed. Such a concept is identical to our definition if the hospitals have substitutable preferences.

[19] As in the case of the term stability, when we explicitly consider a set of matchings that satisfy a particular stability concept, we abuse terminology slightly by, for instance, writing "the doctor-optimal HM-stable matching."

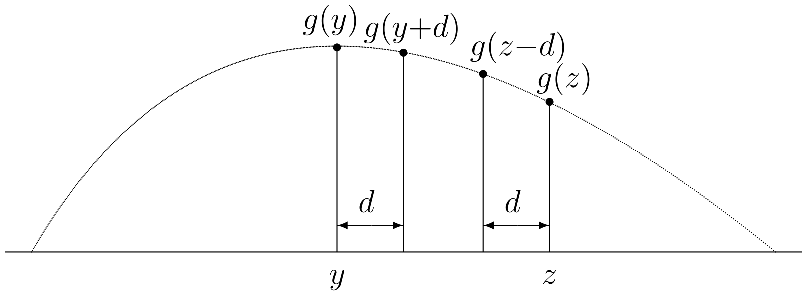

Figure 1: Concavity of a continuous-variable function

Definition 3 (M$^\natural$-concavity (Murota and Shioura, 1999))**.** We say that f is **M$^\natural$-concave** if for all $Y, Z \subseteq X$ and $y \in Y \setminus Z$, there exists $z \in (Z \setminus Y) \cup \{\emptyset\}$ such that $f(Y) + f(Z) \leq f(Y - y + z) + f(Z - z + y)$ holds.

M$^\natural$-concavity is a discrete analogue of concavity of continuous-variable functions. To help develop intuition of M$^\natural$-concavity, consider a continuous-variable function $g : \mathbb{R} \to \mathbb{R}$. We say g is concave if for all $y, z \in \mathbb{R}$ and λ such that $0 \leq \lambda \leq 1$, the following condition holds:

$$g(y) + g(z) \leq g(y + d) + g(z - d),$$

where $d = \lambda(z - y)$.[20] Assume $y < z$. Then, $y + d$ is a point reached from y by moving d to the right, and $z - d$ is a point reached from z by moving d to the left (Figure 1). In a discrete domain, we can interpret $Y - y + z$ is a point reached from Y by moving one-step closer to Z, and $Z - z + y$ is a point reached from Z by moving one-step closer to Y. Thus, M$^\natural$-concavity is a counterpart of concavity, adapted to make sense in the discrete domain.

In our context, M$^\natural$-concavity is a requirement that contracts are substitutable in a particular manner. To be more precise, we can immediately derive the following proposition.

Proposition 1. *Assume f is M$^\natural$-concave. For any $Y \subsetneq X$ and $z \in X \setminus Y$, (i) $Ch_H(Y + z) = Ch_H(Y)$, or (ii) $Ch_H(Y + z) = Ch_H(Y) + z$, or (iii) $Ch_H(Y + z) = Ch_H(Y) - y + z$ for some $y \in Ch_H(Y) \setminus Ch_H(Y + z)$.*

Proof. Let $Y^* = Ch_H(Y)$ and $Z^* = Ch_H(Y + z)$. If $z \notin Z^*$, since $Y^* \subseteq Y$ and $Z^* \subseteq Y \subsetneq Z = Y + z$ hold and f is unique-selecting, $Y^* =$

[20]This definition is equivalent to the most common definition that $g(\lambda y + (1 - \lambda)z) \geq \lambda g(y) + (1 - \lambda)g(z)$ for all $y, z \in \mathbb{R}$ and λ with $0 \leq \lambda \leq 1$.

$\arg\max_{X''\subseteq Y} f(X'') = Z^*$ holds. Thus, let us assume $z \in Z^*$. Since f is M$^\natural$-concave, either (a) there exists $y \in Y^* \setminus Z^*$ such that $f(Z^*) + f(Y^*) \leq f(Z^*-z+y)+f(Y^*-y+z)$ holds, or (b) $f(Z^*)+f(Y^*) \leq f(Z^*-z)+f(Y^*+z)$ holds. Assume (a) and $Z^* \neq Y^*-y+z$ hold. Since $Z^* = \arg\max_{Z'\subseteq Y+z} f(Z')$, $Y^* - y + z \subseteq Y + z$, and f is unique-selecting, $f(Z^*) > f(Y^* - y + z)$ holds. Also, since $Y^* = \arg\max_{Y'\subseteq Y} f(Y')$ and $Z^* - z + y \subseteq Y$, $f(Y^*) > f(Z^* - z + y)$ holds. Thus, $f(Z^*) + f(Y^*) > f(Z^* - z + y) + f(Y^* - y + z)$ holds. This is a contradiction. Thus, if (a) holds, $Z^* = Y^* - y + z$, i.e., $Ch_H(Y + z) = Ch_H(Y) - y + z$, holds. Assume (b) and $Z^* \neq Y^* + z$ hold. Since $Z^* = \arg\max_{Z'\subseteq Y+z} f(Z')$ and $Y^*+z \subseteq Y+z$, $f(Z^*) > f(Y^*+z)$ holds. Also, since $Y^* = \arg\max_{Y'\subseteq Y} f(Y')$ and $Z^* - z \subseteq Y$, $f(Y^*) > f(Z^* - z)$ holds. Thus, $f(Z^*) + f(Y^*) > f(Z^* - z) + f(Y^* + z)$ holds. This is a contradiction. Thus, if (b) holds, $Z^* = Y^* + z$, i.e., $Ch_H(Y + z) = Ch_H(Y) + z$, holds. □

This proposition provides a specific sense in which contracts are viewed as substitutable when the payoff function is M$^\natural$-concave. When a new contract z becomes available, the new chosen set of contracts $Ch_H(Y + z)$ is (i) unchanged from the original chosen set $Ch_H(Y)$ or (ii) obtained by adding z to $Ch_H(Y)$ or (iii) obtained by replacing exactly one contract y in $Ch_H(Y)$ with z. In particular, no contract that is not chosen from the original set is chosen, that is, contracts are substitutable. Note also that at most one contract becomes newly rejected, and that happens only when the new contract is accepted. Therefore, from Proposition 1, we can immediately derive the following properties.[21]

Corollary 1. *If f is M$^\natural$-concave, then Ch_H satisfies substitutability, the law of aggregate demand, and the irrelevance of rejected contracts.*

The converse of Corollary 1 does not hold in general. That is, there exists a choice function Ch_H such that it satisfies irrelevance of rejected contracts, substitutabiilty, and the law aggregate demand, but there exists no M$^\natural$-concave payoff function which represents Ch_H. For example, let us assume that the preference of H on $X = \{a, b, c\}$ is

$$\{b, c\} \succ_H \{a, b\} \succ_H \{a, c\} \succ_H \{a\} \succ_H \{b\} \succ_H \{c\} \succ_H \emptyset \succ_H \{a, b, c\},$$

[21] Fujishige and Tamura (2006) show that substitutability holds if f is M$^\natural$-concave and unique-selecting. Furthermore, Murota and Yokoi (2015) show that the law of aggregate demand holds if f is M$^\natural$-concave and unique-selecting.

and that, for each $X' \subseteq X$, $Ch_H(X')$ is defined as the most preferred subset of X'. It is easy to show that Ch_H satisfies irrelevance of rejected contracts, substitutability and law of aggregate demand. Suppose that Ch_H is described by a payoff function f. In this case, f must satisfy the following inequalities:

$$f(\{b,c\}) > f(\{a,b\}) > f(\{a,c\}) > f(\{a\}) > f(\{b\}) > f(\{c\}) > f(\emptyset) = 0$$

and $f(\{a,b,c\}) = -\infty$. This function f is not M$^\natural$-concave because for $Y = \{b,c\}$, $Z = \{a\}$ and $b \in Y \setminus Z$, the following inequality must hold:

$$f(\{b,c\}) + f(\{a\}) > \max\{f(\{c\}) + f(\{a,b\}), f(\{a,c\}) + f(\{b\})\},$$

which contradicts the definition of M$^\natural$-concavity.

Actually, it is possible to characterize M$^\natural$-concavity based on substitutability (Farooq and Shioura, 2005; Farooq and Tamura, 2004). More precisely, a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with a bounded effective domain is M$^\natural$-concave if and only if for all linear functions $p : \mathbb{Z}^n \to \mathbb{R}$, $f + p$ satisfies substitutability, where the effective domain of f is defined by $\{x \in \mathbb{Z}^n \mid f(x) \neq -\infty\}$. Furthermore, Fujishige and Yang (2003) show that M$^\natural$-concavity is equivalent to the gross substitutes condition due to Kelso and Crawford (1982), as well as the single improvement property due to Gul and Stacchetti (1999).

The generalized DA mechanism (Hatfield and Milgrom, 2005) is a generalized version of the well-known deferred acceptance algorithm (Gale and Shapley, 1962), which is adapted for the 'matching with contracts' model.[22]

Mechanism 1 (Generalized Deferred Acceptance (DA) mechanism)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor offers her most preferred contract which has not been rejected before Stage k. If no remaining contract is acceptable for d, d does not make any offer. Let X' be the set of contracts that are offered in this Step.

Step 2 The hospitals tentatively accept $Ch_H(X')$ and reject all other contracts in X'.

[22] In Hatfield and Milgrom (2005), this mechanism is called *generalized Gale-Shapley* algorithm.

Step 3 If all the contracts in X' are tentatively accepted in Step 2, then let X' be the final matching and terminate the mechanism. Otherwise, go to Stage $k+1$.

Now we are ready to show that the fact that f is M$^\natural$-concave guarantees that the generalized DA mechanism satisfies several desirable properties. The following lemma immediately follows from existing results in discrete convex analysis.

Lemma 1. *Suppose that the preference of the hospitals can be represented by an M$^\natural$-concave function f. Then, the generalized DA mechanism is strategyproof for doctors. Also, it always produces a stable matching, and the obtained matching is the doctor-optimal stable matching.*

Proof. By Corollary 1, if f is M$^\natural$-concave, then Ch_H satisfies the irrelevance of rejected contracts, substitutability, and the law of aggregate demand. Hatfield and Milgrom (2005) show that if Ch_H satisfies these three conditions, then the generalized DA mechanism is strategyproof for doctors, and it obtains the doctor-optimal stable matching. □

This lemma shows that the generalized DA mechanism produces a desirable matching, and incentive compatibility for doctors are guaranteed. These are the key properties emphasized in the theoretical matching literature (see Hatfield and Milgrom (2005) for example) as well as in the literature on practical market design (see, for instance, Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu, Pathak, and Roth (2009)).

Next, we show that M$^\natural$-concavity also guarantees efficient computation.

Theorem 1. *Suppose that the preference of the hospitals can be represented by an M$^\natural$-concave function f. Then, the time complexity of the generalized DA mechanism is proportional to $|X|^2$.*

Proof. At Step 1 in Mechanism 1, the calculation of Ch_D is $O(|X|)$ in total, because each (rejected) doctor selects her most preferred contract which has not been rejected. Hence the time complexity of the generalized DA mechanism depends on calculations of Ch_H. At Step 2 in Mechanism 1, we calculate $Ch_H(X')$ by adding newly offered contracts one by one. More precisely, we use the next relation which is guaranteed by substitutability and irrelevance of rejected contracts,

$$Ch_H(X') = Ch_H(Ch_H(\cdots Ch_H(Ch_H(X'' + y_1) + y_2) + \cdots) + y_k)$$

where X'' and $\{y_1, y_2, \ldots, y_k\}$ are the tentatively accepted contracts at the previous stage (or initially $X'' = \emptyset$) and the newly offered contracts in X' at Step 1, respectively. By Proposition 1 and the fact $X'' = Ch_H(X'')$, $Ch_H(X'' + y_1)$ is determined by calculating f exactly $|X''| + 1$ times, and hence, at most $|X|$ times.[23] In the same way as above, we can determine $Ch_H(X')$ by calculating f at most $k \cdot |X|$ times. Since each contract is selected as a newly offered contract at most once in the generalized DA mechanism, the calculation of Ch_H is $O(T(f) \cdot |X|^2)$ in total, where $T(f)$ denotes the time required to calculate f. Thus, the time complexity of the generalized DA mechanism is $O(T(f) \cdot |X|^2)$. □

This theorem shows that the desired outcome can be easily computed by the algorithm. This property is not guaranteed for the general substitutable preference case. While sometimes de-emphasized in the literature, we emphasize that efficient computability is crucial for the actual implementation of the mechanism in practical market design.

Overall, these results suggest that the generalized DA mechanism is a compelling mechanism if preferences and constraints can be aggregated into an M$^\natural$-concave function. The remainder of this paper demonstrates that such an aggregation is indeed possible in various applied environments.

4 Conditions for M$^\natural$-concavity

In this section, we investigate conditions under which payoff function f becomes M$^\natural$-concave. Without loss of generality, we can assume payoff function f is represented by the summation of two parts, i.e., $f(X') = \widehat{f}(X') + \widetilde{f}(X')$, where $\widehat{f}$ represents hard distributional constraints for hospital-feasibility and $\widetilde{f}$ represents soft preferences over hospital-feasible contracts. More specifically, $\widehat{f}(X')$ returns 0 if X' is hospital-feasible and $-\infty$ otherwise, while $\widetilde{f}(X')$ returns a bounded non-negative value.

Let $\mathrm{dom}\, f = \mathrm{dom}\, \widehat{f} = \{X' \mid X' \subseteq X, \widehat{f}(X') \neq -\infty\}$ be the effective domain of f (or equivalently $\widehat{f}$). In the present context, $\mathrm{dom}\, f$ represents the family of hospital-feasible sets of contracts. In this section, we first show a

[23] To be more precise, $Ch_H(X'' + y_1)$ is either X'', $X'' + y_1$, or $X'' - x + y_1$, where $x \in X''$. Thus, to obtain $Ch_H(X'' + y_1)$, it is sufficient to apply f to these $|X''| + 1$ candidates and to choose the one that maximizes f, because $f(X'')$ has been calculated at the previous stage.

necessary condition on $\widehat{f}$, namely, the effective domain of $\widehat{f}$ (or equivalently f) must constitute a mathematical structure called matroid. Next, we identify three sufficient conditions so that f becomes M$^\natural$-concave, assuming the effective domain of $\widehat{f}$ constitutes a matroid.

Let us first introduce the concept of matroid (Oxley, 2011).

Definition 4 (matroid)**.** Let X be a finite set, and $\mathcal{F}$ be a family of subsets of X. We say a pair $(X, \mathcal{F})$ is a **matroid** if it satisfies the following conditions.

1. $\emptyset \in \mathcal{F}$.

2. If $X' \in \mathcal{F}$ and $X'' \subseteq X'$, then $X'' \in \mathcal{F}$ holds.

3. If $X', X'' \in \mathcal{F}$ and $|X'| > |X''|$, then there exists $x \in X' \setminus X''$ such that $X'' + x \in \mathcal{F}$.

The term "matroid" is created from "matrix" and "-oid", i.e., a matroid is something similar to a matrix, and the concept of a matroid is an abstraction of some properties of matrices. To get an idea, suppose that A is a matrix and X is the set of column vectors of A. Let us assume $\mathcal{F}$ is a family of subsets of X, such that for each $X' \in \mathcal{F}$, all column vectors in X' are linearly independent. It is clear that conditions 1 and 2 of the above definition hold. Also, if X' has more elements than X'', we can always choose $x \in X' \setminus X''$ such that $X''+x$ becomes linearly independent. Therefore condition 3 is also satisfied, which shows that $(X, \mathcal{F})$ in this example is a matroid.

The concept of matroid has been utilized in matching theory. For example, Roth, Sönmez, and Ünver (2005) show that the sets of simultaneously matchable patients induces a matroid. As we will see in this paper, matroids play an essential role in our analysis of matching under constraints.

Now we are ready to present one of the connections between matroids and our theory of matching with constraints. The following lemma holds.

Lemma 2. *Under the assumption* $\emptyset \in \mathrm{dom}\,\widehat{f}$*,* $\widehat{f}$ *is M*$^\natural$*-concave if and only if* $(X, \mathrm{dom}\,\widehat{f})$ *is a matroid.*[24]

Intuitively, if any of the matroid conditions is violated, we can create a situation where $\widehat{f}(X') = \widehat{f}(X'') = 0$, but either $\widehat{f}(X'-x+y)$ or $\widehat{f}(X''+x-y)$ becomes $-\infty$ for some $X', X'', x \in X' \setminus X''$, and $y \in X'' \setminus X' \cup \{\emptyset\}$.

[24]Murota and Shioura (1999) show that the effective domain $\mathrm{dom}\, f$ of an M$^\natural$-concave function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ forms a generalized polymatroid. The "only if" part of Lemma 2 is a special case of this result.

This lemma suggests that the matroid structure plays an important role in our analysis. In particular, it is easy to see that the "only if" part implies that a matroid structure is needed in order for the function $f = \widehat{f} + \widetilde{f}$ to be M$^\natural$concave. This means that, in order to utilize the theory of M$^\natural$-concavity in our analysis of matching with constraints, it is necessary for the sets of hospital-feasible contracts to constitute a matroid.

Now that we have found the necessity of a matroid structure for our analysis, let us turn to sufficient conditions. More specifically, we assume that $(X, \mathrm{dom}\, f) = (X, \mathrm{dom}\, \widehat{f})$ is a matroid and examine conditions on $\widetilde{f}$, i.e., the soft preference of the hospitals, for guaranteeing that $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave. This task would have been easy if the sum of two M$^\natural$-concave functions were always M$^\natural$-concave, since the hard constraint part $\widehat{f}$ is M$^\natural$-concave if $\mathrm{dom}\, \widehat{f}$ is a matroid (Lemma 2). However, the following example demonstrates that the sum of two M$^\natural$-concave functions is not guaranteed to be M$^\natural$-concave.

Example 1. Assume $X = \{x_1, x_2, x_3\}$. $f_1(X')$ is 0 if either $X' \subseteq \{x_1, x_2\}$ or $X' \subseteq \{x_1, x_3\}$, and otherwise, $-\infty$. $f_2(X')$ is 0 if either $X' \subseteq \{x_2, x_3\}$ or $X' \subseteq \{x_1, x_3\}$, and otherwise, $-\infty$. Since $(X, \mathrm{dom}\, f_1)$ and $(X, \mathrm{dom}\, f_2)$ are matroids, both f_1 and f_2 are M$^\natural$-concave from Lemma 2. However, $f = f_1 + f_2$ is not M$^\natural$-concave, since $(X, \mathrm{dom}\, f)$, where $\mathrm{dom}\, f = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_3\}\}$, is not a matroid. To see this, observe that when $X' = \{x_1, x_3\}, X'' = \{x_2\}$, we have $X', X'' \in \mathrm{dom}\, f$ and $|X'| > |X''|$, but there exists no $x \in X' \setminus X''$ such that $X'' + x \in \mathrm{dom}\, f$ holds (note that if f_2 has some special form, e.g., $f_2(X')$ is of the form $\sum_{x \in X'} v(x)$, we can guarantee that $f = f_1 + f_2$ is M$^\natural$-concave as long as f_1 is M$^\natural$-concave, as shown in Condition 1).

The above example shows the mere fact that both $\widehat{f}$ and $\widetilde{f}$ are M$^\natural$-concave is not sufficient for guaranteeing M$^\natural$-concavity of $f = \widehat{f} + \widetilde{f}$. Nevertheless, we demonstrate that a number of simple sufficient conditions exist when the hard constraint part induces a matroid. More specifically, we assume $(X, \mathrm{dom}\, \widehat{f})$ constitutes a matroid, and introduce three sufficient conditions to guarantee that f is M$^\natural$-concave: (1) $\widetilde{f}$ is a sum of contract values, (2) $\widehat{f}$ is symmetric for groups G and $\widetilde{f}$ is order-respecting for G, and (3) $(X, \mathrm{dom}\, \widehat{f})$ is a structure called a laminar matroid on a laminar family and $\widetilde{f}$ is a laminar concave function on it. If one of these conditions hold, from Lemma 1 and Theorem 1, we obtain: (i) the generalized DA mechanism is strategyproof for doctors, (ii) it produces the doctor-optimal stable matching, and (iii) its time complexity is proportional to $|X|^2$. As will be seen in Section 5 and the online appendix, most stability concepts in existing works can be understood as stability with respect to one of these M$^\natural$-concave functions.

Let us introduce the first condition, which provides a simple but very general method for obtaining an M$^\natural$-concave function.

Definition 5 (sum of contract values)**.** We say $\widetilde{f}$ is a **sum of contract values** if $\widetilde{f}(X') = \sum_{x \in X'} v(x)$, where $v : X \to (0, \infty)$ is a function such that $x \neq x'$ implies $v(x) \neq v(x')$.

As indicated by the name, function $\widetilde{f}$ in this definition is written as a sum of values of individual contracts, where $v(x)$ is interpreted as the value of contract x. Note that we assume each value is positive, and different contracts are assigned different values.

Now we are ready to present our first condition.

Condition 1. *Assume* $(X, \operatorname{dom} \widehat{f})$ *is a matroid (or equivalently,* $\widehat{f}$ *is M*$^\natural$*-concave and* $\emptyset \in \operatorname{dom} \widehat{f}$*) and* $\widetilde{f}(X')$ *is a sum of contract values, then* $f = \widehat{f} + \widetilde{f}$ *is M*$^\natural$*-concave.*

Proof. Let us assume $\widetilde{f}(X')$ is represented as $\sum_{x \in X'} v(x)$. Since $\widehat{f}$ is M$^\natural$-concave, for any $Z, Y \subseteq X$, for any $y \in Y \setminus Z$, there exists $z \in (Z \setminus Y) \cup \{\emptyset\}$ such that $\widehat{f}(Y) + \widehat{f}(Z) \leq \widehat{f}(Y - y + z) + \widehat{f}(Z - z + y)$ holds. On the other hand, we have $\sum_{x \in Y - y + z} v(x) + \sum_{x \in Z - z + y} v(x) = \sum_{x \in Y} v(x) + \sum_{x \in Z} v(x)$. Thus, $f(Y) + f(Z) \leq f(Y - y + z) + f(Z - z + y)$ holds, so f is M$^\natural$-concave. □

This class of functions is equivalent to the class of *weighted matroids* (Edmonds, 1971). Ostrovsky and Paes Leme (2015) show that a weighted matroid satisfies the gross substitute condition.

