# Forbidden zones for the expectation. New mathematical results for behavioral and social sciences

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10 May 2018

**Forbidden zones for the expectation.**
**New mathematical results for behavioral and social sciences**

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A forbidden zones theorem, mathematical approach and model are proposed in the present article. In particular, the approach supposes that people decide as if there were some biases of the expectations of measurement data. The article is motivated by the need of a theoretical support for the practical analysis performed for the purposes of utility and prospect theories, behavioral economics, psychology, decision and social sciences. Possible general consequences and applications of the theorem and approach for a noise and biases of measurement data are preliminary considered as well.

MSC codes: 60A86; 62C86
JEL codes: C1; C02; D8; D81
Keywords: probability; variance; noise; bias; measurement; utility theory; prospect theory; behavioral economics; psychology; decision sciences; social sciences;

Contents

## 1. Introduction

In the present article, random variables are analyzed whose supports are located in finite intervals. Some information about the variables, namely non-zero values of the minimal variances, and finite sizes of their support intervals are used in an existence theorem to establish the existence of some non-zero bounds on their expectations near the boundaries of the intervals and to estimate these bounds. The obtained bounds for the expectation can be considered as conditions for some allowed zone in the center and forbidden zones near the boundaries of the interval.

The main attention is paid to applied and practical aspects such as consequences and applications of the theorem, new mathematical approach and models. Questions are especially emphasized those concern the practical analysis of the problems of utility and prospect theories (see, e.g., Harin 2012a, 2012b, 2015) that has motivated the present article.

Section 1 is devoted to the review of the literature and sources of the article. Section 2 is the proof of the theorem. Section 3 is devoted to consequences and explanations of the theorem, to new mathematical approach and models. Section 4 is devoted to applications of the theorem and approach. In section 5 the main conditions, deductions and remaining questions are summarized and discussed. Appendix is devoted to the proofs of the lemmas for the theorem.

## 1.1. Functions, moments, utility, noise and bounds. Review of literature

Various bounds for moments and functions of random variables are considered in a number of works, see, e.g., the following citations.

Continuous random variables on infinite interval are analyzed in Moriguti (1952). The expression for lower bounds of the $n$-th probability moments of any continuous distribution is obtained under the condition of finite variance. Madansky (1959) considers moment spaces of multivariate distributions and derives upper and lower bounds on the expectation of a convex function of a vector valued random variable. Chernoff (1981) considers a normally distributed $X$ with density $\varphi(x)$, mean $0$, variance $1$ and an absolutely continuous function $g(X)$ that has finite variance. The inequality (upper bound) is obtained for the variance of $g(X)$ in terms of its derivative. Cacoullos (1982) obtains also the lower bound for the variance and extends these bounds for other distributions, including discrete ones. Bounds for the probabilities and expectations of convex functions of discrete random variables with finite support are studied in Prékopa (1990). Inequalities for the expectations of functions are studied in Prékopa (1992). These inequalities are based on information of the moments of discrete random variables. A class of lower bounds on the expectation of a convex function using the first two moments of the random variable with a bounded support is considered in Dokov and Morton (2005). Sharma et al (2009) derive bounds on the extreme deviation of a finite interval in terms of its range and standard deviation. They refine the Brunk and Samuelson inequalities. Sharma, Gupta and Kapoor (2010) derive bounds on the variance of a finite interval. Bounds on the exponential moments of $min(y,X)$ and $XI\{X<y\}$ using the first two moments of the random variable $X$ are considered in Pinelis (2011). Sharma and Bhandari (2014) obtain upper bounds on the variance of discrete unimodal distributions.

Prékopa (1990), Prékopa (1992), Dokov and Morton (2005), Sharma et al (2009), Sharma, Gupta and Kapoor (2010), Pinelis (2011) and Sharma and Bhandari (2014) consider the situations some of that are, in the purely mathematical aspects, the most similar to the situation which is analyzed here. Additionally, a discrete part of the proof in the Appendix of the present article can be considered as another variant of the proof of Bhatia and Davis (2000) used in Sharma et al (2009), and Sharma, Gupta and Kapoor (2010). The continuous and mixed parts of the Appendix can be considered as its development.

Mathematical aspects of the utility and prospect theories are considered in a number of works, see, e.g., the following citations.

The classical work of Von Neumann and Morgenstern (1944) founds the mathematical basis of the game theory and introduces the Von Neumann and Morgenstern utility. Debreu (1960) considers the concept of cardinal utility. This gives a topological characterization of families of parallel straight lines in a plane. Kramkov and Schachermayer (1999) consider the problem of maximizing the expected utility of terminal wealth in the framework of a general incomplete semimartingale model of a financial market. Becherer (2006) considers bounded solutions to backward stochastic equations driven by random measures. The solutions are applied to solve different stochastic optimization problems with exponential utility in models where the underlying filtration is noncontinuous. Aczél and Luce, (2007) consider a modified axiomatic condition on a weighting function $W$ for $W(1) = 1$. The modification yields the generalized Prelec function with $W(1) \neq 1$. Steingrimsson and Luce (2007) formulate behavioral equivalents to power and to Prelec functions, argue that either the mathematical form or the assumption $W(1) = 1$ is wrong, explore the alternate that $W(1) \neq 1$, formulate and experimentally test behavioral axioms. Biagini and Frittelli (2008) consider a stochastic financial incomplete market where the price processes are described by a vector-valued semimartingale that is possibly non-locally bounded. The embedding of the utility maximization problem in Orlicz spaces permits to formulate the problem in a unified way. Delong and Klupelberg (2008) consider an optimal investment and consumption problem for an investor who trades in a Black–Scholes financial market with stochastic coefficients driven by a non-Gaussian Ornstein–Uhlenbeck process. Horst et. al. (2014) consider the utility maximization problem with a general utility function and reduce the utility maximization problem with general utility to the study of a fully-coupled Forward-Backward Stochastic Differential Equation. Santacroce and Trivellato (2014) consider the problem of maximizing the expected utility. The optimal strategy is characterized in terms of a semimartingale forward backward system of equations. Vostrikova (2017) considers the expected utility maximization problem for exponential Lévy models and HARA utilities in the presence of illiquid assets in a portfolio. As applications, Black-Scholes models are considered with correlated Brownian motions and also Black-Scholes models with jump part represented by a Poisson process. Choulli and Ma (2017) deals with forward performances of HARA type. Precisely, for a market model in which stock price processes are modeled by a locally bounded $d$-dimensional semimartingale, the authors elaborate a complete and explicit characterization for this type of forward utilities.

Mathematical aspects of the utility and prospect theories are considered in the present article as well. In particular the works Aczél and Luce (2007) and Steingrimsson and Luce (2007) constitute one of the start points for considerations of the next subchapter.

A noise and its influence are the items of a wealth of works.

Channel capacity and noise are considered in a number of works, see, e.g., Shannon (1949), Shannon (1956), Smith (1971), Wolfowitz (1975), Ahlswede et. al. (2013), Cheraghchi (2013), Khanzadi at al (2015).

The above allowed zone is in a sense similar to the above channel capacity. The more the noise, the less the channel capacity. The more the minimal variance, the less the allowed zone.

A noise and equations are considered as well, see, e.g., Caraballo el al. (2007), Hu (2015), Xie (2016), Balan and Conus (2016), Chong (2017), Foondun et al (2017).

Some qualitative influences of a noise are analyzed as well.

For example, stabilization and synchronization by a noise is considered in a number of works, see, e.g., Arnold et. al. (1983), Scheutzow (1993), Kwiecinska (1999), Crauel et. al. (2003), Cerrai (2005), Appleby and Rodkina (2005), Barbu (2009), Hua et. al. (2009), Applebaum and Siakalli (2010), Flandoli et. al. (2017), Ma and Kang (2018).

For example, a noise as a possible cause of periodic behavior is considered in some works, see, e.g., Scheutzow (1985), Giacomin and Poquet (2015),

So the cited articles devoted to a noise and stabilization and periodic behavior and also, in a sense, the present article show that a noise can exert not only quantitative but also qualitative influence.

## 1.2. Practical needs of consideration

### 1.2.1. Problems of probable and sure outcomes

A man is a key subject of economics and other sciences. There are a number of problems concerned with the mathematical description of the behavior of a man. Examples of them are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, etc.

The essence of these problems consists in biases of preferences and choices of people (subjects) for the probable and sure outcomes in comparison with the predictions of the probability theory. The biases are maximal near the boundaries of the probability scale, that is, at high and low probabilities. These problems are well-known, basic and fundamental. They are the most important in behavioral economics in utility and prospect theories and also in decision sciences, social sciences and psychology.

The above basic problems are pointed out in a wealth of works.

For example, we see in Kahneman and Thaler (2006) p. 222:

> *"A long series of modern challenges to utility theory, starting with the paradoxes of Allais (1953) and Ellsberg (1961) and including framing effects, have demonstrated inconsistency in preferences"*

For example, we see in Kahneman and Tversky (1979) p. 265:

*"PROBLEM1: Choose between*

| | | |
|---|---|---|
| *A:* | *2,500 with probability* | *.33,* |
| | *2,400 with probability* | *.66,* |
| | *0 with probability* | *.01;* |
| *B:* | *2,400* | *with certainty.* |
| *N = 72* | *[18]* | *[82]"* |

For example, we see in Starmer and Sugden (1991) p. 974:

> *"… a choice between two lotteries R' (for "riskier") and S' (for "safer"). R' gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); S' gave £7.00 for sure."*

R' gives £10.00×0.2 + £7.00×0.75 = £7.25. S' gives £7.00×1 = £7.00. Here R' = £7.25 > S' = £7.00. The results are: 13 choices for R' and 27 choices for S'.

For example, we see in Barberis (2013) p.177 (after Gonzalez and Wu 1999) the median cash equivalents (in dollars) for the following non-mixed prospect:

Outcomes (0 or $100); Probability .90; Equivalent $63.

