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Abstract

This paper proves a nonparametric identification result for a stochastic dynamic discrete-choice game of incomplete information. The joint distribution of the private information and the stage game payoffs of the players are both assumed unknown for the econometrician and the private information across alternatives is allowed to have different distributions and be dependent. This setup poses a circularity problem in the identification strategy that has not been solved for dynamic games. This paper proposes a solution through exclusion restrictions and implied properties of the unknown functions. Under the assumptions that the distribution of the private shocks for the outside option is known and the outside option's shocks are independent of other shocks, the results jointly identify the stage game payoffs and the joint distribution of the private information.

Keywords: dynamic multinomial choice games, dynamic Markov game, Markov decision processes, multiple choice models, econometric identification, incomplete information, dynamic discrete choice, discrete decision process, decision model.

JEL Classification: C14, C33, C51, C57.

I Introduction

Discrete-time dynamic programming games are particularly attractive to estimate empirical models. These games can be represented by structural models with a finite number of players who are forward-looking and choose among available alternatives to optimize their own outcomes. The players face an individual decision-making process in which their intertemporal outcomes are affected by other players' actions and

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random events that depend on joint actions of all players. Models with similar features can be found in Aguirregabiria and Mira (2007), Bajari, Benkard, and Levin (2007), Hotz and Miller (1993), Rust (1987) and a review of related literature in Ackerberg, Benkard, Berry, and Pakes (2007) and Aguirregabiria and Mira (2010).

This paper studies a stochastic dynamic discrete-choice game of incomplete information with heterogeneous players in an infinite horizon set up[1] and proves a new nonparametric identification result. The game has at least a dynamic equilibrium and the players know all the game's objects but the econometrician does not. To estimate similar models, the literature has proposed different identification strategies but there are still no results for dynamic polychotomous games with both, the stage game payoffs and the distribution of the shocks, unknown.[2] If this type of structural models are estimated under the wrong assumptions their estimators are usually not consistent and the policy implications derived from their results might not be supported by the observations.

The new nonparametric identification result in this paper allows for unknown stage game payoff functions for all but one alternative and also unknown joint distribution of the private shocks with shocks that can be dependent across alternatives[3] and have different marginal distributions. The stage game playoffs for the alternatives are allowed to be different and the outside option's payoff is interpreted as a reference of profitability in exogenous alternatives that are unaffected by the players moves. The outside option's payoff is assumed determined by state variables that also have an impact on the stage game payoffs of other alternatives, which considers its interpretation as an alternative of reference. There is at least one exclusion restriction for each players' alternatives but one, the econometrician has full information about this alternative (i.e. the "outside option"), and the private shocks are independent over time and across players.[4]

In similar setups, there are known nonparametric non-identification results (see Rust 1994 and Magnac and Thesmar 2002), but in this paper the sources of identification depart from the available ones in the literature, hence there are not contradictory results. Some key different assumptions are for instance a stage game payoff for the outside option that cannot be 0 everywhere (or in general a constant) and unobservable shocks that are independent of observable state variables.

In this paper, the identification result is constructive under the assumption that all observations correspond to the same Markov Perfect Equilibrium (MPE). To the best of my knowledge there are not results on the existence of MPE in stochastic dynamic games with unbounded returns and a generalization of the proof of existence is beyond the scope of this paper.[5] Therefore this paper follows the same approach of the literature that studies the prolific applications of similar games, that is, assuming the existence of a MPE (see for instance Bajari, Benkard, and Levin 2007).

Despite a theoretical gap on the existence of a MPE in the setup of this game, there are results for dynamic discrete choice models that are useful in this research because they must also be valid under

[1] If the number of players is 1 the model has similarities to the setup in Rust (1987).

[2] Even under i.i.d. assumptions over the shocks.

[3] Only the outside option's private shock is assumed independent of the shocks of other alternatives.

[4] The specification of the objects for the "outside option" can be selected based on econometric methods applied to a sample of payoffs for exogenous alternatives of reference.

[5] A brief review of the literature on this topic can be found in the Appendix.

a MPE. For instance, Bhattacharya and Majumdar (1989) and Norets (2010) propose conditions under which optimal unbounded lifetime payoffs satisfy a Bellman equation and there is an optimal Markovian policy that solves such equation. Similar conditions are useful for this proposed game because they allow to construct a "pseudo-game" where players' beliefs about the behavior of other players are given by the players' choice probabilities under the MPE actually played. Thus in equilibrium, each player problem can be seen as a game against nature. The assumption that a unique MPE is played does not prevent the researcher from using the result in this paper to investigate the possibility of multiple equilibriums, but this would require better databases and information about the correspondence between the samples and the different equilibriums.

Among the related literature on identification of discrete choice models, there are papers that focus on static setups such as Manski (1975, 1985) and Matzkin (1993). Other literature such as Magnac and Thesmar (2002) study dynamic discrete decision processes under the assumption that the distribution of unobservables is known and propose nonparametric identification results for the payoffs. Heckman and Navarro (2007) focus on semiparametric identification of structural dynamic discrete choice models applied to the choice of a treatment time and its dynamic effects. Bajari, Chernozhukov, Hong, and Nekipelov (2009) prove a nonparametric identification result for period payoffs in a stationary, infinite-horizon game given that the discount factor, the distribution of the choice-specific shocks and the period payoff of one alternative are known. Aguirregabiria (2005) and Aguirregabiria (2010) propose a nonparametric identification strategy of the effects of a counterfactual policy intervention in the context of dynamic discrete decision processes. Blevins (2014) studies conditions for nonparametric identification of dynamic decision processes with discrete alternatives and a nested continuous choice assuming that there are exclusion restrictions for all alternatives, the period payoff of one alternative is known and the conditional median of the unobservable states in differences is 0.

Kasahara and Shimotsu (2009) study dynamic discrete choice models with unobserved persistent state variables that induce dynamic selection on unobservable characteristics. Under a Markov framework, the selection problem can be reduced to the identification of finite mixtures (over transition densities and conditional choice probabilities) and Kasahara and Shimotsu (2009) propose sufficient conditions to identify the components of the mixture. Kasahara and Shimotsu's method is meant to be applied before proceeding with the identification of the payoffs and choice-specific value functions of a standard model, i.e. a model without selection on unobserved and persistent state variables. In this sense, the identification result in Kasahara and Shimotsu (2009) could be used jointly with the methodology developed in this paper and cannot replace it.

The paper is organized as follows. Section II places the contribution of this paper among the related literature. Section III describes the setup of the model, the problem from a player' point of view and examples. The identification arguments are organized under a constructive approach along with the player's problem. Section IV contains the identification strategy and there is a brief description of the main arguments at the beginning of the section. The identification strategy is separated in two broad "steps", section IV.1 develops the arguments to identify the choice specific value functions and section IV.2 uses the objects identified in section IV.1 to derive the identification of the joint distribution of the private shocks and the stage game payoff functions. Finally, section V summarizes the conclusions and general thoughts about the contribution of the paper and section V provides the proofs of the propositions stated in the main text.

II Related literature

The game has a finite number of players who each period choose among a given number of alternatives to optimize their individual outcome. When there is only one player, the model is similar to Rust (1987, 1994). The actions of a player have effects on other players' outcomes and the joint actions of the players affect the future evolution of certain observable state variables. In this game, the unobservable states do not produce selection, but there are not restrictions on the possibility of extending the model to add other unobservables that can generate dynamic selection mechanisms. See for instance Abbring (2010) for a review of models with dynamic selection.

The main contribution of this paper is to prove that under some modifications of the standard setup the model can be nonparametrically identified when all the stage-game payoff functions but one and the joint distribution of the unobservable shocks are unknown for the econometrician. Some particular assumptions on the distribution of the shocks are popular in the literature because they simplify the identification and estimation of the model, for example normally distributed shocks are assumed in Keane and Wolpin (1994), Imai, Jain, and Ching (2009), Norets (2009) and Pesendorfer and Schmidt-Dengler (2003); while extreme value i.i.d. shocks are assumed in the nested fixed point estimator of Rust (1987) and the conditional choice probability estimator of Hotz and Miller (1993). Assuming that the shocks are i.i.d. extreme value of type I allows closed form solutions for the probabilities of choosing an alternative which reduces considerably the computational burden. The simplifications have a cost, in the later example the restrictive property of independence of irrelevant alternatives holds for static models,[6] while in the dynamic framework the property can be somehow avoided only through the continuation values of the choice-specific value functions, thus it might still be introducing a bias in the estimation.

Norets and Tang (2013) propose a semiparametric estimator for dynamic binary choice models that does not require the distribution of unobservables to be known, but such unknown distribution is not identified. In their setup when the distribution of the unobservable states is unknown, per-period payoffs and counterfactual conditional choice probabilities are only set identified (even under parametric restrictions), thus Norets and Tang (2013) characterize the identified sets for these objects. Their estimation experiments show that the parametric assumptions about the distribution of the unobserved states can affect the estimates of per-period payoffs.

There are several estimators with their associated identification results available for dynamic games, however to the best of my knowledge, none of them have nonparametric point-identified results when there are unknown stage game payoff functions and the joint distribution of the unobservable states is unknown. Unless otherwise noticed, the following literature studies the problem assuming that the period payoffs are parametric and the distribution of the shocks is known.

Rust (1987) proposes a nested fixed point algorithm to estimate a parametric dynamic decision problem by maximum likelihood, and Hotz and Miller (1993) develop a two-step approach that avoids solving the full dynamic optimization problem. Pakes, Ostrovsky, and Berry (2007) create a two-step estimator for entry/exit games using the returns actually earned by the firms. Aguirregabiria and Mira (2002) propose a nested pseudo likelihood estimator that iterates over consistent estimates of conditional choice

[6] Under i.i.d. extreme value of type I shocks the ratio of probabilities of choosing any two alternatives is independent of the features or the existence of a third alternative.

probabilities and has the estimators of Rust (1987) and Hotz and Miller (1993) as two extreme cases. Aguirregabiria and Mira (2007) generalize the nested pseudo likelihood estimator in Aguirregabiria and Mira (2002) to games with a finite number of players. Kasahara and Shimotsu (2012) propose modifications to the nested pseudo likelihood algorithm to improve its convergence properties. Pesendorfer and Schmidt-Dengler (2003) study the identification problem under the assumption that the profitability shocks are i.i.d. standard normals and propose an estimation procedure for infinite horizon Markovian games. The estimator in Pesendorfer and Schmidt-Dengler (2003) does not require a second stage of maximum likelihood estimation, nor a nested fixed point algorithm.

Bajari, Benkard, and Levin (2007) generalize the simulation-based estimators in Hotz, Miller, Sanders, and Smith (1994) to a multiple agent setting and use inequality restrictions to construct the estimator.[7] Pesendorfer and Schmidt-Dengler (2008) study a class of asymptotic least squares estimators for dynamic discrete games defined by equilibrium conditions and provide a unified framework for these estimators (among which include those proposed by Hotz and Miller (1993) and Aguirregabiria and Mira (2002)).

Srisuma and Linton (2012) generalize the methodology of Pesendorfer and Schmidt-Dengler (2008) to a continuous observable state space when the transition distribution of the observed state variables is not specified parametrically. Srisuma and Linton (2012) show that the policy value functions are solutions to some type II integral equations with a well-posed inverse problem under conditions that ensure bounded real-valued functions. Srisuma (2013) estimator is obtained by minimizing the distance between distributions of actions observed from the data and predicted by a pseudo-model without needing to solve for the underlying dynamic optimization problem. Srisuma (2013) proposes a two step estimator, in the first step the nonparametric beliefs are estimated and used to simulate the best responses implied by the pseudo-model, and in the second stage the fit of the model is evaluated comparing the simulated distributions with the nonparametric distributions under a suitable metric. The dynamic games that can be estimated by Srisuma (2013)'s estimator are required to have a unique solution almost surely and the result for the pseudo-game is obtained under conditions that ensure bounded real-valued functions.