The second sufficient condition for M$^\natural$-concavity is based on an idea of grouping contracts. We begin by formally introducing the concept of a group of contracts.

Definition 6 (group of contracts)**.** Let $G = \{g_1, \ldots, g_n\}$ be a partition of X, i.e., $g \cap g' = \emptyset$ for any $g, g' \in G$ with $g \neq g'$ and $\bigcup_{g \in G} g = X$. We refer to each element g of G as a **group of contracts** (or simply a group) in G, and G as **groups**.

One division of contracts into groups that we use in this paper is based on hospitals, that is, we let each g_i represent the set of contracts related to hospital h_i.

Kamada and Kojima (2014) introduce a concept called an order-respecting preferences, which models a variety of preferences of the hospitals. Using the concept of groups of contracts, we now introduce a class of payoff functions that represent this class of preferences.

Definition 7 (order-respecting payoff function)**.** For groups G, an **order-respecting payoff function** $\widetilde{f}$ is given as follows:

$$\widetilde{f}(X') = \sum_{g \in G} V_g(|X' \cap g|) + \sum_{x \in X'} v(x),$$

where $v : X \to (0, \infty)$ is a function such that $x \neq x'$ implies $v(x) \neq v(x')$, and $V_g : \mathbb{Z}_+ \to (0, \infty)$ is a concave function.

Now, we introduce a further condition on a matroid so that $\widehat{f} + \widetilde{f}$ becomes $\mathrm{M}^\natural$-concave, when $\widetilde{f}$ is an order-respecting payoff function.

Definition 8 (symmetry of groups)**.** Let $(X, \mathcal{F})$ be a matroid and G be a partition of X. We say that G is **symmetric** in $(X, \mathcal{F})$ if for all $g \in G$, for all $x, x' \in g$, and for all $X' \subsetneq X$ such that $\{x, x'\} \cap X' = \emptyset$, $X' + x \in \mathcal{F}$ if and only if $X' + x' \in \mathcal{F}$ holds.

Now, we are ready to define our second condition.

Condition 2. *If $(X, \mathrm{dom}\, \widehat{f})$ is a matroid, G is symmetric in $(X, \mathrm{dom}\, \widehat{f})$, and $\widetilde{f}$ is an order-respecting payoff function for G, then $f = \widehat{f} + \widetilde{f}$ is $M^\natural$-concave.*

Finally, we introduce the third sufficient condition. The crucial concept we use is the *laminar family* defined below.

Definition 9 (laminar family)**.** $\mathcal{T}$ is a **laminar family** of subsets of X if for any $Y, Z \in \mathcal{T}$, one of the following conditions holds:

1. $Y \cap Z = \emptyset$,
2. $Y \subseteq Z$, or
3. $Z \subseteq Y$.

We say $f(X') = \sum_{T \in \mathcal{T}} f_T(|X' \cap T|)$ is a **laminar concave function** if $\mathcal{T}$ is a laminar family and each f_T is a univariate concave function.

In words, a family of sets $\mathcal{T}$ is said to be a laminar family if it has a structure that can be described as layers or a hierarchy. More specifically, for any pair of sets in this family, either they are disjoint from each other or one of them is a subset of the other. Laminar families have been used for mechanism design in two-sided matching (Biro, Fleiner, Irving, and Manlove, 2010; Kamada and Kojima, 2017a), indivisible object allocation (Budish, Che, Kojima, and Milgrom, 2013), and auction (Milgrom, 2009).

A laminar family of subsets of contracts naturally induces a matroid, a structure we call a laminar matroid, as defined below.

Definition 10 (laminar matroid)**.** We say $(X, \mathcal{F})$ is a **laminar matroid** on a laminar family $\mathcal{T}$ if it is constructed as follows:

- For each $T \in \mathcal{T}$, a positive integer q_T is given.
- $\mathcal{F}$ is defined as $\{X' \subseteq X \mid |X' \cap T| \leq q_T \ (\forall T \in \mathcal{T})\}$.

To show a laminar matroid is a matroid, let us introduce a simple matroid and methods for creating new matroids. $(X, \mathcal{F})$ is said to be a **uniform matroid** if $\mathcal{F} = \{X' \mid X' \subseteq X, |X'| \leq k\}$ for some non-negative integer k.

For a set of matroids $(X_1, \mathcal{F}_1), \ldots, (X_k, \mathcal{F}_k)$, where each X_i is disjoint, their **direct sum** is defined as $(X, \mathcal{F})$, where $X = \bigcup_{1 \leq i \leq k} X_i$, $\mathcal{F} = \{X' \mid X' = \bigcup_{1 \leq i \leq k} X'_i$, where $X'_i \in \mathcal{F}_i\}$. Assume $(X, \mathcal{F})$ is a matroid and k is a non-negative integer. Then, its k**-truncation** is defined as $(X, \tilde{\mathcal{F}})$, where $\tilde{\mathcal{F}} = \{X' \in \mathcal{F} \mid |X'| \leq k\}$.

It is obvious that conditions 1 and 2 of a matroid hold in the above three cases. For a uniform matroid, if $X', X'' \in \mathcal{F}$ and $|X'| > |X''|$ hold, then for any $x \in X' \setminus X''$, it follows that $|X'' + x| \leq |X'| \leq k$, so $X'' + x \in \mathcal{F}$ holds. Thus, a uniform matroid is a matroid. For a direct sum, if $X', X'' \in \mathcal{F}$ and $|X'| > |X''|$, then for some i, $|X' \cap X_i| > |X'' \cap X_i|$ holds. By this and the assumption that $(X_i, \mathcal{F}_i)$ is a matroid, there exists $x \in (X' \cap X_i) \setminus (X'' \cap X_i)$ such that $(X'' \cap X_i) + x \in \mathcal{F}_i$ holds. Therefore $X'' + x \in \mathcal{F}$ holds, showing that a directed sum of matroids is a matroid. Finally, for k-truncation, if $X', X'' \in \tilde{\mathcal{F}}$ and $|X'| > |X''|$, then there always exists $x \in X' \setminus X''$ such that $X'' + x \in \mathcal{F}$, since $X', X'' \in \mathcal{F}$ holds. Since we also have $|X'' + x| \leq |X'| \leq k$, it follows that $X'' + x \in \tilde{\mathcal{F}}$. Thus, a k-truncation of a matroid is a matroid.

A laminar matroid is a matroid, since it is obtained from uniform matroids by repeatedly taking directed sums and truncations. With this concept at hand, we are ready to state the last sufficient condition.

Condition 3. *Assume* $(X, \operatorname{dom} \widehat{f})$ *is a laminar matroid on a laminar family* $\mathcal{T}$, *and* $\widetilde{f}$ *is a laminar concave function on* $\mathcal{T}$, *then* $f = \widehat{f} + \widetilde{f}$ *is* $M^\natural$*-concave.*

5 Applications

As emphasized in Introduction, one of the main motivations of our work is to provide an easy-to-use recipe, or a toolkit, for organizing matching mechanisms under constraints. To demonstrate the effectiveness our recipe/toolkit, we examine numerous applications of our method to two-sided, many-to-one matching problems. Specifically, this section presents one existing application (regional maximum quotas) as well as one new application (distance constraints), while relegating many others to the online appendix.[25] We show that the sufficient conditions described in Section 4 hold in these cases. For existing applications, these findings allow us to reproduce key results and, for some applications, show stronger results.

Before describing how to apply our framework to particular applications, let us summarize our recipe. Consider a mechanism designer who is faced with a matching problem with constraints, and imagine that she has some initial ideas on what hard distributional constraints exist and what kind of stability properties are desired. We suggest the mechanism designer to perform the following two steps.

1. Check whether $(X, \mathcal{F})$, where $\mathcal{F}$ is the family of hospital-feasible sets of contracts, is a matroid. If not, modify distributional constraints so that $(X, \mathcal{F})$ becomes a matroid.

2. Compose $\widetilde{f}$, which reflects stability, such that it satisfies one of the sufficient conditions described in this paper. Modify the stability definition as necessary, by adding more desirable properties, relaxing excessively demanding requirements, or simply introducing tie-breaking.

[25] Specifically, the online appendix examines the following existing applications: the standard model without distributional constraints (Gale and Shapley, 1962), matching markets with regional minimum quotas (Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo, 2016), diversity requirements in school choice (Ehlers, Hafalir, Yenmez, and Yildirim, 2014), the student-project allocation problem (Abraham, Irving, and Manlove, 2007), and the cadet-branch matching problem(Sönmez and Switzer, 2013). It also studies a new application where there are regional maximum quotas and regionally prioritized doctors.

If these two steps are successful, the job of the mechanism designer is done, because she can use an off-the-shelf mechanism, i.e., the generalized DA mechanism. More specifically, our analysis from the preceding sections guarantees that the generalized DA mechanism satisfies desirable properties. The cases we discuss in this section illustrate how the above recipe works. Furthermore, in Section 6, we discuss how to modify constraints and stability definitions when applying this recipe.

The advantages of using this recipe over using other general frameworks (e.g., Hatfield and Milgrom (2005), Hatfield and Kojima (2009, 2010)) are as follows. To use our recipe, basically all one needs to show is that the given hard distributional constraints produce a matroid, since the requirements over soft preferences are usually elementary as described in Conditions 1–3. There exists a vast literature on matroid theory, and what kinds of constraints produce a matroid is well-understood. Therefore, it usually suffices to show that the hard distributional constraints can be mapped into existing results in matroid theory. Furthermore, by using our recipe, the time complexity of the generalized DA mechanism is guaranteed to be polynomial.

5.1 Regional maximum quotas (Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014; Kamada and Kojima, 2015)

5.1.1 Model

A market is a tuple $(D, H, X, R, (\succ_d)_{d\in D}, (\succ_h)_{h\in H}, (q_h)_{h\in H}, (q_r)_{r\in R})$. D is a finite set of doctors and H is a finite set of hospitals. X is a finite set of contracts. A contract $x \in X$ is a pair (d, h), which represents a matching between doctor d and hospital h. $(\succ_d)_{d\in D}$ is a profile of doctors' preferences, i.e., each $\succ_d$ represents the strict preference of each doctor d over acceptable contracts in $X_d = \{(d, h) \in X \mid h \in H\}$. $(\succ_h)_{h\in H}$ is a profile of hospitals' preferences, i.e., each $\succ_h$ represents the preference of each hospital h over the contracts that are related to it. $(q_h)_{h\in H}$ is a profile of hospitals' maximum quotas, i.e., each q_h represents the maximum quota of hospital h. We assume hospitals are grouped into regions $R = \{r_1, \ldots, r_n\}$, where each region r is a subset of hospitals. $(q_r)_{r\in R}$ is a profile of regional maximum quotas, i.e., each q_r represents the regional maximum quota of r. We assume each hospital h

is included in exactly one region, that is, regions partition H.[26]

5.1.2 Feasibility

For each $r \in R$, let X'_r denote $\bigcup_{h \in r} X'_h$. We say X' is hospital-feasible if $|X'_h| \leq q_h$ for all $h \in H$, and $|X'_r| \leq q_r$ for all $r \in R$. We say $X' \subseteq X$ is doctor-feasible if X'_d is acceptable for all d. Then, we say X' is feasible if it is doctor- and hospital-feasible.

5.1.3 Stability

First, let us define the concept of a blocking pair.

Definition 11. For a matching X', we say $(d, h) \in X \setminus X'$ is a **blocking pair** if (i) (d, h) is acceptable for d and $(d, h) \succ_d X'_d$, and (ii) either $|X'_h| < q_h$ or there exists $(d', h) \in X'$ such that $(d, h) \succ_h (d', h)$.[27]

We say a matching X' is strongly stable (Kamada and Kojima, 2017b) if the following condition holds: if (d, h), where $h \in r$, is a blocking pair (Definition 11) then (i) $|X'_r| = q_r$, (ii) $(d', h) \succ_h (d, h)$ for all $(d', h) \in X'_h$, and (iii) if $(d, h') \in X'$, then $h' \notin r$. In words, a matching is strongly stable if satisfying the desire of a blocking pair by matching them results in a violation of a regional maximum quota. To be more precise, assume (d, h) is a blocking pair. Then, if $(d, h') \in X'$, then $h' \notin r$ (condition (iii)). Moreover, moving d to h necessarily involves filling a vacant seat at h because no existing doctor at h is less preferred to d (condition (ii)). Thus, if we move d to h from her current match (which can be h' or the option of being unmatched), the regional maximum quota of r is violated since $|X'_r|$ is already q_r (condition (i)).

A strongly stable matching does not necessarily exist (Kamada and Kojima, 2017b). Thus, we need to consider a weaker definition of stability in order to guarantee the existence. Kamada and Kojima (2015) introduce a weaker stability concept, which we call Kamada-Kojima (KK)-stability. We

[26] Kamada and Kojima (2017a) consider a more general case where regions are hierarchical. We can generalize our results to such a case by utilizing the fact that contracts related to each region form a laminar family.

[27] Note that we denote $X'_d \succ_d X''_d$ if either (i) $X'_d = \{x'\}$, $X''_d = \{x''\}$, and $x' \succ_d x''$ for some $x', x'' \in X_d$ that are acceptable for d, or (ii) $X'_d = \{x'\}$ for some $x' \in X_d$ that is acceptable for d and $X''_d = \emptyset$.

say a matching X' is **KK-stable** if the following condition holds: if (d, h), where $h \in r$, is a blocking pair then (i) $|X'_r| = q_r$, (ii) $(d', h) \succ_h (d, h)$ for all $(d', h) \in X'_h$, and (iii) if $(d, h') \in X'$, then either $h' \notin r$ or $|X'_{h'}| - |X'_h| \leq 1$. The second part of condition (iii) accounts for the difference between KK-stability and strong stability; (d, h) is not regarded as a legitimate blocking pair if h and h' are in the same region and moving d from h' to h does not strictly decrease the imbalance of doctors between these hospitals.

Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) assume there exists a total preference ordering $\succ_H$ over X, i.e., $x_1 \succ_H x_2 \succ_H x_3 \succ_H \ldots$. Here, we assume $\succ_H$ respects each $\succ_h$, i.e., if $(d, h) \succ_h (d', h)$, then $(d, h) \succ_H (d', h)$ holds. Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) introduce a weaker stability concept than strong stability based on this ordering, which we call contract-order-stability.[28] We say a matching X' is **contract-order-stable** (Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014) if the following condition holds: if (d, h), where $h \in r$, is a blocking pair then (i) $|X'_r| = q_r$ and (ii) $(d', h') \succ_H (d, h)$ for all $h' \in r$ and $(d', h') \in X'_{h'}$. We note that the condition (ii) includes the cases where $d' = d$ or $h' = h$. When $d = d'$, the condition (ii) means that (d, h) is not regarded as a legitimate blocking pair when the hospitals prefer (d, h') to (d, h).

5.1.4 Mechanism

Fix a round-robin ordering among hospitals; without loss of generality, denote it as $h_1, h_2, \ldots h_{|H|}$. Kamada and Kojima (2015) present a mechanism called the Flexible Deferred Acceptance (FDA) mechanism, which utilizes this ordering. Roughly speaking, the FDA mechanism allows each hospital to sequentially accept one contract at a time according to the given round-robin ordering, subject to regional maximum quotas. Formally, the FDA mechanism is defined as follows.[29]

Mechanism 2 (FDA)**.**
Apply the following stages from $k = 1$.

[28] A contract-order stable matching is identical to a regionally fair and regionally non-wasteful matching defined in Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014).

[29] To be more precise, Kamada and Kojima (2015) allow *a target capacity* for each hospital such that each hospital gets priority in accepting doctors up to its target capacity. For simplicity, here we consider a case where these target capacities are identical for all hospitals that belong to the same region, but allowing for more general target capacities is straightforward.

Stage $k \geq 1$

Step 1 Each doctor applies to her most preferred hospital by which she has not been rejected before Stage k. If no remaining hospital is acceptable for d, d does not apply to any hospital. Reset X' as $\emptyset$.

Step 2 For each r, iterate the following procedure until all doctors applying to hospitals in r are either tentatively accepted or rejected:

1. Choose the hospital with the smallest index in the region first, the hospital with the second-smallest index second, and so forth and, after the last hospital, go back to the first hospital.
2. Choose doctor d who is applying to h and is not tentatively accepted or rejected yet, and is the most preferred according to $\succ_h$ among the current applicants. If there exists no such doctor, then go to the procedure for the next hospital.
3. If $|X'_h| < q_h$ and $|X'_r| < q_r$, d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next hospital.
4. Otherwise, d is rejected by h. Then go to the procedure for the next hospital.

Step 3 If all the doctors are tentatively accepted in Step 2, then let X' be the final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

Kamada and Kojima (2015) show that the FDA mechanism is strategyproof for doctors and obtains a KK-stable matching.

Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) introduce a different mechanism, called Priority-List based Deferred Acceptance (PLDA) mechanism, which utilizes the total preference ordering $\succ_H$. Formally, the PLDA mechanism is defined as follows.

Mechanism 3 (PLDA)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor applies to her most preferred hospital by which she has not been rejected before Stage k. If no remaining hospital is acceptable for d, d does not apply to any hospital. Reset X' as $\emptyset$.

Step 2 For each r, iterate the following procedure until all doctors applying to hospitals in r are either tentatively accepted or rejected:

1. Choose (d, h), where d is applying to h, d is not tentatively accepted or rejected yet, and (d, h) has the highest priority according to $\succ_H$ among the current applicants to hospitals in r.
2. If $|X'_h| < q_h$ and $|X'_r| < q_r$, d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next pair.
3. Otherwise, d is rejected by h. Then go to the procedure for the next pair.

Step 3 If all the doctors who make applications in this stage are tentatively accepted in Step 2, then let X' be the final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) show that the PLDA mechanism is strategyproof for doctors and obtains a contract-order-stable matching.

5.1.5 Representation in our model

Let us define $\widehat{f}(X')$ as 0 if X' is hospital-feasible, i.e., $|X'_h| \leq q_h$ for all h and $|X'_r| \leq q_r$ for all r, and otherwise, $-\infty$. Then, $(X, \mathrm{dom}\,\widehat{f})$ is a laminar matroid, since $\mathcal{T} = \{X_{r_1}, X_{r_2}, \ldots, X_{r_n}, X_{h_1}, X_{h_2}, \ldots, X_{h_{|H|}}\}$ is a laminar family of X.

First, we study KK-stability. As in Kamada and Kojima (2015), fix a round-robin ordering over hospitals, $h_1, h_2, \ldots, h_{|H|}$. Let $v_{h_i}(j)$ denote the value associated with the j-th choice of hospital h_i. Then, define $v_{h_i}(j)$ as $C(C - |H| \cdot j - i)$ where C is a large positive constant. Let $V_h(k) := \sum_{j=1}^{k} v_h(j)$. It is clear that V_h is concave. Then, define $\widetilde{f}(X')$ as follows:

$$\widetilde{f}(X') = \sum_{h \in H} V_h(|X'_h|) + \sum_{x \in X'} v(x), \tag{1}$$

where $v(\cdot)$ is a function representing contract values as in Definition 5 (where, for each h, $v((d, h)) > v((d', h))$ if and only if $(d, h) \succ_h (d', h)$), and $C \gg v(x)$ for all $x \in X$. By choosing G as $\{X_{h_1}, X_{h_2}, \ldots, X_{h_{|H|}}\}$, it is clear that G is symmetric in $(X, \mathrm{dom}\,\widehat{f})$ and $\widetilde{f}$ defined by equation (1) is an order-respecting payoff function for G. Thus, we can apply Condition 2.

The FDA mechanism is identical to the generalized DA mechanism where Ch_H is defined as the maximizer of function f defined above.

The following proposition holds.

Proposition 2. *HM-stability (based on $\widetilde{f}$ in equation* (1)*) implies KK-stability.*

Proof. To show that HM-stability implies KK-stability, assume X' is not KK-stable, i.e., there exists a blocking pair (d, h), where $h \in r$ and (i) $|X'_r| < q_r$, or (ii) there exists $(d', h) \in X'_h$ such that $(d, h) \succ_h (d', h)$ holds, or (iii) there exists $(d, h') \in X'$, where $h' \in r$ and $|X'_{h'}| - |X'_h| > 1$ holds. In any of these cases, clearly $(d, h) \in Ch_D(X' + (d, h))$. In case (i), obviously $(d, h) \in \arg\max_{X'' \subseteq X' + (d,h)} f(X'')$ because adding (d, h) to X' does not violate the regional maximum quota for f. In case (ii), $(d, h) \in \arg\max_{X'' \subseteq X' + (d,h)} f(X'')$ because adding (d, h) and subtracting $(d', h) \in X'_h$ such that $(d, h) \succ_h (d', h)$ from X', the resulting matching does not violate the regional maximum quota for r. In case (iii), by the construction of f, $(d, h) \in \arg\max_{X'' \subseteq X' + (d,h)} f(X'')$. Thus, for each of the cases (i)–(iii), it follows that $(d, h) \in Ch_H(X' + (d, h))$, which implies X' is not HM-stable. Thus, HM-stability implies KK-stability. □

We note that KK-stability does not imply HM-stability. To see this, let us consider the following case. There are two hospitals h_1 and h_2, both of them belong to region r, and $q_r = 1$. There are two doctors d_1 and d_2. We assume $h_1 \succ_{d_1} h_2$, $h_2 \succ_{d_2} h_1$, $d_1 \succ_{h_1} d_2$, and $d_2 \succ_{h_2} d_1$ hold. The round-robin ordering over hospitals is defined as h_1, h_2. $X' = \{(d_2, h_2)\}$ is clearly KK-stable, but it is not HM-stable since $(d_1, h_1) \in Ch_D(X' + (d_1, h_1))$ and $(d_1, h_1) \in Ch_H(X' + (d_1, h_1))$ hold.