### 1.2.2. Problems of varied domains

Moreover, an additional and, maybe, more hard problem is the inverse behavior of the people in different domains. For instance, there are a number of warrants (at the high probabilities) of risk aversion in the domain of gains but risk seeking in the domain of losses.

For example, we see in Thaler (2016), p. 1582 (the **boldface** is my own):

> "*We observe a pattern that was frequently displayed: subjects were* ***risk averse*** *in the* ***domain of gains*** *but* ***risk seeking*** *in the* ***domain of losses***.

For example, we see in Kahneman and Tversky (1979) p. 268 Table 1:

| | | | |
|---|---|---|---|
| *"Problem 3:* | *(4,000, .80)* | < | *(3,000).* |
| | *[20]* | | *[80]* |
| *Problem 3':* | *(-4,000, .80)* | > | *(-3,000).* |
| | *[92]* | | *[8]"* |

For example, we see in Tversky and Kahneman (1992) p. 307 in Table 3 median cash equivalents (in dollars) for the following non-mixed prospects:

Outcomes (0 or \$50); Probability .90; Equivalent \$37.

Outcomes (0 or -\$50); Probability .90; Equivalent -\$39.

Outcomes (0 or \$200); Probability .90; Equivalent \$131.

Outcomes (0 or -\$200); Probability .90; Equivalent -\$155.

These and similar examples will be simplified and considered below in the next sections.

Note that subjects change their preferences and choices from aversion to seeking and vice versa not only when the domain are changed from gains to losses but from high to low probabilities as well. Such domains will be considered in future articles by means the approach and models proposed here.

The present article is motivated in large measure by the need of rigorous mathematical support for the already performed analysis of the influence of scattering and noisiness of data. The idea of the theorem considered here has explained, at least partially, the above problems (see, e.g., Harin 2012a, Harin 2012b, Harin 2015).

## 1.3. Two ways. Variance, expectation and forbidden zones

Many efforts were applied to explain the above basic problems of behavioral economics and other sciences.

One of possible ways to explain them is widely discussed, e.g., in Schoemaker and Hershey (1992), Hey and Orme (1994), Chay et al (2005), Butler and Loomes (2007). The essence of this way consists in a proper attention to uncertainty, imprecision, noise, incompleteness and other reasons that might cause dispersion, scattering and spread of data.

Another possible way to explain these problems is to consider the vicinities of the borders of the probability scale, e.g. at $p{\sim}1$. Steingrimsson and Luce (2007) and Aczél and Luce (2007) emphasized a fundamental question: whether Prelec's weighting function $W(p)$ (see Prelec, 1998) is equal to $1$ at $p=1$.

In any case, one may suppose that a synthesis of the above two ways can be of some interest. This idea of the synthesis turned out to be useful indeed. It has been successful to explain, at least partially, the underweighting of high and the overweighting of low probabilities, risk aversion, and some other problems (see, e.g., Harin 2012a, Harin 2012b and Harin 2015). There exist also works providing experimental support of this synthesis (see, e.g., Starmer and Sugden 1991, Harin 2014, Cox, Sadiraj and Schmidt 2015).

In the present article some information about the variance of a random variable that takes on values in a finite closed interval is used to estimate bounds on its expectation. It is proven that if there is a non-zero lower bound on the variance of the variable, then non-zero bounds or forbidden zones for its expectation exist near the boundaries of the interval.

The role of a noise, as a possible cause of these forbidden zones and their possible influence on results of measurements near the boundaries of intervals are preliminary considered as well.

Keeping in mind the above bounds on functions of random variables, functions of the expectation of a random variable can be also investigated.

## 2. Theorem

### 2.1. Preliminaries

The practical need of the article is a discrete random variable taking the finite number of values. This corresponds to usual finite numbers of measurements in the behavioral economics. A general case will be considered here nevertheless.

Let us consider a probability space *(Ω, Æ, P)* and a random variable $X$, such that $\Omega \rightarrow R$. Suppose that the support of $X$ is an interval $[a,b]: 0 < (b-a) < \infty$. Suppose that $X$ can have a continuous part and a discrete part and at least one of these parts is not identically equal to zero.

Let us denote the possible discrete values of $X$ as $\{x_k\}$, $k = 1,2,...,K$, where $K \geq 1$, and $a \leq x_k \leq b$, the possible continuous values of $X$ as $x \in [a,b]$. Let us denote the possible probability mass function as $p(x_k)$ and probability density function as $f(x)$.

Under the condition

$$\sum_{k=1}^{K} p(x_k) + \int_{-\infty}^{+\infty} f(x)dx = \sum_{x_k \in [a,b]} p(x_k) + \int_{a}^{b} f(x)dx = 1, \qquad (1)$$

let us consider the expectation $\mu$ of $X$, its variance $\sigma$ and their interrelationships.

### 2.2. Conditions of the variance maximality

The maximal value of the variance of a random variable of any type is intuitively equal to the variance of the discrete random variable whose probability mass function has only two non-zero values located at the boundaries of the interval. This statement is nevertheless proven for the discrete distributions in Bhatia and Davis (2000) and for the general case in lemmas in the Appendix.

Such a probability mass function can be represented by the two values: $f_X(a) = (b-\mu)/(b-a)$ and $f_X(b) = (\mu-a)/(b-a)$. The following inequality is consequently true for the variance of the considered random variable $X$

$$E[X-\mu]^2 \leq (\mu - a)(b - \mu). \qquad (2)$$

## 2.3. Existence theorem

Due to the convenience of abbreviations and consonant with the usage in previous works, the terms "bound" and "forbidden zones" will sometimes be referred to with the term "restriction," especially in mathematical expressions, using its first letter "$r$" or "$R$," for example "$r_{Expect}$" or "$r_\mu$" or "$R$."

**Theorem**. Suppose a random variable $X$ takes on values in an interval $[a, b]$, $0 < (b-a) < \infty$. If there is some non-zero minimal variance $\sigma^2_{Min} > 0 : E[X-\mu]2 \geq \sigma^2_{Min}$, then some non-zero bounds (restrictions) $r_\mu \equiv r_{Expect} \equiv r_{Restrict.Expect} > 0$ exist on its expectation $\mu \equiv E[X]$ near the boundaries of the interval $[a, b]$, that is

$$a < (a + r_\mu) \leq \mu \leq (b - r_\mu) < b. \quad (3).$$

**Proof.** It follows from (2) and the hypotheses of the theorem that

$$0 < \sigma^2_{Min} \leq E[X - \mu]^2 \leq (\mu - a)(b - \mu).$$

For the boundary $a$ this leads to the inequalities $\sigma^2_{Min} \leq (\mu - a)(b - a)$ and

$$\mu \geq a + \frac{\sigma^2_{Min}}{b - a}. \quad (4).$$

For the boundary $b$ the consideration is similar and gives the inequality

$$\mu \leq b - \frac{\sigma^2_{Min}}{b - a}. \quad (5).$$

Determining the bounds (restrictions) $r_\mu$ on the expectation $\mu$ as

$$r_\mu \equiv \frac{\sigma^2_{Min}}{b - a}, \quad (6)$$

and using (4) and (5), we obtain the generalized inequalities

$$a + r_\mu \leq \mu \leq b - r_\mu .$$

Therefore, if the inequalities $0 < (b-a) < \infty$ and $\sigma^2_{Min} > 0$ are true, then the non-zero bounds (restrictions) $r_\mu > 0$ exist, such that the inequalities (3)

$$a < (a + r_\mu) \leq \mu \leq (b - r_\mu) < b$$

are satisfied, which proves the theorem.

# 3. Consequences of the theorem. Examples

## 3.1. General consequences

### 3.1.1. Practical need. General implication. Mathematical support

The initial reason of the above theorem was to provide the mathematical support for the analysis of the practical experiments in behavioral economics.

Due to the need of financial incentives for subjects of the experiments and to the finiteness of financial possibilities of experimenter's teams, the numbers of experimental results are necessarily finite.

The theorem meets this practical need. It provides the mathematical support for the analysis of the above experiments. It proves the possibility of existence of the forbidden zones for the discrete random variables that take a limited number of values that were used in the above analysis. It determines also the conditions of their existence and their minimal width.

In addition to this particular practical value, the theorem proves that this result is true for any random variable. The examples below and earlier works (see, e.g., Harin 2012b) prove that the theorem supports the analysis in more than one domain, moreover.

### 3.1.2. Minimal variance. Data scattering. Noise

The theorem states that the factor which leads to the forbidden zones and determines their widths is the non-zero minimal variance. It is exactly the minimal variance, not the variance itself.

There can be a wealth of causes of this non-zero minimal variance. It can be caused evidently by any non-zero scattering and spread of data. The list of such causes is rather wide. It includes a noise, imprecision, errors, incompleteness, various types of uncertainty, etc. Such causes are considered in a lot of works, e.g., Schoemaker and Hershey (1992), Hey and Orme (1994), Chay et al (2005), Butler and Loomes (2007).

A noise can be one of usual sources of the non-zero minimal variance.

There are many types and subtypes of noise. A hypothetic task of determining of an exact relationship between a level of noise and a non-zero minimal variance of random variables can be a rather complicated one.

If, nevertheless, a noise leads to some non-zero minimal variance of the considered random variable, then, due to the theorem, such a noise leads evidently to the above non-zero forbidden zones. If a noise leads to some increasing of the value of this minimal variance then the value of these zones increase as well.

So the theorem can provide a new mathematical tool for description of the influence of at least some types of a noise near the boundaries of intervals.

## 3.2. Practical examples of existence

### 3.2.1. Practical example of existence. Ships and waves

Suppose the calm or mirror-like sea. Suppose a small rigid boat or any other small rigid floating body which is at rest in the mirror-like sea. Suppose that this boat or the body rests in the mirror-like sea right against (or be constantly touching) the moorage wall (which is also rigid).