Regarding the advances in nonparametric identification for dynamic decision models, Magnac and Thesmar (2002) study dynamic discrete decision processes under the assumption that the distribution of unobservables is known and propose nonparametric identification results for the payoffs. Heckman and Navarro (2007) consider semiparametric identification of primitive structural functions when the per period payoff does not depend on the agent's choices of the contemporaneous period.[8] The time in this model is finite and the researcher observes agents' actions, some state variables, the earnings, a component of the costs and "latent factors".[9] There is an assumed informational structure that combined with the observed earnings and latent factors allows for the nonparametrically identification at infinity of the primitive structural functions and the distribution of the unobservables without solving or using the whole structural model. In a second step, the identification of the full structural model is achieved sequentially applying arguments of identification at infinity starting from the last period and using the objects identified in the first step. Other literature on finite time models is Taber (2000) where the continuation value associated with one of the alternatives is known and it is possible to apply an argument of identification at infinity to identify utilities up to a monotonic transformation of the observables.

[7] Srisuma (2013) points out that this type of estimator constructed on inequality restrictions might lose identifying information.

[8] The per period payoff is given by earnings net of the costs.

[9] The "latent factors" are not state variables.

Aguirregabiria (2010) studies conditions for the nonparametric identification of the choice probability functions and the integrated value functions before and after policy interventions, when primitives of the model such as the payoff functions are not identified. The counterfactual experiments that can be evaluated are those for which the researcher knows how the policy affects the period payoffs, i.e. the amount of change in the utility function per alternative. In Aguirregabiria (2010) the per period payoff of an alternative is assumed to be the sum of two random variables that are independent conditioning on the observed states, one of these variables is the outcome which is assumed observable and can directly identify the "outcome function". The model in Aguirregabiria (2010) has a finite time and the identification proof is recursive, but there is a proof for an infinite horizon version of the model in Aguirregabiria (2005). In this later case, the identification argument requires both, unobservable states with zero conditional median and an unique solution for a fixed point mapping of the conditional choice probabilities. This identification strategy can hardly be useful in dynamic games because it is known that even under parametric assumptions there are values of the structural parameters for which the games usually have multiple equilibriums. The nonparametric identification in this paper is based on a different strategy which does not require uniqueness of the possible equilibriums, the strategy only requires that the observations belong to the same equilibrium.

Blevins (2014) studies conditions for nonparametric identification in dynamic decision models where an agent chooses firstly among discrete alternatives and secondly a nested continuous amount. Blevins (2014)'s nonparametric identification of the distribution of the shocks (in differences) requires exclusion restrictions for each alternative, also special regressors that must be conditionally independent across time, and shocks in differences with 0 conditional median. The payoff functions are assumed bounded and the identification result is partial if the errors in differences are unbounded or have bounds that exceed the bounds of the choice specific value functions in differences. The assumptions in this paper are different to the ones in Blevins (2014) and the identification strategy can deal with polychotomous dynamic discrete games even if there are not exclusion restrictions for all alternatives or a known median for the shocks in differences. Furthermore, in this paper the payoffs are unbounded and the number of players can be larger than 1.

Among the related literature, most of the advances in nonparametric identification have been done for dynamic decision processes but there is a noticeable exception. Bajari, Chernozhukov, Hong, and Nekipelov (2009) prove the nonparametric identification of period payoffs for a stationary, infinite-horizon game with known distribution of the choice-specific shocks, discount factor and period payoffs for one alternative of reference.

III The model

Consider a dynamic discrete game of incomplete information with a finite number $N \geq 1$ of players and an infinite horizon.[10] Denote random variables by capital letters and their realizations by the corresponding lowercase letters. Every period t an observable state of the game $S_{\cdot t}$ is realized, player i observes his information I_{it}, $\iota_{it} \in \mathcal{I}$ and takes an action, A_{it}. For instance, player's action a might be "offer a product or service of type a", or "make a risky investment of type a". The set of actions is $\mathcal{A} = \{0, \dots, J\}$, where

[10] If $N = 1$ the game can be seen as against "nature", or as a dynamic decision process.

0 is the outside option whose stage game payoffs do not depend on the alternatives chosen by the players. The game is dynamic because the actions that players take every period t affect both their immediate payoffs and the evolution of the state variables, which end up influencing future actions and payoffs of all players.

Z_{ita} denotes random attributes of alternative $a \neq 0$ for player i at time t with $z_{ita} \in \mathcal{Z}_a \subset \mathbb{R}^{d_{za}}$ and Z_{ita} affects the stage game payoff function of player i for alternative a only. Every period player i observes the random attributes of his alternatives $Z_{it\cdot} = (Z'_{it1}, \ldots, Z'_{itJ})'$ before choosing one of them.[11] Let a subindex $-i$ indicate that all elements are included but the corresponding to i. From player i's point of view, the other players' attributes, $Z_{-it\cdot}$, are not known at t but they can be learned when the competitors release this information at $t+1$, so assume that the distribution of $Z_{-it\cdot}$ is common knowledge.[12] The attributes of alternative $a = 0$ are denoted by Y_t, they are common knowledge, invariant across players and may affect the payoff functions of other alternatives $a \neq 0$.[13]

In order to motivate some of the assumptions, the reader can find the following example of a "banking game" useful. The example is divided in parts along with the presentation of the model. The model is flexible enough to be useful for the analysis of different industries, but the "banking game" highlights some of its main advantages, i.e. there is no need of providing parametric assumptions on key objects.[14] Therefore, the researcher can consider the the intertemporal complexity of the banking business without reducing it to some particular parametric assumptions. The business of the banks is mainly intertemporal, they pay costs of screening today to evaluate potential customers who later might default on their loans. A considerable part of the screening information that the banks produce for internal use becomes cheaper and available to their competitors. Moreover, under different "states" the banks could find it profitable to allocate certain amount of loanable funds to a selected group of individuals that require a minimum of screening costs, or could be willing to compete through their screening technologies and interest rates to broaden their range of lending services and attract new customers.

[11]The dot notation "$B.$"means that all its relevant $k = 1, \ldots, K$ elements are included, i.e. $B. = (B'_1, \ldots, B'_K)'$. The notation is used when there are additional subscripts that need to be specified, such as t, and it is omitted when there are not ambiguities.

[12]The final prices of the products are not considered attributes, they are endogenous in the sense that if the player chooses to offer product a, he will be able of producing certain quantity during the period and adjusting the price in order to sell it.

[13]Note that Y_t is also a state variable, but it is not included in $S_{\cdot t}$ because it plays a particular role in the identification strategy.

[14]The reduced form stage game payoff functions and the joint distribution of the shocks that might affect the payoffs.

Example 1 (A banking game)**.** Consider a group of N commercial banks that are competing in the market of loans. Let the sample contain a group of variables relevant to classify the lending services of the banks into categories. For instance, the banks can grant different types of loans with different requirements to be approved, costs of screening and target populations. In principle, any group of categories that provides a complete classification of the lending services of the banks can be useful to construct the set of alternatives $\mathcal{A}$. But to keep the example simple, assume that there are only two possible categories that correspond to two types of loans offered, real estate (l_r) and consumer loans (l_c). Thus, the set of alternatives for the banks is $\mathcal{A} = \{0, l_r, l_c\}$, where 0 denotes the outside option.
Every period a bank has a certain amount of loanable funds which can be allocated among the alternatives in $\mathcal{A}$. The consequent changes in the portfolio of the banks reveal the outcomes of these choices.

The information of player i is summarized in the vector $I_{it} = \left(S'_{it}, S'_{-it}, Y'_t, Z'_{it\cdot}, \varepsilon'_{it\cdot}\right)'$, which includes the observable state of the game for player i, S_{it} with $s_{it} \in \mathcal{S}$, and for all other players but i, S_{-it}, the attribute of the outside option, Y_t, with $y_t \in \mathcal{Y}$, the attributes of player i's other alternatives $Z_{it\cdot}$, with $z_{it\cdot} \in \mathcal{Z}$ and the payoff relevant private shocks $\varepsilon_{it\cdot} = (\varepsilon_{it0}, \ldots, \varepsilon_{itJ})'$, with realizations $\epsilon_{it\cdot} \in \mathbb{R}^{J+1}$. The shocks $\varepsilon_{it\cdot}$ can be interpreted as private information with respect to the ability of the players to realize gains in each alternative. The distribution function of $\varepsilon_{it\cdot}$ is denoted by q and assumed known to the players, but unknown to the researcher.

Example 2 (A banking game, cont.)**.** Regarding the state variables of the banking game, consider those that can be affected by the type of loans granted and their corresponding target populations. These state variables can be for instance the number of branches, ATMs, the geographic presence, the share of highly qualified staff and the number of employees per branch.
An additional state variable to consider is the default rate. It is possible that similar screening technologies are available for all banks and its selection is endogenous to the players' problem. Thus the default rate of each bank could be affected by the types of loans granted and the competition for such loans.
The passive interest rates that a bank must pay in order to increase its loanable funds may also be considered a state variables. Indicators of bancarization, such as the percentage of people with certain access to credit lines, are also relevant state variables in some cases. This is a proxy for potential costumers that could likely be granted a profitable loan with low screening costs. The indicators of bancarization might grow if the banks are willing to grant loans to clients with scarce credit history.[a] If a bank performs a customized screening to an individual that is new for the set of banks, this bank might pay some additional costs, but the whole set of banks will benefit from the information about the performance of this new client. This argument explain why this type of state variable is relevant across banks and is affected by all players' moves.
Finally, there is a group of state variables that are less likely to be affected by the banks' choices, but they may instead affect the choices. Among these variables one can consider the type of ownership (public or private, foreign or local), and its relative size (i.e. large, medium or small).

[a] In this case, the expected income for having new customers or charging higher interest rates should compensate additional screening costs and increased previsions for non-performing loans.

Every period t, player i observes I_{it}, takes an action $a_{it} \in \mathcal{A}$ and receives a stage game payoff $\breve{u}_{a_{it}}$ that does not depend on S_{-it}, i.e. $\breve{u}_{a_{it}}\left(S_{it}, Y_t, Z_{ita_{it}}, A_{-it}, \varepsilon_{ita_{it}}\right)$.

Assumption A. *(Exclusion restrictions). The stage game payoff of player i at time t, $\breve{u}_{a_{it}}$, depends on $S_{it}, Y_t, Z_{ita_{it}}, A_{-it}$ and $\varepsilon_{ita_{it}}$ when $a_{it} \neq 0$. When $a_{it} = 0$, $\breve{u}_{a_{it}}$ depends on Y_t and ε_{it0} only.*

Assumption A has a simple interpretation, if the researcher accounts for the effects of S_{it}, Y_t, Z_{ita} and A_{-it} on the stage game payoff of alternative a at time t, then the variables included in S_{-it} and $Z_{-it\cdot}$ do not affect the payoffs of alternative a. Assumption A also ensures that $\breve{u}_0$ is not affected by the moves of the players, and this is a characteristic feature of the outside option of the game. Equation (1) informally summarizes these exclusion restrictions.

$$\breve{u}_{a_{it}}\left(\cdot\right) = \begin{cases} \breve{u}_{a_{it}}\left(s_{it}, y_t, z_{ita_{it}}, a_{-i}, \epsilon_{ita}\right) & \text{if } a_{it} \neq 0 \\ \breve{u}_0\left(y_t, \epsilon_{it0}\right) & \text{otherwise} \end{cases} \tag{1}$$

Example 3 (A banking game, cont.)**.** It is reasonable to assume that the banks have access to information on the performance of their own portfolios before it is released to the public. Such information can be compared to market trends or forecasts for competitors' variables that are already public to construct attributes for the alternatives, z_{il_st} and z_{il_ct} for all i. The attributes can be proxies for the ability of a bank to do business in each alternative or for the ability to discriminate and attract applicants. For instance, some attributes can be the variation in non-performing loans (per alternative) minus the corresponding expected variation for competitor banks, a comparison of the loan to value ratio for mortgages or the denial rates between a bank and its competitors.