The FDA (and the generalized DA) mechanism is not guaranteed to obtain the doctor-optimal KK-stable matching. In fact, there is a case where the doctor-optimal KK-stable matching does not even exist. Note that the main focus of Kamada and Kojima (2015) is two-sided matching such as labor market matching, so optimality for one side of the market is not the main requirement. Note also that, despite the fact that the FDA (and hence the generalized DA) mechanism does not lead to doctor-optimality, the mechanism is still strategyproof for doctors.

Alternatively, we can define $\widetilde{f}(X')$ as $C_1 \cdot |X'| - \sum_{h \in H} C_2 \cdot |X'_h|^2 + \sum_{h \in H} C_h \cdot |X'_h| + \sum_{x \in X'} v(x)$, where $C_1 \gg C_2 \gg C_h \gg v(x)$ for all h and x, and $C_{h_1} \gg C_{h_2} \gg \ldots$. This is a laminar concave function on $X \cup \{X_{r_1}, X_{r_2}, \ldots, X_{r_n}, X_{h_1},$

$X_{h_2}, \ldots, X_{h_{|H|}}\}$. Thus, we can apply Condition 3. It is clear that Proposition 2 still holds when $\widetilde{f}$ is defined in this way.

Next, we study contract-order-stability. As in Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014), let us assume there exists a total preference ordering $\succ_H$ over X, i.e., $x_1 \succ_H x_2 \succ_H x_3 \succ_H \ldots$. Then assume a positive value $v(x)$ for each x is defined with the property that $v(x) > v(x')$ when $x \succ_H x'$. Let us assume $\widetilde{f}(X')$ is given as follows:

$$\widetilde{f}(X') = \sum_{x \in X'} v(x). \tag{2}$$

Thus, we can apply Condition 1. The PLDA mechanism is identical to the generalized DA mechanism where Ch_H is defined by the maximizer of this function f.

The following proposition holds.

Proposition 3. *HM-stability (based on $\widetilde{f}$ in equation (2)) is equivalent to contract-order-stability.*

Proof. Assume X' is not contract-order-stable, i.e., there exists a blocking pair (d, h), where $h \in r$ and either (i) $|X'_r| < q_r$ or (ii) there exists $(d', h') \in X'$ such that $h' \in r$ and $(d, h) \succ_H (d', h')$ hold. In either case, $(d, h) \in Ch_D(X' + (d, h))$ and $(d, h) \in Ch_H(X' + (d, h))$ hold.

If X' is contract-order-stable, then the first condition for HM-stability, namely $X' = Ch_H(X') = Ch_D(X')$, is obvious. Assume there exists $(d, h) \in X \setminus X'$ such that $(d, h) \in Ch_D(X' + (d, h))$ and $(d, h) \in Ch_H(X' + (d, h))$ hold. Then, it is clear that (d, h) is a blocking pair, and either (i) $|X'_r| < q_r$ or (ii) there exists $(d', h') \in X'$ such that $h' \in r$ and $(d, h) \succ_H (d', h')$ hold. Thus, X' is not contract-order-stable. □

From Proposition 3, we can guarantee that the generalized DA mechanism obtains the doctor-optimal contract-order-stable matching, so the generalized DA mechanism and the PLDA mechanism obtain the same outcome. Note that this fact can be derived without checking whether these two mechanisms behave exactly in the same way.

5.2 Distance constraints

In this subsection, we investigate a new application domain, where hospital-feasibility is defined based on the distance from an ideal distribution. For example, assume $|D|/|H| = k$, and a set of contracts that allocates exactly k doctors to each hospital is regarded as ideal, but for a set of contracts X', as long as its distribution is *close enough* to this ideal distribution, then X' is regarded as hospital-feasible. We call such constraints *distance constraints*. Although such a constraint seems to be natural, to our knowledge, distance constraints have never been studied before.

The closest to our model is Echenique and Yenmez (2015) who analyze a choice function of a school that chooses a subset of students that gives a distribution of students of different types closest to a given ideal distribution. Their model is different from ours in a number of ways, however. First, they consider a choice function of a single school, whereas our model is concerned about distributional constraints of multiple hospitals/schools. Second, the metrics used in their paper and ours are different. Lastly, and perhaps most importantly, Echenique and Yenmez (2015) provide a choice function that minimizes the distance from an ideal point while our feasibility constraint only requires the distance to be at most a given amount. Given these differences, there is no logical relationship, and the connection appears to be tangential at best. We show a more formal discussion on Echenique and Yenmez (2015) in the end of this subsection. Erdil and Kumano (2012) propose a preference class of one school called *substitutable priorities with ties* and show that a preference that is similar to Echenique and Yenmez (2015) can be represented as an instance of this preference class. The preference of the hospitals in our model is not an instance of this preference class, since the former is concerned about distributional constraints over contracts, which can be related to multiple hospitals.

5.2.1 Model and desirable properties

A market is a tuple $(D, H, X, (\succ_d)_{d \in D}, (\succ_h)_{h \in H}, (q_h)_{h \in H}, \xi^*, \delta, \epsilon)$. The definitions of $D, H, X, \succ_d, \succ_h$, and $(q_h)_{h \in H}$ are identical to those in Section 5.1. We assume each doctor can accept any hospital and each hospital can accept any doctor. Thus, $X = D \times H$. For notation simplicity, let us denote $|H| = m$ and $|D| = n$, and $M = \{1, 2, \ldots, m\}$. $\xi^* \in (\mathbb{Z}_+)_m$ is a given vector, which is interpreted as the ideal distribution of doctors across the hospitals. We assume $\sum_{i \in M} \xi^*_i = n$ and $\xi^*_i \leq q_{h_i}$ holds for all $i \in M$, i.e., the ideal distribution satisfies maximum quotas. $\delta : (\mathbb{Z}_+)^m \times (\mathbb{Z}_+)^m \to \mathbb{R}_+$ is a distance function, which returns the distance between two vectors. We consider two representative distance functions: (i) the Manhattan distance (or L^1 distance), which is defined as $\delta(\xi, \xi') = \sum_{i \in M} |\xi_i - \xi'_i|$, and (ii) the Chebyshev distance (or L^∞ distance), which is defined as $\delta(\xi, \xi') = \max_{i \in M} |\xi_i - \xi'_i|$. $\epsilon \geq 0$ represents the maximal allowed distance.

For a set of contracts X', let $\xi(X')$ denote a vector of m non-negative integers $(|X'_{h_1}|, |X'_{h_2}|, \dots, |X'_{h_m}|)$. Also, let $\xi_h(X')$ denote $|X'_h|$. The value $\xi(X')$ describes how doctors are distributed among hospitals at X'.

We say X' is hospital-feasible if the following conditions hold:

(i) $|X'| = n$,

(ii) $|X'_h| \leq q_h$ for all $h \in H$, and

(iii) $\delta(\xi(X'), \xi^*) \leq \epsilon$.

At a matching X', a doctor d where $(d, h) \in X'$ has a **justified envy** towards another doctor d' where $(d', h') \in X'$, if $(d, h') \succ_d (d, h)$ and $(d, h') \succ_{h'} (d', h')$ hold. We say a matching is **fair** if no doctor has justified envy.

At a matching X', a doctor d where $(d, h) \in X'$ **claims an empty seat** of h' if the following conditions hold: (i) $(d, h') \succ_d (d, h)$, and (ii) $X'' = X' - (d, h) + (d, h')$ is hospital-feasible. We say a matching is **nonwasteful** if no doctor claims an empty seat.

In general, fairness and nonwastefulness are incompatible, i.e., there exists a case where no matching is fair and nonwasteful. To see this point, let us consider the following example. There are six doctors $d_1, \dots, d_6$ and three hospitals h_1, h_2, and h_3, where $q_h = 3$ for all $h \in H$. The ideal distribution is $\xi^* = (2, 2, 2)$, δ is the Manhattan distance, and $\epsilon = 2$.

The preferences of doctors are given as follows:

- $(d_1, h_2) \succ_{d_1} (d_1, h_3) \succ_{d_1} (d_1, h_1)$,
- $(d_2, h_3) \succ_{d_2} (d_2, h_2) \succ_{d_2} (d_2, h_1)$,
- $(d_3, h_2) \succ_{d_3} (d_3, h_3) \succ_{d_3} (d_3, h_1)$,
- $(d_4, h_2) \succ_{d_4} (d_4, h_3) \succ_{d_4} (d_4, h_1)$,
- $(d_5, h_3) \succ_{d_5} (d_5, h_2) \succ_{d_5} (d_5, h_1)$,
- $(d_6, h_3) \succ_{d_6} (d_6, h_2) \succ_{d_6} (d_6, h_1)$.

The preferences of hospitals are given as follows:

- $(d_1, h_1) \succ_{h_1} (d_2, h_1) \succ_{h_1} (d_3, h_1) \succ_{h_1} (d_4, h_1) \succ_{h_1} (d_5, h_1) \succ_{h_1} (d_6, h_1)$,
- $(d_3, h_2) \succ_{h_2} (d_4, h_2) \succ_{h_2} (d_2, h_2) \succ_{h_2} (d_1, h_2) \succ_{h_2} (d_5, h_2) \succ_{h_2} (d_6, h_2)$,
- $(d_5, h_3) \succ_{h_3} (d_6, h_3) \succ_{h_3} (d_1, h_3) \succ_{h_3} (d_2, h_3) \succ_{h_3} (d_3, h_3) \succ_{h_3} (d_4, h_3)$.

Assume X' is fair and nonwasteful. Since $\epsilon = 2$, at least one doctor must be assigned to each hospital. Note that h_1 is the least preferred hospital for all doctors. Thus, only one doctor can be assigned to h_1 because, otherwise, any doctor assigned to h_1 claims an empty seat of either h_2 or h_3. Then, remaining five doctors must be assigned to h_2 and h_3. Since the maximum quota of each hospital is three, at least two doctors must be assigned to h_2 and h_3. Since (d_3, h_2) is the most preferred contract for both d_3 and h_2, it must be included in X' (otherwise, d_3 has justified envy). Also, (d_5, h_3) is the most preferred contract for both d_5 and h_3, so it must be included in X'. Furthermore, since h_2 accepts at least two doctors, (d_4, h_2) must be included in X', since it is the most preferred contract for d_4 and the second most preferred contract for h_2. Similarly, (d_6, h_3) must be included in X'. Then, either d_1 or d_2 must be assigned to h_1. Assume d_1 is assigned to h_1. Then, X' must assign d_2 to her most preferred hospital h_3 because, otherwise, d_2 will claim an empty seat of h_3. However, this implies that d_1 has justified envy towards d_2, since $(d_1, h_3) \succ_{h_3} (d_2, h_3)$ holds. Then, assume d_2 is assigned to h_1. Then, X' must assign d_1 to her most preferred hospital h_2 because, otherwise, d_1 will claim an empty seat of h_2. However, this implies that d_2 has justified envy towards d_1, since $(d_2, h_2) \succ_{h_2} (d_1, h_2)$ holds.

Since fairness and nonwastefulness are incompatible in general, let us introduce weaker requirements related to nonwastefulness.

As in Section 5.1, let us assume the hospitals have a strict preference $\succ_H$ over contracts X. We assume $(d, h) \succ_h (d', h)$ implies $(d, h) \succ_H (d', h)$, i.e., $\succ_H$ respects each $\succ_h$. Then, at a matching X', we say a doctor d where $(d, h) \in X'$ **strongly claims an empty seat** of h' **based on a contract order** if the following conditions hold: (i) $(d, h') \succ_d (d, h)$, (ii) $X'' = X' - (d, h) + (d, h')$ is hospital-feasible, and (iii) $(d, h') \succ_H (d, h)$. We say a matching is **weakly nonwasteful based on a contract order** if no doctor strongly claims an empty seat based on a contract order.

Similarly, at a matching X', we say a doctor d where $(d, h) \in X'$ **strongly claims an empty seat** of h' **based on hospital equality** if the following conditions hold: (i) $(d, h') \succ_d (d, h)$, (ii) $X'' = X' - (d, h) + (d, h')$ is hospital-feasible, and (iii) $|X'_h| - |X'_{h'}| \geq 2$. We say a matching is **weakly**

nonwasteful based on hospital equality if no doctor strongly claims an empty seat based on hospital equality.

5.2.2 Representation in our model

It is clear that the family of all hospital-feasible contracts does not form a matroid since $\emptyset$ is not hospital-feasible. Thus, let us introduce a weaker condition than hospital-feasibility. We say X' is **semi-hospital-feasible** if it is a subset of (or equal to) a hospital-feasible matching. Then, it is clear that the family of all semi-hospital-feasible contracts forms a matroid.[30] The following proposition enables us to apply our methodology for this environment.

Proposition 4. *Let δ be the Manhattan distance function or the Chebyshev distance function, and $\mathcal{F}$ be the family of semi-hospital-feasible contracts. Then $(X, \mathcal{F})$ is a matroid.*

Thanks to Proposition 4, we can use the off-the-shelf mechanism (i.e., the generalized DA) by appropriately designing f. Let us assume $\widehat{f}(X')$ is 0 if $X' \in \mathcal{F}$, and otherwise, $-\infty$. Also, let us define contract values $v : X \to (0, \infty)$ where $(d, h') \succ_H (d, h)$ implies $v((d, h')) > v((d, h))$.

We can obtain an appropriate f to apply Condition 1. Let us assume $\widetilde{f}(X')$ is given as follows:

$$\widetilde{f}(X') = \sum_{x \in X'} v(x). \tag{3}$$

Thus, we can apply Condition 1. It is clear that HM-stability (based on $\widetilde{f}$ in equation (3)) implies fairness as well as weak nonwastefulness based on a contract order.

Furthermore, we can obtain another f to apply Condition 2. As in Section 5.1, fix a round-robin ordering over hospitals, $h_1, h_2, \ldots, h_m$. Let $v_{h_i}(j)$ denote the value associated with the j-th choice of hospital h_i. Then, define $v_{h_i}(j)$ as $C(C - |H| \cdot j - i)$ where C is a large positive constant. Let $V_h(k) := \sum_{j=1}^{k} v_h(j)$. It is clear that V_h is concave. Then, define $\widetilde{f}(X')$ as follows:

$$\widetilde{f}(X') = \sum_{h \in H} V_h(|X'_h|) + \sum_{x \in X'} v(x), \tag{4}$$

[30] Since we assume all hospitals/doctors are acceptable to each other, when we apply the generalized DA, the obtained contracts are guaranteed to be hospital-feasible.

where $C \gg v(x)$ for all $x \in X$. By choosing G as $\{X_{h_1}, X_{h_2}, \ldots, X_{h_m}\}$, it is clear that G is symmetric in $(X, \operatorname{dom} \widehat{f})$ and $\widetilde{f}$ defined by equation (4) is an order-respecting payoff function for G. Thus, we can apply Condition 2. It is clear that HM-stability (based on $\widetilde{f}$ in equation (4)) implies fairness as well as weak nonwastefulness based on hospital equality.

In the model of Echenique and Yenmez (2015), the set of students is partitioned into finite types $T = \{t_1, \ldots, t_k\}$, as in the model discussed in Section 5.4. Each school c has its own ideal distribution of types $\zeta^* \in (\mathbb{Z}_+)^k$ such that $\sum_{i=1}^{k} \zeta_i^* \leq q_c$ where q_c is the maximum quota of school c. For given X', the choice function of school c first chooses a feasible distribution $\hat{\zeta}$ based on X' that is closest to ζ^*. More specifically, let $\mathrm{Z} = \{\zeta \mid \zeta \in (\mathbb{Z}_+)^k, \forall i \in \{0, \ldots, k\}, \zeta_i \leq \min(\zeta_i^*, |X'_{c,t_i}|), \sum_{i=1}^{k} \zeta_i \leq q_c\}$, where X'_{c,t_i} represents the set of contracts in X' which are related to school c and type t_i students. Then, $\hat{\zeta}$ is given as $\arg\min_{\zeta \in \mathrm{Z}} \delta(\zeta, \zeta^*)$, where δ can be any L^p distance function except for $p = \infty$ (i.e., the Chebyshev distance). Actually, given the restriction that $\sum_{i=1}^{k} \zeta_i^* \leq q_c$, $\hat{\zeta}$ is simply given by $\hat{\zeta}_i = \min(\zeta_i^*, |X'_{c,t_i}|)$ for all $i \in \{1, \ldots, k\}$, regardless of the definition of the distance function. Therefore, this setting is isomorphic to the following model in which a school is divided into t type-specific sub-schools. For each sub-school corresponding to type t_i, its maximum quota is ζ_i^*. There exist no distributional constraints over these sub-schools. In contrast to theirs, our distance constraints only requires the distance from the ideal point to be at most a given amount. Thus, our constraint allows for some flexibility in the sense that the chosen set of contracts does not need to minimize the distance. Instead, the choice depends on the soft preference subject to the constraint that its distance from the ideal distribution is within the given bound.

6 Discussion

In Section 5, we introduced a two-step recipe for applying our framework: (1) check whether distributional constraints form a matroid; if not, modify distributional constraints, and (2) compose soft preferences, which reflect stability, such that they are represented by an M$^\natural$-concave function; modify the stability definition as necessary. In this section, we discuss how to modify constraints and stability definitions when applying our recipe.

6.1 Modifying constraints

When distributional constraints do not form a matroid, one approach is to transform the given distributional constraints into different ones that form a matroid and ensure the original hospital-feasibility. This implies that the family of hospital-feasible contracts must be smaller than the original one, which means we lose the flexibility specified by the original distributional constraints to some extent, and this can in turn result in, among other things, some loss in efficiency. Thus, in this subsection we analyze when it is necessary to resort to modifying constraints and, if so, how to "minimize" the loss of flexibility, as formalized below.

Let us introduce a setting where distributional constraints do not necessarily form a matroid. A market is represented by $(D, H, X, R, (\succ_d)_{d\in D}, (\succ_h)_{h\in H}, (q_h)_{h\in H}, (q_r)_{r\in R})$, which is basically identical to the model presented in Section 5.1, but the set of regions R is not necessarily a partition of H, i.e., for two regions r and $r' \neq r$, the set $r \cap r'$ can be nonempty. To exclude trivial cases, we assume D and H are nonempty, $q_h > 0$ and $X_h \neq \emptyset$ for each $h \in H$.

The following proposition shows that the fact that R is a laminar family (Definition 9) is a necessary and sufficient condition for guaranteeing the distributional constraints to form a matroid.

Proposition 5. *Let D and H be the sets of doctors and hospitals, respectively, and R be a family of subsets of H.*

1. *If R is a laminar family, then for any $(q_h)_{h\in H}$ and $(q_r)_{r\in R}$, the distributional constraints form a matroid.*

2. *If R is not a laminar family, then for any $(q_h)_{h\in H}$, there exist $(q_r)_{r\in R}$ such that the distributional constraints do not form a matroid.*

As highlighted in Proposition 5, the matroid imposes a meaningful, if reasonably mild, restriction on the constraints. Given this, we consider how we modify a given non-matroid constraint into a matroid such that any matching that satisfies the latter always satisfies the former. In the context of regional maximum quotas, the simplest transformation method would be to transform the distributional constraints into individual maximum quotas. This method is called the artificial cap approach and has been applied to handle various distributional constraints, including those that form a matroid (Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014; Kamada and Kojima, 2015).

For example, if R is not a laminar family, we can use artificial individual maximum quotas $(\hat{q}_h)_{h \in H}$ that satisfy the following conditions: $\forall r \in R, \sum_{h \in r} \hat{q}_h \leq q_r$, and $\forall h \in H, \hat{q}_h \leq q_h$. It is clear that if X' satisfies these artificial quotas, it also satisfies the original regional maximum quotas. It is also clear that we lose the flexibility specified by the regional maximum quotas.

By transforming the original constraints into less restrictive matroid constraints, we can retain more flexibility. For example, let us assume $\hat{R} = \{\hat{r}_1, \hat{r}_2, \ldots\}$ is a partition of H, and for each $r \in R$, there exists $\hat{R}' \subseteq \hat{R}$, such that $r = \bigcup_{\hat{r} \in \hat{R}'} \hat{r}$ holds. For each $r \in R$, let $\hat{R}(r)$ denote $\hat{R}' \subseteq \hat{R}$, such that $r = \bigcup_{\hat{r} \in \hat{R}'} \hat{r}$. Intuitively, $\hat{R}$ is a fine-grained partition, which can exactly cover each of $r \in R$ by $\hat{R}(r)$. Let us define $(q_{\hat{r}})_{\hat{r} \in \hat{R}}$, i.e., the regional maximum quotas for $\hat{R}$, such that for each $r \in R$, $\sum_{\hat{r} \in \hat{R}(r)} q_{\hat{r}} \leq q_r$. It is clear that if X' satisfies these regional maximum quotas, it also satisfies the original regional maximum quotas. Furthermore, if $q_{\hat{r}} \geq \sum_{h \in \hat{r}} q_{\hat{h}}$ holds for each $\hat{r} \in \hat{R}$ (it is always possible to choose such $q_{\hat{r}}$), then this method is (weakly) more flexible than the artificial cap approach, in the sense that the family of hospital-feasible matching in this method is a superset of that in the artificial cap approach.

To summarize, if the family of hospital-feasible contracts do not form a matroid, we can transform the distributional constraints into different constraints that form a matroid. The artificial cap and partition approaches described above are such examples in the context of regional maximum quota setting. More generally, such a transformation is always possible as long as constraints are *hereditary*, i.e., if X' is hospital-feasible, any $X'' \subseteq X'$ is also feasible.[31] In the worst case, we can just choose one element in the family, and assume only the subset of this element is hospital-feasible. Clearly, this newly defined family forms a matroid. To retain more flexibility, the newly defined family should be large enough. Ideally, it should be maximal, i.e., for the original family $\mathcal{F}$ and the newly defined family $\widehat{\mathcal{F}} \subsetneq \mathcal{F}$, there exists no family $\mathcal{F}' \subsetneq \mathcal{F}$ such that $\mathcal{F}' \supsetneq \widehat{\mathcal{F}}$ and $\mathcal{F}'$ forms a matroid.

