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Player splitting, players merging, the Shapley set value and the Harsanyi set value

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Abstract

Shapley (1953a) introduced the weighted Shapley values as a family of values, also known as Shapley set. For each exogenously given weight system exists a seperate TU-value. Shapley (1981) and Dehez (2011), in the context of cost allocation, and Radzik (2012), in general, presented a value for weighted TU-games that covers the hole family of weighted Shapley values all at once. To distinguish this value from a weighted Shapley value in TU-games we call it Shapley set value. This value coincides with a weighted Shapley value only on a subdomain and allows weights which can depend on coalition functions.

Hammer (1977) and Vasil'ev (1978) introduced independently the Harsanyi set, also known as selectope (Derks, Haller and Peters, 2000), containing TU-values which are referred to as Harsanyi-payoffs. These values are obtained by distributing the dividends from all coalitions by a sharing system that is independent from the coalition function.

In this paper we introduce the Harsanyi set value that, similar to the Shapley set value, covers the hole family of Harsanyi payoffs at once, allows not exogenously given share systems and coincides thus also with non linear values on some subdomains. We present some new axiomatizations of the Shapley set value and the Harsanyi set value containing a player splitting or a players merging property respectively as a main characterizing element that recommend these values for profit distribution and cost allocation.

Keywords Cost allocation · Profit distribution · Player splitting · Players merging · Shapley set value · Harsanyi set value

1. Introduction

Usually coalition functions don't exist in a vacuum and in some games there are personal weights assigned to the players and can change from game to game. For such games where the weights are given exogenously Shapley (1953a) introduced the weighted Shapley values. The family of all such TU-values is also known as the Shapley set.

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But in many cases the weights can also depend on the given coalition functions as handled in some problems of profit distribution (earnings per share) or cost allocation (see e. g. Moriarity (1975)). If for example the weights are the singleton worths of the players one can use the formula of a weighted Shapley value and has to change the weighted Shapley value for each new coalition function. This was the idea of the "Independent Cost Proportional Scheme (ICPS)" in Gangolly (1981). But if you do so you use instead of a weighted Shapley value a new value, called proportional Shapley Value (Besner 2016; Béal et al. 2017; Gangolly 1981) which differ in the axiomatizations from the weighted Shapley values.

In practice it does not matter if someone uses for calculating a weighted Shapley value, even though the weights are not given exogenously, but if someone wants to know which axiomatizations the used value has, for example to argue why a certain value should be selected, she has to know the right value. Also that the value has to be changed if the underlying weights are changing is not a convincing argument to choice a value for calculating. That we can use the same value in cost games and cost saving games or in profit games and cooperation benefit games in the same situation seems naturally. Here the Shapley value is significantly superior to proportional variants in TU-games.

All these mentioned lacks do not occur by the Shapley set value (Shapley 1981; Dehez 2011; Radzik 2012) At our knowledge, the first one who gave attention to the fact that a value regards a coalition function with assigned personally given weights are Shapley (1981) and Dehez (2011)[1]. They introduced weighted cost games and presented particular axiomatizations of a "weighted value" in this regard in the context of cost allocation. (Radzik, 2012) formulated this value in general, denoted there by ϕ_* and ϕ_0 respectively. This value is not a TU-value in the originally sense, because it takes into account not only the coalition function but also players' weights. Radzik introduced there weighted coalition functions, defined weighted TU-games and shaped out a value which is axiomatized by a relatively large group of axioms which are not logically independent. Then Radzik called this value also weighted Shapley value, because this value satisfies adapted well-known axiomatizations of the weighted Shapley values in Nowak and Radzik (1995).

For this value we consider the naming "weighted Shapley value" as not accurate in some respects: On the one hand to avoid name conflicts with the weighted Shapley values for TU-games and here, in our opinion, not the value is weighted, it is the coalition function, on the other hand this value satisfies different axioms as the weighted Shapley values, particularly in case of the use of sub domains. Casajus (2017) introduced a huge class of weighted values φ^ω. There the weights are given by a function that can depend on coalition functions and exogenously given players' weights in the same time. It turns out that the Shapley set value covers not only the weighted Shapley values, but also all values from Casajus' class on associated sub domains.