In general, due to regulatory requirements each bank foresees possible losses associated to bad performance of its portfolio of loans and make provisions which need to be adjusted according to the new available information.[a] For instance, some loan loss indicators are non-performing loans, allowances for loan losses, expected loan losses and net loan charge-offs. Changes in the percentages of these indicators (by loan type) that are not explained by the public state variables can be interpreted as "surprises" that have an impact on the stage game payoffs. Thus, they can be random attributes per alternative.

Other group of attributes can be constructed with interactions between indicators of performance of a bank and proxies for market opportunities weighted by the location of its branches. For instance, interactions between the growth of granted loans by type and indexes of consumption or the price volatility of the houses weighted by the location of the branches.

[a] A group of these indicators (by loan type) are required to be reported by the U.S. Securities and Exchange Commission (SEC).

Assumption B ensures that there is enough variability in S_{-it} for each $s_i \in \mathcal{S}_i$.

Assumption B. *(Support condition). For any t, the conditional distribution of the states S_{-it} given $s_{it} \in \mathcal{S}_i$, has at least $(J+1)^{N-1}$ points in its support with probability* 1.

The probability measure of $(S'_{\cdot t}, Y'_t, Z'_{\cdot t \cdot})'$ is denoted by $\mathcal{G}$, and assumption B introduces restrictions on $\mathcal{G}$ which are useful for the identification of $\breve{u}_{\cdot}$.

Assumption C. *(Additive separability). For all $a \in \mathcal{A}$, $\breve{u}_a$ is additively separable,*

$$\breve{u}_a(\cdot, \varepsilon_{ita}) = u_a(\cdot) + \varepsilon_{ita}.$$

The additive separability of the random private shocks in assumption C is a standard simplification that is particularly useful to model players' choices.

Assumption D. *(Known outside option payoff). u_0 is a known function.*

Assumption D ensures that u_0 is known, which provides information to identify the unknown stage game payoffs of other alternatives in levels. Assumption E imposes restrictions on q, the density function of $\varepsilon_{it\cdot}$.

Assumption E. *(Distribution of private information). For all i and t, $\varepsilon_{it\cdot}$ has a distribution with density q. Moreover, the distribution function of ε_{it0}, denoted by q_0, is known.*

When $\mathcal{A}$ contains more than 2 elements, the literature usually assumes a known distribution of all shocks (up to some parameters' values). Other standard assumptions are shocks equally distributed and conditionally independent across alternatives. These assumptions seem unreasonable, in particular for applications involving alternatives that represent different types of investment opportunities and risks.

A noticeable feature of the setup in this paper is that q_{-0}, the density of $\varepsilon_{it,-0}$, is assumed unknown and the shocks ε_{itj} are allowed to be dependent among alternatives $j > 0$. The specifications of the choice specific payoffs of alternatives other than 0 are also unknown, but the outside option's payoff is assumed known and observable with some delay. The outside option represents the gains that can be obtained from exogenous alternatives of reference and under the given assumptions, its payoff can be estimated in advance.

This paper proposes a different identification strategy that uses the available information on the outside option and allows for the nonparametric specification of the remaining unknown objects of the model. Therefore, it is in principle possible to check results on policy implications without imposing parametric assumptions on key objects.

Assumption F. *(Stationarity).* $(S'_{\cdot t}, Y'_t, Z'_{\cdot t \cdot}, \varepsilon'_{\cdot t \cdot})'$ *is stationary.*

The players' actions taken at $t-1$ become public at the beginning of period t and affect the evolution of $S_{\cdot t}$ through a known first order Markov process. The states evolve as Markov chains of first degree, while the whole set of private shocks $\varepsilon_{it\cdot}$ are independent and identically distributed across time and players. The new realizations of the public state variables and the new private information are revealed at the beginning of every period.

Assumption G. *(Identical distributions).* $S'_{it}|(S'_{it-1}, S'_{-it-1}, Y'_{t-1}, A_{it-1}, A'_{-it-1}), Y'_t|Y'_{t-1}$, $Z'_{it\cdot}|(S'_{it}, S'_{-it}, Y'_t)$ *and* $\varepsilon_{it\cdot}$ *are identically distributed across* i *and* t.

Example 4 (A banking game, cont.)**.** The model assumes that the banks can invest in exogenous alternative activities summarized by the "outside option", with attribute y_t. For instance, y_t can be the return of some procyclical, liquid and risky financial portfolio of reference. The stage game payoff for the outside option is the same for all players, but the strategies under which each player could pursue alternative investments can be different, and the variable that accounts for such differences is the private information ε_{i0t}.
Under assumption E, the distribution function of ε_{it0}, q_0, is known and can be estimated from the historic returns of portfolios with different compositions and ex-antes similar risk, or from the time series residuals between expected and actual returns of the portfolio of reference.

The $Z_{it\cdot}$ are shocks to the alternatives that are independent across t and i conditional on the observable states $S_{\cdot t}, Y_t$. The players know the distributions of the private shocks $\varepsilon_{it\cdot}$ and the attributes $Z_{it\cdot}$. Besides, the conditional independence assumptions in assumption H ensure that the players' inferences for $\varepsilon_{-it\cdot}$ and $Z_{-it\cdot}$ cannot improve with their private information.

Assumption H. *(Independence). For all* t, i, $k \neq i$ *and* $h < t$,
$S'_{\cdot t} \perp\!\!\!\perp (S'_{\cdot h}, Y'_h, Z'_{\cdot h}, \varepsilon'_{\cdot h \cdot}, A'_{\cdot h})|(S'_{\cdot t-1}, Y'_{t-1}, A'_{\cdot t-1})$, $Y_t \perp\!\!\!\perp (S'_{\cdot h}, Y'_h, Z'_{\cdot h}, \varepsilon'_{\cdot h \cdot}, A'_{\cdot h})|Y_{t-1}$,
$Z'_{it\cdot} \perp\!\!\!\perp (S'_{\cdot h}, Y'_h, Z'_{\cdot h}, \varepsilon'_{\cdot h \cdot}, A'_{\cdot h})|(S'_{\cdot t}, Y'_t)$ *and* $Z'_{it\cdot} \perp\!\!\!\perp (Z'_{kt\cdot}, \varepsilon'_{kt\cdot})|(S'_{\cdot t}, Y'_t)$. $\varepsilon_{it\cdot}$ *is independent across* i, t

and with respect to $(S'_{\cdot t}, Y'_t, Z'_{\cdot t})$. Finally, for all t and $a \neq 0$, the outside option's private shock, $\varepsilon_{\cdot t0}$ is independent of $\varepsilon_{\cdot ta}$.

The random factors that affect the stage game payoffs of the players for alternatives $j > 0$, $\varepsilon_{it,-0}$, are likely to be related with each other. Assumptions G and H let $\varepsilon_{it,-0}$ be independent of ε_{it0} and stationary with a distribution invariant across players, and unknown for the econometrician.

Considering the Markovian structure of the game, the relevant information for player i at t is independent of any history beyond $t-1$. Then, for convenience and clarity, the time index is omitted and the superscript '$^\circ$' points out next period variables. Under assumptions F to H, let the Markov transition probability function of Y be $F_{Y^\circ|Y}$. In addition, denote by $F_{Z|S.Y}$, $F_{S^\circ|S.Y A.}$, Q, Q_0 and Q_{-0} the transition probability functions for $Z_{it\cdot}$, $S_{\cdot t}$, $\varepsilon_{it\cdot}$, ε_{it0}, and $\varepsilon_{it,-0}$ respectively.

Assumption I. *The functions $F_{Y^\circ|Y}$, $F_{Z|S.Y}$, $F_{S^\circ_\cdot|S.Y A.}$ are directly identified from the observations.*

The players are risk neutral and do not observe the actions or private information of other players nor the future realization of the relevant variables. Therefore, they choose among alternatives based on their beliefs, which determine their expected payoffs and are correct in equilibrium.

Assumption J. *(Strategies and single equilibrium). There exist at least a Markov perfect equilibrium in the game which is symmetric in unobservable variables. Among these possible equilibriums, only one is played. The players follow (stationary) Markov strategies $\sigma : \mathcal{I} \to \mathcal{A}$ and their beliefs about other players' strategies are correct.*

Under assumption J, the equilibrium choice probabilities of the players are identified from the players' actions and the observable variables. Assumption J implies that if $I_{it} = I_{ih}$, player i's actions at periods t and h are the same, and if $I_{it} = I_{kt}$, player i and k's actions at period t are the same. Under player i's beliefs, $\sigma_- = (\sigma, \ldots, \sigma) : \mathcal{I}^{N-1} \to \mathcal{A}^{N-1}$ and assumptions F to J, the transition probability function for $S_{\cdot t}$ from a player's perspective is given by

$$F_{S^\circ_\cdot|S.Y A_i}(s^\circ_\cdot|s_\cdot, y, a_i) = \sum_{a_{-i} \in \mathcal{A}_{-i}} F_{S^\circ_\cdot|S.Y A_i A_{-i}}(s^\circ_\cdot|s_\cdot, y, a_i, a_{-i}) \\ \text{Prob}\left\{ \sigma_- \left(\iota_{-i}\right) = a_{-i} \middle| \, s_i, s_{-i}, y \right\}. \tag{2}$$

Then, in equilibrium, the transition probability function of I_{it} under beliefs σ_- is

$$F_{I^\circ_i|I_i A_i} = F_{Z^\circ_\cdot|S^\circ_\cdot Y^\circ} \, F_{S^\circ_\cdot|S.Y A_i} \, F_{Y^\circ|Y} \, Q. \tag{3}$$

Let $\bar{u}_a$ in equation (4) define the conditional expectation of u_a over the moves of players other than i and note that the dependence of σ_- is left implicit. For all $a \neq 0$,

$$\begin{aligned} \bar{u}_a\left(S_i, S_{-i}, Y, Z_{ia}\right) &= \sum_{a_{-i} \in \mathcal{A}_{-i}} u_a\left(S_i, Y, Z_{ia}, a_{-i}\right) \text{Prob}\left\{ \sigma_-\left(\iota_{-i}\right) = a_{-i} \middle| \, S_i, S_{-i}, Y \right\} \\ &= E\left[u_a\left(S_i, Y, Z_{ia}, a_{-i}\right) \middle| \, S_i, S_{-i}, Y, Z_{ia}, a \right]. \end{aligned} \tag{4}$$

In equilibrium, player i maximizes his expected payoffs given σ_- and I_{it}. Under regularity conditions for the existence of this maximum (implied by assumption J) and provided assumption F, player i's discounted expected payoffs can be written as a dynamic programming problem in its recursive form. Indeed,

$$V\left(I_i\right) = \max_{a \in \mathcal{A}} \left\{ \bar{u}_a\left(S_i, S_{-i}, Y, Z_{ia}\right) + \varepsilon_{ia} + \beta \, E\left[V\left(I^\circ_i\right) \middle| \, I_i, a \right] \right\} \tag{5}$$

for some known $\beta < 1$ and $\bar{u}_0 = u_0$ according to assumptions A and C. Now, for all $a \in \mathcal{A}$ with $a \neq 0$, define the choice-specific value functions, v_a, as in equation (6).