We present an algorithm to find such a maximal family. It deals with the largest elements of $\mathcal{F}$ called *bases*, and selects a subset of bases such that it induces a matroid. Let us assume $X = \{x_1, \ldots, x_n\}$. Let $k = \max\{|Y| \mid Y \in \mathcal{F}\}$. Without loss of generality, let us assume $\{x_1, x_2, \ldots, x_k\} \in \mathcal{F}$. Let $\mathcal{B}$

[31] As shown in Section 5.2, even if constrains are not hereditary, e.g., $\emptyset$ is not hospital-feasible, we can apply our approach by considering semi-hospital-feasible contracts.

denote $\{Y \mid Y \in \mathcal{F}, |Y| = k\}$. Also, without loss of generality, we assume $\mathcal{F}$ is hereditary with respect to $\mathcal{B}$, i.e., for all $B \in \mathcal{B}$ and all $B' \subseteq B$, $B' \in \mathcal{F}$ holds.[32] For $\widehat{\mathcal{B}} \subseteq \mathcal{B}$, let $\widehat{\mathcal{F}}$ denote $\{B' \mid B' \subseteq B \in \widehat{\mathcal{B}}\}$. It is well-known that $\widehat{\mathcal{F}}$ forms a matroid if and only if $\widehat{\mathcal{B}}$ satisfies the following condition (Oxley, 2011):

$$\forall Y, Z \in \widehat{\mathcal{B}}, \forall y \in Y \setminus Z, \exists z \in Z \setminus Y \text{ such that } Y - y + z \in \widehat{\mathcal{B}} \text{ holds.} \tag{5}$$

It is also well-known that (5) is equivalent to the following stronger condition (Oxley, 2011):

$$\begin{aligned} &\forall Y, Z \in \widehat{\mathcal{B}}, \forall y \in Y \setminus Z, \exists z \in Z \setminus Y \\ &\quad \text{such that } Y - y + z \in \widehat{\mathcal{B}} \text{ and } Z + y - z \in \widehat{\mathcal{B}} \text{ hold.} \end{aligned} \tag{6}$$

Given $\mathcal{B}$, the following algorithm returns $\widehat{\mathcal{B}} \subseteq \mathcal{B}$, such that (5) holds.

Algorithm 1.

Step 1: Set $\widehat{\mathcal{B}}_k$ to $\{\{1, 2 \ldots, k\}\}$ and j to $k + 1$.

Step 2: If $j > n$, return $\widehat{\mathcal{B}}_{j-1}$ as $\widehat{\mathcal{B}}$.

Step 3: Let $\mathcal{Y} = \{Y \in \mathcal{B} \mid |Y| = k, Y \subseteq \{x_1, \ldots, x_j\}, \text{ and } x_j \in Y\}$.

Step 4: If there exist $Y \in \mathcal{Y}$ and $Z \in \widehat{\mathcal{B}}_{j-1}$ such that the following condition holds,[33] then remove Y from $\mathcal{Y}$ and go to Step 4.

$$\exists y \in Y \setminus Z, \forall z \in Z \setminus Y, Y - y + z \notin \widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y} \text{ or } Z + y - z \notin \widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y},$$

or

$$\exists z \in Z \setminus Y, \forall y \in Y \setminus Z, Y - y + z \notin \widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y} \text{ or } Z + y - z \notin \widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y}.$$

Step 5: Otherwise, set $\widehat{\mathcal{B}}_j$ to $\widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y}$, set j to $j + 1$, and go to Step 2.

The following proposition holds.

Proposition 6. *For each $j = k, \ldots, n$, $\widehat{\mathcal{B}}_j$ obtained by Algorithm 1 satisfies (5), and $\widehat{\mathcal{F}}$ (obtained from $\widehat{\mathcal{B}}$) gives a maximal matroid for a given family $\mathcal{F}$.*

[32] If some B violates this condition, we can simply remove B from $\mathcal{B}$.

[33] When this condition holds, $Y, Z \in \widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y}$ violate (6).

This proposition guarantees that Algorithm 1 produces a maximal matroid subset of any given constraints. This is appealing because there is no other matroid subset that is larger in the set inclusion sense than the one found by the algorithm, so there is a sense in which we find one of the most flexible constraints among those satisfying the matroid property while guaranteeing the original constraints. Note, however, that there may be multiple maximal matroid subsets of given constraints, and in some specific examples, the particular maximal subset found in our algorithm may be regarded undesirable by some policy makers. We address this issue further in Appendix D.3 by proposing another class of algorithms and offering a further generalization of our algorithms.

6.2 Modifying stability

When distributional constraints are imposed, the standard stability concept may be inappropriate or just infeasible. Thus, some modification is called for. Since there is no absolute criterion of the most "natural" or "appropriate" stability definition (that would be highly application-dependent), our recipe aims to provide a flexible tool that is versatile enough to cope with many (if not all) definitions in the literature and those that may appear in future works. We showed that our tool is actually useful in various application domains described in Section 5 and the online appendix. In this subsection, we examine in more detail how to modify stability definitions when using our tool.

When distributional constraints are imposed, one natural way in extending the standard stability definition would be to require that if (d, h) is a blocking pair (Definition 11), then moving doctor d to hospital h causes a violation of a distributional constraint. However, a stable matching often fails to exist with this definition. This is true for regional maximum quotas; as described in Section 5.1, the above requirement–called strong stability–may lead to non-existence and, in fact, conditions for the existence are extremely restrictive (Kamada and Kojima, 2017b). Given this fact, one natural approach is to weaken the stability requirement. Section 5.1 presented two such weakened concepts, KK-stability and contract-order stability. By crafting an appropriate $\widetilde{f}$, an HM-stable matching is guaranteed to satisfy KK-stability or contract-order stability.

As in the case for the aforementioned stability concepts under regional maximum quotas, in general our modified stability requirement must be weaker than (or identical to) HM-stability. However, this is not a drawback for the following reason. As mentioned earlier, there is no absolute appropriate stability concept (recall, for instance, at least two alternative stability concepts have been proposed under regional maximum quotas). In our recipe, the mechanism designer first decides an appropriate stability concept given her environment and policy goal and then implements the concept by choosing an appropriate function $\widetilde{f}$. In that sense, function $\widetilde{f}$, as well as HM-stability based on $\widetilde{f}$, is merely a tool for achieving a given stability property, and not a goal per se. Thus, the fact that the chosen stability property is weaker than HM-stability is not a limitation; if the mechanism designer wants to make the stability requirement more demanding, she can do so by choosing an appropriate function $\widetilde{f}$ to implement it. Then, HM-stability under that $\widetilde{f}$ may also become stronger and could be strictly more demanding than the given desired stability concept, but this is not problematic because HM-stability for the chosen $\widetilde{f}$ per se is not the desired property anyway.

With that said, it is often possible to strengthen a given stability concept to make it equivalent to HM-stability. To see this point, we modify KK-stability under regional maximum quotas as in Section 5.1. Specifically, we change condition (iii) of KK-stability to: *(iii) if* $(d, h') \in X'$, *then either (iii-a)* $h' \notin r$, *(iii-b)* $|X'_{h'}| - |X'_h| < 1$, *or (iii-c)* $|X'_{h'}| - |X'_h| = 1$ *and* h *appears earlier than* h' *in the round-robin ordering over hospitals.* This definition is stronger than the original one as the requirement for a legitimate blocking pair is weaker. Intuitively, this concept uses a round-robin ordering as a tie-breaking rule to resolve the competition among doctors who apply to different hospitals within the same region. It is straightforward to verify that with this modification, KK-stability is equivalent to HM-stability.

In some applications, the standard stability definition is simply inappropriate. For example, in the controlled school choice (Ehlers, Hafalir, Yenmez, and Yildirim, 2014), doctors (students) are partitioned into a finite number of types $\{t_1, \dots, t_k\}$. A type may represent race, income, gender, or any socioeconomic status. Let us assume for each hospital h and for each type t, a type-specific minimum quota p_h^t is defined.[34] Here, this type-specific minimum quota is a soft bound and does not affect feasibility. Intuitively, for each type t, up to p_h^t doctors of type t can be assigned to hospital h without competing against other types. If less than p_h^t doctors are applying to h,

[34] We show a more detailed discussion in online appendix C.3, where type-specific maximum quotas, as well as type-specific minimum quotas, are introduced.

assigning less than p_h^t doctors of type t is considered to be feasible. Also, more than p_h^t doctors of type t can be assigned, as long as they get through the competition against other types. In this setting, each hospital (or a policy maker) hopes to treat different types of doctors differently. Thus, the standard stability definition, which does not consider the types of doctors, is inappropriate.

One approach to formalize such a policy goal is to modify the condition of a blocking pair $(d, h) \in X \setminus X'$, where the type of d is t, as follows: *(i)* (d, h) *is acceptable for* d *and* $(d, h) \succ_d X'_d$*, and one of the following conditions holds (ii-a)* $|X'_h| < q_h$*, (ii-b)* $|X'_{h,t}| < p_h^t$ *(where* $X'_{h,t}$ *is the set of contracts in* X'*, which are related to* h *and type* t *students), or (ii-c) there exists* $(d', h) \in X'$ *such that* $(d, h) \succ_h (d', h)$ *and the type of* d' *is* t*.* Let us compare this definition with Definition 11. On the one hand, condition (ii-b) is added. Thus, the set of legitimate blocking pairs becomes larger. On the other hand, condition (ii-c) is more demanding than the corresponding condition in the original definition. Thus, the set of legitimate blocking pair becomes smaller. Condition (ii-c) means, if doctor d claims that she should be accepted to hospital h instead of another doctor d', such a claim is considered to be legitimate only if d and d' are of the same type.[35] As we show in online appendix C.3, we can further refine this stability requirement, such that it reflects how the competition among different types of doctors should be resolved.

To summarize, under distributional constraints, the standard stability definition is often inapplicable or inappropriate. Then, the mechanism designer needs to modify the stability definition and craft $\widehat{f}$. The modification can be done by adding more desirable properties (e.g., condition (ii-b) in the controlled school choice), relaxing excessively demanding requirements (e.g., KK-stability, contract-order stability, condition (ii-c) in the controlled school choice), or simply introducing tie-breaking (e.g., the modified definition of KK-stability). Since an appropriate stability definition can be highly context dependent, it is not our aim to argue one particular stability definition is always the "best" in all cases. Instead, our aim is to provide a flexible tool to accommodate a wide variety of existing and new stability concepts. As illustrated throughout the paper, the approach based on discrete convex analysis seems to achieve this goal.

[35] This property is called fairness for same type of students (Ehlers, Hafalir, Yenmez, and Yildirim, 2014).

7 Conclusion

This paper studied two-sided matching problems in which certain distributional constraints are imposed. We demonstrated that if the preference of the hospitals can be represented by an $M^\natural$-concave function, then the generalized DA mechanism is strategyproof for doctors and finds the doctor-optimal stable matching. Building on this result, we derived sufficient conditions under which the generalized DA mechanism satisfies desirable properties, and established that these sufficient conditions are satisfied in various applied settings. By utilizing these conditions, we obtained various results in the existing literature as well as new ones as corollaries of our results. Because our sufficient conditions are easy to verify in many cases, they provide a recipe for non-experts in matching theory or discrete convex analysis to develop desirable mechanisms that handle distributional constraints.

Our aim was to demonstrate that this paper's methodology is flexible enough to address various types of constraints. For this purpose, we showed that existing results in the literature can be derived with our methodology. Furthermore, we exploited our methodology to establish new results. A related point is that all of our mechanisms are instances of the generalized DA mechanism. This is a major advantage of our approach in that it allows a policy maker to adopt a mechanism off the shelf rather than searching for an entirely new solution from scratch. We envision that our approach will prove useful when the match organizer is challenged by new kinds of constraints. We verified this conjecture to a certain extent in Section 5. Further verifying whether this conjecture is true and, if so, in what applications, is left for future research.

While many application domains have matroid structures and are amenable to our method, there are other important applications that are outside of the scope of our present method. For example, Goto, Kojima, Kurata, Tamura, and Yokoo (2017) introduce a problem called the student-project-room assignment problem. In this problem, students and a room must be assigned to each project, and the maximum quota of a project is determined by the capacity of the room assigned to the project. Such distributional constraints do not have a matroid structure. Goto, Kojima, Kurata, Tamura, and Yokoo (2017) develop a strategyproof mechanism whose outcome is nonwasteful while satisfying such constraints. Their mechanism, however, does not produce a fair outcome in the sense of eliminating justified envy, one of the main desiderata achieved in the present paper. A possible step in future research is to develop a strategyproof and fair mechanism that can handle constraints beyond the matroid structure.

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Appendix

A Properties of matroids

We introduce three properties related to matroids that are used in our proofs.

Property 1 (simultaneous exchange property)**.** *Let* $(X, \mathcal{F})$ *be a matroid. Then for all* $Y, Z \in \mathcal{F}$ *and* $y \in Y \setminus Z$*, there exists* $z \in (Z \setminus Y) \cup \{\emptyset\}$ *such that* $Y - y + z \in \mathcal{F}$ *and* $Z - z + y \in \mathcal{F}$ *hold.*

Proof. Let $\mathcal{B}$ be the set of maximal elements in $\mathcal{F}$ with respect to set inclusion. $\mathcal{B}$ is called the family of **bases**. For $\mathcal{B}$, the following property holds: For all $\hat{Y}, \hat{Z} \in \mathcal{B}$ and $y \in \hat{Y} \setminus \hat{Z}$, there exists $z \in \hat{Z} \setminus \hat{Y}$ such that $\hat{Y} - y + z \in \mathcal{B}$ and $\hat{Z} - z + y \in \mathcal{B}$ hold. This property is known as the simultaneous exchange property for bases of matroids (see Theorem 39.12 of Schrijver (2003) or Condition (B) on page 69 of Murota (2003) for its proof). For Y and Z in $\mathcal{F}$, let us choose $\hat{Y}, \hat{Z} \in \mathcal{B}$ such that $\hat{Y} \supseteq Y$ and $\hat{Z} \supseteq Z$ hold. For any $y \in Y \setminus Z$, either $y \in Y \setminus \hat{Z}$ or $y \in \hat{Z}$ holds.

1. If $y \in Y \setminus \hat{Z}$, y is also included in $\hat{Y} \setminus \hat{Z}$. Thus, from the simultaneous exchange property for bases, there exists $z \in \hat{Z} \setminus \hat{Y}$ such that $\hat{Y} - y + z \in \mathcal{B}$ and $\hat{Z} - z + y \in \mathcal{B}$ holds. If $z \in Z$, both $Y - y + z$ (which is a subset of $\hat{Y} - y + z$) and $Z - z + y$ (which is a subset of $\hat{Z} - z + y$) are elements in $\mathcal{F}$. If $z \in \hat{Z} \setminus Z$, both $Y - y$ (which is a subset of $\hat{Y} - y + z$) and $Z + y$ (which is a subset of $\hat{Z} - z + y$) are elements in $\mathcal{F}$. In either case, we have established that there exists $z \in (Z \setminus Y) \cup \{\emptyset\}$ such that $Y - y + z \in \mathcal{F}$ and $Z - z + y \in \mathcal{F}$, as desired.

2. If $y \in \hat{Z}$, both $Y - y$ (which is a subset of $\hat{Y}$) and $Z + y$ (which is a subset of $\hat{Z}$) are elements in $\mathcal{F}$. Thus the desired property holds with respect to $z = \emptyset$.

□

Property 2 (laminar concave function)**.** *Assume* $\mathcal{T}$ *is a laminar family of subsets of* X*, and* $f(X')$ *is given by a laminar concave function* $\sum_{T \in \mathcal{T}} f_T(|X' \cap T|)$*. Then,* $f(X')$ *is* $M^\natural$*-concave.*

This property is the $M^\natural$-concave version of Note 6.11 of Murota (2003).

Finally, we show a property that connects a network flow problem and a matroid. Let (V, E, c, S, t) be a directed graph, where V is a set of vertexes and E is a set of directed edges. Here, $S \subsetneq V$ is a set of start vertexes and $t \in V \setminus S$ is a unique terminal vertex. Let $c \in \mathbb{Z}^E$ represent the capacities of each edge, i.e., $c((u, v))$ is the capacity of the directed edge from u to v, where $u, v \in V$ and $\mathbb{Z}$ is the set of all integers.

A network flow is represented by $\rho \in \mathbb{Z}^E$. $\rho((u, v))$ is the flow for edge $(u, v) \in E$, i.e., the flow from u to v. For flow ρ, $\delta\rho(v)$ represents the boundary at vertex v, which is defined as $\sum_{(u,v)\in E} \rho((u, v)) - \sum_{(v,u)\in E} \rho((v, u))$, i.e., the difference between the inflow to v and the outflow from it.

We say a flow ρ is valid if for all $e \in E, \rho(e) \leq c(e)$ holds, and there exists $S' \subseteq S$ such that for all $v \in S', \delta\rho(v) = -1$, $\delta\rho(t) = |S'|$, and for all $v' \in V \setminus (S' \cup \{t\}), \delta\rho(v') = 0$ hold. We say S' is the sources of ρ.

The following property holds.

Property 3. *Let (V, E, c, S, t) be a directed graph. Then $(S, \mathcal{F})$, where $\mathcal{F} = \{S' \mid S' \subseteq S, \exists \rho \in \mathbb{Z}^E,$ such that ρ is a valid flow where S' is its sources$\}$, is a matroid.*

Proof. This matroid is a variant of gammoids; see Oxley (2011) on gammoids. This is also a special case with zero costs of the network flow problem defined in Section 9.6 in Murota (2003). Theorem 9.26 in Murota (2003) shows that a function $f(S')$, which is defined as 0 if $S' \in \mathcal{F}$, and $-\infty$ otherwise, is M$^\natural$-concave. Since $\emptyset \in \mathcal{F}$, from Lemma 2, $(S, \mathcal{F})$ is a matroid.

Here, we provide a more elementary proof. It suffices to show the last condition of matroids (the other conditions are obvious). Let $S_1, S_2 \in \mathcal{F}$ with $|S_1| < |S_2|$. Let us add an artificial source s and edges (s, v) of capacity 1 for each vertex v in $S_1 \cup S_2$ to a given directed graph. We denote the new directed graph by $\hat{N} = (V \cup \{s\}, \hat{E}, \hat{c}, s, t)$, where $\hat{E} = E \cup \{(s, v) \mid v \in S_1 \cup S_2\}$ and $\hat{c}(u, v)$ is 1 if $u = s$ and $c(u, v)$ otherwise. For $i = 1, 2$, we can construct a valid flow ρ_i in $\hat{N}$ such that $\rho_i(s, v) = 1$ for all $v \in S_i$, $\delta\rho_i(t) = |S_i| = -\delta\rho_i(s)$ and $\delta\rho_i(u) = 0$ for $u \neq s, t$. By an optimality criterion of maximum flows in networks (e.g., see Corollary 10.2a of Schrijver (2003)), a valid flow $\rho \in \mathbb{Z}^{\hat{E}}$ from s to t in $\hat{N}$ is maximum (i.e., maximizes $\delta\rho(t)$ among valid flows from s to t in $\hat{N}$) if there is no directed path from s to t in the auxiliary directed graph $\tilde{N}_\rho = (V \cup \{s\}, \tilde{E}, \tilde{c}, s, t)$, where

$$\begin{aligned} E_f &= \{(u,v) \mid (u,v) \in \hat{E},\ c(u,v) > \rho(u,v)\}, \\ E_b &= \{(v,u) \mid (u,v) \in \hat{E},\ \rho(u,v) > 0\}, \\ \tilde{E} &= E_f \cup E_b, \\ \tilde{c}(u,v) &= c(u,v) - \rho(u,v) \qquad ((u,v) \in E_f), \\ \tilde{c}(v,u) &= \rho(u,v) \qquad ((v,u) \in E_b). \end{aligned}$$

Since ρ_1 is not maximum by $\delta\rho_1(t) < \delta\rho_2(t)$, there is a directed path P from s to t in $\tilde{N}_{\rho_1}$. By modifying ρ_1 into ρ_1' by

$$\rho_1'(u,v) = \begin{cases} \rho_1(u,v) + 1 & ((u,v) \in P \cap E_f) \\ \rho_1(u,v) - 1 & ((v,u) \in P \cap E_b) \\ \rho_1(u,v) & \text{(otherwise)}, \end{cases}$$

we have $\delta\rho_1'(t) = \delta\rho(t) + 1$. Furthermore, since there is no edge from s to S_1 in $\tilde{N}_{\rho_1}$, P must pass through a vertex v in $S_2 \setminus S_1$. We can construct a valid flow in the original directed graph from ρ_1' where $S_1 + v$ is its source, that is, $S_1 + v$ is a member of $\mathcal{F}$.

□

B Proofs

B.1 Proof of Lemma 2

The "if" part is obtained immediately from Property 1. The proof of the "only if" part is given as follows. Let us denote dom $\widehat{f}$ by $\mathcal{F}$. The condition $\emptyset \in \mathcal{F}$ holds because of the hypothesis of this lemma. To show the second condition of Definition 4, let us consider $X' \in \mathcal{F} \setminus \{\emptyset\}$ and $x \in X'$. From the M$^\natural$-concavity of $\widehat{f}$, we have

$$\widehat{f}(X') + \widehat{f}(\emptyset) \leq \widehat{f}(X' - x) + \widehat{f}(\{x\}).$$

This inequality implies that $X' - x \in \mathcal{F}$ because $\widehat{f}(X') = \widehat{f}(\emptyset) = 0$ by assumption, so the left hand side of the above inequality is 0, which holds only if $\widehat{f}(X' - x) = 0$. By repeatedly using the above argument, any subset of X' is also included in $\mathcal{F}$.