As long as the sea is calm, the expectation of its side can touch the wall.

Suppose the heavy sea. Suppose a small rigid boat or any other small rigid floating body which oscillates on waves in the heavy sea. Suppose that this boat or the body oscillates on waves near the rigid moorage wall.

When the boat is oscillated by sea waves, then its side oscillates also (both up-down and left-right) and it can touch the wall only in the nearest extremity of the oscillations. Therefore, the expectation of the side cannot touch the wall (if the oscillations are non-zero). Therefore, the expectation of the side is biased from the wall.

So, one can say that, in the presence of the waves, a forbidden zone exists between the expectation of the side and the wall.

This forbidden zone biases and separates the expectation from the wall. The width of the forbidden zone is roughly about a half of the amplitude of the oscillations.

### 3.2.2. Practical examples of existence. Washing machine, drill, …

Suppose a washing machine that can vibrate when pressing bed linen. Suppose this washing machine near a rigid wall. Suppose an edgeless side of a drill or any other rigid body that can vibrate is located near a rigid surface or wall.

If the washing machine or the drill is at rest, then the expectation of its edgeless side can be located right against (be constantly touching) the wall.

If the washing machine or the drill vibrates, then the expectation of its edgeless side is biased and kept away from the rigid wall due to its vibrations.

## 3.3. General example

### 3.3.1. Rigidness

The same is true for any other rigid body near any rigid surface or wall:

If the body is at rest, then the expectation of its side can be located right against the wall (be constantly touching the wall). If the body vibrates, then the expectation of its side is biased and kept away from the wall by the vibrations.

In other words, a forbidden zone arises between the rigid wall (surface) and the expectation of the side of the rigid body, when the body vibrates. The width of the forbidden zone is roughly about a half of the amplitude of the vibrations.

The above rigid boat near rigid moorage wall, rigid washing machine near rigid wall and rigid drill near rigid surface were the examples of a rigid body that can vibrate or oscillate near a rigid boundary (a rigid surface).

What do the conditions of “rigid” body and “rigid” boundary mean?

If either the body or the boundary or the both are not rigid, then the vibrations and oscillations can be suppressed partially or even totally. Hence the forbidden zone can be suppressed also.

### 3.3.2. Vibrations suppression. Sure outcomes

Vibrations, oscillations can be suppressed by some efforts. Such efforts can be, e.g., physical in the case of the physical vibrations of the body. A vibrating rigid body can be pressed by some drawing or pressing force exerted by some means. The suppressing means and their principles of action can be of different kinds, e.g., a flexible or inextensible cord, a pressure plate, etc. The forbidden zone can be suppressed either partially or even totally, depending on the parameters of the suppression and suppression means.

This suppression can correspond to the case of sure outcomes in behavioral economics, decision and social sciences and psychology.

Let us compare probable and sure outcomes and corresponding biases.

The term "sure" presumes usually that some efforts are applied to guarantee this sure outcome in comparison with the probable ones. This leads to some qualitative difference between these probable and sure outcomes. This qualitative difference can lead to some quantitative difference between the widths of the forbidden zones and hence the biases for the expectations of data for these probable and sure outcomes.

Due to the guaranteeing efforts, the width of the forbidden zones and hence the bias for sure outcomes can be less than the width and biases for the probable outcomes. The width for the sure outcomes can even be equal to zero, which means that the cause of the forbidden zones is too weak to overcome the guaranteeing efforts.

So, sure outcomes are guaranteed by some guaranteeing efforts. Due to these efforts, minimal variance $\sigma^2_{Sure}$, the forbidden zones and the bias for the sure outcomes can be suppressed and reduced.

The nature of these guaranteeing efforts can nevertheless vary for various cases. Therefore in the case of the sure outcomes, a consideration of the minimal variance $\sigma^2_{Sure}$ and even of the forbidden zones can be more complicated than in the case of the probable outcomes.

## 4. Mathematical approach of biases of expectations

### 4.1. Preliminary considerations. Two main presuppositions

First of all, the above problems of these theories have been analyzed many times by various teams of researchers but have not been adequately solved nevertheless. For example, Kahneman and Thaler (2006) noted (see p. 222):

> *"A long series of modern challenges to utility theory, starting with the paradoxes of Allais (1953) …, have demonstrated inconsistency in preferences"*

In other words, the problem that was revealed in 1953 was not adequately solved during more than a half of century (the available literature testifies that it was not adequately solved even in 2017). In addition, the modern utility and prospect theories undoubtedly constitute a complex set of the data, rules, suppositions etc.

All the circumstances and reasons lead to the deduction that an essential and elaborated contribution to the modern utility and prospect theories needs the elaborated work of a sufficient number of research teams. So it cannot be made by a single researcher and all the more by a single theorem and single article.

Therefore the leading principle of the approach should be "stage by stage and step by step." Consequently the approach that can be based on the proposed theorem and its consequences and can be proposed in the present single article should be only a preliminary stage for subsequent changes, modifications and refinements by some research teams.

So there is no sense and possibility for this single article to build a thorough and well-composed construction of rigorous statements proven by a wealth of experimental and theoretical works. So for such a preliminary stage it is sufficient to propose only the above theorem with its consequences and a collection of some suppositions and relationships.

Secondly, due to the theorem, the non-zero minimal variance of measurement data leads to the existence of the forbidden zones for the expectation of the data near the boundaries of the intervals of the data. These forbidden zones evidently lead to the biases of the expectations, at least right against the boundaries.

The above examples of this chapter evidently illustrate such forbidden zones. Similar examples are widespread and usual in the practical real life. Due to this prevalence, the subjects can keep in mind the feasibility of such forbidden zones and the biases of the expectations caused by the zones. This can influence subjects' behavior and choices.

Due to all these considerations, the two main presuppositions can be proposed for the approach:

1. **Biases of expectations**. The subjects make their choices (at least to a considerable degree) as if there were some biases of the expectations of the outcomes.

(This presupposition can be supported by the thought that such biases may be proposed and tested even from some purely formal point of view. The mathematical approach of biases of the expectations is to explain not only the objective situations but also and mainly the subjective behavior and choices of subjects. The analysis of the literature shows that this presupposition is new)

2. **Explanation by theorem**. These biases (real biases or subjective reaction and choices of the subjects) can be explained (at least to a considerable degree) with the help of the forbidden zones of the theorem.

(The analysis of the literature shows that the forbidden zones statement of the theorem is new)

## 4.2. Denotations

I denote the expectations of the probable and sure outcomes as

$$\mu_{\mathrm{Pr}ob} \equiv \mu_{\mathrm{Pr}obable} \qquad \text{and} \qquad \mu_{Sure}\,.$$

Due to the first presupposition, the subjects make their choices as if there were some biases of the expectations of the outcomes. The real measurement data represent the set of the choices of the subjects. Using this set, one can estimate the biases of the expectations of the data for the probable and sure outcomes that are required to obtain the data corresponding to these choices. I denote them as

$$\Delta_{\mathrm{Pr}ob} \equiv \Delta_{\mathrm{Pr}obable} \equiv \Delta_{Choice.\mathrm{Pr}obable} \quad \text{and} \quad \Delta_{Sure} \equiv \Delta_{Choice.Sure}\,.$$

Let us consider some abstract mode 1 and mode 2 of outcomes. Irrespective of these numbers, one of these modes corresponds to the probable outcomes (this may be either mode 1 or mode 2) and the other – to the sure ones. The corresponding expectations are $\mu_1$ and $\mu_2$ and the biases are $\Delta_1$ and $\Delta_2$.

One can introduce also the two more designations:

a) the difference between the expectations of the compared modes

$$d_\mu \equiv \mu_2 - \mu_1\,,$$

b) the difference

$$d_{Choice} \equiv \Delta_2 - \Delta_1$$

that is required to obtain the data corresponding to the revealed choices.

The simplicity of the mathematical calculations and transformations allows to omit further the most of intermediate mathematical manipulations.

## 4.3. General mathematical relationships

Let us consider some essential features of the examined situations and, using the above denotations, develop some mathematical relationships.

**1. Necessary condition for approach**. Due to the first presupposition, the approach can evidently be useful only if some non-zero difference between the biases for the choices exists

$$\exists\, d_{Choice} : |\, d_{Choice} | > 0 \quad \text{or} \quad \operatorname{sgn} d_{Choise} \neq 0 \,. \tag{7}$$

**2. Forbidden zones as, at least, one of origins of biases**. The biases of the expectations may be introduced and considered purely formally. The question is not only whether these biases can explain the problems. Due to the second presupposition, these biases themselves should be explained by the theorem.

First of all, the theorem should be applicable. This condition is satisfied if

$$\sigma^2{}_{Min} > 0 \,.$$

Further let us denote the biases caused by the forbidden zones of the theorem by $\Delta_{Theorem}$ and the difference that can be explained by the theorem as $d_{Theorem}$. The sign of the difference for the choice should coincide with that for the theorem

$$\operatorname{sgn} d_{Choice} = \operatorname{sgn} d_{Theorem} \,.$$

Then the conditions for the explanation can be represented as $d_{Theorem} \approx d_{Choice}$, in the case when the forbidden zones of the theorem are the main source of the biases. If the forbidden zones of the theorem are one of the essential source of the biases, then the conditions for the explanation can be represented as $d_{Theorem} = O(d_{Choice})$. So the relationships of the explanation can be represented as

$$d_{Theorem} \approx d_{Choise} \quad \text{or at least} \quad d_{Theorem} = O(d_{Choise}) \,. \tag{8}$$

The examples considered below prove that the theorem predicts the right signs of the difference and there is no need to state the concerned additional supposition.

The above considerations, suppositions and formulas may be used in more general situations as well. Let us consider a particular supposition.

**4. Biases of sure outcomes**. The above considerations about the noise suppression and sure outcomes lead to the deduction that the sure outcomes are guaranteed by some guaranteeing efforts. Due to these efforts, the biases of the sure outcomes can be suppressed and reduced. They can be moreover equal to zero.