The weighted Shapley values are obtained by distributing the dividends by a weight system where a player owns for each coalition containing her the same weight. Hammer (1977) and Vasil'ev (1978) introduced the Harsanyi set, also known as selectope (Derks, Haller and Peters, 2000), a set of TU-values called Harsanyi payoffs. For each player exists for each coalition containing him a possibly different weight that has to be independent from the coalition function. In this system of weights, called sharing system, the weights

[1] Shapley (1981) notes only a comment at an accounting conference that was worked out and proved later by Dehez (2011).

of the players for each coalition are not negative, sum up to one and the dividends of the game have to be distributed between the players according to these weights.

Similar to the Shapley set value we introduce the Harsanyi set value that covers all Harsanyi payoffs at once and is defined on sharing TU-games which consist of a TU-game together with a sharing system at a time. This value also coincides on subdomains with values which are not in the Harsanyi set and the sharing systems can depend on coalition functions.

In the main part of this paper we introduce axiomatizations of the Shapley set value and the Harsanyi set value. Within these axiomatizations a player splitting property and a players merging property play a decisive role and can help to close the gap using cooperative game theory in profit distribution and cost allocation not only in theory but also in practice.

The paper is organized as follows. Section 2 contains some preliminaries. In section 3 we give two motivating examples. Section 4 presents axioms for the Shapley set value, give axiomatizations for this value and reveals as a side-benefit a new axiomatization of the Shapley value. In section 5 we transfer our proceeding with the Shapley set value in the previous section to the Harsanyi set value. Section 6 summarizes the results and gives a short conclusion. An appendix (section 7) provides all the proofs and shows logical independence of the axioms used for axiomatization.

2. Preliminaries

We denote by $\mathbb{N}$ the natural numbers, by $\mathbb{R}$ the real numbers, by $\mathbb{R}_{++}$ the set of all positive real numbers and by $\mathbb{Q}_{++}$ the set of all positive rational numbers. Let $\mathfrak{U}$ be a countably infinite set, the universe of all players, and denote by $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N is the power set of N. We refer to a TU-game also by v. The subsets $S \subseteq N$ are called **coalitions**, $v(S)$ is the **worth** of coalition S and the set of all nonempty subsets of S is denoted by Ω^S. The set of all TU-games with player set N is denoted by $\mathbb{V}(N)$ and, if $v(\{i\}) > 0$ for all $i \in N$, by $\mathbb{V}_0(N)$. The **restriction** of (N, v) to the player set $S \in \Omega^N$ is denoted by (S, v).

Let $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$ and $S \subseteq N$. The **dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), & \text{if } S \in \Omega^N, \text{ and} \\ 0, & \text{if } S = \emptyset. \end{cases}$$

A game (N, u_S), $S \in \Omega^N$, with $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise, $T \subseteq N$, is called an **unanimity game**. It is well-known that any $v \in \mathbb{V}(N)$ has an unique presentation

$$v = \sum_{S \in \Omega^N} \Delta_v(S) u_S. \tag{1}$$

The **marginal contribution** $MC_i^v(S)$ of player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. Player $i \in N$ is called a **dummy player** if $v(S \cup \{i\}) =$

$v(S)+v(\{i\})$, $S \subseteq N\backslash\{i\}$; if in addition $v(\{i\})=0$, then i is called a **null player**; players $i,j \in N$, $i \neq j$, are called **symmetric** in v if $v(S\cup\{i\}) = v(S\cup\{j\})$, they are called (mutually) **dependent** (Nowak and Radzik, 1995) in v if $v(S\cup\{i\}) = v(S) = v(S\cup\{j\})$ and **weakly dependent** (Besner, 2017a) in v if $v(S\cup\{k\}) = v(S)+v(\{k\})$, $k \in \{i,j\}$, for all $S \subseteq N\backslash\{i,j\}$; a coalition $Q \subseteq N$ is called a **partnership** (Kalai and Samet, 1987) in v if $v(S\cup T) = v(S)$ for all $S \subseteq N\backslash Q$ and $T \subsetneq Q$.