To show the third condition of Definition 4, let us consider $X', X'' \in \mathcal{F}$ with $|X'| > |X''|$. It follows from $|X'| > |X''|$ that there exists $x \in X' \setminus X''$. The M$^\natural$-concavity of $\widehat{f}$ guarantees that (a) $X' - x, X'' + x \in \mathcal{F}$ or (b) there exists $y \in X'' \setminus X'$ such that $X' - x + y, X'' + x - y \in \mathcal{F}$ because $0 = \widehat{f}(X') + \widehat{f}(X'') \leq \widehat{f}(X' - x) + \widehat{f}(X'' + x)$ or $0 = \widehat{f}(X') + \widehat{f}(X'') \leq \widehat{f}(X' - x + y) + \widehat{f}(X'' + x - y)$ for some $y \in X'' \setminus X'$. In case (a), we have $X'' + x \in \mathcal{F}$. In case (b), let $\hat{X}' := X' - x + y \in \mathcal{F}$. We note that $|\hat{X}'| = |X'|$ and $\hat{X}' \setminus X''$ is a proper subset of $X' \setminus X''$. We replace X' by $\hat{X}'$, and continue the above discussion. After a finite number of iterations, the above (a) must occur by $|X'| > |X''|$. Hence the third condition in Definition 4 holds.

B.2 Proof of Condition 2

It suffices to show that the function $\acute{f}$ defined by $\acute{f}(X') = \widehat{f}(X') + \sum_{g \in G} V_g(|X' \cap g|)$ is M$^\natural$-concave, because f is equal to the sum of $\acute{f}$ and a linear function $\sum_{x \in X'} v(x)$ (see the proof of Condition 1). Let $Y, Z \in \mathrm{dom}\, \widehat{f}$ and $y \in Y \setminus Z$. We assume that y belongs to a group $g \in G$.

For any $z' \in (Z \setminus Y) \cap g'$ with $|Z \cap g'| \leq |Y \cap g'|$ and $g' \neq g$, by the symmetry of G in $(X, \mathrm{dom}\, \widehat{f})$, there exists $y' \in (Y \setminus Z) \cap g'$ such that $Z - z' + y' \in \mathrm{dom}\, \widehat{f}$ and $\acute{f}(Z - z' + y') = \acute{f}(Z)$. By repeatedly using the above argument, without loss of generality, we can assume that Z satisfies $Z \cap g' \subseteq Y \cap g'$ whenever $|Z \cap g'| \leq |Y \cap g'|$ and $g' \neq g$.

If $|Y \cap g| \leq |Z \cap g|$ then there exists $z \in g \cap (Z \setminus Y)$. By the symmetry of G in $(X, \mathrm{dom}\, \widehat{f})$, $\acute{f}(Y) = \acute{f}(Y - y + z)$ and $\acute{f}(Z) = \acute{f}(Z - z + y)$, so the desired inequality for M$^\natural$-concavity holds with equality.

For the remainder of the proof, we suppose that $|Y \cap g| > |Z \cap g|$. Since $(X, \mathrm{dom}\, \widehat{f})$ is a matroid, Property 1 guarantees that either (i) $Y - y, Z + y \in \mathrm{dom}\, \widehat{f}$ or (ii) there exists $z \in (Z \setminus Y)$ such that $Y - y + z, Z - z + y \in \mathrm{dom}\, \widehat{f}$. In case (i), we have $\acute{f}(Y) + \acute{f}(Z) \leq \acute{f}(Y - y) + \acute{f}(Z + y)$, because

$$\begin{aligned} &\Big(\acute{f}(Y - y) + \acute{f}(Z + y)\Big) - \Big(\acute{f}(Y) + \acute{f}(Z)\Big) \\ &= \left(V_g(|Z \cap g| + 1) - V_g(|Z \cap g|)\right) - \left(V_g(|Y \cap g|) - V_g(|Y \cap g| - 1)\right). \end{aligned}$$

This value must be non-negative since V_g is concave and $|Z \cap g| < |Y \cap g|$.

In case (ii), z must belong to a group g' with $|Y \cap g'| < |Z \cap g'|$ and

$$\acute{f}(Y) + \acute{f}(Z) \leq \acute{f}(Y - y + z) + \acute{f}(Z - z + y)$$

holds, because

$$
\begin{aligned}
&\Big(\acute{f}(Y-y+z)+\acute{f}(Z-z+y)\Big)-\Big(\acute{f}(Y)+\acute{f}(Z)\Big)\\
=&\,(V_g(|Z\cap g|+1)-V_g(|Z\cap g|))-(V_g(|Y\cap g|)-V_g(|Y\cap g|-1))\\
&+(V_{g'}(|Y\cap g'|+1)-V_{g'}(|Y\cap g'|))-(V_{g'}(|Z\cap g'|)-V_{g'}(|Z\cap g'|-1)).
\end{aligned}
$$

This value must be non-negative since V_g is concave, $|Z\cap g|<|Y\cap g|$, and $|Y\cap g'|<|Z\cap g'|$. Hence $\acute{f}$ is M$^\natural$-concave.

B.3 Proof of Condition 3

Assume $\widetilde{f}(X')$ is given as $\sum_{T\in\mathcal{T}}\widetilde{f}_T(|X'\cap T|)$. Then, $f=\widehat{f}+\widetilde{f}$ can be written as $\sum_{T\in\mathcal{T}}f_T(|X'\cap T|)$, where $f_T(k)=\widehat{f}_T(k)$ if $k\le q_T$, and otherwise, $-\infty$. This is also a laminar concave function, since each f_T is a univariate concave function. Thus, f is M$^\natural$-concave from Property 2.

B.4 Proof of Proposition 4

Let $\mathcal{B}$ be the set of maximal elements in $\mathcal{F}$ with respect to set inclusion, i.e., the set of all hospital-feasible contracts. To prove the claim, it is enough to show that $\mathcal{B}$ satisfies the following property: For all $Y,Z\in\mathcal{B}$ and $y\in Y\setminus Z$, there exists $z\in Z\setminus Y$ such that $Y-y+z\in\mathcal{B}$ (e.g., see Theorem 39.6 of Schrijver (2003)).

Let $Y,Z\in\mathcal{B}$, $y\in Y\setminus Z$ and $y\in Y_h$ for some hospital $h\in H$. If $|Y_h|\le|Z_h|$ then there exists $z\in Z_h\setminus Y_h$ which also satisfies $Y-y+z\in\mathcal{B}$ because $\xi(Y)=\xi(Y-y+z)$. In the rest of the proof, we assume that $|Y_h|>|Z_h|$. Since $|Y|=|Z|$, there exist a hospital $h'\in H$ and a contract z such that $|Y_{h'}|<|Z_{h'}|$ and $z\in Z_{h'}\setminus Y_{h'}$. Let $Y'=Y-y+z$. Obviously, Y' satisfies $|Y'|=n$ and $|Y'_i|\le q_i$ for all $i\in H$. If δ is the Chebyshev distance function, Y' is hospital-feasible because $|\xi_h(Y')-\xi^*_h|\le\max\{|\xi_h(Y)-\xi^*_h|,|\xi_h(Z)-\xi^*_h|\}$ and $|\xi_{h'}(Y')-\xi^*_{h'}|\le\max\{|\xi_{h'}(Y)-\xi^*_{h'}|,|\xi_{h'}(Z)-\xi^*_{h'}|\}$. For the remainder of the proof, we assume δ represents the Manhattan distance. If $|Y_h|>\xi^*_h$ then Y' is hospital-feasible because $|\xi_h(Y')-\xi^*_h|=|\xi_h(Y)-\xi^*_h|-1$. Let us assume that $\xi^*_h\ge|Y_h|>|Z_h|$. If $|Y_{h'}|<\xi^*_{h'}$ then Y' is hospital-feasible because $\delta(\xi(Y'),\xi^*)=\delta(\xi(Y),\xi^*)$. We finally assume that for each $i\in H$ if $|Y_i|<|Z_i|$ then $\xi^*_i\le|Y_i|$. By this assumption together with $|Y|=|Z|=\sum_i\xi^*_i$, we have

$\{i : |Y_i| < \xi_i^*\} \cup \{h\} \subseteq \{i : |Z_i| < \xi_i^*\}$ and

$$\delta(\xi(Y), \xi^*) = 2 \sum_{i:|Y_i|<\xi_i^*} (\xi_i^* - |Y_i|) \leq 2 \sum_{i:|Y_i|<\xi_i^*} (\xi_i^* - |Z_i|)$$
$$\leq 2 \sum_{i:|Z_i|<\xi_i^*} (\xi_i^* - |Z_i|) = \delta(\xi(Z), \xi^*),$$

where the first equality follows from the assumption $|Y| = \sum_i \xi_i^*$, the first inequality results from the assumption that $|Y_i| \geq |Z_i|$ for all $i \in H$ with $\xi_i^* > |Y_i|$, the second inequality follows from the relation $\{i : |Y_i| < \xi_i^*\} \cup \{h\} \subseteq \{i : |Z_i| < \xi_i^*\}$, and the last equality results from the assumption that $|Z| = \sum_i \xi_i^*$. Also, note that at least one of the above inequalities holds strictly. This implies $\delta(\xi(Y'), \xi^*) = \delta(\xi(Y), \xi^*) + 2 \leq \delta(\xi(Z), \xi^*)$, that is, the hospital-feasibility of Y'.

B.5 Proof of Proposition 5

Proof. If R is a laminar family, the family of hospital-feasible contracts is given as: $\{X' \subseteq X \mid |X'_r| \leq q_r \ (\forall r \in R), |X'_h| \leq q_h \ (\forall h \in H)\}$. This is a laminar matroid (Definition 10).

If R is not a laminar family, there exist two regions r, r' such that each of $r \cap r'$, $r \setminus r'$, and $r' \setminus r$ is non-empty. Let us choose h_1, h_2, and h_3 from $r \cap r'$, $r \setminus r'$, and $r' \setminus r$, respectively. Also, let us choose $q_r = 1$, $q_{r'} = 1$, $(d, h_1) \in X_{h_1}$, $(d', h_2) \in X_{h_2}$, and $(d'', h_3) \in X_{h_3}$. Then, $X' = \{(d', h_2), (d'', h_3)\}$ and $X'' = \{(d, h_1)\}$ are hospital-feasible. However, there exists no $x \in X' \setminus X''$ such that $x + X''$ becomes hospital-feasible. □

B.6 Proof of Proposition 6

Proof. We prove the first assertion by induction on j. Clearly, $\widehat{\mathcal{B}}_k = \{\{x_1, x_2 \ldots, x_k\}\}$ satisfies (5). We assume that $j > k$ and $\widehat{\mathcal{B}}_{j-1}$ satisfies (5). Let us consider the situation just before Step 5. For each pair of $Z \in \widehat{\mathcal{B}}_{j-1}$ and $Y \in \mathcal{Y}$, (6) holds for $\widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y}$. Thus, for each $Y \in \mathcal{Y}$, there exists $Z \in \widehat{\mathcal{B}}_{j-1}$ with $|Z \cap Y| = k-1$. It is sufficient to show that for all $Y_1, Y_2 \in \mathcal{Y}$ with $|Y_1 \setminus Y_2| \geq 2$, and for all $x_i \in Y_1 \setminus Y_2$, there exists $x_{i'} \in Y_2 \setminus Y_1$ with $Y_1 - x_i + x_{i'} \in \mathcal{Y}$. We suppose $Y_1 \setminus Y_2 = \{x_{i_1}, \ldots, x_{i_h}\}$, and $Y_2 \setminus Y_1 = \{x_{i'_1}, \ldots, x_{i'_h}\}$.

From symmetry, it is sufficient to prove the case where $i = i_1$. Let us choose $Z_1, Z_2 \in \widehat{\mathcal{B}}_{j-1}$, such that $Z_1 = Y_1 - x_j + x_{i_0}$, $Z_2 = Y_2 - x_j + x_{i'_0}$, and $|Z_1 \cap Y_1| = |Z_2 \cap Y_2| = k - 1$ hold. We consider the following three cases.

Case 1: $i_0 \in \{i'_1, \dots, i'_h\}$. Without loss of generality, we assume $i_0 = i'_h$. Since $Z_1 \in \widehat{\mathcal{B}}_{j-1}$, $Y_2 \in \mathcal{Y}$, and $x_{i_1} \in Z_1 \setminus Y_2$, there exists $x_{i'} \in Y_2 \setminus Z_1 = \{x_{i'_1}, \dots, x_{i'_{h-1}}, x_j\}$ such that $Z'_1 = Z_1 - x_{i_1} + x_{i'} \in \widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y}$ by (5). If $i' = j$ then $Z'_1 = Y_1 - x_{i_1} + x_{i'_h} \in \mathcal{Y}$ satisfies the required condition. Thus, we assume that $i' \in \{i'_1, \dots, i'_{h-1}\}$. Without loss of generality, we assume $i' = i'_1$. Since $Y_1 \in \mathcal{Y}$, $Z'_1 \in \widehat{\mathcal{B}}_{j-1}$, and $x_j \in Y_1 \setminus Z'_1$, there exists $x_\ell \in Z'_1 \setminus Y_1$ such that $Z'_1 + x_j - x_\ell \in \widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y}$ by (6). Since ℓ is either i'_1 or i'_h, $Z'_1 + x_j - x_\ell$ is either $Y_1 - x_{i_1} + x_{i'_1}$ or $Y_1 - x_{i_1} + x_{i'_h}$, each of which is in $\mathcal{Y}$ and satisfies the required condition.

Case 2: $i_0 = i'_0$. Since $Z_1, Z_2 \in \widehat{\mathcal{B}}_{j-1}$, and $x_{i_1} \in Z_1 \setminus Z_2$, there exists $x_{i'} \in Z_2 \setminus Z_1$ such that $Z'_1 = Z_1 - x_{i_1} + x_{i'} \in \widehat{\mathcal{B}}_{j-1}$ by (5). Without loss of generality, we assume $i' = i'_h$. Furthermore, since $Z'_1 \in \widehat{\mathcal{B}}_{j-1}$, $Y_1 \in \mathcal{Y}$, and $x_{i_0} \in Z'_1 \setminus Y_1$, there exists $x_\ell \in Y_1 \setminus Z'_1 = \{x_j, x_{i_1}\}$ such that $Z''_1 = Z'_1 - x_{i_0} + x_\ell \in \widehat{\mathcal{B}}_{j-1} \cup \mathcal{Y}$ by (5). If $\ell = j$ then $Z''_1 = Y_1 - x_{i_1} + x_{i'_h} \in \mathcal{Y}$ satisfies the required condition; otherwise we have $Z''_1 = Z_1 - x_{i_0} + x_{i'_h} = Y_1 - x_j + x_{i'_h} \in \widehat{\mathcal{B}}_{j-1}$, and hence, we induce **Case 1**.

Case 3: $i_0 \neq i'_0$ and $i_0 \notin \{i'_1, \dots, i'_h\}$. Since $Z_1, Z_2 \in \widehat{\mathcal{B}}_{j-1}$, and $x_{i_0} \in Z_1 \setminus Z_2$, there exists $x_{i'} \in Z_2 \setminus Z_1$ such that $Z'_1 = Z_1 - x_{i_0} + x_{i'} \in \widehat{\mathcal{B}}_{j-1}$ by (5), and hence, we induce **Case 1** or **Case 2**.

To show the maximality of $\widehat{\mathcal{B}}$, assume for contradiction that there exists $\mathcal{B}' \supsetneq \widehat{\mathcal{B}}$, such that $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{B}'$ forms a matroid. Let Y' be the first element of $\mathcal{B}' \setminus \widehat{\mathcal{B}}$ removed by Algorithm 1, and x_h be the element of Y' with the maximum subscript. Then there exists $Z \in \widehat{\mathcal{B}}_{h-1} \subseteq \mathcal{B}'$ such that (6) does not hold for Y', Z and $\widehat{\mathcal{B}}_{h-1} \cup \mathcal{Y}'$, where $\mathcal{Y}'$ is $\mathcal{Y}$ in Algorithm 1 just before Y' is removed. By the definition of Y', if $Y \subsetneq \{x_1, \dots, x_h\}$ is an element of $\mathcal{B}'$, it is also an element of $\widehat{\mathcal{B}}_{h-1} \cup \mathcal{Y}'$. Thus (6) does not hold for Y', Z and $\mathcal{B}'$, which contradicts the assumption that $\mathcal{B}'$ is a matroid. $\square$

C Applications

In this section, we examine existing and new models of matching and show that the sufficient conditions described in Section 4 hold in these cases. For existing applications, this connection allows us to reproduce key results and, for some applications, show stronger results. This enables us to provide mechanisms that are strategyproof for doctors and produce stable matchings.

C.1 Standard model (Gale and Shapley, 1962)

C.1.1 Model

A market is a tuple $(D, H, X, (\succ_d)_{d\in D}, (\succ_h)_{h\in H}, (q_h)_{h\in H})$. D is a finite set of doctors and H is a finite set of hospitals. X is a finite set of contracts. A contract $x \in X$ is a pair (d, h), which represents a matching between doctor d and hospital h. $(\succ_d)_{d\in D}$ is a profile of doctors' preferences, i.e., each $\succ_d$ represents the strict preference of each doctor d over acceptable contracts in $X_d = \{(d, h) \in X \mid h \in H\}$. $(\succ_h)_{h\in H}$ is a profile of hospitals' preferences, i.e., each $\succ_h$ represents the preference of each hospital h over the contracts that are related to it. $(q_h)_{h\in H}$ is a profile of hospitals' maximum quotas, i.e., each q_h represents the maximum quota of hospital h.

C.1.2 Feasibility

We say $X' \subseteq X$ is hospital-feasible if $|X'_h| \leq q_h$ for all h, where $X'_h = \{(d, h) \in X' \mid d \in D\}$. We say $X' \subseteq X$ is doctor-feasible if X'_d is acceptable for all d (we say X'_d is acceptable for d if either (i) $X'_d = \{x\}$ and x is acceptable for d, or (ii) $X'_d = \emptyset$ holds). We say X' is feasible if it is doctor- and hospital-feasible.

C.1.3 Stability

Here, let us reprint the definition of a blocking pair in Section 5.1. For a matching X', we say $(d, h) \in X \setminus X'$ is a **blocking pair** if (i) (d, h) is acceptable for d and $(d, h) \succ_d X'_d$, and (ii) either $|X'_h| < q_h$ or there exists $(d', h) \in X'$ such that $(d, h) \succ_h (d', h)$.

We say a matching X' is **Gale-Shapley (GS)-stable** if there exists no blocking pair (Gale and Shapley, 1962).

C.1.4 Mechanism

The standard Deferred Acceptance (DA) mechanism (Gale and Shapley, 1962) is defined as follows.

Mechanism 4 (standard DA)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$: Each doctor d applies to her most preferred hospital by which she has not been rejected before Stage k (if no remaining hospital is acceptable for d, d does not apply to any hospital). Each hospital h tentatively accepts doctors applying to h up to q_h according to $\succ_h$. The rest of doctors are rejected. If no doctor is rejected by any hospital, terminate the mechanism and return the current tentatively accepted pairs as the final matching. Otherwise, go to Stage $k + 1$.

The standard DA mechanism is strategyproof for doctors (Dubins and Freedman, 1981; Roth, 1982) and obtains a GS-stable matching (Gale and Shapley, 1962).

C.1.5 Representation in our model

Let us define $\widehat{f}(X')$ as 0 if X' is hospital-feasible, i.e., $|X'_h| \leq q_h$ for all h, and otherwise as $-\infty$. Then, $(X, \mathrm{dom}\,\widehat{f})$ is a laminar matroid, since $\{X_h \mid h \in H\}$ is a laminar family and we require $|X' \cap X_h| \leq q_h$ for each h.

Let us assume a positive value $v(x)$ is defined for each $x = (d, h)$ with the property that $v((d, h)) > v((d', h))$ when $(d, h) \succ_h (d', h)$ holds.[36] With $\widetilde{f}(X') = \sum_{x \in X'} v(x)$, we can apply Condition 1. The standard DA mechanism is identical to the generalized DA mechanism where Ch_H is defined as the maximizer of function f defined above.

The following proposition holds.

Proposition 7. *HM-stability is equivalent to GS-stability.*

[36] We note that although we assume a value $v(x)$ is given for each contract and f is defined by the sum of these values, $Ch_H(X')$ is determined only by the relative ordering of the values among the contracts that belong to the same hospital. Thus, the specific cardinal choice of these values, or the relative ordering among contracts for different hospitals, is not important.

Proof. To show that HM-stability implies GS-stability, assume for contradiction that a feasible matching X' is not GS-stable. Then there exists a blocking pair (d, h). Because (d, h) is acceptable for d and $(d, h) \succ_d X'_d$, it immediately follows that $(d, h) \in Ch_D(X' + (d, h))$ by definition of Ch_D. Because either $|X'_h| < q_h$ or there exists $(d', h) \in X'$ such that $(d, h) \succ_h (d', h)$, by the definition of f, in either case it follows that $(d, h) \in \arg\max_{X'' \subseteq X' + (d,h)} f(X'') = Ch_H(X' + (d, h))$. Therefore X' is not HM-stable.

To show that GS-stability implies HM-stability, assume for contradiction that a feasible matching X' is not HM-stable. $X' = Ch_D(X')$ because X' is a matching and hence doctor-feasible, and $X' = Ch_H(X')$ because X' is a matching and hence hospital-feasible, that is, $|X'_h| \leq q_h$ for all h in the current model, and $v((d, h)) > 0$ for all $(d, h) \in X'$ by the definition of $v(\cdot)$. These facts and the assumption that X' is not HM-stable imply there exists a doctor-hospital pair (d, h) such that $(d, h) \in Ch_H(X' + (d, h))$ and $(d, h) \in Ch_D(X' + (d, h))$ hold. Then $(d, h) \succ_d X'_d$ by the definition of Ch_D and, by the definition of f, either $|X'_h| < q_h$ or there exists $(d', h) \in X'$ such that $(d, h) \succ_h (d', h)$. Therefore (d, h) is a blocking pair, showing that X' is not GS-stable. Thus HM-stability is equivalent to GS-stability. □

From Proposition 7, we can guarantee that the generalized DA mechanism obtains the doctor-optimal GS-stable matching, so the generalized DA mechanism and the standard DA mechanism obtain the same outcome. Note that this fact can be derived without checking whether the two mechanisms behave exactly in the same way.