In accordance with this deduction, I suppose that the bias of the measurement data for the sure outcomes is equal to zero or, more generally, is strictly less than the bias for the probable outcomes.

The application of the condition (7) of non-zero difference between the biases for the choices enables to deduce that the absolute value of the bias for the probable outcomes should be non-zero.

This is supported by the examples considered below. They prove that the theorem predicts the true signs of the bias for the probable outcomes. So there is no need to state the additional supposition about the signs.

The relationships of sure and probable outcomes and choices can be formulated as

$$|\Delta_{\mathrm{Pr}ob}| > |\Delta_{Sure}| \quad \text{or} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\mathrm{Pr}ob} . \qquad (9)$$

## 4.4. First stage of the approach. Qualitative problems and explanations

A first stage of the approach can be a qualitative one. This means that the approach can both deal with qualitative problems and give qualitative explanations.

The preliminary statements of the first stage of the approach can be formulated as follows:

**Qualitative analysis**. Only qualitative analysis will be performed.

**Qualitative problems**. Only qualitative problems will be considered.

**Qualitative explanation**. Only qualitative explanation of the existing problems will be given. No predictions will be made in the scope of this first stage of the approach.

**Choices of subjects**. The approach will explain mainly the subjective behavior and choices of subjects.

## 5. Qualitative mathematical models

Let us consider a possible qualitative mathematical model for the analysis of the above problems in the scope of the first stage of the approach. First of all let us consider possible restrictions and questions.

### 5.1 Restrictions on models. Main question

#### 5.1.1 Theorem bound for the bias

Let us estimate the limits for the biases of the expectations with the help of the theorem.

Due to (6), the minimal value of the width of the forbidden zone (of the restriction $r_\mu$) is

$$r_\mu = \frac{\sigma^2{}_{Min}}{b-a} \quad \text{and we have} \quad \frac{\sigma_{Min}}{b-a} = \sqrt{\frac{r_\mu}{b-a}}\,.$$

Due to the evident limit

$$\frac{\sigma_{Max}}{b-a} \le \frac{1}{2} \quad \text{we have} \quad \frac{r_\mu}{b-a} \le \frac{1}{4}\,.$$

This is some rough estimate for the maximal width of the forbidden zone. More exact estimates will be given in next articles. In any case it is not more than *(b-a)/2*.

The bias of the expectation cannot be more than the width of the forbidden zone. The obtained estimate for the maximal width is therefore the estimate for the maximal bias. It should be noted that, for example, if one considers some normal distribution that is located near the boundary at the distance of three sigma from its expectation, then there is no need to use such an estimate.

Nevertheless this estimate of *0.25(b-a)* can be used as some secure upper bound for the bias. We can denote this secure upper bound as $\Delta_{Sequre}$ and write

$$\Delta_{Secure} \le \frac{b-a}{4}\,.$$

### 5.1.2. Certainty equivalents. Relative biases

Let us consider the real experimental data and normalize the values of the biases to the values of the gains/losses. These normalized values can represent the relative biases of the expectations or probabilities.

Let us consider the practical numerical examples of certainty equivalents.

For instance, we see in the above example of Barberis (2013):

The probable outcomes give $100*.9 = 90$. The median cash equivalent gives $63*1 = 63$. The expectations are

$$90 > 63,$$

but the subjects manifest the equivalent choices. To provide the equivalent choices, the difference between the biases of the expectations for the probable and sure outcomes should be equal to $\Delta_{Prob} - \Delta_{Sure} = 27$. That is the bias for the probable outcome should not be less than $\Delta_{Prob} \geq 27$.

For instance, we see in the above examples of Tversky and Kahneman (1992):

1. Gain. The probable outcomes give $50*.9 = 45$. The median cash equivalent gives $37*1 = 37$. The expectations are

$$45 > 37,$$

but the subjects manifest the equivalent choices. The bias for the probable outcome should not be less than $\Delta_{Prob} \geq 8$.

Loss. The probable outcomes give $-50*.90 = -45$. The median cash equivalent gives $-39*1 = -39$. The expectations are

$$-45 < -39,$$

but the subjects manifest the equivalent choices. The bias for the probable outcome should not be less than $\Delta_{Prob} \geq -6$.

2. Gain. The probable outcomes give $200*.90 = 180$. The median cash equivalent gives $131*1 = 131$. The expectations are

$$180 > 131,$$

but the subjects manifest the equivalent choices. The bias for the probable outcome should not be less than $\Delta_{Prob} \geq 49$.

Loss. The probable outcomes give $-200*.90 = -180$. The median cash equivalent gives $-155*1 = -155$. The expectations are

$$-180 < -155,$$

but the subjects manifest the equivalent choices. The bias for the probable outcome should not be less than $\Delta_{Prob} \geq -35$.

Let us estimate the biases of the expectations for the probable outcomes in the scope of the approach.

The values of the considered biases differ essentially from each other. Let us normalize them to the values of the gain/loss. These normalized values can represent the relative biases of the expectations or the relative biases of the probabilities. So we obtain:

Barberis (2013): The relative bias is $\Delta_{Prob} \geq 30/100 = 0.3$.

Tversky and Kahneman (1992):

1. Gain. The relative bias is $\Delta_{Rel} \geq 8/50 = 0.16$.

Loss. The relative bias is $\Delta_{Rel} \geq -6/(-50) = 0.12$.

2. Gain. The relative bias is $\Delta_{Rel} \geq 49/200 = 0.245$.

Loss. The relative bias is $\Delta_{Rel} \geq -35/(-200) = 0.175$.

So sometimes the relative biases are comparable or even more than the above secure upper relative bound $0.25$.

Therefore, and also from general and formal points of view, the following supposition can be stated:

"In general cases, along with the non-zero minimal variance of the measurement data, another source or sources of the biases can exist and cannot be excluded so far."

Therefore, only some general formal qualitative mathematical model can be considered so far.

### 5.1.3. Main question

Due to the second presupposition, the approach implies that the biases are caused by the forbidden zones of the theorem. The forbidden zones are, in turn, caused by the non-zero minimal variance of the random variable. Due to the above high experimental values of the biases, the main question is to determine whether the forbidden zones can lead to such high values of the biases. This question leads to another one about the widths of the forbidden zones for various types of distributions.

So, the main question of future research is to analyze the widths of the forbidden zones for various types of distributions.

## 5.2. Basics of general formal qualitative mathematical model

Keeping in the mind the above restrictions and question, let us analyze possible basics of the general formal qualitative mathematical model.

The model should deal with qualitative problems. There can be only three combinations: the expectation for the probable outcome can be more, less or equal to that for the sure ones.

The inalienable feature of the qualitative problems is that the signs of the differences for the choices do not coincide with the signs of the differences for the expectations of the probable and sure outcomes.

That is when the difference of the expectations for the probable and sure outcomes is, e.g., positive, then the corresponding difference for subjects' choices is negative. Due to (7), the difference for subjects' choices should not equal zero. This feature of the qualitative problems can be represented mathematically as

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_{\mu}. \tag{10}$$

That is: for example, if the difference $d_\mu$ between the expectations of the compared modes is undoubtedly positive (that is the sign of $d_\mu$ is $sgn\ d_\mu > 0$), then the revealed choice of the subjects is such that the difference $d_{Choice}$, that is required to obtain the data corresponding to this choice, should be undoubtedly negative (that is the sign of $d_{Choice}$ is $sgn\ d_{Choice} < 0$).

These qualitative types of the above problems are chosen as the examples that are usual in experiments (see, e.g., Kahneman and Tversky 1979, Starmer and Sugden 1991, Tversky and Kahneman 1992, Thaler 2016). They can manifest clear qualitative representations of the above problems and can be a background for some further generalizations.

To change the difference of the expectations for the probable and sure outcomes to another qualitative situation, the bias of choices should be evidently not less than this difference, that is

$$|d_{Choice}| \geq |d_{\mu}|.$$

This relationship implies that for the problems of certainty equivalents

$$|d_{Choice}| = |d_{\mu}| \quad \text{and, due to (10),} \quad d_{Choice} = -d_{\mu},$$

and for the other problems

$$|d_{Choice}| > |d_{\mu}|.$$

### 5.2.1. Trial examples of applications of general formal qualitative mathematical model

Let us test the above examples of Section 1 by the general formal qualitative mathematical model.

In the above citation from Kahneman and Tversky (1979) p. 265 the difference between the expectations is *2,500*.33 + 2,400*.66 - 2,400 = 2,400 - 2,400*.01 + 100*.33 - 2,400 = - 24 + 33 = 9.* The difference between the choices should be more than *9*. Let it be equal, for example, to *15*.

So the subjects decide if the resulting difference between the expectations was *15 – 9 = 6* in favor of the sure outcome.

The qualitative result is supported by the experiment. That is *82%* in favor of the sure outcome.

In the above citation from Starmer and Sugden (1991) p. 974 the difference between the expectations is *10.00*.2 + 7.00*.75 - 7.00 = 2.00 + 5.25 - 7.00 = +0.25*. The difference between the choices should be more than *0.25* and should be at least partially caused by a noise. Let it be equal, for example, to *0.4*.

So the subjects decide if the resulting difference between the expectations was *0.4 – 0.25 = 0.15* in favor of the sure outcome.

The qualitative result is supported by the experiment. That is *27/(13+27) = 27/40 = 87.5%* in favor of the sure outcome.

In the above citation from Barberis (2013) the difference between the expectations is *100*0.9 - 63 = 27.* The difference for the choices should be equal to *27* as well.

So the subjects decide if the resulting difference between the expectations was *27* in favor of the sure outcome. The qualitative result is supported by the experiment.

In the above citation from Tversky and Kahneman (1992) we can find:

1. Gain. The difference between the expectations is $50*0.9 - 37 = 8$. The difference for the choices should be equal to $8$ as well.