$$v_a\left(S_i, S_{-i}, Y, Z_{ia}\right) = \bar{u}_a\left(S_i, S_{-i}, Y, Z_{ia}\right) + \beta\, E\left[V\left(I_i^\circ\right) \middle| S_\cdot, Y, a\right]. \tag{6}$$

Notice that the continuation value in equation (6) can be rewritten in terms of v_a.

$$\begin{aligned} v_a\left(S_i, S_{-i}, Y, Z_{ia}\right) &= \bar{u}_a\left(S_i, S_{-i}, Y, Z_{ia}\right) \\ &\quad + \beta\, E\left[\max_{a\in\mathcal{A}}\left\{v_a\left(S_i^\circ, S_{-i}^\circ, Y^\circ, Z_{ia}^\circ\right) + \varepsilon_{ia}^\circ\right\} \middle| S_\cdot, Y, a\right]. \end{aligned} \tag{7}$$

The outside option is assumed to be a benchmark of possible gains that the players are missing for being involved in the game. Once that a player chooses the outside option, it is assumed that he will do so indefinitely.[15] Under assumptions A and C to H the stage game payoff for the outside option and the distribution of its shocks are known and independent so the choice-specific value function for the outside option is given in equation (8).

$$v_0\left(Y\right) = u_0\left(Y\right) + \beta\, E\left[v_0\left(Y^\circ\right) + \varepsilon_{i0}^\circ | Y\right]. \tag{8}$$

The policy function of player i can be characterized as follows

$$\sigma\left(I_i\right) = \arg\max_{a\in\mathcal{A}}\left\{v_a\left(S_i, S_{-i}, Y, Z_{ia}\right) + \varepsilon_{ia}\right\}. \tag{9}$$

The probability that player i chooses alternative a is denoted by $\mathbb{P}_a\left(S_i, S_{-i}, Y, Z_{i\cdot}\right)$, and this probability can be identified from the observations because the players' moves and $\left(S_i', S_{-i}', Y', Z_{i\cdot}'\right)$ are observables. Besides,

$$\mathbb{P}_a\left(S_i, S_{-i}, Y, Z_{i\cdot}\right) = E\left[\mathbb{I}\left\{\sigma\left(I_i\right) = a\right\} \middle| S_i, S_{-i}, Y, Z_{i\cdot}\right], \tag{10}$$

with the expectation on the right hand side of equation (10) taken over the unobservable private shocks. Under assumptions C and J, equation (10) can be rewritten as

$$\begin{aligned} &\mathbb{P}_a\left(s_i, s_{-i}, y, z_{i\cdot}\right) \\ &= \text{Prob}\left\{ v_a\left(S_i, S_{-i}, Y, Z_{ia}\right) + \varepsilon_{ia} \geq \max_{k\in\mathcal{A}, k\neq a}\left[v_k\left(S_i, S_{-i}, Y, Z_{ik}\right) + \varepsilon_{ik}\right] \middle| s_i, s_{-i}, y, z_{i\cdot}\right\}. \end{aligned} \tag{11}$$

To shorten the notation, let $\Delta\varepsilon_i^{(a)} = \left(\Delta_0\varepsilon_i^{(a)}, \ldots, \Delta_J\varepsilon_i^{(a)}\right) \in \mathbb{R}^J$ be the vector of shocks in differences,[16] where $\Delta_k\varepsilon_i^{(a)} = \varepsilon_{ik} - \varepsilon_{ia}$. Similarly, let $\Delta v^{(a)} : \mathcal{S}^N \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}^J$,[17] then

$$\mathbb{P}_a\left(s_i, s_{-i}, y, z_{i\cdot}\right) = \text{Prob}\left\{\Delta\varepsilon^{(a)} \leq \Delta v^{(a)}\left(s_i, s_{-i}, y, z_{i\cdot}\right)\right\}.$$

In view of assumptions E, F and H, for all $a \neq 0, J$, the right hand side of equation (11) becomes

$$\begin{aligned} &\text{Prob}\left\{\Delta\varepsilon^{(a)} \leq \Delta v^{(a)}(\cdot)\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\Delta_0 v^{(a)}+\epsilon_a} \ldots \int_{-\infty}^{\Delta_J v^{(a)}+\epsilon_a} q\left(\epsilon_0, \ldots, \epsilon_a, \ldots, \epsilon_J\right)\, d\epsilon_0 \ldots d\epsilon_{a-1} d\epsilon_{a+1} \ldots d\epsilon_J d\epsilon_a. \end{aligned} \tag{12}$$

[15] It is possible to accommodate "re-entry" allowing it in an additional alternative, i.e. suspending operations.

[16] The element $\varepsilon_{ik} - \varepsilon_{ik}$ is always excluded in the vector $\Delta\varepsilon_i^{(k)}$.

[17] Where $\Delta v^{(a)}$ has functional elements, $\Delta_k v^{(a)} = v_a - v_k$.

Let $P = (P_0, P_1, ..., P_J)$ be the cumulative distribution of $\Delta\varepsilon^{(a)}$, such that for all $a \in \mathcal{A}$,

$$\mathbb{P}_a\left(S_i, S_{-i}, Y, Z_{i\cdot}\right) = P_a\left(\Delta v^{(a)}\left(S_i, S_{-i}, Y, Z_{i\cdot}\right)\right). \tag{13}$$

At this point, the main elements of the setup were presented. The next section describes the identification strategy from an arbitrary player' point of view.

IV Identification

The identification argument starts with the assumption that q_0 and u_0 are known, so the choice specific value function for the outside option that satisfies equation (8), i.e. v_0^*, can be identified. Under certain conditions this unique solution v_0^* is continuous and unbounded, which becomes useful in the next steps. The identification of the remaining v_j^* requires a different strategy because equation (7) depends on the unknown q and u_j for all $j \geq 1$. The next object to identify is P^*, followed by the choice specific value functions in differences, $\Delta v^{*(j)}$'s, and the identification of the v_j^*'s in levels through v_0^*.

The identification of P^* is firstly achieved on a subspace of its domain using the already identified v_0^* and properties derived for the v_j^*'s. The partially identified P^*, exclusion restrictions on the Z_{ij}'s and properties of P^* are useful to identify the $\Delta v^{*(j)}$'s on a larger subspace. Then, the identification strategy uses the partially identified $\Delta v^{*(j)}$'s to span the domain of P^* and complete its identification. The invertibitily of P^* allows to fully identify the $\Delta v^{*(j)}$'s and finally, the v_j^*'s in levels are identified through v_0^*.

An additional proof establishes that the unknown q^* can be identified from the distribution of the private shocks in differences (in particular from P_0^*) and the independence of $\varepsilon_{\cdot t0}$ with q_0 known. At this point all the objects in equation (7) are identified with the exception of the u_j's. In this final step, a key element is assumption B which provides conditions for the support of the conditional distribution of the states $S_{-i}|s_i$. Thus, the u_j's can be identified through a linear system of equations.

IV.1 Identification of v^* and P^*

Under assumption J, assumption K provides additional conditions to ensure that given some u_j's and Q, the v_j's are continuous and have a unique solution. This result is stated in lemma 1 and its proof is in section V. Lemma 2 specializes the result for v_0 in equation (8) which can be solved (independently of V) for some given u_0 and q_0. The particular assumptions required for the elements of I_i are irrelevant to prove these first results, so to keep the notation simpler the results are given in terms of the general transition probability function in equation (4), i.e. $F_{I^\circ|I,A}$, and its density with respect to a generic measure ϱ, $f_{I^\circ|I,A}$.[18]

The functions $\breve{u}_j$'s are allowed to be unbounded and then a key element to prove the results is the choice of a suitable norm that defines a particular Banach space. Let a function w be involved in the

[18]The subindexes are omitted here to simplify the notation.

definition of $\mathbb{B}_{w^\alpha}$, the normed linear space of w^α-bounded measurable functions on $\mathbb{R}$. A real-valued function υ in $\mathbb{B}_{w^\alpha}$ satisfies

$$\|\upsilon\|_{w^\alpha} \equiv \sup_{\iota \in \mathcal{I}} |\upsilon(\iota)| \, w(\iota)^{-\alpha} < \infty.$$

for some positive integer α. The function w^α is assumed to be known and chosen such that assumption K holds. The conditions in assumption K and the main arguments to prove uniqueness and continuity of V are similar to those found in Hernández-Lerma (1999), Lippman (1975) and Norets (2010).

Assumption K.

i. *$\mathcal{I}$ is a nonempty, complete, and separable metric space*
ii. *For all $j \in \mathcal{A}$, $\breve{u}_j(\iota, a_{-i})$ is continuous in $\iota \in \mathcal{I}$*
iii. *Prob$\{a_{-i}|\iota\}$ is continuous in $\iota \in \mathcal{I}$*
iv. *There exists a continuous function $w : \mathcal{I} \to \mathbb{R}$, such that for all $\iota \in \mathcal{I}$, $1 \leq w(\iota) < \infty$*
v. *There exists a positive integer α such that for $\dot{\breve{u}}_j(\iota) \equiv \max_{a_{-i} \in \mathcal{A}_{-i}} |\breve{u}_j(\iota, a_{-i})|$*

$$\|\max_{j \in \mathcal{A}} \dot{\breve{u}}_j\|_{w^\alpha} < \infty$$

vi. *There is a scalar $b \in (0, \infty)$ such that for all $\iota \in \mathcal{I}$, $n = 1, 2, \dots \alpha$*

$$\max_{j \in \mathcal{A}} \int w(\iota^\circ)^n f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \leq (w(\iota) + b)^n$$

vii. *For all $j \in \mathcal{A}$, $F_{I^\circ|I,A}$ is weakly continuous on $\mathcal{I}$, i.e. for all h in the set of continuous and bounded functions on $\mathcal{I}$, $\mathbb{C}_b(\mathcal{I})$, $\int h(\iota^\circ) \, F_{I^\circ|I,A}(d\iota^\circ|\iota, j)$ is continuous.*

Considering assumptions A and H note that the notation in assumption K is general, in the sense that Prob$\{a_{-i}|I_i\}$ in assumption K.iii makes reference to Prob$\{a_{-i}|S_i, S_{-i}, Y\}$ because the private information of player i is unknown for other players and then it does not have a direct role in their choices. This probability Prob$\{a_{-i}|S_i, S_{-i}, Y\}$ is assumed to be observable since it can be directly identified from the observations.

Lemma 1. *Under assumption K, for some positive integer J the operator Γ^J constructed with the Bellman operator*

$$\Gamma(V)(\iota) = \max_{a \in \mathcal{A}} \left\{ \sum_{a_{-i} \in \mathcal{A}_{-i}} \breve{u}_a(\iota, a_{-i}) \, Prob\{a_{-i}|\iota\} + \beta \int V(\iota^\circ) \, f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \tag{14}$$

is a contraction on $\mathbb{B}_{w^\alpha}$. Then the corresponding Bellman equation has a unique solution in $\mathbb{B}_{w^\alpha}$ such that

$$\Gamma(V^*) = V^*.$$

Moreover, $V(\iota)$ and $E[V(I_i^\circ)|\iota, a]$ are continuous in ι.