We define by $\Lambda^N := \{f\colon N \to \mathbb{R}_{++}\}$ with $\lambda_i := \lambda(i)$ for all $\lambda \in \Lambda^N$, $i \in N$, the collection of all positive **weight systems** on N and by $\Lambda^N_{\mathbb{Q}} := \{f\colon N \to \mathbb{Q}_{++}\}$ the collection of all positive rational weight systems on N. Let $v \in \mathbb{V}(N)$ and $\lambda \in \Lambda^N$. Then $v^\lambda := (\lambda, v)$ is called a **weighted coalition function**, (N, v^λ) or short also v^λ is said to be a **weighted TU-game**[2] (**WTU-game**), the set of all WTU-games with player set N is denoted by $\mathbb{V}^\Lambda(N)$ if $\lambda \in \Lambda^N$ and by $\mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$ if $\lambda \in \Lambda^N_{\mathbb{Q}}$.

The collection Γ^N on N of all **sharing systems** $\gamma \in \Gamma^N$ is given by

$$\Gamma^N := \left\{\gamma = (\gamma_{S,i})_{S\in\Omega^N,\, i\in S} \,\middle|\, \sum_{i\in S}\gamma_{S,i} = 1 \text{ and } \gamma_{S,i} \geq 0 \text{ for each } S \in \Omega^N \text{ and all } i \in S\right\}$$

and the collection $\Gamma^N_{\mathbb{Q}}$ on N of all positive rational sharing systems by

$$\Gamma^N_{\mathbb{Q}} := \left\{\gamma = (\gamma_{S,i})_{S\in\Omega^N,\, i\in S} \,\middle|\, \sum_{i\in S}\gamma_{S,i} = 1 \text{ and } \gamma_{S,i} \in \mathbb{Q}_{++} \text{ for each } S \in \Omega^N \text{ and all } i \in S\right\}.$$

Let $v \in \mathbb{V}(N)$ and $\gamma \in \Gamma^N$. Then $v^\gamma := (\gamma, v)$ is called a **sharing coalition function**, (N, v^γ) or short also v^γ is said to be a **sharing TU-game** (**STU-game**), the set of all STU-games with player set N is denoted by $\mathbb{V}^\Gamma(N)$ if $\gamma \in \Gamma^N$, by $\mathbb{V}^{\Gamma_{\mathbb{Q}}}(N)$ if $\gamma \in \Gamma^N_{\mathbb{Q}}$ and by $\mathbb{V}^\Gamma_0(N)$ if $\gamma \in \Gamma^N$ and $v \in \mathbb{V}_0(N)$.

A **TU-value** φ is an operator that assigns to any $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ a payoff vector $\varphi(N,v) \in \mathbb{R}^N$, a **weighted TU-value** (**WTU-value**) φ^Λ is an operator that assigns to any $v^\lambda \in \mathbb{V}^\Lambda(N)$ a payoff vector $\varphi^\Lambda(N, v^\lambda) \in \mathbb{R}^N$ and a **sharing TU-value** (**STU-value**) φ^Γ is an operator that assigns to any $v^\gamma \in \mathbb{V}^\Gamma(N)$ a payoff vector $\varphi^\Gamma(N, v^\gamma) \in \mathbb{R}^N$.

Let $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$. The (simply) **weighted Shapley Value**[3] Sh^λ (Shapley, 1953a) is defined by

$$Sh^\lambda_i(N,v) := \sum_{S\subseteq N,\, S\ni i} \frac{\lambda_i}{\sum_{j\in S}\lambda_j}\Delta_v(S) \ \text{ for all } \ i \in N \text{ and } \lambda \in \Lambda. \tag{2}$$

The **Shapley set** is the set of all weighted Shapley values. A special case of a weighted Shapley value, all weights are equal, is the **Shapley value** Sh (Shapley, 1953b), given by

$$Sh_i(N,v) := \sum_{S\subseteq N,\, S\ni i} \frac{\Delta_v(S)}{|S|} \ \text{ for all } \ i \in N.$$

Let $v \in \mathbb{V}_0(N)$. The **proportional Shapley Value** Sh^p (Besner 2016; Béal et al. 2017; Gangolly 1981) is defined by

$$Sh^p_i(N,v) := \sum_{S\subseteq N,\, S\ni i} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) \ \text{ for all } \ i \in N.$$