So the subjects decide if the resulting difference between the expectations was $8$. This qualitative result is supported by the experiment.

Loss. The difference between the expectations is $-50*0.9 – (-39) = -6$. The difference for the choices should be equal to $-6$ as well.

So the subjects decide if the resulting difference between the expectations was $-6$. This qualitative result is supported by the experiment.

2. Gain. The difference between the expectations is $200*.90 - 131 = 49$. The difference for the choices should be equal to $49$ as well.

So the subjects decide if the resulting difference between the expectations was $49$. This qualitative result is supported by the experiment.

Loss. The difference between the expectations is $-200*.90 - (-155) = -35$. The difference for the choices should be equal to $-35$ as well.

So the subjects decide if the resulting difference between the expectations was $-35$. This qualitative result is supported by the experiment.

In all the above examples the difference between the choices should be at least partially caused by the non-zero minimal variance of the data. These examples of applications of the general formal qualitative mathematical model are trial because there is so far too little information about what part of the difference between the choices is caused by the non-zero minimal variance of the data.

## 5.3. Special qualitative mathematical model

Let us consider the qualitative problems under some special condition

$$d_{\mu} = 0 \quad \text{or} \quad d_{\mu} = 0 \quad \text{or} \quad \mu_{\Pr obable} = \mu_{Sure}\,. \tag{11}$$

That is the expectations of the probable and sure outcomes are equal to each other. Due to this condition, the difference for the choices should be, in accordance with (7) either negative or positive.

This special situation enables to avoid the constraints of the secure upper bound $\Delta_{Sequre}$ for the bias and to make the special qualitative model less formal. The biases can be selected much less than $\Delta_{Sequre}$ and suppositions will be more simple. This special qualitative model can be considered as a first step of the first stage of the approach and of an explanation of the above problems. The model will be applied to practical numerical examples in the next section.

The relationships of the special qualitative mathematical model can be summarized as follows:

The relationship (7) of the non-zero difference between the biases for the choices

$$\exists\, d_{Choice} : |\, d_{Choice} \,| > 0 \quad \text{or} \quad \operatorname{sgn} d_{Choise} \neq 0\,.$$

The relationships (8) of the theorem and choices

$$\sigma^2{}_{Min} > 0 \quad \text{and} \quad d_{Theorem} \approx d_{Choise} \quad \text{or at least} \quad d_{Theorem} = O(d_{Choise})\,.$$

The relationships (9) of the probable and sure outcomes and choices

$$|\, \Delta_{\Pr ob} \,| > |\, \Delta_{Sure} \,| \quad \text{and} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\Pr ob}\,.$$

The relationships (11) of the special qualitative problems

$$\operatorname{sgn} d_{\mu} = 0 \quad \text{or} \quad d_{\mu} = 0 \quad \text{or} \quad \mu_{\Pr obable} = \mu_{Sure}\,.$$

## 6. Applications of the theorem and approach. Newness

### 6.1. Practical applications in behavioral economics and decision sciences

The idea of the considered forbidden zones was applied, e.g., in Harin (2012b). This work was devoted to the well-known problems of utility and prospect theories and was performed for the purposes of utility and prospect theories, behavioral economics, psychology, decision and social sciences. Such problems were pointed out, e.g., in Kahneman and Thaler (2006).

In Harin (2012b), some examples of typical paradoxes were studied. The studied and similar paradoxes may concern problems such as the underweighting of high and the overweighting of low probabilities, risk aversion, etc.

The dispersion and noisiness of the initial data can lead to the forbidden zones for the expectations of these data. This should be taken into account when dealing with these kinds of problems. The above forbidden zones explained, at least partially, the analyzed examples of paradoxes.

The concrete numerical examples of analysis and explanation of such problems by the proposed special qualitative model will be considered below. To emphasize the uniformity of the proposed models, the parameters and analysis will be the same for the different domains.

## 6.2. Practical numerical example. First domain. Gains

The special qualitative mathematical model enables to use small and convenient biases. In particular, it is convenient to consider integer numbers. The minimal non-zero integer for the bias for the sure outcome is *\$1*. Hence the minimal integer for the bias for the probable outcomes is *\$2*. Suppose that the parameters of the special model for the gains are: the bias for the probable outcomes is equal to *\$2*, and for the sure outcome the bias is equal to *\$1* or to zero.

The above examples can be simplified to the special qualitative ones similar to Harin (2012b):

Imagine that you face the following pair of concurrent decisions.

Choose between:

A) A sure gain of \$99.

B) 99% chance to gain \$100 and 1% chance to gain or lose nothing.

### 4.2.1. Ideal case

In the ideal case, without taking into account the dispersion of the data, the expected values for the probable and sure outcomes are

$$\$99 \times 100\% = \$99 ,$$

$$\$100 \times 99\% = \$99 .$$

Here, the ideal expected values are exactly equal to each other

$$\$99 = \$99 .$$

Therefore the both outcomes should be equally preferable.

So in the ideal case, without taking into account the dispersion of the data, the probable and sure outcomes should be equally preferable.

### 6.2.2. Forbidden zones and biases

In the real case, one should take into account the dispersion of the data, some minimal non-zero variance caused by this dispersion and the forbidden zones caused by this variance. These forbidden zones can lead to the biases of the expectations, at least for the probable outcomes. Let us consider the case of the non-zero variance of the data, corresponding forbidden zones and biases.

Let the bias be equal to, say, $\Delta_{Prob} = \$2$ for the probable outcomes.

Let us consider the case when the bias for the sure outcome is equal to $\$1$. We have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$1 = \$98,$$
$$\$100 \times 99\% - \Delta_{Prob} = \$99 - \$2 = \$97.$$

Here, the probable expected value is biased more than the sure one and we have

$$\$98 > \$97.$$

Let us consider the case when the bias for the expectations of data for the sure outcome is equal to zero. We have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$0 = \$99,$$
$$\$100 \times 99\% - \Delta_{Prob} = \$99 - \$2 = \$97.$$

Here, the probable expected value is biased but the sure expected value is not and we have

$$\$99 > \$97.$$

In all the cases, the probable expected value is biased more than the sure one. The bias decreases the advantage (preference) of the outcome. Therefore the probable gain is (due to the obvious difference between the expected values) less preferable than the sure one.

We see the clear and evident difference between the expected values and the corresponding salient and unequivocal preferences and choices.

So the theorem provides the mathematical support for the above analysis in the domain of gains.

So, the forbidden zones and their natural difference for probable and sure outcomes can predict the experimental fact that the subjects are risk averse in the domain of gains. They explain, at least qualitatively or partially, the analyzed example of Thaler (2016) and many other similar results.

The theorem provides the mathematical support for the analysis in the domain of gains.

### 6.3. Practical numerical example. Second domain. Losses

The case of gains has been explained many times in a lot of ways. The uniform explanation for both gains and losses, without additional suppositions, as, e.g., Kahneman and Tversky (1979), has not been recognized nevertheless by the author of the present article (see a slightly similar work Egozcue et. al. 2011). The theorem, approach and models occur to be useful for such a uniform explanation.

Let us consider the case of losses under the same suppositions as gains.

Imagine you face the following pair of concurrent decisions. Choose between:

A) A sure loss of $99.

B) 99% chance to loss $100 and 1% chance to gain or lose nothing.

#### 6.3.1. Ideal case

In the ideal case without the forbidden zones, the expected values for the probable and sure outcomes are

$$-\$99 \times 100\% = -\$99,$$

$$-\$100 \times 99\% = -\$99.$$

Here, the expected values are exactly equal to each other

$$-\$99 = -\$99.$$

Therefore the both outcomes should be equally preferable.

So in the ideal case, without taking into account the dispersion of the data, the probable and sure outcomes should be equally preferable.

### 6.3.2. Forbidden zones and biases

Let us consider the case of the forbidden zones and biases under the same suppositions as for the gains. That is for the same parameters of the models.

The forbidden zone biases the expectation from the boundary of the interval to its middle. The bias is subtracted from the absolute value for the both cases of gains and losses therefore. That is, due to the opposite signs of the values for gains and losses, the bias is subtracted from the expected values for the gains and added to the expected values for the losses. It should be emphasized that this is not a supposition but a rigorous conclusion. Therefore the applications of the special qualitative mathematical model are naturally uniform for more than one domain.

The parameters of the special model for the gains are: the bias for the probable outcomes is equal to *\$2*, and for the sure outcome to *\$1* or to zero.

Let us consider the case when the bias for the sure outcome is equal to *\$1*

$$-\$99 \times 100\% + \Delta_{Sure} = -\$99 + \$1 = -\$98,$$

$$-\$100 \times 99\% + \Delta_{\Pr ob} = -\$99 + \$2 = -\$97.$$

Here, the probable expected value is biased more than the sure one and we have

$$-\$98 < -\$97.$$

Let us consider the case when the width of the forbidden zones for the expectations of data in the sure outcome is equal to zero. We have

$$-\$99 \times 100\% + \Delta_{Sure} = -\$99 + \$0 = -\$99,$$

$$-\$100 \times 99\% + \Delta_{\Pr ob} = -\$99 + \$2 = -\$97.$$

Here, the probable expected value is biased but the sure expected value is not and

$$-\$99 < -\$97.$$

In all the cases, the probable expected value is biased more than the sure one as in the case of gains, but here the bias increases the advantage (preference) of the outcome and the probable loss is (due to the obvious difference between the expected values) more preferable than the sure one.

We see the clear and evident difference between the expected values and the corresponding salient and unequivocal preferences and choices.

So the special qualitative mathematical model can be naturally, uniformly and successfully applied in the domain of losses as well. Instead of the seeming simplicity of these applications, the author has not revealed such successful and uniform applications in more than one domain in the literature.