Lemma 1 shows that the player's problem has a unique solution when the stage game payoffs are unbounded under the usual norm but bounded in w-norm. Intuitively the w-norm bound is a growth restriction on the payoff functions under which there is some finite positive integer J such that Γ^J is a contraction mapping. The contraction property is proven by an induction argument in Lippman (1975) which relies on assumption K.vi to bound in w-norm the distance between any two functions in $\mathbb{B}_{w^\alpha}$. Provided that $(\mathbb{B}_{w^\alpha}, \|\cdot\|_{w^\alpha})$ is a complete metric space, it follows by the Contraction Mapping Theorem that Γ^J has a unique fixed point in $\mathbb{B}_{w^\alpha}$. The continuity proof shows that the mapping in equation (14) maps the set of continuous functions in $\mathbb{B}_{w^\alpha}$ into itself. Since this set is closed, the unique fixed point is also a continuous function.[19]

Example 5. Assume that there is a scalar state variable, s_i, that evolves as an independent AR(1) process, $s_i^\circ \sim N(\rho_s s_i, \sigma_s^2)$ for parameters ρ_s and σ_s. Similarly let $y^\circ \sim N(\rho_y y, \sigma_y^2)$ for parameters ρ_y and σ_y, $z_{ij} \sim N(\mu_{z_j}, \sigma_{z_j}^2)$ for μ_{z_j}, σ_{z_j} and $j = 1, 2, \ldots, J$, and $\varepsilon_{ij} \sim N(\mu_{\varepsilon_j}, \sigma_{\varepsilon_j}^2)$ for $\mu_{\varepsilon_j}, \sigma_{\varepsilon_j}$ and $j = 0, 1, \ldots, J$.
Let the stage game payoffs be $\breve{u}_j(s_i, y, z_{ij}, a_{-i}, \varepsilon_{ij}) = \mathfrak{u}_j(a_{-i}) + s_i + y + z_{ij} + \varepsilon_{ij}$ with $|\mathfrak{u}_j(a_{-i})| \leq B < \infty$ for $j \geq 1$ and $\breve{u}_0(y, \varepsilon_{i0}) = y + \varepsilon_{i0}$.
Note that $\bar{u}_j(s_i, s_{-i}, y, z_{ij}) = \sum_{a_{-i} \in \mathcal{A}_{-i}} \mathfrak{u}_j(a_{-i}) \text{Prob}\{a_{-i} | s_i, s_{-i}, y\} + s_i + y + z_{ij}$ where $\text{Prob}\{a_{-i} | s_i, s_{-i}, y\} = \text{Prob}\{a_{-i}|\iota\}$ because the private information of player i is not observed by players other than i and assumption H holds. If $\alpha = 1$ and $w(s_i, s_{-i}, y, z_{i\cdot}, \varepsilon_{i\cdot}) = B + 1 + s_i^2 + y^2 + \sum_{j=1}^J z_{ij}^2 + \sum_{j=0}^J \varepsilon_{ij}^2$ then assumption K.iv and K.v hold.
To see that assumption K.vi also holds, note that for ρ_s, $\rho_y \in [-1, 1]$ and finite variances

$$\begin{aligned}
&\int w(\iota^\circ) f_{I^\circ|I,A}(\iota^\circ|\iota, j)\varrho(d\iota^\circ) \\
&= B + 1 + (\rho_s s_i)^2 + \sigma_s^2 + (\rho_y y)^2 + \sigma_y^2 + \sum_{j=1}^{J}[\mu_{z_j}^2 + \sigma_{z_j}^2] + \sum_{j=0}^{J}[\mu_{\varepsilon_j}^2 + \sigma_{\varepsilon_j}^2] \\
&\leq B + 1 + s_i^2 + y^2 + b \leq (w(\iota) + b)
\end{aligned}$$

where $b = \sigma_s^2 + \sigma_y^2 + \sum_{j=1}^J[\mu_{z_j}^2 + \sigma_{z_j}^2] + \sum_{j=0}^J[\mu_{\varepsilon_j}^2 + \sigma_{\varepsilon_j}^2] < \infty$.[a]

[a] This example is partially borrowed from Norets (2010).

The present value of indefinitely choosing the outside option, v_0^*, can be solved without identifying the v_j's for all $j > 0$ and a variation of lemma 1 also applies to v_0 in equation (8). This guarantee that v_0 also has a unique and continuous solution. Lemma 2 states these results for v_0.

Lemma 2. *Let $\breve{u}_0$ and q_0 be known functions (assumptions C to E). Let $F_{Y^\circ|Y}$ be directly identified (assumption I) and assume that assumption K holds when applied to the relevant objects for v_0, i.e.*

i. *$\mathcal{Y}$ is a nonempty, complete, and separable metric space*
ii. *$\breve{u}_0 : \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ is continuous in $(y, \varepsilon_0) \in \mathcal{Y} \times \mathbb{R}$*
iii. *There exist a continuous function $w_0 : (\mathcal{Y} \times \mathbb{R}) \to \mathbb{R}$, such that for all $(y, \varepsilon_0) \in (\mathcal{Y} \times \mathbb{R})$, $1 \leq w_0(y, \varepsilon_0) < \infty$*
iv. *There exist a positive integer α such that $\|\breve{u}_0\|_{w^\alpha} < \infty$*

[19] Bhattacharya and Majumdar (1989) and Norets (2010) prove that the infinite summation of discounted expected payoffs of an agent satisfies the Bellman equation, which is left out of the scope of this paper.

v. *There is a* $b \in (0, \infty)$ *such that for all* $(y, \varepsilon_0) \in (\mathcal{Y} \times \mathbb{R})$*,* $n = 1, 2, \ldots \alpha$

$$\int w_0(y^\circ, \varepsilon_0^\circ)^n f_{Y^\circ|Y}(y^\circ|y)\, q_0(\varepsilon_0^\circ)\varrho_0(dy^\circ, d\varepsilon_0^\circ) \leq (w(y, \varepsilon_0) + b)^n$$

vi. $F_{Y^\circ|Y}$ *is weakly continuous on* $\mathcal{Y}$*.*

Then v_0 *in equation* (8) *has a unique solution* v_0^**. Moreover,* v_0^* *is a continuous function.*

At this point v_0^* is identified by lemma 2. A formal definition of identification helps to see this and the task ahead. Recall that $P_a : \mathbb{R}^J \to [0, 1]$ for all $a \in \mathcal{A}$. Denote by $\boldsymbol{\mathcal{P}}$ the space of functions with typical element $P = (P_0, \ldots, P_J)'$, and let the set of relevant choice specific value functions be $\boldsymbol{\mathcal{V}} = (\boldsymbol{\mathcal{V}}_0, \ldots, \boldsymbol{\mathcal{V}}_J)$ defined by equation (15).

$$\boldsymbol{\mathcal{V}} = \left\{ (v_0, \ldots, v_J) : \begin{array}{c} v_0 \text{ is the unique solution for equation (8), and} \\ \text{for all } a \geq 1,\ v_a \text{ satisfies equation (7) for some } P \in \boldsymbol{\mathcal{P}} \end{array} \right\}. \tag{15}$$

Denote by $\boldsymbol{\mathcal{U}}$ the set of stage game payoffs:

$$\boldsymbol{\mathcal{U}} = \{(u_0, \ldots, u_J) : \text{for all } j, u_j(\cdot) \text{ satisfies assumptions A, C, D and K}\}.$$

Definition 1 (Identification)**.** Let $(v^*, P^*, u^*) \in (\boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{U}})$ be the true functions to be identified and $\mathcal{D}$ be the probability measure on $\left(\mathcal{S} \times \mathcal{S}^{N-1} \times \mathcal{Y} \times \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_J\right)$.

Then identification is obtained when:

- $v_j(\cdot) = v_j^*(\cdot)$ a.s. with respect to the probability measure $\mathcal{D}$.
- $u_j(\cdot, a_{-i}) = u_j^*(\cdot, a_{-i})$ a.s. with respect to the probability measure $\mathcal{D}$ for all $a_{-i} \in \mathcal{A}_{-i}$.
- $P = P^*$, i.e. for all $j \in \mathcal{A}$, and $\ell \in R^J$, $P_j(\ell) = P_j^*(\ell)$ a.e. with respect to the Lebesgue measure in $\mathbb{R}^J$ and a.s. with respect to the probability measure $\mathcal{D}$.

Theorem 1 shows that under additional assumptions it is possible to identify a unique pair of functions $(P^*, v^*) \in (\boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{V}})$. The result depends on invertibility properties of P, which are established in lemma 3.

Lemma 3. *Let* $P_j : \mathbb{R}^J \to \mathbb{R}$*,* $a, b_j \in \mathbb{R}$ *for* $j = 0, \ldots, J$*,* $\overrightarrow{a}_0 = (a, \ldots, a)'$*,* $\overrightarrow{a}_j = \overrightarrow{a}_1 = (-a, 0, \ldots, 0)'$ *for* $j > 0$*, and*

$$\overrightarrow{b}_j = (b_j - b_0, \ldots, b_j - b_{j-1}, b_j - b_{j+1}, \ldots, b_j - b_J)',$$

for any $0 < j < J$*, with the corresponding changes for* $j = 0, J$*.*

Denote $(\lambda_1, \ldots, \lambda_J) \gtrsim (\gamma_1, \ldots, \gamma_J)$ *when for all* $j > 0$*,* $\lambda_j \geq \gamma_j$*, and for at least a* $k \in \{1, \ldots, J\}$*,* $\lambda_k > \gamma_k$*. Let* $m \gtrsim m'$ *imply that* $P_j(m) \neq P_j(m')$ *for any* j*.*

Then, $P_j(\overrightarrow{b}_j) = P_j(\overrightarrow{a}_j)$ *for all* $j = 0, \ldots, J$*, implies that* $b_0 - b_j = a$ *for all* $j = 1, \ldots, J$*.*

Assumptions L and M ensure properties of the primitives of the model.

Assumption L. *There is some* $\bar{s}_i \in \mathcal{S}$*,* $\bar{s}_{-i} \in \mathcal{S}^{N-1}$ *such that*

i. *For* $j > 0$*, any* $y \in \mathcal{Y}$*, and* $a_{-i} \in \mathcal{A}^{N-1}$*,* $u_j(\bar{s}_i, y, \mathcal{Z}_j, a_{-i}) = \mathbb{R}$*.*

ii. *If for some $j \neq 0$, $y \in \mathcal{Y}$ and $a_{-i} \in \mathcal{A}^{N-1}$ there is some $z_{ij}, \tilde{z}_{ij} \in \mathcal{Z}_j$, such that*

$$u_j\left(\bar{s}_i, y, z_{ij}, a_{-i}\right) \leq u_j\left(\bar{s}_i, y, \tilde{z}_{ij}, a_{-i}\right), \tag{16}$$

then for all $\tilde{a}_{-i} \in \mathcal{A}^{N-1}$ equation (16) holds with $a_{-i} = \tilde{a}_{-i}$.

iii. *If for some $j \neq 0$, $z_{ij} \in \mathcal{Z}_j$, and $a_{-i} \in \mathcal{A}^{N-1}$ there is some $y, \tilde{y} \in \mathcal{Y}$, such that*

$$u_j\left(\bar{s}_i, y, z_{ij}, a_{-i}\right) \leq u_j\left(\bar{s}_i, \tilde{y}, z_{ij}, a_{-i}\right), \tag{17}$$

then for all $\tilde{a}_{-i} \in \mathcal{A}^{N-1}$ equation (17) holds with $a_{-i} = \tilde{a}_{-i}$.

Assumption L provides sufficient conditions to guarantee that the choice specific value functions cover a range sufficiently large to fully identify P^*.

Assumption M. *Assume that for all $P \in \boldsymbol{\mathcal{P}}$, and all $a \in \mathcal{A}$,*

i. $P_a\left(\cdot \mid S_\cdot, Y, Z_{i\cdot}\right) = P_a\left(\cdot\right)$.
ii. *P_a is continuous and strictly increasing.*

Assumption M imposes restrictions on P^* that remove a possible direct dependence from the observable variables and allow for its invertibility.