[2] Radzik (2012) called such a game a "transferable utility weighted game in characteristic function form".
[3] We desist from possibly null weights as in Shapley (1953a) or Kalai and Samet (1987)

As a WTU-value, the **Shapley set value** Sh^Λ (Shapley 1981/Dehez 2011; Radzik 2012)[4] is defined by

$$Sh_i^\Lambda(N, v^\lambda) := \sum_{S \subseteq N,\, S \ni i} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \Delta_v(S) \ \text{ for all } \ i \in N \text{ and } v^\lambda \in \mathbb{V}^\Lambda(N). \tag{3}$$

The set of the following TU-values is called **Harsanyi set** (Hammer, 1977; Vasil'ev, 1978), also known as **selectope** (Derks, Haller and Peters, 2000), where the payoffs are obtained by distributing the dividends with the help of a sharing system γ. The TU-values H^γ in this set, titled **Harsanyi payoffs**, are defined by

$$H_i^\gamma(N, v) := \sum_{S \subseteq N,\, S \ni i} \gamma_{S,i} \Delta_v(S), \gamma \in \Gamma^N,\, i \in N.$$

As an STU-value we introduce the **Harsanyi set value** H^Γ defined by

$$H_i^\Gamma(N, v^\gamma) := \sum_{S \subseteq N,\, S \ni i} \gamma_{S,i} \Delta_v(S) \text{ for all } \ i \in N \text{ and } v^\gamma \in \mathbb{V}^\Gamma(N). \tag{4}$$

3. Profit and cost allocation and motivating examples

A cooperative game with coalition function v represents commonly profits or savings for all possible coalitions of the player set. Thus in the following we denote by v a **profit game**. The profit game is given by a **payout function** p by

$$v(S) := p(S) - \sum_{i \in S} f(\{i\}) \text{ for each } S \in \Omega^N, \tag{5}$$

where $f(\{i\})$ stands for the (financial) **involvement** of player i. Another application for coalition functions are games where the worth of coalitions represent costs. We denote such **cost games** by c. Closely related with cost games are **cost saving games** u which give the savings obtained by forming coalitions and are defined by

$$u(S) := \sum_{i \in S} c(\{i\}) - c(S) \text{ for each } S \in \Omega^N.$$

In addition, similar as cost savings games, we introduce **cooperation benefit games** q which are related to profit games v by

$$q(S) := v(S) - \sum_{i \in S} v(\{i\}) \text{ for each } S \in \Omega^N \tag{6}$$

and present the profit of cooperating towards to be lone fighters.

E. g., Amer et al., (2007) claim a coherent solution should exist for both cost and saving (related) problems, so that all players are indifferent between sharing costs and sharing savings. That means for a player i we should have

$$\varphi_i(c) = c(\{i\}) - \varphi_i(u) \text{ for all } i \in N. \tag{7}$$

[4] Shapley (1981) and Dehez (2011) denoted this value as "weighted value" and used it in the context of cost games, Radzik (2012) denoted this value as "weighted Shapley value". We also desist from possibly null weights.

In the same sense we should have that players are indifferent between sharing profits and sharing cooperation benefits

$$\varphi_i(v) = v(\{i\}) + \varphi_i(q) \text{ for all } i \in N. \tag{8}$$

In the following we give two examples why the usage of the Shapley set value, or, if the weights of players are in different proportion to each other in some coalitions, the Harsanyi set value, is recommended in profit and cost allocation instead of a simple TU-values.

3.1. Example for profit allocation

An entrepreneur wants to bridge a short-term need for finance of 50 millions monetary units (MMU) for one year. He is willing to pay 5 percent interest in this year. Additionally he wants as few financiers as possible. Thus the deposits must be multiples of 5 millions and the entrepreneur pays a bonus of $100,000$ if the deposit amounts not less then 30 millions and $500,000$ if the deposit amounts exactly the sum of 50 millions. Therefore we have a payout function p (in MMU) given by

$$p(f) := \begin{cases} 1.05f, & \text{for } 5 \leq f < 30, \\ 1.05f + 0.1, & \text{for } 30 \leq f < 50, \\ 1.1f + 0.5, & \text{if } f = 50, \end{cases} \tag{9}$$

where f is the deposit of a financier under the restriction that $f = 5k$, $k \in \mathbb{N}$.