## 6.4. Newness

Due to, e.g., Harin (2012b), the forbidden zones and their natural difference for probable and sure outcomes can predict the experimental fact that the subjects are risk seeking in the domain of gains but risk seeking in the domain of losses. They explain, at least qualitatively or partially, the analyzed examples of Thaler (2016) and many other similar results.

The important feature is that, due to, e.g., Harin (2012b), the described forbidden zones can explain the problems and explain experimental results not only in the domains of the gains and losses. Hence the forbidden zones and their natural difference for probable and sure outcomes can qualitatively or, at least, partially predict the experimental facts and explain the problems in various domains.

There are a lot of real examples of the forbidden zones. The idea of such zones helps in the analysis of the well-known problems. The existence theorem provides the mathematical description of the forbidden zones and the mathematical support for this analysis. The mathematical approach is an application of the theorem to these problems. The qualitative mathematical models are the first stage of the approach and the special qualitative mathematical model is its first step.

Unfortunately, the analysis of the literature, comments of comments of journals' editors and reviewers on similar articles and on the previous versions of the present article and more than 10-years experience of the editorship in NEP reports on utility and prospect theories allow to state that the idea, theorem and its support of the above analysis, the approach and models have not been described before. So they are new.

Why did not such an evident and widespread phenomenon as these forbidden zones be mathematically described before? The long absence of such a description can be probably explained by reasons that such phenomena, those are similar to the forbidden zones between ships boards and moorage wall, washing machines and walls, etc., are evident, can be as a rule easily estimated as approximately a half of the amplitude of the vibrations and need not more detailed research. In the above problems and paradoxes, such phenomena are hidden by other details of experiments (see, e.g., Harin 2014) and hence are non-evident. In addition, the well-known law of diminishing marginal utility proposes another ways of the analysis.

## 6.5. Possible applications

### 6.5.1. Possible applications. Noise

Let us preliminary consider possible applications of the theorem to a noise.

If a noise leads to some non-zero minimal variance of the considered random variable, then this non-zero minimal variance and, consequently, this noise leads to the above non-zero forbidden zones for the expectation of this variable. If a noise leads to some increasing of the value of this minimal variance then the width of these forbidden zones increases also.

The proposed theorem, approach and model enable to make a step to develop possible new mathematical tools for description of the possible influence of noise near the boundaries of finite intervals. In particular, if a noise leads to a non-zero minimal variance $\sigma^2_{Min} : \sigma^2 > \sigma^2_{Min} > 0$ of a random variable, then the theorem predicts (6) the forbidden zones having the width $r_{Noise}$ which is not less than

$$r_{Noise} \geq \frac{\sigma^2_{Min}}{b-a}.$$

So, the presented theorem can be some preliminary step to a general mathematical description of the possible influence of a noise near the boundaries of finite intervals.

Some general questions concerning this item can arise. For example, general definition and determination of level, strength, power, etc. of a noise are needed. They should lead to general definition and determination of the non-zero noise. Questions about specification of common widespread types of the non-zero noise of a measurement, those surely lead to the non-zero minimal variance of the measurement data in the common circumstances and environment, can arise as well.

Due to the general character of the above questions and demand of widespread experimental support, there is a need of a wide variety of research teams to give solutions reliable answers for these questions.

### 6.5.2. Possible applications. Biases of measurement data

Let us preliminary consider possible applications of the theorem to possible biases of measurement data.

The considered forbidden zones can evidently lead to some biases in measurements. We can preliminary consider this a bit closer. Suppose some measurements are performed on a finite interval and their result is a set of the measurement data and its expectation. Suppose some forbidden zones arise near the boundaries of the interval due to the minimal variance of the data.

The expectations of the data of the measurements cannot be indeed located inside the forbidden zones. They cannot be located closer to the boundaries of the interval than the width of the forbidden zone.

So the above forbidden zones can cause biases for the expectations of the data of measurements. The biases are directed from the boundaries to the middle of the interval. The biases have the opposite signs near the opposite boundaries of the interval. The absolute values of the biases decrease from the boundaries to the middle of the interval.

When the minimal variance of the data is equal to zero, then the expectations of the data of measurements can touch the boundaries of the interval. When the above forbidden zones are not taken into the consideration then the estimated results are also located closer to the boundaries than the real case. Hence the estimated results are biased in the comparison with the real ones.

Particular example of the biases. If the minimal variance of the data $\sigma^2_{Min}$ is non-zero, that is if $\sigma^2 > \sigma^2_{Min} > 0$, then the theorem predicts (6) that near the boundaries of intervals, the absolute value $\Delta_{Bias}$ of the biases is not less than

$$|\Delta_{Bias}| \geq \frac{\sigma^2_{Min}}{b-a}.$$

So, the presented theorem and approach their consequences and applications can be considered as some preliminary step to a general mathematical description of the biases of measurement data near the boundaries of finite intervals.

## 7. Conclusions and discussions

The article can be concluded by the five main and some additional items:

**1) Problems.** There are the well-known problems of prospect theories (see, e.g., Hey and Orme 1994, Kahneman and Thaler 2006, Thaler 2016): The choices of the subjects (people) don't correspond to the expectations of the outcomes.

Some of the typical problems consist in the comparison of sure and probable outcomes (see, e.g., Kahneman and Tversky 1979, Thaler 2016). They are the most pronounced near the boundaries of intervals. Some of them have opposite solutions for different domains. For example, Thaler (2016) states (the **boldface** is my own):

"*We observe a pattern that was frequently displayed: subjects were **risk averse** in the domain of gains **but risk seeking** in the domain of losses.*"

These problems can be represented in the simplified and demonstrable form by the qualitative and special qualitative problems (or that of the equal expectations for the probable and sure outcomes) that are considered in the present article similar to Harin (2012b). The special qualitative problems are:

First domain. Gains. Choose between:

A) A sure gain of *$99*.

B) *99%* chance to gain *$100* and *1%* chance to gain or lose nothing.

The expectations are

$$\$99 \times 100\% = \$99 = \$99 = \$100 \times 99\% .$$

Second domain. Losses. Choose between:

A) A sure loss of *-$99*.

B) *99%* chance to loss *-$100* and *1%* chance to gain or lose nothing.

The expectations are

$$-\$99 \times 100\% = -\$99 = -\$99 = -\$100 \times 99\% ,$$

The expected values are exactly equal to each other in the both domains. A wealth of experiments (see, e.g. Kahneman and Tversky 1979, Starmer and Sugden 1991, Thaler 2016) proves nevertheless that the choices of the subjects are essentially biased. Moreover as is pointed out, e.g., in Thaler (2016), they are biased in the opposite directions for gains and losses. These are the well-known and fundamental problems that are usual in behavioral economics and other sciences.

**2) Analysis of the problems.** A new analysis of these problems was developed in recent years (see, e.g., Harin 2012a, Harin 2012b, Harin 2015). The analysis is founded on the idea of the non-zero forbidden zones studied here and enables at least qualitative explanation of these problems (see, e.g., Harin 2012b).

**3) Mathematical support for the analysis.** The forbidden zones theorem is proven in the present article. The theorem states that, for a finite interval $[a, b]$ under the condition of existence of some non-zero minimal variance $\sigma^2_{Min} : \sigma^2 \geq \sigma^2_{Min} > 0$, the expectation $\mu$ of the measurement data is separated from the boundaries $a$ and $b$ of the interval $[a, b]$ by the non-zero forbidden zones

$$a < a + \frac{\sigma^2_{Min}}{b-a} \leq \mu \leq b - \frac{\sigma^2_{Min}}{b-a} < b .$$

In other words, the theorem proves the possibility of existence of the non-zero forbidden zones that were used in the above analysis. The forbidden zones can exist near the boundaries of the intervals of the measurement data. The theorem also determines the conditions of the existence of the zones and their minimal width.

**4) Mathematical approach for the analysis**. The mathematical approach of the biases of the expectations (or, simpler, approach of biases, or, simple, approach) is founded on the theorem and is to explain not only the objective situations but also and mainly the subjective behavior and choices of subjects.

The two main presuppositions of the approach are:

1. The subjects make their choices (at least to a considerable degree) as if there were some biases of the expectations of the outcomes.

(This presupposition can be supported, at least formally: such biases may be proposed and tested even only from the purely formal point of view)

2. These biases (real biases or subjective reaction and choices of the subjects) can be explained (at least to a considerable degree) with the help of the theorem.

The supposed main general relationships of the approach can be accumulated into the three groups (partially corresponding to the above presuppositions):

1) The relationship (7) of the non-zero difference between the biases for the choices

$$\exists\, d_{Choice} : |\, d_{Choice} | > 0 \quad \text{or} \quad \operatorname{sgn} d_{Choise} \neq 0 \,.$$

2) The relationships (8) of the theorem and biases of the choices

$$\sigma^2{}_{Min} > 0 \quad \text{and} \quad d_{Choise} = O(d_{Theorem}) \,.$$

3) The relationships (9) of the probable and sure outcomes and choices

$$|\, \Delta_{\Pr ob} | > | \Delta_{Sure} | \quad \text{or} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{\Pr ob} \,.$$

Here $\Delta_{Prob}$ , $\Delta_{Sure}$ and $d_{Choise} \equiv \Delta_{Prob}$ - $\Delta_{Sure}$ – are appropriately the biases of the expectations of the data for the probable and sure outcomes and their difference, that is required to obtain the data corresponding to these choices; $d_{Choise}$ – is the difference that can be obtained by the theorem.

The first stage of the approach consists in the qualitative mathematical explanation of the qualitative problems.

**5) Mathematical models for the analysis**.

**5.1) Basics of general qualitative model**. The basics of the general formal preliminary qualitative mathematical model are developed in the present article.

The supposed main general relationships additional to the approach are

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_{\mu} \quad \text{and} \quad |d_{Choice}| \geq |d_{\mu}|,$$

where $d_{\mu} \equiv \mu_{Prob} - \mu_{Sure}$ – is the observed difference between the expectations.