Theorem 1. *Assume that $\breve{u}_0$ and q_0 are known functions (assumptions C to E). Let assumption K hold and $F_{Y^\circ|Y}$ be directly identified (assumption I). Then by lemma 2,*

- *For all $v_0 \in \boldsymbol{\mathcal{V}}_0$, $v_0 = v_0^*$.*

If in addition the known v_0^* is such that $v_0^*\left(\mathcal{Y}\right) = \mathbb{R}$, and assumptions A, F to H, J, L and M hold, then

- *$\left(v^*, P^*\right)$ is identified within $\left(\boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{P}}\right)$*

The proof of theorem 1 first shows that P^* is identified on a subspace in which the v_j's attain 0 for $j \geq 1$. This argument holds by the range of the functions, properties in assumption L and the already identified v_0^*. The v_j^*'s can be identified on a larger subspace mainly using the partial identification of P^*, the properties of P^* in lemma 3 and the broad range attained by v_0^*. Provided the exclusion restrictions on the Z_j's (assumption A) and the conditional independence assumptions (assumption H), the variation of the choice-specific value functions, $\Delta v^{(j)}$'s, on the mentioned larger subspace suffices to span the domain of P^* and allow its identification. Finally, P^* can be inverted to identify $\Delta v^{*(j)}$ on the whole space.

IV.2 Identification of u_j^*'s and q^*

By assumption E, q_0^* is known and by theorem 1, P^* is identified, then it is also possible to identify q^* and lemma 4 states this result.

Lemma 4. *Under assumption H, if P_0^* and q_0^* are provided, q^* is identified.*

The problem in lemma 4 reduces to a deconvolution argument because equation (12) holds and $\varepsilon_{\cdot 0}$ is independent of $\varepsilon_{\cdot,-0}$ by assumption H. Knowing P_0^* and q_0^*, the characteristic function of q_{-0}^* can be identified and then q_{-0}^* and q^* become known.

The argument to identify the stage game payoff u_j^* for $j > 0$ uses the (v^*, P^*) already identified in theorem 1. Equation (7) shows that the expected stage game payoff, $\bar{u}_j^*$ for $j > 0$ can be identified using the q^* identified in lemma 4 and the choice specific value functions, v^*.

There are two key assumptions to identify u_j, the exclusion restrictions in assumption A and the support conditions in assumption B. Assumption A implies that the stage game payoffs of player i does not depend on the state variables of other players, S_{-i} after considering their actions, i.e. a_{-i}. Then, following Theorem 1 in Bajari, Chernozhukov, Hong, and Nekipelov (2009), the identification problem can be reduced to a linear system of unknown u_j's, as in equation (18), which has a unique solution if assumption B holds, i.e. the conditional distribution of the states $S_{-i}|s_i$ has at least $(J+1)^{N-1}$ points in its support.

$$\begin{aligned} \bar{u}_{a_i}(s_i, s_{-i}, y, z_{ia_i}) &= E\left[u_{a_i}(s_i, y, z_{ia_i}, A_{-i}) \mid s_i, s_{-i}, y, z_{ia_i}, a_i\right] \\ &= \sum_{a_{-i} \in \mathcal{A}_{-i}} u_{a_i}(s_i, y, z_{ia_i}, a_{-i}) \, \text{Prob}\left\{a_{-i} | s_i, s_{-i}, y\right\}, \end{aligned} \tag{18}$$

where $z_{i\cdot}$ is not observed by players other than i.[20] Theorem 2 states the results in this section.

Theorem 2. *Let assumption H hold and assume that v^*, P_0^* and q_0^* are known, then the expected stage game payoffs, $\bar{u}_j^*$ for all $j > 0$ are identified.*

If in addition, assumptions A to C hold and equation (18) defines at least $(J+1)^{N-1}$ linearly independent equations almost surely, then the stage game payoffs u_j for all $j > 0$ are identified for almost every point in their support.

The proof of theorem 2 is omitted because it follows directly from the result in lemma 4 and equation (7). Besides, the final statement in theorem 2 is analogous to the rank condition of an ordinary least square regression with Prob $\{a_{-i}|s_i, \cdot, y\}$ for all a_{-i} as explanatory variables and $\bar{u}_{a_i}(s_i, \cdot, y, z_{ia_i})$ as the dependent one.

Theorems 1 and 2 summarize the identification strategy for the objects of interest in definition 1.

V Conclusions

The setup of this model has a variety of applications where stochastic dynamic discrete choice models are useful to study agents' simultaneous intertemporal decisions in an infinite horizon. There is a finite number of players who are forward-looking and choose among available alternatives to optimize their own payoffs. This is a game of incomplete information where the players receive private information, i.e. the "shocks", and can be heterogeneous. Every period, the players face an individual decision-making

[20]
$$\begin{aligned} \text{Prob}\left\{a_{-i} | s_i, s_{-i}, y\right\} = &\prod_{n=1}^{N-1} \prod_{b_n \in \mathcal{A}} \mathbb{I}((b_1, \ldots, b_n, \ldots, b_{N-1}) = a_{-i}) \\ &\int P_{b_n}^*\left(\Delta v^{*(b_n)}(s_\cdot, y, z_{n\cdot})\right) F_{Z\cdot|S.Y}(dz_{n\cdot}|s.\, y). \end{aligned}$$

process after observing their own "shocks" and the public information. The current actions of a player affect his own current and future payoffs, which are also affected by other players' actions and the random evolution of state variables that may depend on the joint (past) actions of all players.

This paper presents a new nonparametric identification result that allows for unknown stage game payoff for all but one alternative and also unknown joint distribution of the private information with shocks that can be dependent across alternatives and have different marginal distributions. The result is proved under the assumption that players are playing a single equilibrium and the private shocks are independent over time and across players.

This new nonparametric identification strategy allows the researcher to disregard arbitrary assumptions on the distribution of the private information across alternatives. Thus, the model can represent different and unknown distributions of the shocks for each alternative, which for instance can be more relevant when the alternatives represent different types of business strategies.

Instead of assuming parametric payoff functions and known distributions of the shocks, some key information for the identification strategy comes from the outside option. The payoff for this alternative is assumed to respond to exogenous variations in the "states of the economy", for instance the reader can think of an economy with macroeconomic fluctuations. In this sense, the assumptions required in this model are more general than the available ones in the literature, with some variations used to overcome the nonparametric non-identification results in Rust (1994) and Magnac and Thesmar (2002).

The identification strategy relies on exclusion restrictions across the payoff functions, the known outside option's payoffs and properties of the joint distribution of the private information in differences. The model does not need parametric assumptions on the unknown stage game payoff functions of the alternatives, except for the assumptions on the outside option's payoffs.

This identification strategy allows the researcher to explore questions that would not be possible to answer under parametric methods. For instance, if a simplifying assumption about the joint distribution of the private information across alternatives would yield similar results or policy implications to the more general (unknown) case. Thus instead of assuming a joint distribution of the private information or imposing parametric assumptions for all payoff functions, the researcher can learn the shapes of these objects among a large class of functions. Since the consistency of the estimators derived from structural models usually depends on the accuracy of their assumptions, this nonparametric identification result makes possible to reduce some of these concerns checking for robustness.

Finally, the result is an example of a nonparametric identification strategy that relaxes assumptions hard to justify in applications. In this type of dynamic discrete choice models there is an already known nonparametric non-identification result, hence under the standard setup all assumptions cannot be jointly relaxed and the identification strategy in this paper proposes key modifications to overcome the difficulties that the literature has found. Different assumptions could be more plausible than others and changes in the setup of the model lead to customized identification strategies that can shed light on questions that would otherwise be difficult to answer.

Appendix

Proofs

Lemma 1

Proof. The proof of the contraction property of Γ^J is given in Lippman (1975) and extended by Norets (2010). The main arguments of the proof are as follow. Pick $V_1, V_2 \in \mathbb{B}_{w^\alpha}$

$$\begin{aligned}
|[\Gamma(V_1) - \Gamma(V_2)](\iota)| &= \left| \max_{a\in\mathcal{A}} \left\{ \bar{u}_a(\iota) + \beta \int V_1(\iota^\circ) f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \right. \\
&\quad \left. - \max_{a\in\mathcal{A}} \left\{ \bar{u}_a(\iota) + \beta \int V_2(\iota^\circ) f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \right| \\
&\leq \max_{a\in\mathcal{A}} \left| \left\{ \bar{u}_a(\iota) + \beta \int V_1(\iota^\circ) f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \right. \\
&\quad \left. - \left\{ \bar{u}_a(\iota) + \beta \int V_2(\iota^\circ) f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \right| \\
&\leq \beta \max_{a\in\mathcal{A}} \left\{ \int |V_1 - V_2|(\iota^\circ) f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \\
&\leq \beta \max_{a\in\mathcal{A}} \left\{ \int \|V_1 - V_2\|_{w^\alpha} w(\iota^\circ)^\alpha f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\}
\end{aligned}$$

The first inequality follows from a property of the maximum operator, the following line is true due to a property of the absolute values of integrals and the last inequality applies by the definition of the w-norm.

$$\begin{aligned}
|[\Gamma(V_1) - \Gamma(V_2)](\iota)| &\leq \beta\ \|V_1 - V_2\|_{w^\alpha} \max_{a\in\mathcal{A}} \left\{ \int w(\iota^\circ)^\alpha f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \\
&\leq \beta\ \|V_1 - V_2\|_{w^\alpha} (w(\iota) + b)^\alpha
\end{aligned} \tag{19}$$

The last inequality follows by assumption K.vi. Lippman (1975), pag. 1228-1229, shows by induction that equation (19) implies

$$\left\|\Gamma^J(V_1) - \Gamma^J(V_2)\right\|_{w^\alpha} \leq \beta^J (1 + Jb)^\alpha\ \|V_1 - V_2\|_{w^\alpha}.$$

To see the argument let $\left|\Gamma^2 \circ \ldots \circ \Gamma^{n+1}(V_1 - V_2)\,(\iota)\right| \leq \beta^n\ \|V_1 - V_2\|_{w^\alpha}\ (w(\iota) + nb)^\alpha$, then

$$\begin{aligned}
\left|\Gamma[\Gamma^2 \circ \ldots \circ \Gamma^{n+1}(V_1 - V_2)]\,(\iota)\right| &\leq \beta \max_{a\in\mathcal{A}} \left\{ \int \left|\Gamma^2 \circ \ldots \circ \Gamma^{n+1}(V_1 - V_2)\right|(\iota^\circ) \right. \\
&\left. f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \\
&\leq \beta^{n+1}\ \|V_1 - V_2\|_{w^\alpha} \max_{a\in\mathcal{A}} \left\{ \int (w\,(\iota^\circ) + nb)^\alpha \right. \\
&\left. f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \\
&= \beta^{n+1}\ \|V_1 - V_2\|_{w^\alpha} \max_{a\in\mathcal{A}} \left\{ \sum_{i=0}^{\alpha} \binom{\alpha}{i} (nb)^{\alpha - i} \int w\,(\iota^\circ)^i \right. \\
&\left. f_{I^\circ|I,A}(\iota^\circ|\iota, a)\varrho(d\iota^\circ) \right\} \\
&\leq \beta^{n+1}\ \|V_1 - V_2\|_{w^\alpha} \sum_{i=0}^{\alpha} \binom{\alpha}{i} (nb)^{\alpha - i}\, (w(\iota) + b)^i \\
&= \beta^{n+1}\ \|V_1 - V_2\|_{w^\alpha}\ (w(\iota) + (n+1)b)^\alpha\,,
\end{aligned}$$

where the last inequality follows by assumption K.vi and finally the induction hypothesis is verified.