3.1.1. Situation 1

Three investors, investor A, B and C, will cooperate to obtain a share of the bonus too. Therefore they must occur as one single financier. A wants to make a involvement about 20 millions, B and C want to invest in each case 16 millions. The investors agree, that in the case of a consensus how to share the benefits, the investor with the larger investment possibilities waives the not needed share of the optimal deposit (Table 1).

Table 1: Deposits f (MMU)

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	$\{A, B, C\}$
$f(S)$	20	15	15	19+16=35	19+16=35	15+15=30	18+16+16=50

By (9), we establish a payout function p on $N = \{A, B, C\}$ where the worth of a coalition $S \subseteq N$ is the payout of the coalition S. By (5), we obtain Table 2 for the related profit game v and, by (6), Table 3 for the cooperation benefit game q.

Table 2: Profit game v (MMU)

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	$\{A, B, C\}$
$v(S)$	1	0.75	0.75	1.85	1.85	1.6	3

Table 3: Cooperation benefit game q (MMU)

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$q(S)$	0	0	0	0.1	0.1	0.1	0.5

The three investors want to use cooperative game theory to share the benefits. B and C propose at first the Shapley value Sh. But A disagrees. A argues that the Shapley value prefers players with lower deposits and points to the following situation.

3.1.2. Situation 2

Investor A splits up in two new investors A_1 and A_2. With investors A_1, A_2, B and C the deposits $\tilde{f}$ should be given as shown in Table 4.

Table 4: Deposits $\tilde{f}$ (MMU)

S	$\{A_1\}$	$\{A_2\}$	$\{B\}$	$\{C\}$	$\{A_1,A_2\}$	$\{A_1,B\}$	$\{A_1,C\}$	$\{A_2,B\}$	$\{A_2,C\}$
$\tilde{f}(S)$	10	10	15	15	10+10=20	10+15=25	10+15=25	10+15=25	10+15=25

S	$\{B,C\}$	$\{A_1,A_2,B\}$	$\{A_1,A_2,C\}$	$\{A_1,B,C\}$	$\{A_2,B,C\}$	$\{A_1,A_2,B,C\}$
$\tilde{f}(S)$	15+15	9.5+9.5+16=35	9.5+9.5+16=35	10+15+15=40	10+15+15=40	9+9+16+16=50

We have a new payout function $\tilde{p}$ on $\tilde{N} = \{A_1, A_2, B, C\}$ and obtain the profit game $\tilde{v}$ in Table 5 and the cooperation benefit game $\tilde{q}$ given in Table 6.

Table 5: Profit game $\tilde{v}$ (MMU)

S	$\{A_1\}$	$\{A_2\}$	$\{B\}$	$\{C\}$	$\{A_1,A_2\}$	$\{A_1,B\}$	$\{A_1,C\}$	$\{A_2,B\}$	$\{A_2,C\}$
$\tilde{v}(S)$	0.5	0.5	0.75	0.75	1	1.25	1.25	1.25	1.25

S	$\{B,C\}$	$\{A_1,A_2,B\}$	$\{A_1,A_2,C\}$	$\{A_1,B,C\}$	$\{A_2,B,C\}$	$\{A_1,A_2,B,C\}$
$\tilde{v}(S)$	1.6	1.85	1.85	2.1	2.1	3

Table 6: Cooperation benefit game $\tilde{q}$ (MMU)

S	$\{A_1\}$	$\{A_2\}$	$\{B\}$	$\{C\}$	$\{A_1,A_2\}$	$\{A_1,B\}$	$\{A_1,C\}$	$\{A_2,B\}$	$\{A_2,C\}$
$\tilde{q}(S)$	0	0	0	0	0	0	0	0	0

S	$\{B,C\}$	$\{A_1,A_2,B\}$	$\{A_1,A_2,C\}$	$\{A_1,B,C\}$	$\{A_2,B,C\}$	$\{A_1,A_2,B,C\}$
$\tilde{q}(S)$	0.1	0.1	0.1	0.1	0.1	0.5

Here is a special kind of "dependency":

- All coalitions which include both players A_1 and A_2 have the same worth in $\tilde{v}$ and $\tilde{q}$ respectively as the related coalitions in v and q which content the player A and players A_1 and A_2 spend together in these coalitions the same investment as player A in the old coalition.