C.2 Regional minimum quotas (Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo, 2016)

C.2.1 Model

A market is a tuple $(D, H, X, R, (\succ_d)_{d\in D}, (\succ_h)_{h\in H}, (q_h)_{h\in H}, (p_h)_{h\in H}, (p_r)_{r\in R})$. As in the standard model, D is a finite set of doctors and H is a finite set of hospitals. $(q_h)_{h\in H}$ is a profile of maximum quotas of hospitals. $X := D \times H$ is the set of contracts. Here, we assume every doctor is acceptable to every hospital and vice versa; without this assumption, satisfying all minimum quotas can be impossible.[37]

We assume hospitals are grouped into regions $R = \{r_1, \dots, r_n\}$, where each region r is a subset of hospitals. Here, we allow these regions to overlap. To be more precise, we assume R is a laminar family of H, i.e., these regions have a hierarchical structure. Without loss of generality, we assume $H \in R$ holds. We assume each region, as well as each individual hospital, has its minimum quota. More specifically, for each $h \in H$, p_h represents the minimum quota of hospital h, and for each $r \in R$, p_r represents the regional minimum quota of region r.

Since R is a laminar family of subsets of H, R has a tree structure, where H is the root node, and each $h \in H$ is a leaf node (as shown in Figure 2 (a)). In a tree, we say region r_p is the **parent** of another region r if $r_p = \arg\min_{r' \supsetneq r} |r'|$. Similarly, we say region r_p is the parent of hospital h if $r_p = \arg\min_{r' \ni h} |r'|$. We say region r (or hospital h) is a **child** of region r_p if r_p is a parent of r (or h). For each node r, let children(r) denote the set of all children of r. Without loss of generality, we assume for each r, $p_r \geq \sum_{r' \in \text{children}(r)} p_{r'}$ holds (if this inequality does not hold, then one can redefine $p_r := \sum_{r' \in \text{children}(r)} p_{r'}$ and the constraints are unchanged). We also assume $p_H = |D|$, i.e., the minimum quota of the root is equal to the number of doctors.

The model presented in Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2016) is a special case of this model in which minimal quotas are imposed only on individual hospitals.

C.2.2 Feasibility

We say $X' \subseteq X$ is hospital-feasible if $p_h \leq |X'_h| \leq q_h$ for all h, and $p_r \leq |X'_r|$ for all r. We say X' is doctor-feasible if $|X'_d| = 1$ for all d. Then, we say X' is feasible if it is doctor- and hospital-feasible. Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo (2016) show that if $p_r \leq \sum_{h \in r} q_h$ and $p_h \leq q_h$

[37] This assumption is motivated by some real-life applications. For example, in many universities in Japan, an undergraduate student who majors in engineering must be assigned to a laboratory to conduct a project, and the project is required for graduation. In this setting, every student can be assumed to be acceptable to every laboratory and vice versa. In particular, since 2011, the third author has been applying a mechanism based on Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2016) to assign students to laboratories in Department of Electrical Engineering and Computer Science, School of Engineering, Kyushu University, where every student is acceptable to every laboratory and vice versa.

hold for all r and h, then a feasible matching always exists. In the rest of this section, we assume the above conditions hold.

X' is hospital-feasible only if $|X'| = |D|$ since $p_H = |D|$. Thus, feasibility of X' requires that all doctors be matched to some hospital. It is clear that the family of all hospital-feasible contracts does not form a matroid since $\emptyset$ is not hospital-feasible. However, we can apply a technique similar to the one used in Section 5.2, i.e., we say X' is semi-hospital-feasible if it is a subset of (or equal to) a hospital-feasible matching.

C.2.3 Stability

Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo (2016) introduce several concepts that are related to stability. First, in a matching X', a doctor d where $(d, h) \in X'$ has a **justified envy** towards another doctor d' where $(d', h') \in X'$, if $(d, h') \succ_d (d, h)$, and $(d, h') \succ_{h'} (d', h')$ hold.

Second, in a matching X', a doctor d where $(d, h) \in X'$ **claims an empty seat** of h' if the following conditions hold: (i) $(d, h') \succ_d (d, h)$, and (ii) $X'' = X' - (d, h) + (d, h')$ is feasible.

Third, in a matching X', a doctor d where $(d, h) \in X'$ **strongly claims an empty seat** of h' if the following conditions hold: (i) $(d, h') \succ_d (d, h)$, (ii) $X'' = X' - (d, h) + (d, h')$ is feasible, and (iii) $|X'_h| - |X'_{h'}| \geq 2$. The intuitive meaning of condition (iii) is similar to KK-stability; the claim of doctor d for moving her from h to h' is justified if such a movement strictly decreases the imbalance of doctors between these hospitals, but not otherwise.

We say a matching is **fair** if no doctor has justified envy. We say a matching is nonwasteful if no doctor claims an empty seat, and **weakly nonwasteful** if no doctor strongly claims an empty seat. In general, fairness and nonwastefulness are incompatible, i.e., there exists a case where no matching is fair and nonwasteful (Ehlers, Hafalir, Yenmez, and Yildirim, 2014). Fleiner and Kamiyama (2016) gave an efficient algorithm for checking whether there exists a fair and nonwasteful matching or not. On the other hand, Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo (2016) show that a fair and weakly nonwasteful matching always exists.

C.2.4 Mechanism

Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo (2016) present a mechanism based on the FDA mechanism called Round-robin Selection Deferred

Acceptance mechanism for Regional Minimum Quotas (RSDA-RQ), which is defined as follows.[38]

Mechanism 5 (RSDA-RQ)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor applies to her most preferred hospital by which she has not been rejected before Stage k. Reset X' as $\emptyset$.

Step 2 For each r, iterate the following procedure until all doctors applying to hospitals in r are either tentatively accepted or rejected:

1. Choose hospital h based on the round-robin ordering.
2. Choose doctor d who is applying to h and is not tentatively accepted or rejected yet, and has the highest priority according to $\succ_h$ among the current applicants to h. If there exists no such doctor, then go to the procedure for the next hospital.
3. If $X' + (d, h)$ is semi-hospital-feasible, d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next hospital.
4. Otherwise, d is rejected by h. Then go to the procedure for the next hospital.

Step 3 If all the doctors are tentatively accepted in Step 2, then let X' be a final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

This procedure is almost identical to the FDA mechanism. The only difference is that in the RSDA-RQ mechanism, when making a decision whether to tentatively accept (d, h) or not, the RSDA-RQ mechanism checks whether $X' + (d, h)$ is semi-hospital-feasible or not. Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo (2016) introduce a computationally efficient method to

[38] To be more precise, Goto, Iwasaki, Kawasaki, Kurata, Yasuda, and Yokoo (2016) deal with a more generalized model where regional maximum quotas are imposed as well as regional minimum quotas. Throughout this section, we consider a simplified setting where only regional minimum quotas exist.

check semi-hospital-feasibility. The matching obtained by the RSDA-RQ mechanism is fair and weakly nonwasteful.

Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2016) present a mechanism based on the deferred acceptance mechanism called Extended Seat Deferred Acceptance (ESDA) mechanism. The ESDA mechanism is a special case of the RSDA-RQ mechanism for an environment in which minimal quotas are imposed only on individual hospitals. Thus, the ESDA mechanism is fair and weakly nonwasteful in that setting.

C.2.5 Representation in our model

In the face of nontrivial minimum quotas, the family of hospital-feasible sets of contracts cannot be a matroid. This is because $\emptyset$ is not hospital-feasible. Also, since hospital-feasibility requires that $|X'| = |D|$ holds, no proper subset of X' can be hospital-feasible. Thus, conditions 1 and 2 in Definition 4 are violated. Here, we consider the family of semi-hospital-feasible sets of contracts. By definition, the family of semi-hospital-feasible sets of contracts satisfies conditions 1 and 2 in Definition 4.

We create a network flow problem (Cormen, Leiserson, Rivest, and Stein, 2009) that represents these regional constraints as follows. For notational simplicity, let q_r denote $\sum_{h \in r} q_h$.

- We set the set of start vertexes as X.
- There exists a unique terminal vertex t.
- For each h, we create an intermediate vertex v-h. There exists a directed edge from each (d, h) to this vertex, whose capacity is 1. Also, from v-h, there exists a directed edge to t, whose capacity is p_h.
- For each r, we create one intermediate vertex v-r. There exists a directed edge from each v-r', where $r' \in \text{children}(r)$, to v-r, whose capacity is $q_{r'} - p_{r'}$. From v-r, there exists a directed edge to t, whose capacity is $p_r - \sum_{r' \in \text{children}(r)} p_{r'}$.

We assume $\widehat{f}(X') = 0$ if there exists a **valid** network flow from X', i.e., a flow from X' to the terminal vertex t that satisfies capacity constraints of edges, and otherwise, $-\infty$.

To illustrate our construction, consider the following example. Assume there are four hospitals $h_1, \ldots, h_4$. Their maximum and minimum quotas

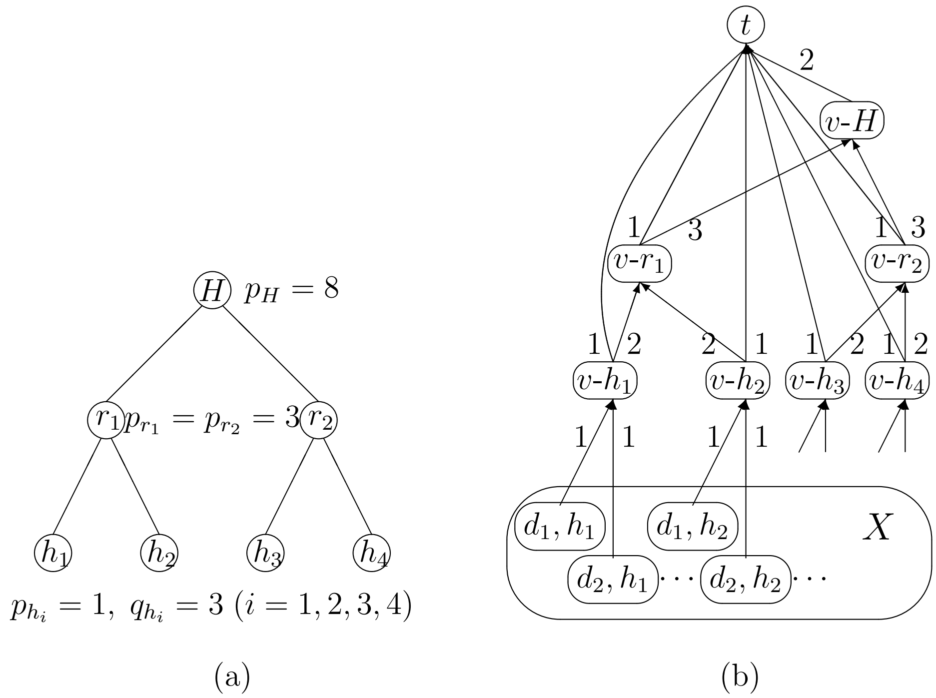

Figure 2: Example of regional minimum quotas (a) and an associated network flow problem (b)

are 3 and 1, respectively. They are divided into two regions r_1, r_2. Their minimum quotas are 3. Thus, we require at least one doctor is assigned to both h_1 and h_2, and one additional doctor is assigned to either h_1 or h_2. There are 8 doctors. Thus, p_H is set at 8 (Figure 2 (a)).

Now we are ready to illustrate our construction of the associated network flow problem. For h_1, we create one intermediate vertex v-h_1. There exists a directed edge from each contract related with h_1 to v-h_1. Also, from v-h_1, there exists a directed edge to the terminal node t, whose capacity is $p_{h_1} = 1$. Similarly, for h_2, we create one intermediate vertex v-h_2. There exists a directed edge from each contract related to h_2 to v-h_2. Also, from v-h_2, there exists a directed edge to the terminal node t, whose capacity is $p_{h_2} = 1$. The construction related to h_3 and h_4 is symmetric.

Furthermore, for r_1, we create one intermediate vertex v-r_1. There exist edges from nodes representing hospitals in children(r_1), i.e., v-h_1 and v-h_2, whose capacities are $q_{h_1} - p_{h_1} = q_{h_2} - p_{h_2} = 2$. Also, from v-r_1, there exists a directed edge to the terminal node t, whose capacity is $p_{r_1} - \sum_{r' \in \text{children}(r_1)} p_{r'} = 3 - (1+1) = 1$. The construction related to r_2 is symmetric. Also, for H, we create one intermediate node v-H. There exists a directed edge from v-r_1 (which is in children(H)) to v-H, whose capacity is $q_{r_1} - p_{r_1} = 3 + 3 - 3 = 3$. Similarly, there exists a directed edge from v-r_2 to v-H, whose capacity is 3. There exists a directed edge from v-H to t, whose capacity is $|D| - \sum_{r \in \text{children}(H)} p_r = 8 - (3+3) = 2$.

Figure 2 (b) shows the network flow problem of this example. Here, the number associated to a directed edge represents its capacity.

The following proposition holds.

Proposition 8. *X' is feasible if and only if $\widehat{f}(X') = 0$ and $|X'| = |D|$. X' is semi-hospital-feasible if and only if $\widehat{f}(X') = 0$.*

Proof. First, we show that X' is feasible if and only if $\widehat{f}(X') = 0$ and $|X'| = |D|$. Assume X' is feasible. From the fact X' is hospital-feasible, $|X'| = |D|$ holds. Let us define a flow in which X' is the source as follows. For each edge from $(d, h) \in X'$ to v-h, we set its flow as 1. Since X' is hospital feasible, $p_h \leq |X'_h| \leq q_h$ holds. Thus, the total input flow to v-h is at least p_h and at most q_h. Then, for each $h \in H$, we set the flow from v-h to t as p_h, and the flow from v-h to its parent r as $|X'_h| - p_h$. This is at most $q_h - p_h$. Thus, it is within the capacity. Also, for each $r \in R - H$, the total input flow to v-r is $|X'_r| - \sum_{r' \in \text{children}(r)} p_{r'}$, which is at least $p_r - \sum_{r' \in \text{children}(r)} p_{r'}$ and at most $q_r - \sum_{r' \in \text{children}(r)} p_{r'}$. Then, we set the flow from v-r to t as $p_r - \sum_{r' \in \text{children}(r)} p_{r'}$, and the flow from v-r to its parent region r' as $|X'_r| - p_r$. This is at most $q_r - p_r$. Thus, it is within the capacity. Finally, for H, the total input flow to v-H is $|D| - \sum_{r \in \text{children}(H)} p_r$. Then, we set the flow from v-H to t as $|D| - \sum_{r \in \text{children}(H)} p_r$. It is clear that the flow defined as above is valid. Thus, $\widehat{f}(X') = 0$ holds.

Next, we show that if $\widehat{f}(X') = 0$ and $|X'| = |D|$, then X' is hospital-feasible. The total capacity of edges toward t is $(|D| - \sum_{r \in \text{children}(H)} p_r) + \sum_{r \in R-H} (p_r - \sum_{r' \in \text{children}(r)} p_{r'}) + \sum_{h \in H} p_h = |D|$. Thus, if $\widehat{f}(X') = 0$ and $|X'| = |D|$, each of these edges is saturated, i.e., its flow is equal to its capacity. Thus, for each hospital $h \in H$, $|X'_h|$ is at least p_h since the edge from v-h to t is saturated. Also, since the flow is valid, at the edge from v-h to r, where r is h's parent region, its flow is at most $q_h - p_h$. Thus, $|X'_h|$ is at most q_h. Similarly, for each region $r \in R$, we can recursively show that $|X'_r|$ is at least p_r and at most q_r. Thus, X' is feasible.

For matroid $(X, \mathcal{F})$, recall that we say $X' \in \mathcal{F}$ is a base if there exists no $X'' \in \mathcal{F}$ such that $X'' \supset X'$, i.e., X' is maximal. From Definition 4, it is clear that all bases have the same size. Also, since we assume every doctor is acceptable for all hospitals, the size of a base is $|D|$.

Now, let us show that if $\widehat{f}(X') = 0$, X' is semi-hospital-feasible. We have already shown that if $\widehat{f}(X') = 0$ and $|X'| = |D|$, then X' is hospital-feasible. If $\widehat{f}(X') = 0$ and $|X'| < |D|$, let us choose a base X'' such that $X'' \supset X'$ holds. Then, $|X''| = |D|$ and $\widehat{f}(X'') = 0$. Thus, X'' is hospital-feasible and hence X' is semi-hospital-feasible.

Finally, let us show that if X' is semi-hospital-feasible, then $\widehat{f}(X') = 0$. Since X' is semi-hospital-feasible, there exists $X'' \supseteq X'$ such that X'' is hospital-feasible. Then, $\widehat{f}(X'') = 0$ and from Definition 4, $\widehat{f}(X') = 0$ holds. $\square$

Define $\widetilde{f}(X')$ in the same way as in equation (1) in Section 5.1.5. Then, from Property 3, $(X, \mathrm{dom}\,\widehat{f})$ is a matroid. Thus, we can apply Condition 2. With the help of these results, we are ready to show the following claim.

Proposition 9. *HM-stability implies fairness and weak nonwastefulness.*

Proof. Suppose X' is HM-stable. If doctor d prefers (d, h) to X'_d, then no (d', h) with $(d, h) \succ_h (d', h)$ is in X' by the definition of HM-stability and the definition of the payoff function f. Thus, HM-stability implies fairness. Also, if doctor d prefers (d, h') to (d, h) and $(d, h) \in X'$ while $X' - (d, h) + (d, h')$ is feasible, then $|X'_h| - |X'_{h'}| \leq 1$ must hold, i.e., moving d from h to h' does not strictly decrease the imbalance of doctors between h and h', by the definition of HM-stability and the construction of f. Thus, HM-stability implies weak nonwastefulness. $\square$

We note that fairness and weak nonwastefulness do not imply HM-stability. To see this, let us consider the following case. There are three hospitals h_1, h_2 and h_3 and two doctors d_1 and d_2. The minimum quota of h_1 is 1 and the minimum quotas of the other hospitals are 0. No regional minimum quota is imposed. We assume $h_1 \succ_{d_1} h_2 \succ_{d_1} h_3$ and $h_2 \succ_{d_2} h_3 \succ_{d_2} h_1$ hold. The round-robin ordering over hospitals is defined as h_1, h_2, h_3. All hospitals prefer d_1 over d_2. We assume the individual maximum quota of each hospital is large enough. $X' = \{(d_1, h_1), (d_2, h_3)\}$ is fair and weakly nonwasteful. In particular, (d_2, h_2) satisfies conditions (i) and (ii) for strongly claiming an empty seat, but condition (iii) does not hold since $|X'_{h_3}| = 1$ and

$|X'_{h_2}| = 0$. However, X' is not HM-stable since $(d_2, h_2) \in Ch_D(X' + (d_2, h_2))$ and $(d_2, h_2) \in Ch_H(X' + (d_2, h_2))$ hold.

The RSDA-RQ mechanism is identical to the generalized DA mechanism where Ch_H is defined as the maximizer of f described above. The ESDA is identical to the generalized DA mechanism where Ch_H is defined as the maximizer of f described above, when minimal quotas are imposed only on individual hospitals.

C.3 Controlled school choice (Ehlers, Hafalir, Yenmez, and Yildirim, 2014)

C.3.1 Model

This section studies a model of matching with diversity constraints. Although the original contribution by Ehlers, Hafalir, Yenmez, and Yildirim (2014) frames the model in the context of student placement in schools, we stick to our terminology of doctors and hospitals.

A market is a tuple $(D, H, X, (\succ_d)_{d \in D}, (\succ_h)_{h \in H}, (q_h)_{h \in H}, T, \tau, (\underline{q}_h^T)_{h \in H}, (\overline{q}_h^T)_{h \in H})$. The definitions of $D, H, X, \succ_d, \succ_h$, and q_h are identical to the standard model.

One major difference between this model and the standard ones is that we assume each doctor d has her type $\tau(d) \in T = \{t_1, \ldots, t_k\}$. A type of a doctor may represent race, income, gender, or any socioeconomic status. Furthermore, each hospital has soft minimum and maximum bounds for each type t, i.e., $(\underline{q}_h^T)_{h \in H}$ and $(\overline{q}_h^T)_{h \in H}$, where $\underline{q}_h^T = (\underline{q}_h^t)_{t \in T}$ and $\overline{q}_h^T = (\overline{q}_h^t)_{t \in T}$. Each $\underline{q}_h^t$ and $\overline{q}_h^t$ represent minimum and maximum bounds for type t at hospital h. We assume $\sum_{t \in T} \underline{q}_h^t \leq q_h$ holds, i.e., the minimum bounds for all types in h can be satisfied without violating the maximum quota of the hospital. For $X' \subseteq X$, let $X'_{h,t}$ denote $\{(d, h) \in X' \mid d \in D, \tau(d) = t\}$.

C.3.2 Feasibility

The bounds $\underline{q}_h^t$ and $\overline{q}_h^t$ are *soft* bounds and do not affect feasibility. We say $X' \subseteq X$ is hospital-feasible if $|X'_h| \leq q_h$ for all h. We say $X' \subseteq X$ is doctor-feasible if X'_d is acceptable for all d. Then, we say X' is feasible if it is doctor- and hospital-feasible.

C.3.3 Stability

Ehlers, Hafalir, Yenmez, and Yildirim (2014) introduce a stability concept, which we call Ehlers-Hafalir-Yenmez-Yildirim (EHYY)-stability. For a matching X', we say $(d,h) \in X \setminus X'$ is an EHYY-blocking pair, where $\tau(d) = t$, if (d,h) is acceptable for d and $(d,h) \succ_d X'_d$, and any one of the following conditions holds:

(i) $|X'_h| < q_h$,

(ii) $|X'_{h,t}| < \underline{q}^t_h$, or

(iii) there exists another doctor d', such that $(d',h) \in X'$ and $\tau(d') = t'$ hold, and any one of the following conditions holds:

  (a) $t = t'$ and $(d,h) \succ_h (d',h)$,

  (b) $t \neq t'$, $\underline{q}^t_h \leq |X'_{h,t}| < \overline{q}^t_h$, $\underline{q}^{t'}_h < |X'_{h,t'}| \leq \overline{q}^{t'}_h$, and $(d,h) \succ_h (d',h)$,

  (c) $t \neq t'$, $\underline{q}^t_h \leq |X'_{h,t}| < \overline{q}^t_h$, and $|X'_{h,t'}| > \overline{q}^{t'}_h$, or

  (d) $t \neq t'$, $|X'_{h,t}| \geq \overline{q}^t_h$, $|X'_{h,t'}| > \overline{q}^{t'}_h$, and $(d,h) \succ_h (d',h)$.