By assumption K.iv, $\left\|\Gamma^1 \circ \ldots \circ \Gamma^n(V_1 - V_2)\right\|_{w^\alpha} \leq \beta^n\ \|V_1 - V_2\|_{w^\alpha}\ (1 + nb)^\alpha$ thus Γ^J is a contraction for some finite J, since for J large enough $\beta^J\,(1 + Jb)^\alpha < 1$. By the Contraction Mapping Theorem, assumption K.i and provided that $(\mathbb{B}_{w^\alpha}, \|\cdot\|_{w^\alpha})$ is a complete metric space (see Proposition 7.2.1 in Hernández-Lerma, 1999), then Γ^J has a unique fixed point in $\mathbb{B}_{w^\alpha}$.

To show continuity, let $\mathbb{C}$ be the set of continuous functions and pick $V \in \mathbb{C} \cap \mathbb{B}_{w^\alpha}$. To see that the unique fixed point of the mapping Γ^J is a continuous function it is enough to show that $\Gamma : \mathbb{C} \to \mathbb{C}$ because $\mathbb{C}$ is closed and $\mathbb{C} \subset \mathbb{B}_{w^\alpha}$. Thus for $V \in \mathbb{C}$ and $\iota_1, \iota_2 \in \mathcal{I}$ note that

$$\begin{aligned}
|\Gamma(V)\,(\iota_1) - \Gamma(V)\,(\iota_2)| &\leq \max_{a\in\mathcal{A}} \left| \left\{ \bar{u}_a\,(\iota_1) + \beta \int V\,(\iota^\circ)\, f_{I^\circ|I,A}(\iota^\circ|\iota_1, a)\varrho(d\iota^\circ) \right\} \right. \\
&\left. - \left\{ \bar{u}_a\,(\iota_2) + \beta \int V\,(\iota^\circ)\, f_{I^\circ|I,A}(\iota^\circ|\iota_2, a)\varrho(d\iota^\circ) \right\} \right| \\
&\leq \max_{a\in\mathcal{A}} |\bar{u}_a\,(\iota_1) - \bar{u}_a(\iota_2)| \\
&+ \beta \max_{a\in\mathcal{A}} \left| \int V\,(\iota^\circ)\, f_{I^\circ|I,A}(\iota^\circ|\iota_1, a)\varrho(d\iota^\circ) \right. \\
&\left. - \int V\,(\iota^\circ)\, f_{I^\circ|I,A}(\iota^\circ|\iota_2, a)\varrho(d\iota^\circ) \right|
\end{aligned} \tag{20}$$

Provided assumptions K.ii and K.iii, $\bar{u}_a$ is continuous for all $a \in \mathcal{A}$. Therefore, given any $\epsilon > 0$ there is a δ_1 such that $\|\iota_1 - \iota_2\| < \delta_1$ implies $\max_{a\in\mathcal{A}} |\bar{u}_a(\iota_1) - \bar{u}_a(\iota_2)| < \epsilon/2$.

Arguments similar to the proof of Lemma 8.5.5 (a) in Hernández-Lerma (1999) are useful to show that the continuation value, $\int V\,(\iota^\circ)\, F_{I^\circ|I,A}(d\iota^\circ|\iota, a)$, is a continuous function. Define $\upsilon_w \equiv V + \|V\|_{w^\alpha}\, w^\alpha$. By definition of the norm and assumption K.iv, υ_w is a non-negative function in $\mathbb{B}_{w^\alpha}$, then there is a nondecreasing sequence of measurable bounded functions υ_k such that $\upsilon_k \uparrow \upsilon_w$. By assumption K.vii, $F_{I^\circ|I,A}$ is weakly continuous then let $\iota_n \to \iota$ to see that

$$\begin{aligned}\liminf_{n\to\infty} \int \upsilon_w(\iota^\circ) F_{I^\circ|I,A}(d\iota^\circ|\iota_n, a) &\geq \liminf_{n\to\infty} \int \upsilon_k(\iota^\circ) F_{I^\circ|I,A}(d\iota^\circ|\iota_n, a) \\ &= \int \upsilon_k(\iota^\circ) F_{I^\circ|I}(d\iota^\circ|\iota, a).\end{aligned}$$

Let $k \to \infty$, then by the Monotone Convergence Theorem

$$\liminf_{n\to\infty} \int \upsilon_w(\iota^\circ) F_{I^\circ|I,A}(d\iota^\circ|\iota_n, a) \geq \int \upsilon_w(\iota^\circ) F_{I^\circ|I,A}(d\iota^\circ|\iota, a)$$

Therefore $\int \upsilon_w(\iota^\circ) F_{I^\circ|I,A}(d\iota^\circ|\cdot, a)$ is l.s.c. which together with assumption K.iv and K.vii gives that for all $V \in \mathbb{C} \cap \mathbb{B}_{w^\alpha}$, $\int V(\iota^\circ) F_{I^\circ|I,A}(d\iota^\circ|\iota, a)$ is l.s.c. on $\mathcal{I}$. To see that the continuation value is upper semi-continuous, pick $-V$ (in place of V) and note that by the later result $\int -V(\iota^\circ) F_{I^\circ|I,A}(d\iota^\circ|\iota, a)$ is also l.s.c. on $\mathcal{I}$. Since, the continuation value is both upper and l.s.c., then it is continuous on $\mathcal{I}$.

Pick any $\epsilon > 0$, then by continuity of the continuation value there is a δ_2 such that $\|\iota_1 - \iota_2\| < \delta_2$ implies $\max_{a\in\mathcal{A}} \left|\int V(\iota^\circ) f_{I^\circ|I,A}(\iota^\circ|\iota_1, a)\varrho(d\iota^\circ) - \int V(\iota^\circ) f_{I^\circ|I,A}(\iota^\circ|\iota_2, a)\varrho(d\iota^\circ)\right| < \frac{\epsilon}{2\beta}$. Coming back to equation (20), note that for any $\epsilon > 0$ there is a $\delta = \min\{\delta_1, \delta_2\}$ such that $\|\iota_1 - \iota_2\| < \delta$ implies $|\Gamma(V)(\iota_1) - \Gamma(V)(\iota_2)| < \epsilon$. Thus $\Gamma(V)$ is a continuous function. ■

Lemma 2

Proof. The proof of lemma 2 is analogous to the more general case proved for lemma 1. ■

Lemma 3

Proof. Suppose by contradiction that $b_0 - b_j \neq a$ for some j and without loss of generality (wlog) let $j = 2$. If $\max_j(b_0 - b_j) \leq a$ and $b_0 - b_2 < a$ then $P_0(\overrightarrow{b}_0) \neq P_0(\overrightarrow{a}_0)$ by assumption. If $\max_j(b_0 - b_j) > a$ and wlog $b_0 - b_1 = \max_j(b_0 - b_j)$, then $P_1(\overrightarrow{b}_1) \neq P_1(\overrightarrow{a}_1)$ because $b_1 - b_j \leq 0$ for $j > 2$ and $b_1 - b_0 < -a$. Therefore, it cannot be true that $b_0 - b_2 \neq a$ when $P_j(\overrightarrow{b}_j) = P_j(\overrightarrow{a}_j)$ for all $j = 0, \ldots, J$. ■

Theorem 1

Proof. Suppose that $(v, P) \in (\boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{P}})$ and for all $j \in \mathcal{A}$,

$$\mathbb{P}_j(s_i, s_{-i}, y, z_{i\cdot}; (v, P)) = \mathbb{P}_j(s_i, s_{-i}, y, z_{i\cdot}; (v^*, P^*)) \text{ a.s.} (\mathcal{D}).$$

Then by assumption M.i,

$$P_j\left(\Delta v^{(j)}(s_i, s_{-i}, y, z_{i\cdot})\right) = P_j^*\left(\Delta v^{*(j)}(s_i, s_{-i}, y, z_{i\cdot})\right) \text{ a.s.} (\mathcal{D}).$$

By Lemmas 1 and 2, for all j, v_j is continuous on $\mathcal{I}$. Since P_j and P_j^* are continuous (assumption M.ii) and v, v^* are continuous, it follows that for all $(s_\cdot, y, z_{i\cdot}) \in (\mathcal{S}_\cdot \times \mathcal{Y} \times \mathcal{Z}_\cdot)$,

$$P_j\left(\Delta v^{(j)}(s_\cdot, y, z_{i\cdot})\right) = P_j^*\left(\Delta v^{*(j)}(s_\cdot, y, z_{i\cdot})\right). \tag{21}$$

Provided assumption A and assumptions L.i and L.ii, it follows that $\bar{u}_j(\bar{s}_i, \bar{s}_{-i}, y, \mathcal{Z}_j) = \mathbb{R}$ for any $j \in \mathcal{A}$, $y \in \mathcal{Y}$. Note that the continuation value in equation (7) does not depend on $Z.$, see assumption H, then for all j, it follows that $v_j(\bar{s}_i, \bar{s}_{-i}, y, \mathcal{Z}_j) = \mathbb{R}$. Therefore, for each j there is at least a function $\zeta_{v_j} : \mathcal{Y} \to \mathbb{R}$, such that for each $y \in \mathcal{Y}$,

$$v_j\left(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v_j}(y)\right) = 0. \tag{22}$$

Consider a first case for all $(\bar{s}., y, \zeta_{v.}(y)) \in \left(\{\bar{s}.\} \times \mathcal{Y} \times \prod_{j=1}^{J}\left\{\zeta_{v_j}(\mathcal{Y})\right\}\right)$, then by equation (21) it follows that for all $j \in \mathcal{A}$,

$$P_j\left(\Delta v^{(j)}(\bar{s}., y, \zeta_{v.}(y))\right) = P_j^*\left(\Delta v^{*(j)}(\bar{s}., y, \zeta_{v.}(y))\right), \tag{23}$$

where by equation (22),

$$\begin{aligned}\Delta v^{(0)}(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v.}(y)) &= \left(v_0(y) - v_1(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v_1}(y)), \ldots, v_0(y) - v_J(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v_J}(y))\right) \\ &= (v_0(y), \ldots, v_0(y)),\end{aligned}$$

and for all $J > j > 1$,

$$\begin{aligned}\Delta v^{(j)}(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v.}(y)) =& (v_j(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v_j}(y)) - v_0(y), v_j(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v_j}(y)) \\ &- v_1(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v_1}(y)), \ldots, v_j(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v_j}(y)) \\ &- v_J(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v_J}(y))) \\ =& (-v_0(y), 0, \ldots, 0).\end{aligned} \tag{24}$$

The analogous to equation (24), with the straightforward modifications, also holds for $j = 1, J$.

By Lemma 2, for all $v_0 \in \boldsymbol{\mathcal{V}}_0$, $v_0 = v_0^*$, thus equations (22) and (23) imply that for all $(\bar{s}., y, \zeta_{v.}(y)) \in \left(\{\bar{s}.\} \times \mathcal{Y} \times \prod_{j=1}^{J}\left\{\zeta_{v_j}(\mathcal{Y})\right\}\right)$ and $j \in \mathcal{A}$,

$$P_j\left(\Delta v^{*(j)}(\bar{s}., y, \zeta_{v.}(y))\right) = \mathbb{P}_j\left(\bar{s}_i, \bar{s}_{-i}, y, \zeta_{v.}(y); (v^*, P^*)\right).$$

Therefore, P^* is identified on $\left(\{\bar{s}.\} \times \mathcal{Y} \times \prod_{j=1}^{J}\left\{\zeta_{v_j}(\mathcal{Y})\right\}\right)$.