- In each game the marginal contributions of player A_1 or player A_2 to any coalition which does not contain the respective other player are only the singleton worths of these players.[5]
- Coalitions which are the same in both situations have the same worth in v/q and $\tilde{v}/\tilde{q}$.

In the sum in situation 2 there is no effect by splitting player 1 to the other players. A argues that the payoff to not splitted players should not change in this case. But this is not true for the Shapley value (Table 7). So the investors reject the Shapley value. A

Table 7: Profits (MMU)

Investor	A	A_1	A_2	B	C
Shapley value $Sh(v)$	1.167	-	-	0.917	0.917
Shapley value $Sh(\tilde{v})$	-	0.617	0.617	0.883	0.883
Proportional Shapley value $Sh^p(v)$	1.194	-	-	0.903	0.903
Proportional Shapley value $Sh^p(\tilde{v})$	-	0.597	0.597	0.903	0.903
Shapley set value $Sh^\Lambda(v^\lambda)$	1.194	-	-	0.903	0.903
Shapley set value $Sh^\Lambda(\tilde{v}^{\tilde{\lambda}})$	-	0.597	0.597	0.903	0.903
Harsanyi set value $H^\Gamma(v^\gamma)$	1.181	-	-	0.910	0.910
Harsanyi set value $H^\Gamma(\tilde{v}^{\tilde{\gamma}})$	-	0.590	0.590	0.910	0.910

makes the proposal to use the proportional Shapley value Sh^p instead that provide the desired result for profit games.

Now B opposed that also (8) should be satisfied where Sh^p completely fails. C suggests a weighted Shapley value Sh^λ. The Investors agree that the singleton involvements should be the weights. So they can use formula (2) but they recognize that then the weights are not given exogenously as required for a weighted Shapley value, the coalition functions depend on the involvements.

Using the Shapley set value Sh^Λ they get the same result as by Sh^p. But now occurs another problem. Apparently, the investments of the players differ in some coaltions and so the weights have to be adapted. The investors recognize that they must use a sharing system γ in the STU-game $(\{A, B, C\}, v^\gamma)$, given by

$$\gamma_{S,i} := \frac{f_i(S)}{f(S)},$$

and a sharing system $\tilde{\gamma}$ in the in the STU-game $\left(\{A_1, A_2, B, C\}, \tilde{v}^{\tilde{\gamma}}\right)$, given by

$$\tilde{\gamma}_{S,i} := \frac{\tilde{f}_i(S)}{\tilde{f}(S)},$$

where $f_i(S), \tilde{f}_i(S)$ are the respective shares of player i in the deposit of coalition S. But if they use a Harsanyi payoff H^γ they have the same problem as before by the weighted Shapley value, the coalition functions and the sharing systems are not independent. Finally all investors reached a consensus to use the Harsanyi set value H^Γ that meets all requirements and it is easy to verify that also (8) is satisfied.

[5]Note that players A_1 and A_2 are dependent in $\tilde{q}$ and weakly dependent in $\tilde{v}$.

3.2. Example for cost allocation

We refer to the example in Besner (2017a). There also the Shapley value Sh fails if we regard situation 2 after situation 1. The proportional Shapley value Sh^p gives the unsplitted players for the cost game in both situations the same payoffs. But if we regard (7) as a desirable requirement Sh^p completely fails because in the related cost saving games the singletons have a worth of zero. A weighted Shapley value Sh^λ, too, can not be used because the weights are the singleton worths from the cost game and depend so on the coalition function. Finally the Shapley set value Sh^Λ satisfies all requirements we demand. We get for the cost game the same payoffs as by Sh^p and it can be easily seen that the players are indifferent between sharing costs and sharing savings. If there, unlike as in this example, the weights differ in some coalitions so that we have a sharing system γ, the Harsanyi set value H^Γ is recommended for the usage if γ depends on the coalition function.