We say a matching X' is **EHYY-stable** if there exists no EHYY-blocking pair.

Intuitively, this stability concept means that for each type t, up to $\underline{q}^t_h$ doctors of type t can be assigned to hospital h without competing against doctors of other types. Then, if more type t doctors hope to be assigned to h, they can be assigned up to $\overline{q}^t_h$ but these doctors must compete against doctors of other types. Furthermore, if more type t doctors beyond $\overline{q}^t_h$ hope to be assigned to h, they can be assigned only when q_h is not filled yet by accepting doctors of type $t' \neq t$ up to $\overline{q}^{t'}_h$.

C.3.4 Mechanism

Ehlers, Hafalir, Yenmez, and Yildirim (2014) present a mechanism called the Deferred Acceptance Algorithm with Soft Bounds (DAASB), whose outcome satisfies EHYY-stability. The DAASB mechanism is defined as follows.

Mechanism 6 (DAASB)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor applies to her most preferred hospital by which she has not been rejected before Stage k. If no remaining hospital is acceptable for d, d does not apply to any hospital. Reset X' as $\emptyset$.

Step 2 For each hospital h, iterate the following procedure until all doctors applying to h are either tentatively accepted or rejected:

Phase 1:
1. Choose doctor d who is applying to h and is not tentatively accepted, rejected, or postponed to the next phase yet, and has the highest priority according to $\succ_h$ among current applicants to h. If there exists no such doctor, then go to the procedure for the next phase.
2. If $|X'_{h,\tau(d)}| < \underline{q}_h^{\tau(d)}$, then d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next doctor.
3. Otherwise, the decision on d is postponed to the next phase. Go to the procedure for the next doctor.

Phase 2:
1. Choose doctor d who is applying to h and is not tentatively accepted, rejected, or postponed to the next phase yet, and has the highest priority according to $\succ_h$ among current applicants to h. If there exists no such doctor, then go to the procedure for the next phase.
2. If $|X'_h| = q_h$ then reject all doctors applying to h who have not been tentatively accepted yet. Go to the procedure for the next hospital. If $|X'_{h,\tau(d)}| < \overline{q}_h^{\tau(d)}$, then d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next doctor.
3. Otherwise, the decision on d is postponed to the next phase. Go to the procedure for the next doctor.

Phase 3:
1. Choose doctor d who is applying to h and is not tentatively accepted, or rejected yet, and has the highest priority according to $\succ_h$ among current applicants to h. If there exists no such doctor, then go to the procedure for the next hospital.
2. If $|X'_h| = q_h$ then reject all doctors applying to h who are not tentatively accepted yet. Go to the procedure for the next hospital.

3. Otherwise, d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next doctor.

Step 3 If all the doctors are tentatively accepted in Step 2, then let X' be a final matching and terminate the mechanism. Otherwise, go to Stage $k+1$.

C.3.5 Representation in our model

Let us consider an extended market $(D, H, \tilde{X}, (\tilde{\succ}_d)_{d \in D}, f)$. Here, a contract $x \in \tilde{X}$ is represented as (d, h, s), where $d \in D$, $h \in H$, and $s \in \{0, 1, 2\}$. $s = 0, 1, 2$ are interpreted to mean that doctor d is accepted at hospital h for type $\tau(d)$'s priority seat, normal seat, and extended seat, respectively. As described later, we introduce such a seat distinction to satisfy EHYY-stability. From the matching in the extended market $\tilde{X}'$, the matching in the original market X' is obtained by mapping each contract (d, h, s) to (d, h). Let $\tilde{X}'_{h,t,s}$ denote $\{(d, h, s) \in \tilde{X}' \mid d \in D, \tau(d) = t\}$.

We define the modified preference of each doctor d, denoted $\tilde{\succ}_d$ such that $(d, h, s) \tilde{\succ}_d (d, h', s')$ holds for any $h \neq h'$, s, and s' if $(d, h) \succ_d (d, h')$, and $(d, h, 0) \tilde{\succ}_d (d, h, 1) \tilde{\succ}_d (d, h, 2)$ holds for any h, i.e., for each h, doctor d prefers h's priority seat over its normal seat, and its normal seat over its extended seat.

For the extended market, let us assume for each x, its value $v(x)$ is defined. We assume $v((d, h, 0)) > v((d', h, 1))$ and $v((d, h, 1)) > v((d', h, 2))$ hold for any d, d', and h, i.e., hospitals first try to fill their priority seats, then normal seats, and finally extended seats. Also, we assume $v((d, h, s)) > v((d', h, s))$ if $(d, h) \succ_h (d', h)$, i.e., the preference of an individual hospital over doctors is respected, as long as doctors have the same type of seat.

Let us define $\widehat{f}(\tilde{X}')$ as 0 when $|\tilde{X}'_h| \leq q_h$, $|\tilde{X}'_{h,t,0}| \leq \underline{q}^t_h$, and $|\tilde{X}'_{h,t,1}| \leq \overline{q}^t_h - \underline{q}^t_h$ hold for all $h \in H$ and $t \in T$, and otherwise, $-\infty$. Intuitively, these definitions mean that the number of priority seats of hospital h for type t doctors is $\underline{q}^t_h$, and the number of normal seats is $\overline{q}^t_h - \underline{q}^t_h$.

Also, let us define $\widetilde{f}(\tilde{X}')$ as $\sum_{x \in \tilde{X}'} v(x)$. $(\tilde{X}, \mathrm{dom}\,\widehat{f})$ is a laminar matroid, since $\mathcal{T} = \{\tilde{X}_{h,t,s} \mid h \in H, t \in T, s \in \{0, 1, 2\}\} \cup \{\tilde{X}_h \mid h \in H\}$ is a laminar family of $\tilde{X}$. Thus, we can apply Condition 1. The matching obtained by the DAASB mechanism is identical to the matching in the original market mapped from the outcome of the generalized DA mechanism where Ch_H is defined as the maximizer of function f.

The following proposition holds.

Proposition 10. *HM-stability of $\tilde{X}'$ in the extended market implies EHYY-stability of X' in the original market.*

Proof. Assume (d, h) is an EHYY-blocking pair in the original market, where $\tau(d) = t$. If condition (i) of an EHYY-blocking pair holds, by choosing $x = (d, h, 2)$, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. Also, if condition (ii) holds, $|\tilde{X}'_{h,t,0}| < \underline{q}^t_h$ holds. Thus, by choosing $x = (d, h, 0)$, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. Then, in either case, $\tilde{X}'$ is not HM-stable.

Assume condition (iii) holds, so there exists another doctor d' such that $\tau(d') = t'$ and d' is assigned to h. If condition (a) holds, $(d', h, s) \in \tilde{X}'$ and $v((d, h, s)) > v((d', h, s))$ must hold. Thus, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. If condition (b) holds, $(d', h, 1) \in \tilde{X}'$, $|\tilde{X}'_{h,t,1}| < \overline{q}^t_h - \underline{q}^t_h$, and $v((d, h, 1)) > v((d', h, 1))$ must hold. Thus, by choosing $x = (d, h, 1)$, $x \in Ch_D(\tilde{X}'+x)$ and $x \in Ch_H(\tilde{X}'+x)$ hold. If condition (c) holds, $(d', h, 2) \in \tilde{X}'$, $|\tilde{X}'_{h,t,1}| < \overline{q}^t_h - \underline{q}^t_h$, and $v((d, h, 1)) > v((d', h, 2))$ must hold. Thus, by choosing $x = (d, h, 1)$, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. If condition (d) holds, $(d', h, 2) \in \tilde{X}'$ and $v((d, h, 2)) > v((d', h, 2))$ must hold. Thus, by choosing $x = (d, h, 2)$, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. Thus, in any of these cases, $\tilde{X}'$ is not HM-stable. □

For a matching in the extended market, the corresponding matching in the original market is uniquely determined, but multiple matchings in the extended market can be mapped onto an identical matching in the original market. Thus, whether EHYY-stability in the original market implies HM-stability in the extended market or not depends on how to determine the mapping from the original market to the extended market. However, since the generalized DA mechanism is identical to the DAASB mechanism, it obtains the doctor-optimal EHYY-stable matching.

C.4 Student-project allocation (Abraham, Irving, and Manlove, 2007)

C.4.1 Model

In the Student-Project Allocation (SPA) problem, a market is represented as a tuple $(S, P, L, X, (\succ_s)_{s \in S}, (\succ_l)_{l \in L}, (q_l)_{l \in L}, (q_p)_{p \in P})$. S is a finite set of students, P is a finite set of projects, and L is a finite set of lecturers. Each project $p \in P$ is offered by some lecturer $l \in L$. Let P_l denote the set of projects offered by lecturer l. Each contract $x \in X$ is a pair (s, p), which represents the assignment of student s to project p. For $X' \subseteq X$, let X'_s denote $\{(s, p) \in X' \mid p \in P\}$ and X'_l denote $\{(s, p) \in X' \mid s \in S, p \in P_l\}$. For each $s \in S$, $\succ_s$ represents the preference of student s over acceptable contracts in X_s. For each $l \in L$, $\succ_l$ represents the preference of lecturer l over S, and q_l represents the maximum quota of lecturer l. For each $p \in P$, q_p represents the maximum quota of project p.

C.4.2 Feasibility

We say $X' \subseteq X$ is student-feasible if X'_s is acceptable for each $s \in S$. We say $X' \subseteq X$ is lecturer-feasible if $|X'_l| \leq q_l$ holds for all $l \in L$, and $|X'_p| \leq q_p$ holds for all $p \in P$. Then, we say X' is feasible if it is student- and lecturer-feasible.

C.4.3 Stability

For a matching X', a contract $(s, p) \in X \setminus X'$, where $p \in P_l$, is an **Abraham-Irving-Manlove (AIM) blocking pair** of X' if (i) (s, p) is acceptable for s, (ii) $(s, p) \succ_s X'_s$, and (iii) one of the following conditions holds:

(a) $|X'_p| < q_p$ and $|X'_l| < q_l$.

(b) $|X'_p| < q_p$ and $|X'_l| = q_l$, and either $X'_s = \{(s, p')\}$ and $p' \in P_l$, or there exists $(s', p'') \in X'$, such that $p'' \in P_l$ and $s \succ_l s'$.

(c) $|X'_p| = q_p$ and there exists $(s', p) \in X'$, such that $s \succ_l s'$.

We say a matching X' is **Abraham-Irving-Manlove (AIM)-stable** if it has no AIM-blocking pair.

C.4.4 Mechanism

Abraham, Irving, and Manlove (2007) present two mechanisms based on the DA mechanism. One is called SPA-student, in which students make offers, and the other is called SPA-lecturer, in which lecturers make offers. Both mechanisms produce AIM-stable matchings. Although Abraham, Irving, and Manlove (2007) do not examine strategyproofness, the SPA-student is strategyproof for students. The SPA-student is defined as follows.

Mechanism 7 (SPA-student)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each student s applies to her most preferred project by which she has not been rejected before Stage k. If no remaining project is acceptable for s, s does not apply to any project. Reset X' as $\emptyset$.

Step 2 For each lecturer l, iterate the following procedure until all students applying to projects in P_l are either tentatively accepted or rejected:

1. Choose s, where s is applying to some project $p \in P_l$, s has not been tentatively accepted or rejected yet in any previous step of this stage, and s has the highest priority according to $\succ_l$ among the students currently applying to some project offered by lecturer l.
2. If $|X'_p| < q_p$ and $|X'_l| < q_l$, s is tentatively accepted by p and (s, p) is added to X'. Then go to the procedure for the next student.
3. Otherwise, s is rejected by p. Then go to the procedure for the next student.

Step 3 If all the students are tentatively accepted in Step 2, then let X' be a final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

C.4.5 Representation in our model

Let us define $\widehat{f}(X')$ as 0 if X' is lecturer-feasible, i.e., $|X'_p| \leq q_p$ for all $p \in P$ and $|X'_l| \leq q_l$ for all $l \in L$, and otherwise, $-\infty$. Then, $(X, \operatorname{dom} \widehat{f})$ is a laminar matroid, since $\mathcal{T} = \{X_{l_1}, X_{l_2}, \ldots, X_{p_1}, X_{p_2}, \ldots\}$ is a laminar family of X.

Let us assume there exists an ordering $p_1, p_2, \ldots$ among projects within P_l. Then, let us define a positive value $v(x)$ for each $x \in X$ with the following properties: for every $p, p' \in P_l$, $v((s, p)) > v((s', p'))$ if $s \succ_l s'$, and $v((s, p)) > v((s, p'))$ if p appears earlier than p' in the above ordering over P_l.

Let us assume $\widetilde{f}(X')$ is given as $\sum_{x \in X'} v(x)$. Then, we can apply Condition 1. SPA-student is identical to the generalized DA mechanism where the choice function of the lecturers Ch_L is defined as the maximizer of function f.

The following proposition establishes a connection between HM-stability and AIM-stability.

Proposition 11. *AIM-stability implies HM-stability.*

Proof. If X' is AIM-stable, then the first condition for HM-stability, namely $X' = Ch_L(X') = Ch_S(X')$, is obvious. We show that if there exists $(s, p) \in X \setminus X'$ such that $(s, p) \in Ch_L(X' + (s, p))$ and $(s, p) \in Ch_S(X' + (s, p))$ hold, then (s, p) is an AIM-blocking pair.

By way of contradiction, let us assume (s, p) is not an AIM-blocking pair. From the fact that $(s, p) \in Ch_S(X' + (s, p))$, (s, p) is acceptable for s. Also, either $(s, p) \succ_s (s, p')$ holds where $(s, p') \in X'$, or $X'_s = \emptyset$.

Assume $p \in P_l$. Since (s, p) is not an AIM-blocking pair, either $|X'_p| = q_p$ or $|X'_l| = q_l$ holds. Since $(s, p) \in Ch_L(X' + (s, p))$, there exists $(s', p') \in X'$ such that $(s', p') \notin Ch_L(X' + (s, p))$ and $p' \in P_l$ hold (otherwise, $|X'_p| < q_p$ and $|X'_l| < q_l$ hold). Since $(s', p') \notin Ch_L(X' + (s, p))$ and $(s, p) \in Ch_L(X' + (s, p))$, $s \succ_l s'$ or $s = s'$ hold. In either case, (s, p) becomes an AIM-blocking pair. This is a contradiction. □

We note that there exist cases in which a matching X' is HM-stable but not AIM-stable. To see this, assume there exist one student s and two projects p_1 and p_2, and both projects are provided by the same lecturer l. The order on P_l is p_2, p_1. s prefers p_1 to p_2. In this example, $X' = \{(s, p_2)\}$ is stable, since $Ch_L(X' + (s, p_1)) = X'$, but it is not AIM-stable since (s, p_1) is an AIM-blocking pair under the definition of AIM-stability. However, the following proposition holds.

Proposition 12. *The generalized DA mechanism obtains the doctor-optimal AIM-stable matching.*

Proof. Since the generalized DA mechanism obtains the student-optimal HM-stable matching and AIM-stability implies HM-stability, it suffices to show that the student-optimal HM-stable matching satisfies AIM-stability. To show the latter, let us assume by way of contradiction that X' is the student-optimal HM-stable matching but it is not AIM-stable, i.e., there is an AIM-blocking pair $(s, p) \in X \setminus X'$. Let $p \in P_l$. Since X' is HM-stable, cases (a) and (c) in the definition of an AIM-blocking pair in Section C.4.3 are not possible. Thus the only possibility is that $|X'_p| < q_p$, $|X'_l| = q_l$, $X'_s = \{(s, p')\}$, and $p' \in P_l$ hold, i.e., s is assigned to project p', although s prefers another project p, while p' and p are held by the same lecturer l and p is not full. Let $X'' = X' - (s, p') + (s, p)$. It is clear that X'' is HM-stable, s prefers X'' over X', and other students are indifferent between X'' and X'. This contradicts the assumption that X' is the student-optimal HM-stable matching. □

In the above example, the generalized DA mechanism returns $\{(s, p_1)\}$, which is AIM-stable.

C.5 Cadet-branch matching (Sönmez and Switzer, 2013)

C.5.1 Model

A market is a tuple $(I, B, T, X, (\succ_i)_{i \in I}, \succ_B, (q_b)_{b \in B}, (p_b)_{b \in B})$. I is a finite set of cadets and B is a finite set of branches of the military. $T = \{t_0, t_+\}$ is a pair of terms, where t_0 means that a cadet serves for a standard term, and t_+ means that a cadet serves for an extended term, which is longer than the standard term. $X := I \times B \times T$ is the set of contracts. A contract $x = (i, b, t)$ means i is matched with b with term t. For $X' \subseteq X$, let X'_i denote $\{(i, b, t) \in X' \mid t \in T, b \in B\}$ and X'_b denote $\{(i, b, t) \in X' \mid i \in I, t \in T\}$.

For each $i \in I$, $\succ_i$ represents the preference of cadet i over acceptable contracts in X_i. $\succ_B$ is the priority ordering (master-list) over the cadets, which is common to all branches. For each $b \in B$, q_b represents the maximum quota of branch b, and $p_b < q_b$ represents the reserved quota for extended-term contracts at branch b.

C.5.2 Feasibility

We say $X' \subseteq X$ is cadet-feasible if X'_i is acceptable for all i. We say $X' \subseteq X$ is branch-feasible if $|X'_b| \leq q_b$ holds for all $b \in B$. Then, we say X' is feasible if it is cadet- and branch-feasible.

C.5.3 Stability

We say a matching X' is **fair** if for each pair of contracts $(i, b, t), (i', b', t') \in X'$, $(i, b', t') \succ_i (i, b, t)$ implies $i' \succ_B i$ (Sönmez and Switzer, 2013). In other words, fairness requires that a higher-priority cadet never envies the assignment of a lower-priority cadet.

C.5.4 Mechanism

Sönmez and Switzer (2013) present a mechanism called the Cadet-Optimal Stable Mechanism (COSM), which is defined as follows.[39] The COSM always produces a fair matching.

Mechanism 8 (COSM)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each cadet i chooses her most preferred contract (i, b, t) which has not been rejected before Stage k and applies to b with term t. If no remaining contract is acceptable for i, i does not apply to any branch. Reset X' as $\emptyset$.

Step 2 For each b, iterate the following procedure until all cadets applying to b are either tentatively accepted or rejected:

Phase 1: Choose cadet i such that i has the highest priority according to $\succ_B$ among the applicants to b, who are applying to b with term t (t can be either t_0 or t_+) and are not tentatively accepted yet. If there exists no such cadet, then go to the procedure for the next branch. If $|X'_b| < q_b - p_b$, tentatively accept i to b and add (i, b, t) to X', and go to the procedure for the next cadet. Otherwise, go to the procedure for the next phase.

Phase 2: Choose cadet i such that i has the highest priority according to $\succ_B$ among the applicants to b, who are applying to b with the extended term t_+ and are not tentatively accepted yet. If there exists no such cadet, then go to the next phase. If $|X'_b| < q_b$, tentatively accept i to b, add (i, b, t_+) to X', and go to the

[39] To be more precise, the COSM presented in Sönmez and Switzer (2013) is slightly different from the one presented here. The difference would matter when branch b handles two contracts offered by the same cadet i, i.e., (i, b, t_0) and (i, b, t_+). However, this difference does not matter since b handles cadet-feasible contracts only. Allowing multiple contracts between the same pair of agents, as is done in the present paper, enables the preference to satisfy substitutability. This technique has been used in Kamada and Kojima (2015, 2017a) and Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) in the context of matching with constraints. See also Hatfield and Kominers (2009, 2014) who study this issue further.

procedure for the next cadet. Otherwise, reject all cadets applying to b who are not tentatively accepted yet. Go to the procedure for the next branch.

Phase 3: Choose cadet i such that i has the highest priority according to $\succ_B$ among the applicants to b, who are applying to b with the standard term t_0 and are not tentatively accepted yet. If there exists no such cadet, then go to the procedure for the next branch. If $|X'_b| < q_b$, tentatively accept i to b, add (i, b, t_0) to X', and go to the procedure for the next cadet. Otherwise, reject all cadets applying to b who are not tentatively accepted yet. Go to the procedure for the next branch.

Step 3 If all the cadets are tentatively accepted in Step 2, then let X' be a final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

C.5.5 Representation in our model

Let us consider an extended market $(I, B, T, \tilde{X}, \tilde{\succ}_I, f)$. For each contract (i, b, t_+) in X, we create two contracts $(i, b, t_+, 0)$ and $(i, b, t_+, 1)$ in the extended market. Here, $(i, b, t_+, 0)$ and $(i, b, t_+, 1)$ are interpreted to mean i is accepted to b for its priority seat and normal seat, respectively. For each contract (i, b, t_0), we create a single contract $(i, b, t_0, 1)$ in the extended market.

We obtain the modified preference of each cadet i, denoted $\tilde{\succ}_i$, such that $(i, b, t, s)\tilde{\succ}_i(i, b', t', s')$ holds for any b, b', s, s', t, and t', if $(i, b, t) \succ_i (i, b', t')$, and $(i, b, t_+, 1)\tilde{\succ}_i(i, b, t_+, 0)$ holds for any b (as long as (i, b, t_+) is acceptable for i).

From the matching in the extended market $\tilde{X}'$, the matching in the original market X' is obtained by mapping each contract (i, b, t, s) to (i, b, t). For $\tilde{X}' \subseteq \tilde{X}$, let $\tilde{X}'_{b,s}$ denote $\{(i, b, t, s) \in \tilde{X}' \mid i \in I, t \in T\}$.

For each $x \in \tilde{X}$, we define its value $v(x)$. We assume $v(\cdot)$ respects $\succ_B$ in the sense that if $i \succ_B i'$, $v((i, b, t, s)) > v((i', b, t, s))$ holds for all b, t, and s. Also assume $v((i, b, t_+, 0)) > v((i', b', t, 1))$ for all i, i', b, b', and t, i.e., a contract for a priority seat has a larger value than any contract for a normal seat.