Consider a second and last case for some $y \in \mathcal{Y}$ and $z_{i\cdot} \notin \prod_{j=1}^{J}\left\{\zeta_{v_j}(y)\right\}$ such that for all j,

$$P_j\left(\Delta v^{(j)}(\bar{s}_i, \bar{s}_{-i}, y, z_{i\cdot})\right) = \mathbb{P}_j\left(\bar{s}_i, \bar{s}_{-i}, y, z_{i\cdot}; (v^*, P^*)\right).$$

In this second case, w.l.o.g., first proceed identifying $v_1^*(\bar{s}_i, \bar{s}_{-i}, y, z_{i1})$ as follows. Recall that $v_j(\bar{s}_i, \bar{s}_{-i}, y, \mathcal{Z}_j) = \mathbb{R}$, for all j, y and given assumption L.iii, pick for all $j \neq 1$ some $\hat{z}_j \in \mathcal{Z}_j$, $\tilde{y} \in \mathcal{Y}$ and $\tilde{z}_{i\cdot} \in \prod_{j=1}^{J}\left\{\zeta_{v_j}(\tilde{y})\right\}$ such that for all j

$$P_j^*\left(\Delta v^{(j)}(\bar{s}_i, \bar{s}_{-i}, y, z_{i1}, \hat{z}_{i2}, \ldots, \hat{z}_{iJ})\right) = \mathbb{P}_j\left(\bar{s}_i, \bar{s}_{-i}, \tilde{y}, \tilde{z}_{i\cdot}; (v^*, P^*)\right).$$

The result for the first case guarantees that $P_j^*\left(\Delta v^{*(j)}(\bar{s}_i, \bar{s}_{-i}, \tilde{y}, \zeta_{v.}(\tilde{y}))\right) = \mathbb{P}_j\left(\bar{s}_i, \bar{s}_{-i}, \tilde{y}, \tilde{z}_{i\cdot}; (v^*, P^*)\right)$ for all j, so

$$P_j^*\left(\Delta v^{(j)}(\bar{s}_i, \bar{s}_{-i}, y, z_{i1}, \hat{z}_{i2}, \ldots, \hat{z}_{iJ})\right) = P_j^*\left(\Delta v^{*(j)}(\bar{s}_i, \bar{s}_{-i}, \tilde{y}, \tilde{z}_{i\cdot})\right) \tag{25}$$

By lemma 3 and assumption M, equation (25) implies $\Delta v^{(j)}\left(\bar{s}_i, \bar{s}_{-i}, y, z_{i1}, \hat{z}_{i2}, \ldots, \hat{z}_{iJ}\right) = \Delta v^{*(j)}\left(\bar{s}_i, \bar{s}_{-i}, \tilde{y}, \tilde{z}_{i\cdot}\right)$ for all j and in particular $v_1\left(\bar{s}_i, \bar{s}_{-i}, y, z_{i1}\right) - v_0\left(y\right) = v_1^*\left(\bar{s}_i, \bar{s}_{-i}, \tilde{y}, \tilde{z}_{i1}\right) - v_0^*\left(\tilde{y}\right)$. Then $v_1\left(\bar{s}_i, \bar{s}_{-i}, y, z_{i1}\right) = v_0\left(y\right) - v_0^*\left(\tilde{y}\right)$ and the identification result follows since $v_0 = v_0^*$ by Lemma 2. The analogous arguments apply to $v_j\left(\bar{s}_i, \bar{s}_{-i}, y, z_{ij}\right)$ for all j, then all choice specific values functions are identified for arbitrary $y \in \mathcal{Y}$ and $z_{i\cdot} \notin \prod_{j=1}^{J}\left\{\zeta_{v_j}(y)\right\}$.

Combining the first and second cases, $v_j^*(\cdot)$ is identified on $\left(\{\bar{s}_\cdot\} \times \mathcal{Y} \times \mathcal{Z}_\cdot\right)$ for all $j > 0$. Recall that for all $y \in \mathcal{Y}$, $v_j\left(\bar{s}_i, \bar{s}_{-i}, y, \mathcal{Z}_j\right) = \mathbb{R}$, and by assumption, the range of the known v_0^* is also the reals, i.e. $v_0^*\left(\mathcal{Y}\right) = \mathbb{R}$, then P^* is identified. Since P^* is invertible (see proposition 1 in Hotz and Miller 1993) and identified, the mapping $\Delta v^{*(j)}(\cdot)$ is identified for all j on $\left(\mathcal{S}^N \times \mathcal{Y} \times \mathcal{Z}_\cdot\right)$. Finally by lemma 2, v_0^* is known, thus the levels, $v_j^*(\cdot)$ are also identified for all j on $\left(\mathcal{S}^N \times \mathcal{Y} \times \mathcal{Z}_\cdot\right)$. ■

Lemma 4

Proof. Recall that $\varepsilon_{i,-0}$ and ε_{i0} are independent, define $p_0\left(\Delta\nu\right) = \partial^J P_0\left(\Delta\nu\right)/\partial\Delta\nu_1 ... \partial\Delta\nu_J$, i.e. the joint probability density function of $\Delta\varepsilon_i^{(0)} = \varepsilon_{i,-0} - \varepsilon_{i0} \cdot 1_J$, and note that

$$P_0\left(\Delta\nu\right) = \int_{\mathbb{R}}\left(\int_{-\infty}^{\Delta\nu_1+\epsilon_0} ... \int_{-\infty}^{\Delta\nu_J+\epsilon_0} q_{-0}\left(y_{-0}\right) q_0\left(\epsilon_0\right) dy_1 ... dy_J\right) d\epsilon_0$$
$$p_0\left(\Delta\nu\right) = \int_{\mathbb{R}} q_{-0}\left(\Delta\nu_1 + \epsilon_0, \ldots, \Delta\nu_J + \epsilon_0\right) q_0\left(\epsilon_0\right) d\epsilon_0.$$

Hence, the problem of identifying q_{-0} is a deconvolution problem. For an arbitrary density function f, denote its Fourier transform by ϕ_f, thus

$$\begin{aligned}\phi_{p_0}\left(t_1, \ldots, t_J\right) &\equiv \int_{\mathbb{R}^J} e^{i\sum_{a=1}^{J} t_a \Delta\nu_a} p_0\left(\Delta\nu\right) d\Delta\nu = E\left[e^{i\sum_{a=1}^{J} t_a\epsilon_a} e^{i\sum_{a=1}^{J} t_a(-\epsilon_0)}\right] \\ &= E\left[e^{i\sum_{a=1}^{J} t_a\epsilon_a}\right] E\left[e^{i\epsilon_0\left(\sum_{a=1}^{J} - t_a\right)}\right] \\ &= \phi_{q_{-0}}\left(t_1, \ldots, t_J\right) \phi_{q_0}\left(\sum_{a=1}^{J} - t_a\right).\end{aligned}$$

Provided that P_0 and q_0 are known, $\phi_{q_{-0}}$ is identified

$$\phi_{q_{-0}}\left(t\right) = \phi_{p_0}\left(t\right) / \phi_{q_0}\left(\sum_{a=1}^{J} - t_a\right).$$

Finally, q_{-0} is identified from its characteristic function $\phi_{q_{-0}}$ as follows

$$q_{-0}\left(\alpha_1, \ldots, \alpha_J\right) = \frac{1}{\left(2\pi\right)^J} \int_{\mathbb{R}^J} e^{i\sum_{a=1}^{J} t_a\alpha_a} \phi_{q_{-0}}\left(t\right) dt,$$

and $q\left(\alpha_0, \ldots, \alpha_J\right) = q_{-0}\left(\alpha_1, \ldots, \alpha_J\right) q_0\left(\alpha_0\right)$. ■

On the existence of a Markov Perfect Equilibrium in dynamic games with unbounded payoffs

Regarding the theoretical foundations for the existence of equilibriums in dynamic Markov stochastic games of incomplete information, the literature has made relevant progress, although there are not general results that guarantee such existence and consequently no extensions with unbounded stage game payoffs (see Dutta and Sundaram (1998) for further details about some difficulties). The available results about the existence of MPE in stochastic dynamic games require for instance bounded reward functions, compact state spaces (or both), zero-sum frameworks or the so-called maximin strategies. Among the partial results available for the Markov-perfect equilibrium framework, Parthasarathy (1982) studies the existence of Markov perfect equilibrium in discounted stochastic games assuming finite action sets. Under strong separability assumptions on the stage payoffs and continuity of the transition probabilities and the reward functions, Parthasarathy (1982) shows that the existence of a p–equilibrium[21] in Markovian strategies implies the existence of an equilibrium in stationary (Markov) strategies when the state space is uncountable and compact. The results in Parthasarathy (1982) have some remarkable restrictions since the state space is compact, the stage game payoffs and the transition function are additive separable in the actions and a p–equilibrium may not exists if the state space is not countable. Nowak (1985) pursues an alternative approach and proves the existence of an "approximate equilibrium" concept, the ε–equilibria, in stationary strategies in infinite horizon Markovian game. The idea is to approximate the original game by a sequence of finite or countable state games which can be as close as the original game as desired. Nowak (1985)'s result requires bounded reward functions, and a partition of the state space such that its elements are countable with constant values for the transition probabilities and reward functions.[22]

Dynamic Markovian games have been an active topic in applied and theoretical research which is still under development. Recently, Levy (2013) provides an example of a game that satisfies the assumptions used in positive results (for special cases) on the existence of stationary Markov equilibrium and shows that the game has a stationary approximate MPE, but it does not have a MPE. Matkowski and Nowak (2011) develops the idea of k–local contractions in Rincón-Zapatero and Rodríguez-Palmero (2003, 2009)[23] and proves results for the existence of a unique solution to the Bellman equation in a discounted stochastic dynamic programming model with unbounded returns, but this existence result was not extended to Markovian games. Duggan (2012) proves the existence of a stationary Markov perfect equilibrium in noisy stochastic games with bounded stage game playoffs. The noise is assumed public and this is a key element to derive the result in Duggan (2012).[24] Jaśkiewicz and Nowak (2011a) develops an alternative approach to prove the existence of an optimal stationary policy in discounted dynamic programming problems with unbounded returns and uncountable state space. The work of Jaśkiewicz and Nowak (2011a) prove the convergence of a value iteration algorithm to a solution to the Bellman equation, but such solution is not necessarily unique and the results were not extended to Markovian games.

Some existence results are available for Markov zero-sum stochastic games with unbounded payoffs. In a zero-sum two players' game, Jaśkiewicz and Nowak (2006) proves a minimax theorem[25] with optimal stationary strategies for the players. Küenle (2007) proves the existence of an optimal stationary strategy for the maximizing player, and an ε–optimal stationary strategy for the minimizing player. Jaśkiewicz and Nowak (2011b) studies further the stochastic Markov game with unbounded payoffs under maximin

[21]A p–equilibriumis a strategy profile in which each player's strategy is a best-response to the profile from p–almostevery initial state, where $p$is a given probability measure on the state space.

[22]Moreover, the transition probabilities must admit a density function with respect to some measure on the state space and this density function must be continuous in the players actions.

[23]Rincón-Zapatero and Rodríguez-Palmero (2003, 2009) studies a deterministic dynamic programing problem.

[24]The noise in Duggan (2012) is not directly affected by the previous period's state and actions.

[25]Under a minimax setup a player considers a range of models and chooses the alternative that minimizes his loss given by the worst possible case.

strategies and the zero-sum setup. In this later case, Jaśkiewicz and Nowak (2011b) proves that the maximizer has optimal stationary strategy whereas the minimizer has an ε–optimal semi-stationary one for any $\varepsilon > 0$.

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