The general model enables formal solutions of the qualitative problems considered here, but the limits of its applicability need additional research.

**5.2) Special qualitative model**. The special qualitative mathematical model is intended for the practical analysis of the above problems in the special cases when the expectations for the probable and sure outcomes are exactly equal to each other. The additional relationships (11) of these special cases can be written as

$$\operatorname{sgn} d_{\mu} = 0 \quad \text{or} \quad d_{\mu} = 0 \quad \text{or} \quad \mu_{\mathrm{Pr}obable} = \mu_{Sure}.$$

The model can be considered as the first step of the first stage of the approach.

The special qualitative mathematical model implies the application of the forbidden zones theorem under the additional facilitating supposition:

Due to relationships (9), the bias for the probable outcomes $|\Delta_{\mathrm{Pr}obable}| > 0$ should be non-zero but can be as small as possible. Therefore the minimal variance of the measurement data for the probable outcomes can be supposed to be equal to an arbitrary non-zero value that is as small as possible to be evidently explainable in the presence of a common noise and scattering of the data.

**Numerical examples**. In the scope of the special model, suppose that the biases of the expectations are equal, for example, to $\Delta_{Prob} = \$2$ for the probable outcomes and $\Delta_{Sure} = \$1$ for the sure outcomes. Then we have:

**First domain. Gains.** In the case of gains we have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$1 = \$98,$$

$$\$100 \times 99\% - \Delta_{Prob} = \$99 - \$2 = \$97.$$

The probable expected value is biased more than the sure one. The biases are directed from the boundary to the middle of the interval and, hence, decrease the modules of the values and the both values themselves. Therefore the biased sure expected value is more than the biased probable one

$$\$98 > \$97.$$

The sure gain is evidently more preferable than the probable one and this choice is supported by a wealth of experiments.

**Second domain. Losses.** In the case of losses we have

$$-\$99 \times 100\% + \Delta_{Sure} = -\$99 + \$1 = -\$98,$$

$$-\$100 \times 99\% + \Delta_{Prob} = -\$99 + \$2 = -\$97.$$

The probable expected value is biased more than the sure one. The biases are directed from the boundary to the middle of the interval and, hence, reduce the modules of the values but, due to their negative signs, increase the both values. Therefore the biased sure expected value is less than the biased probable one

$$-\$98 < -\$97.$$

The probable loss is evidently more preferable than the sure one and this choice is supported by a wealth of experiments.

So, the special model enables the qualitative analysis and qualitative explanation for the above special problems in more than one domain.

**Main mathematical contributions.** The four main particular applied mathematical contributions of the present article are the mathematical support, approach and special qualitative mathematical model for the above analysis and the successful uniform application of this model in more than one domain.

The author has not revealed in the literature such a natural, uniform and successful application of any model in more than one domain of the discussed problems. Therefore, instead of seeming simplicity, the successful natural and uniform application of the special qualitative mathematical model in more than one domain belongs also to the main contributions.

**Possible additional contributions**. The two more possible additional general applied mathematical contributions can be preliminary mentioned:

**Possible general addition. Noise**. In addition, possible general consequences and applications of the theorem for a noise are preliminary considered.

In particular, suppose that some type of noise leads to a non-zero minimal variance $\sigma^2_{Min} : \sigma^2 > \sigma^2_{Min} > 0$ of a random variable. Then the theorem predicts (6) the existence of the forbidden zones having the width $r_{Noise}$ which is not less than

$$r_{Noise} \geq \frac{\sigma^2_{Min}}{b-a}.$$

The future goal of this consideration is a general mathematical description of the possible influence of a noise near the boundaries of finite intervals.

**Possible general addition. Biases**. In addition, possible general applications of the theorem for biases of measurement data are preliminary considered.

In particular, if the minimal variance of the data $\sigma^2_{Min}$ is non-zero, that is if $\sigma^2 > \sigma^2_{Min} > 0$, then the theorem predicts the biases of measurement data in general cases. The biases have the opposite signs near the opposite boundaries, are maximal near the boundaries and tend to zero in the middles of the intervals. Right against the boundaries of intervals, the absolute value $\Delta_{Bias}$ of the biases (6) is not less than

$$|\Delta_{Bias}| \geq \frac{\sigma^2_{Min}}{b-a}.$$

The future goal of this consideration is a general mathematical description of the biases of measurement data that can be caused by the above forbidden zones.

**Two main future questions**. The first main question for future research is to analyze the widths of the forbidden zones for various types of distributions. The second main future question is to define rigorously the term “non-negligible noise” of measurements and prove that any non-negligible noise of measurements causes some non-zero minimal variance of the measurement data or, at least, to rigorously determine such types of a noise.


**Acknowledgements**

The author wishes to say many words of gratitude to Professor A. A. Novosyolov for his long-term support and methodological tutorship.

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## Appendix. Lemmas of variance maximality conditions

Preliminaries

The initial particular need is the mathematical support for the analysis (see, e.g., Harin 2012a, Harin 2012b and Harin 2015) of the problems of behavioral economics. These problems take place for the discrete finite random variables. The support for the discrete distributions is given in Bhatia and Davis (2000). Let us give an alternative support for the general case.

In the general case, we have for the random variable of subsection 2.1

$$E[X-\mu]^2 = \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k) + \int_a^b (x-\mu)^2 f(x)dx \equiv$$
$$\equiv E_{Discr}[X-\mu]^2 + E_{Contin}[X-\mu]^2 .$$

under the condition (1) that either the probability mass function or probability density function or alternatively both of them are not identically equal to zero

$$\sum_{x_k\in[a,b]} p(x_k) + \int_a^b f(x)dx = 1 .$$

Pairs of values whose mean value coincides with the expectation of the random variable were used, e.g., in Harin (2013). More arbitrary choice of pairs of values was used in Harin (2017). Here every discrete and infinitesimal value will be transformed, namely divided into the pair of values in the following manner:

Let us divide every value $p(x_k)$ into the two values located at $a$ and $b$

$$p(x_k)\frac{b-x_k}{b-a} \quad \text{and} \quad p(x_k)\frac{x_k-a}{b-a} .$$

The total value of these two parts is evidently equal to $p(x_k)$. The center of gravity of these two parts is evidently equal to $x_k$.

Let us divide every value of $f(x)$ into the two values located at $a$ and $b$

$$f(x)\frac{b-x}{b-a} \quad \text{and} \quad f(x)\frac{x-a}{b-a} .$$

The total value of these two parts is evidently equal to $f(x)$. The center of gravity of these two parts is evidently equal to $x$. So these divisions (transformations) do not change the expectation of the random variable.

Let us prove that the variances of the divided parts are not less than those of the initial parts.

## A1. Lemma 1. Discrete part

**Lemma 1. Discrete part lemma.** If the support of a random variable $X$, is an interval $[a,b]$: $0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2 = \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k) + \int_a^b x^2 f(x)dx \equiv \sigma^2,$$

where $p$ is the probability mass function of $X$, $a \le x_k \le b$, $k = 1,2,...,K$, where $K \ge 1$ and $\mu \equiv E[X]$ and

$$\sum_{k=1}^{K} p(x_k) \ge 0,$$

then the inequality

$$\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k) \ge$$
$$\ge \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k).$$

is true.

**Proof.** Let us find the difference between the transformed

$$\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)$$

and initial

$$\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)$$

expressions for the variance.

Let us consider separately the cases of $x_k \ge \mu$ and $x_k \le \mu$.

A.1.1. Case of $x_k \geq \mu$

If $x_k \geq \mu$, then the expression in the square brackets can be simplified

$$\left[(a-\mu)^2 \frac{b-x_k}{b-a} + (b-\mu)^2 \frac{x_k-a}{b-a} - (x_k-\mu)^2\right] \geq$$
$$\geq \left[(b-\mu)^2 \frac{x_k-a}{b-a} - (x_k-\mu)^2\right] = .$$
$$= (b-\mu)^2 \left[\frac{x_k-a}{b-a} - \left(\frac{x_k-\mu}{b-\mu}\right)^2\right]$$

Due to $x_k \leq b$ and

$$0 \leq \frac{x_k-\mu}{b-\mu} \leq 1,$$

it holds true that

$$\left(\frac{x_k-\mu}{b-\mu}\right)^2 \leq \frac{x_k-\mu}{b-\mu}$$

and

$$\frac{x_k-a}{b-a} - \left(\frac{x_k-\mu}{b-\mu}\right)^2 \geq \frac{x_k-a}{b-a} - \frac{x_k-\mu}{b-\mu}$$

and then

$$\frac{x_k-a}{b-a} - \frac{x_k-\mu}{b-\mu} \equiv \frac{(x_k-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)} - \frac{x_k-\mu}{b-\mu}.$$

Due to

$$0 \leq \frac{x_k-a}{b-a} \leq 1 \quad \text{and} \quad \mu-a \geq 0,$$

the inequality

$$\frac{(x_k-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)} \geq \frac{x_k-\mu}{b-\mu}$$

is true and

$$(b-\mu)^2 \left[\frac{x_k-a}{b-a} - \left(\frac{x_k-\mu}{b-\mu}\right)^2\right] \geq 0.$$

So in the case of $x_k \geq \mu$ the difference between the transformed and initial expressions for the variance is non-negative.