Let us define $\widehat{f}(\tilde{X}')$ as 0 when $|\tilde{X}'_b| \leq q_b$ and $|\tilde{X}'_{b,0}| \leq p_b$ hold for all $b \in B$, and $-\infty$ otherwise. Also, let us define $\widetilde{f}(\tilde{X}')$ as $\sum_{x \in \tilde{X}'} v(x)$. Then,

$(\tilde{X}, \operatorname{dom} \widehat{f})$ is a laminar matroid, since $\mathcal{T} = \{\tilde{X}_{b,s} \mid b \in B, s \in \{0,1\}\} \cup \{\tilde{X}_b \mid b \in B\}$ is a laminar family of $\tilde{X}$. Thus, we can apply Condition 1. COSM is identical to the generalized DA mechanism where the choice function of branches Ch_B is defined as the maximizer of function f.

The following proposition holds.

Proposition 13. *HM-stability of $\tilde{X}'$ in the extended market implies fairness of X' in the original market.*

Proof. Assume X' is not fair, i.e., there exist $(i,b,t),(i',b',t') \in X'$ such that $(i,b',t') \succ_i (i,b,t)$ and $i \succ_B i'$ hold. Consider the case where $t' = t_+$ and $(i',b',t_+,s) \in \tilde{X}'$. Then, if we choose $x = (i,b',t_+,s)$, it is clear that $x \in Ch_I(\tilde{X}'+x)$ and $x \in Ch_B(\tilde{X}'+x)$ because $v(i,b',t_+,s) > v(i',b',t_+,s)$. Consider the case where $t' = t_0$ and $(i',b',t_0,1) \in \tilde{X}'$. Then, if we choose $x = (i,b',t_0,1)$, it is clear that $x \in Ch_I(\tilde{X}'+x)$ and $x \in Ch_B(\tilde{X}'+x)$ because $v(i,b',t_0,1) > v(i',b',t_0,1)$. □

On the other hand, HM-stability is not implied by fairness. To see this, let a cadet i hope to be assigned branch b with term t_+ but she is not accepted by b in X', while $|X'_{b,0}| < p_b$. Then X' is not HM-stable, even if each cadet assigned to b has a higher priority than i (thus X' is fair).

Fairness as defined in Section C.5.3 is a mild requirement, and the cadet-optimal fair matching does not always exist. Thus, neither the generalized DA mechanism nor the COSM always produce the cadet-optimal fair matching.

C.6 Regional maximum quotas with regionally prioritized doctors

In this subsection, instead of examining an existing application, we consider a new application. Specifically, we study a situation where several different constraints must be satisfied simultaneously.

Assume hospitals are partitioned into regions as in Section 5.1, each with a regional maximum quota. In addition, each region is associated with "regionally prioritized doctors" who are granted priority for placement in that region over other doctors. More specifically, each hospital in the region is required to accept the regionally prioritized doctors up to a certain number even if the hospital prefers other doctors.

A case in point is Japanese medical residency matching. As described earlier, the Japanese government imposes a regional maximum quota on each region of the country for medical residency matching. Furthermore, there is a discussion on giving priorities to residents who received regional scholarships. More specifically, some regions have scholarship programs to medical students. If a student who received the scholarship from a region becomes a medical resident in that region, she does not need to pay back the scholarship. However, if she fails be a medical resident in that region (even if she wanted to), she needs to pay back the scholarship. Given this nature of the scholarship, there is a discussion among policymakers to give some priority to such students for positions in that region.

Another possible example is centralized admission to public universities. Assume each region has regional maximum quotas due to its budget limitation, for instance. At the same time, the region gives some priority to students who live in the region.

In this subsection, we show how to model such situations in our framework. A market is a tuple

$$(D, H, X, R, (\succ_d)_{d \in D}, \succ_H, (p_h)_{h \in H}, (q_h)_{h \in H}, (q_r)_{r \in R}, \tau).$$

The definitions of $D, H, X, R, \succ_d, q_h$, and q_r are identical to the model presented in Section 5.1. $\tau : D \to R \cup \{\emptyset\}$ returns the region where doctor d is regionally prioritized. If $\tau(d) = \emptyset$, it means that no region regionally prefers d.

For $X' \subseteq X$, let $X'_{r,h}$ denote $\{(d, h) \mid (d, h) \in X', h \in r, r = \tau(d)\}$, i.e., the subset of contracts in X' involving doctors prioritized in region r and $h \in r$. Each hospital $h \in r$ has a minimum quota p_h for regionally prioritized doctors of region r. As in Section C.3, these minimum quotas are soft bounds and do not affect feasibility. We assume $\sum_{h \in r} p_h \le q_r$ holds for all $r \in R$, i.e., the regional maximum quota of r is large enough to satisfy the minimum quotas of regionally prioritized doctors for all $h \in r$. As in the case of contract-order stability, we assume there exists a total order $\succ_H$ over X.

We say X' is doctor-feasible if X'_d is acceptable for all d. We say X' is hospital-feasible if $|X'_h| \le q_h$ and $|X'_r| \le q_r$ hold for all $h \in H$ and $r \in R$. Then, we say X' is feasible if it is doctor- and hospital-feasible.

For a matching X', we say $(d, h) \in X \setminus X'$ (where $h \in r$) is a blocking pair, if (d, h) is acceptable for d and $(d, h) \succ_d X'_d$, and any one of the following conditions holds:

(i) $\tau(d) = r$ and $|X'_{r,h}| < p_h$,

(ii) $|X'_h| < q_h$ and $|X'_r| < q_r$,

(iii) $|X'_h| < q_h$, $|X'_r| = q_r$, there exists $h' \in r$ such that $(d, h') \in X'$, $(d, h) \succ_H (d, h')$, and either (iii-a) $\tau(d) = r$ and $|X'_{r,h'}| > p_{h'}$, or (iii-b) $\tau(d) \neq r$ holds, or

(iv) there exists $(d', h) \in X'$, such that $(d, h) \succ_H (d', h)$, and any one of the following conditions holds:

  (iv-a) $\tau(d') \neq r$,

  (iv-b) $\tau(d) = \tau(d') = r$, or

  (iv-c) $\tau(d) \neq r, \tau(d') = r$, and $|X'_{r,h}| > p_h$.

We say a matching X' is stable if there exists no blocking pair according to the above definition.

In this definition, conditions (i)-(iv) list the cases in which the formation of a matching between doctor d and hospital h is regarded as a "legitimate" blocking in the presence of constraints. More specifically, (i) is a case in which d is a regionally prioritized doctor in r and there is an available seat reserved for regionally prioritized doctors in h; (ii) is a case in which h has an empty seat and its region r has room for another doctor; (iii) is a case in which the region r is full, but d is currently matched with another hospital h' in r, H prefers (d, h) over (d, h'), and moving d away from h' will not cause the minimum quota for regionally prioritized doctors at h' to be violated; (iv) is a case in which h is currently matched with a less preferred doctor d', and moving d' away from h will not cause the minimum quota for regionally prioritized doctors at h to be violated.

To represent this problem using our framework, let us consider an extended market given by tuple

$$(D, H, \tilde{X}, R, (\tilde{\succ}_d)_D, \tilde{\succ}_H, (p_h)_{h \in H}, (q_h)_{h \in H}, (q_r)_{r \in R}, \tau).$$

Each contract $x \in \tilde{X}$ is represented as a triple (d, h, t) where $d \in D$, $h \in H$, and $t \in \{0, 1\}$. We assume that contracts of the form $(d, h, 0)$ are available only to a regionally prioritized doctor such that $h \in \tau(d)$. The triple $(d, h, 0)$ is interpreted as a contract in which $d \in D$ is assigned to a priority seat of hospital h, and $(d, h, 1)$ is interpreted as a contract in which d is assigned to a normal seat of hospital h. For $d \in D$, $\tilde{\succ}_d$ is obtained from $\succ_d$ such that for any $h, h' \in H$, $(d, h, 1)\tilde{\succ}_d(d, h', 1)$ if and only if $(d, h) \succ_d (d, h')$. Also, for each $h \in \tau(d)$, $(d, h, 0)\tilde{\succ}_d(d, h, 1)$ holds. Furthermore, for any $d \in D$ and any $h, h' \in \tau(d)$, $(d, h, 0)\tilde{\succ}_d(d, h', 0)$ if and only if $(d, h) \succ_d (d, h')$. $\tilde{\succ}_H$ is obtained from $\succ_H$ such that for any h, h', d, d', and t, $(d, h, t)\tilde{\succ}_H(d', h', t)$ if and only if $(d, h) \succ_H (d', h')$ holds, and $(d, h, 0)\tilde{\succ}_H(d', h', 1)$ holds for any h, h', d and d', where $h \in \tau(d)$. Let $v(x)$ denote the value of contract x which respects $\tilde{\succ}_H$, i.e., $v(x) > v(x')$ holds if and only if $x\tilde{\succ}_H x'$. Let $\tilde{X}'_{h,t}$ denote $\{(d, h, t) \in X' \mid d \in D\}$.

Let us define $\widehat{f}(\tilde{X}')$ as 0 when $|\tilde{X}'_h| \leq q_h$, $|\tilde{X}'_r| \leq q_r$, and $|\tilde{X}'_{h,0}| \leq p_h$ hold for all $h \in H$ and $r \in R$, and $-\infty$ otherwise. Also, let us define $\widetilde{f}(\tilde{X}')$ as $\sum_{x \in \tilde{X}'} v(x)$.

Then, $(\tilde{X}, \mathrm{dom}\,\widehat{f})$ is a laminar matroid since $\mathcal{T} = \{\tilde{X}_{h,t} \mid h \in H, t \in \{0, 1\}\} \cup \{\tilde{X}_h \mid h \in H\} \cup \{\tilde{X}_r \mid r \in R\}$ is a laminar family of $\tilde{X}$. Thus, we can apply Condition 1.

From the matching in the extended market $\tilde{X}'$, the matching in the original market X' is obtained by mapping each contract (d, h, t) to (d, h). The following proposition holds.

Proposition 14. *HM-stability of $\tilde{X}'$ in the extended market implies that there exists no blocking pair in the original market.*

Proof. Assume (d, h) is a blocking pair and condition (i) holds. Then, by choosing $x = (d, h, 0)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold. If condition (ii) holds, by choosing $x = (d, h, 1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold since $\tilde{X}' + x$ is feasible.

Assume condition (iii) holds. If (iii-a) holds, then there exists at least one regionally prioritized doctor d' in r, who is assigned to a normal seat of h', i.e., $(d', h', 1)$ is included in $\tilde{X}'$, and $(d, h, 1)\tilde{\succ}_H(d', h', 1)$ hold. By choosing $x = (d, h, 1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold since $(d, h, 1)\tilde{\succ}_H(d', h', 1)$ holds. If (iii-b) holds, $(d, h', 1)$ is included in $\tilde{X}'$. By choosing $x = (d, h, 1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold since $(d, h, 1)\tilde{\succ}_H(d, h', 1)$ holds.

Next, assume condition (iv) holds, i.e., there exists $(d', h) \in X'$ such that $(d, h) \succ_H (d', h)$ holds. If condition (iv-a) holds, then $(d', h, 1) \in \tilde{X}'$ and $(d, h, 1)\tilde{\succ}_H(d', h, 1)$ hold. Thus, if we choose $x = (d, h, 1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold. If condition (iv-b) holds, then $(d', h, t) \in \tilde{X}'$ (where t can be either 0 or 1) and $(d, h, t)\tilde{\succ}_H(d', h, t)$ hold. Thus, if we choose

$x = (d, h, t)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold. If condition (iv-c) holds, then there exists at least one regionally prioritized doctor d'' in r (d'' can be equal to d'), who is assigned to a normal seat of h in $\tilde{X}'$ and $(d, h, 1) \tilde{\succ}_H (d'', h, 1)$ hold. Thus, if we choose $x = (d, h, 1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold.

Thus, if there exists a blocking pair, $\tilde{X}'$ is not HM-stable. □

D Discussions

D.1 Relations between applications

The SPA problem in Section C.4 can be represented using the regional maximum quota model in Section 5.1, by letting projects provided by the same lecturer P_l form a region and its regional maximum quota be set at q_l. In the SPA problem, individual projects in P_l do not have their own preferences over students; one can interpret that all projects in P_l use a common preference $\succ_l$. As a result, AIM-stability implies strong stability defined in Section 5.1. Thus, AIM-stability is stronger than KK-stability or contract-order-stability.[40]

The model presented in Section 4 of Biro, Fleiner, Irving, and Manlove (2010) can be regarded as a generalization of the SPA problem, where the constraints have a laminar structure and maximum quotas are imposed at each element of the laminar family. Biro, Fleiner, Irving, and Manlove (2010) show that a stable matching always exists and a modification of the standard DA mechanism obtains a stable matching. Clearly, our analysis in Section C.4 can be generalized to this environment.

D.2 Aggregation of individual hospital preferences

Recall the model of matching with regional maximum quotas in Section 5.1. As illustrated there, there may be several alternative methods for aggregating the preferences of individual hospitals into a single preference of the hospitals. One method is to introduce an order over hospitals to determine a preference

[40] AIM-stability does not coincide with strong stability or KK-stability or contract-order stability in general. Thus our analysis of the SPA problem is not subsumed by our analysis of regional maximum quotas.

over the numbers of accepted contracts at each hospital (as used in KK-stability). Then, which contracts should be accepted at each hospital is determined by the individual preference of the hospital. Another method is to generate an ordering among contracts that respects the preferences of individual hospitals (as in contract-order stability). As seen in Section 5.1, both types of aggregated hospital preferences can be represented by M$^\natural$-concave functions.

Of course, what preference aggregation employ depends on what stability concept one adopts as the solution concept. This amounts to deciding a criterion for socially desirable outcomes. Recommending one criterion over another is not the goal of the present paper, because the decision would involve a value judgment by the members of the society, and it is likely to depend on specific applied contexts. Our contribution is to provide a tool for achieving a desirable outcome *given societal preferences*, and we aimed to accommodate as wide a range of constraints and societal preferences as possible.

In this regard, one advantage of our methodology is that it is general and flexible enough to subsume a wide class of aggregated preferences including both those corresponding to KK-stability and contract-order stability. Recall the function $\widetilde{f}$ representing the hospitals' soft preferences as in KK-stability,

$$\widetilde{f}(X') = \sum_{h \in H} V_h(|X'_h|) + \sum_{x \in X'} v(x), \tag{7}$$

where $V_h(k) = \sum_{j=1}^{k} v_h(j)$ and $v_{h_i}(j) = C(C - |H| \cdot j - i)$ with a constant $C > 0$. Contrary to the case in KK-stability, however, let us relax the assumption $C \gg v(x)$, and allow for arbitrary relations between $v_{h_i}(j)$ and $v(x)$. Even under this relaxation, by Condition 2, it follows that function $f = \widehat{f} + \widetilde{f}$ as defined in Section 5.1.5 is M$^\natural$-concave, and thus all our results hold with respect to this function f, including the existence of an HM-stable matching and strategyproofness for doctors of the generalized DA mechanism.

As mentioned above, the function $\widetilde{f}$ in equation (7) generalizes the function corresponding to KK-stability. Moreover, contract-order stability corresponds to HM-stability with respect to equation (7) for the case in which $v(x) - v(x') \gg C$ for all $x \neq x'$: with this assumption, $\widetilde{f}$ primarily values accepting a contract with a high value. Thus, this paper's methodology enables us to generalize and unify Kamada and Kojima (2015) and Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) in a straightforward manner.

The additional generality of our methodology also enables us to accommodate other possible societal preferences. For instance, consider a situation in which the hospitals mostly try to equalize the numbers of assigned doctors across hospitals as in KK-stability, but some special matchings are given priority. Suppose, for example, a particular hospital h has urgent needs for pediatricians, so matching pediatricians to h takes priority. Such a case can be accommodated by equation (7) if $v(x)$ is very small for most contracts,[41] but $v(x)$ is sufficiently large for any x which represents a matching of a pediatrician to the hospital h.

Another example is a situation in which there is a target capacity for each hospital that needs to be achieved first, but beyond the target capacities, applications are accepted according to a common preference ordering $\succ_H$. This criterion can be expressed by equation (7) by setting $v_h(j) \gg v(x)$ for any h, x, and j as long as j is at most the target capacity for h, while $v_h(j) \ll v(x)$ for any j that is strictly larger than the target capacity.

Both of these cases can be expressed by equation (7), and the function $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave by the preceding argument. Therefore the generalized DA mechanism is strategyproof for doctors and produces the doctor-optimal HM-stable matching. Beyond these specific examples, the major advantage of our methodology is to provide the policy maker with a general and flexible method to set a policy goal and immediately verify if such a goal is achievable and, if achievable, provide an off-the-shelf mechanism that produces a desired outcome.

D.3 Algorithms for finding a maximal matroid

As mentioned in the main text, in some extreme cases, Algorithm 1 may produce undesirable matroids. To see this point, assume there exists only one element $B \in \mathcal{F}$ such that $|B| = \max\{|Y| \mid Y \in \mathcal{F}\}$, i.e., $\mathcal{B} = \{B\}$. Then, $\widehat{\mathcal{B}}$ obtained by Algorithm 1 is $\{B\}$. Thus, the family of hospital-feasible contracts contains only B and its subsets. In summary, in Algorithm 1, the number of bases can be small (while the size of a base can be large).

To address this concern, let us introduce an alternative algorithm to find a maximal family. It starts from a family that contains the smallest element, i.e., an empty set, and gradually expands it by adding larger elements. For a family of sets of contracts $\mathcal{F}$ and a nonnegative integer k, let $\mathcal{F}^k$ denote

[41] More formally, $v_{h_i}(j) = C(C - |H| \cdot j - i), C \gg v(x)$ as in the case of KK-stability.

$\{X' \in \mathcal{F} \mid |X'| = k\}$ (we define $\widehat{\mathcal{F}}^k$ in a similar way). Given $\mathcal{F}$, the following algorithm returns $\widehat{\mathcal{F}} \subseteq \mathcal{F}$, which forms a matroid.

Algorithm 2.

Step 1: Set $\widehat{\mathcal{F}}$ to $\{\emptyset\}$, k to 1, and $\mathcal{Y}$ to $\mathcal{F}^1$.

Step 2: If $\mathcal{Y} = \emptyset$, return $\widehat{\mathcal{F}}$.

Step 3: If there exists $Y \in \mathcal{Y}$ which satisfies either (a) there exists $Y' \subsetneq Y$ such that $|Y'| = k - 1$ and $Y' \notin \widehat{\mathcal{F}}^{k-1}$, or (b) there exists $Z \in \widehat{\mathcal{F}}^{k-1}$, such that for all $x \in Y \setminus Z$, $x + Z \notin \mathcal{Y}$, then remove Y from $\mathcal{Y}$. Go to **Step 2**.

Step 4: Otherwise, set $\widehat{\mathcal{F}}$ to $\widehat{\mathcal{F}} \cup \mathcal{Y}$, $\mathcal{Y}$ to $\mathcal{F}^{k+1}$, and k to $k+1$. Go to **Step 2**.

In this algorithm, initially, $\widehat{\mathcal{F}}$ contains only $\emptyset$. Then, it is gradually expanded by adding larger sets of contracts. When $k = 1$, for each $Y \in \mathcal{Y}$ (where $|Y| = 1$), in Step 3 of Algorithm 2, no Y is removed. Thus, $\widehat{\mathcal{F}}$ includes all originally hospital-feasible contracts that contain exactly one contract.

The following proposition holds.

Proposition 15. *The family of subsets $\widehat{\mathcal{F}}$ obtained by Algorithm 2 is a matroid and maximal.*

Proof. It is clear that $\widehat{\mathcal{F}}$ is a matroid, by the way of adding new sets of contracts in Steps 3-4. To show it is maximal, assume for a contradiction that there exists $\mathcal{F}' \supsetneq \widehat{\mathcal{F}}$, such that $\mathcal{F}' \subsetneq \mathcal{F}$ and $\mathcal{F}'$ forms a matroid. Let X' be an element of $\mathcal{F}' \setminus \widehat{\mathcal{F}}$, where $|X'|$ is smallest. Then, X' must be removed in Step 3 of Algorithm 2. However, this contradicts with the assumption that $\mathcal{F}'$ is a matroid. □

Although this algorithm may perform better than Algorithm 1 in some cases, it does not always performs well. To see this, assume $\{x\} \in \mathcal{F}$ and for each $X' \in \mathcal{F}$ such that $X' \neq \{x\}$, $x \notin X'$ holds, i.e., x is included only in $\{x\}$. Then, in Algorithm 2, nothing can be added for $k \geq 2$. Thus, For each $X' \in \widehat{\mathcal{F}}$, $|X'| \leq 1$ holds, i.e., at most one contract can be accepted. In summary, in Algorithm 2, the size of a base can be small (while the number of bases can be large).

Neither of our algorithms is guaranteed to "work well" in all situations. We think this is somewhat unavoidable. Among other things, our model is ordinal and does not have any intuitive measure of how "good" a given constraint is, except the one based on set inclusion–and in fact, both of our algorithms perform "best" in this sense.

That said, to be useful in practice, it would be important for us to offer a variety of algorithms that could perform reasonably well in all cases. For this purpose, we consider combining Algorithms 1 and 2 to obtain an intermediate result. First, we choose $k' \leq k$. Next, we apply Algorithm 1 for $\mathcal{B} = \{Y \mid Y \in \mathcal{F}, |Y| = k'\}$ and obtain $\widehat{\mathcal{B}}$. Then, we apply Algorithm 2, while we set $\widehat{\mathcal{F}} = \{X' \mid X' \subseteq B \in \widehat{\mathcal{B}}\}$ and k to $k' + 1$ in **Step 1**. It is obvious that we obtain a maximal matroid. By setting an appropriate k', both of the size of a base and the number of bases can be sufficiently large.

Admittedly, one could find an extreme example in which even these hybrid mechanisms fail to perform well. However, even in such cases, since a vast literature on matroid theory has identified a variety of constraints that form a matroid (e.g., a laminar matroid in Definition 10), we expect that finding a good transformation would not be too difficult, as we illustrated for the case of non-laminar regions.