### A.1.2. Case of $x_k \le \mu$

If $x_k \le \mu$, then

$$\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]=$$
$$=\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(\mu-x_k)^2\right]\ge$$
$$\ge\left[(\mu-a)^2\frac{b-x_k}{b-a}-(\mu-x_k)^2\right]=$$
$$=(\mu-a)^2\left[\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\right]$$

Due to

$$0\le\frac{\mu-x_k}{\mu-a}\le 1,$$

we have

$$\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\ge\frac{b-x_k}{b-a}-\frac{\mu-x_k}{\mu-a}.$$

Then

$$\frac{b-x_k}{b-a}-\frac{\mu-x_k}{\mu-a}\equiv\frac{(b-\mu)+(\mu-x_k)}{(b-\mu)+(\mu-a)}-\frac{\mu-x_k}{\mu-a}.$$

Due to

$$0\le\frac{\mu-x_k}{\mu-a}\le 1 \quad \text{and} \quad b-\mu\ge 0$$

we have

$$\frac{(b-\mu)+(\mu-x_k)}{(b-\mu)+(\mu-a)}\ge\frac{\mu-x_k}{\mu-a}$$

and

$$(\mu-a)^2\left[\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\right]\ge 0.$$

So in the case of $x_k \le \mu$ the difference between the transformed and initial expressions for the variance is non-negative as well.

So the difference

$$
\begin{aligned}
&(a-\mu)^2 p(x_k)\frac{b-x_k}{b-a}+(b-\mu)^2 p(x_k)\frac{x_k-a}{b-a}-(x_k-\mu)^2 p(x_k)= \\
&= p(x_k)\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]
\end{aligned}
.
$$

is non-negative.

Let us calculate the difference between the transformed and initial expressions of the discrete part of the variance

$$
\begin{aligned}
&E_{Discr.Transformed}[X-\mu]^2-E_{Discr.Initial}[X-\mu]^2 = \\
&=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)-\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)= \\
&=\sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]p(x_k)
\end{aligned}
.
$$

Every member of a sum is non-negative, as in the above expression. Hence the total sum is non-negative as well. The lemma has been proven.

So for the discrete case the variance is not more than that for the probability mass function which is concentrated in the two boundary points $a$ and $b$.

A.1.4. Theorem of Huygens-Steiner

Besides, in the initial expression of the discrete part of the variance

$$E_{Discr.Initial}[X-\mu]^2 = \sum_{k=1}^{K}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k),$$

in the summand

$$\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k) =$$
$$=[(a-\mu)^2(b-a)+(b-\mu)^2(b-a)]\frac{p(x_k)}{b-a},$$

the expression

$$(a-\mu)^2(b-x_k)+(b-\mu)^2(x_k-a).$$

can be identically rewritten to

$$[(x_k-a)+(\mu-x_k)]^2(b-x_k)+[(b-x_k)+(x_k-\mu)]^2(x_k-a)=$$
$$=[(x_k-a)^2+2(x_k-a)(\mu-x_k)+(\mu-x_k)^2](b-x_k)+$$
$$+[(b-x_k)^2+2(b-x_k)(x_k-\mu)+(x_k-\mu)^2](x_k-a)$$

and

$$[(x_k-a)^2+2(x_k-a)(\mu-x_k)+(\mu-x_k)^2](b-x_k)+$$
$$+[(b-x_k)^2+2(b-x_k)(x_k-\mu)+(x_k-\mu)^2](x_k-a)=$$
$$=(x_k-\mu)^2(b-a)+(x_k-a)^2(b-x_k)+(b-x_k)^2(x_k-a)+$$
$$+2(x_k-a)(b-x_k)[(\mu-x_k)+(x_k-\mu)].$$

This can be transformed to the expression

$$(x_k-\mu)^2(b-a)+(a-\mu)^2(b-x_k)+(b-\mu)^2(x_k-a)$$

and the summand can be rewritten as

$$[(x_k-\mu)^2(b-a)+(a-\mu)^2(b-x_k)+(b-\mu)^2(x_k-a)]\frac{p(x_k)}{b-a}=$$
$$=\left[(x_k-\mu)^2+(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k),$$

that is in a sense analogous to the theorem of Huygens-Steiner (The general possibility of application of the Huygens-Steiner theorem was helpfully pointed out by one of the anonymous referees when the preceding version of the present article was refereed)

## A2. Lemma 2. Continuous part

**Lemma 2. Continuous part lemma.** If the support of a random variable $X$, is an interval $[a,b]$: $0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2=\sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx\equiv\sigma^2,$$

where $f$ is the probability density function of $X$ and $\mu \equiv E[X]$ and

$$\int_a^b f(x)dx\geq 0,$$

then the inequality

$$\int_a^b\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx\geq\int_a^b(x-\mu)^2 f(x)dx.$$

is true.

**Proof.** Let us find the difference between the transformed

$$\int_a^b\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx$$

and initial

$$\int_a^b(x-\mu)^2 f(x)dx$$

expressions for the variance.

Let us consider separately the cases of $x\geq\mu$ and $x\leq\mu$.

A.2.1. Case of $x \geq \mu$

If $x_k \geq \mu$, then the difference can be simplified as

$$\begin{aligned}&\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}-(x-\mu)^2\right]\geq\\&\geq\left[(b-\mu)^2\frac{x-a}{b-a}-(x-\mu)^2\right]=\\&=(b-\mu)^2\left[\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\right]\end{aligned}.$$

Due to $x \leq b$ and

$$0\leq\frac{x-\mu}{b-\mu}\leq 1,$$

it holds true that

$$\left(\frac{x-\mu}{b-\mu}\right)^2\leq\frac{x-\mu}{b-\mu}$$

and

$$\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\geq\frac{x-a}{b-a}-\frac{x-\mu}{b-\mu}$$

and then

$$\frac{x-a}{b-a}-\frac{x-\mu}{b-\mu}\equiv\frac{(x-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)}-\frac{x-\mu}{b-\mu}.$$

Due to

$$0\leq\frac{x-a}{b-a}\leq 1 \quad \text{and} \quad \mu-a\geq 0,$$

we have

$$\frac{(x-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)}\geq\frac{x-\mu}{b-\mu}.$$

and

$$(b-\mu)^2\left[\frac{x-a}{b-a}-\left(\frac{x-\mu}{b-\mu}\right)^2\right]\geq 0.$$

A.2.2. Case of $x \le \mu$

If $x \le \mu$, then the difference can be simplified as

$$\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}-(\mu-x)^2\right]\ge$$
$$\ge\left[(\mu-a)^2\frac{b-x}{b-a}-(\mu-x)^2\right]=$$
$$=(\mu-a)^2\left[\frac{x-a}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\right].$$

Due to

$$0\le\frac{\mu-x}{\mu-a}\le 1,$$

we have

$$\frac{b-x}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\ge\frac{b-x}{b-a}-\frac{\mu-x}{\mu-a}.$$

Then

$$\frac{b-x}{b-a}-\frac{\mu-x}{\mu-a}\equiv\frac{(b-\mu)+(\mu-x)}{(b-\mu)+(\mu-a)}-\frac{\mu-x}{\mu-a}.$$

Due to

$$0\le\frac{\mu-x}{\mu-a}\le 1 \quad \text{and} \quad b-\mu\ge 0$$

we have

$$\frac{(b-\mu)+(\mu-x)}{(b-\mu)+(\mu-a)}\ge\frac{\mu-x}{\mu-a}$$

and

$$(\mu-a)^2\left[\frac{b-x}{b-a}-\left(\frac{\mu-x}{\mu-a}\right)^2\right]\ge 0.$$

### A.2.3. Maximality

Let us calculate the difference between the transformed and initial expressions of the continuous part of the variance

$$E_{Contin.Transformed}[X-\mu]^2 - E_{Contin.Initial}[X-\mu]^2 =$$
$$= \int_a^b \left[ (a-\mu)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} \right] f(x)dx - \int_a^b (x-\mu)^2 f(x)dx = .$$
$$= \int_a^b \left[ (a-\mu)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} - (x-\mu)^2 \right] f(x)dx$$

Due to the integrand of the integral is non-negative for every point in the scope of the limits of integration in this expression, the complete integral is non-negative as well. The difference is therefore non-negative. The lemma has been proven.

So for the continuous case the variance is not more than that for the probability mass function which is concentrated in the two points $a$ and $b$.

## A3. Lemma 3. Mixed case

Let us consider the general mixed case.

**Lemma 3. General mixed case lemma.** If the support of a random variable $X$, is an interval $[a,b]: 0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2 = \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k) + \int_a^b x^2 f(x)dx \equiv \sigma^2,$$

where $p$ is the probability mass function of $X$, $a \le x_k \le b$, $k=1,2,...,K$, where $K \ge 1$ and $f$ is the probability density function of $X$ and $\mu \equiv E[X]$ and (1)

$$\sum_{k=1}^{K} p(x_k) + \int_a^b f(x)dx = 1,$$

then the following inequality is true

$$\sum_{k=1}^{K}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p(x_k)+$$
$$+\int_a^b\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f(x)dx \ge.$$
$$\ge \sum_{k=1}^{K}(x_k-\mu)^2 p(x_k)+\int_a^b x^2 f(x)dx$$

**Proof.** The general mixed case is compiled from the discrete and continuous parts under the condition (1) that at least one of them is not identically equal to zero. The conclusions concerned to these parts are true for their sum as well. The lemma has been proven.

So in any case the variance is maximal for the probability mass function that has only the two values located in the two boundary points $a$ and $b$. The considered transformations (divisions) do not change the expectation of the random variable. The expectation for the probability mass function of these two boundary points is therefore equal to that of the initial random variable. The expectation of any two-points probability mass function determines undoubtedly their values as

$$f(a) = \frac{b-\mu}{b-a} \quad \text{and} \quad f(b) = \frac{\mu-a}{b-a}.$$

and the variance as

$$\begin{aligned} E[X-\mu]^2 &= (\mu-a)^2\frac{b-\mu}{b-a} + (b-\mu)^2\frac{\mu-a}{b-a} = \\ &= (\mu-a)(b-\mu)\frac{\mu-a+b-\mu}{b-a} = (\mu-a)(b-\mu) \end{aligned}.$$

For purely discrete variables, this expression coincides naturally with the result of Bhatia and Davis (2000) and the proof can be treated as another version of it.

So the variance of any random variable that support is located in a finite interval *[a, b]* is not more than

$$E[X-\mu]^2 \le (\mu-a)(b-\mu).$$