4. The Shapley set value

The set of all weighted Shapley values, also known as Shapley set, requires exogenously given weights. Shapley (1981) and Dehez (2011), only in the context of cost games, and Radzik (2012) introduced the Shapley set value together with the related weighted games. But there is no reference that the weights related to a coalition function can here depend on it. The naive reader will take to mean that in the mentioned articles there is just discussed a variety of the weighted Shapley values. The following subsections are intended to demonstrate that this is not the case.

4.1. Axioms for TU-values

We refer to the following standard axioms for TU-values:

Efficiency, E. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, we have $\sum_{i\in N} \varphi_i(N,v) = v(N)$.

Null player, N. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$ and $i \in N$ such that i is a null player in v, we have $\varphi_i(N,v) = 0$.

Linearity, L. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, we have $\varphi(N, \alpha v + w) = \alpha\varphi(N,v) + \varphi(N,w)$.

Additivity, A. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, we have $\varphi(N,v) + \varphi(N,w) = \varphi(N, v+w)$.

Weighted proportionality, WP (Nowak and Radzik, 1995)[6]**.** For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $\lambda \in \Lambda$ and $i, j \in N$ such that i and j are dependent in v, we have

$$\frac{\varphi_i(N,v)}{\lambda_i} = \frac{\varphi_j(N,v)}{\lambda_j}.$$

[6]This is the essential part of the ω-Mutual Dependence axiom in (Nowak and Radzik, 1995).

4.2. Radzik's axioms for WTU-values

Radzik (2012) presented a lot of axioms for WTU-values. For one of these axioms is an additional player set needed. Thus Radzik introduced for a fixed coalition $Q \in \Omega^N$ a new player set $N^Q := (N \backslash Q) \cup \{Q\}$ where Q is regarded as a single player and so $\{Q\}$ is a singleton. Then he defined for this player set a new WTU-game, related to the old game.

Definition 4.1. *(Radzik, 2012) Let $N, N^Q \in \mathcal{N}$, $Q \in \Omega^N$, $q := Q$ if Q is regarded as a player, $N^Q := (N \backslash Q) \cup \{q\}$, $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$, $\big(N^Q, (v^\lambda)^Q\big) \in \mathbb{V}^\Lambda(N^Q)$, $(v^\lambda)^Q := (\lambda^Q, v^Q)$ such that $v^Q(S) := v(S)$ and $v^Q(S \cup \{q\}) := v(S \cup Q)$ for all $S \subseteq N \backslash Q$ and*

$$\lambda_k^Q := \begin{cases} \lambda_k, & \text{if } k \in N^Q \backslash \{q\}, \\ \sum_{i \in Q} \lambda_i & \text{if } k = q. \end{cases}$$

*Then $\big(N^Q, (v^\lambda)^Q\big)$ is called a **merged players WTU-game**[7] (**MPW-game**) to (N, v^λ).*

The following axioms for WTU-values come from Radzik (2012):

Efficiency$^\Lambda$, E$^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, we have $\sum_{i \in N} \varphi_i^\Lambda(N, v^\lambda) = v(N)$.

Null player$^\Lambda$, N$^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $i \in N$ such that i is a null player in v, we have $\varphi_i^\Lambda(N, v^\lambda) = 0$.

Additivity$^\Lambda$ (in the coalition function)**, A$^\Lambda$.** For all $N \in \mathcal{N}$, $v^\lambda, w^\lambda \in \mathbb{V}^\Lambda(N)$, we have $\varphi^\Lambda(N, v^\lambda) + \varphi^\Lambda(N, w^\lambda) = \varphi^\Lambda(N, (v+w)^\lambda)$.

Linearity$^\Lambda$ (in the coalition function)**, L$^\Lambda$.** For all $N \in \mathcal{N}$, $v^\lambda, w^\lambda \in \mathbb{V}^\Lambda(N)$ and $\alpha \in \mathbb{R}$, we have $\varphi^\Lambda(N, (\alpha v + w)^\lambda) = \alpha \varphi^\Lambda(N, v^\lambda) + \varphi^\Lambda(N, w^\lambda)$.

Weight proportionality$^\Lambda$, WP$_Q^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, $Q \in \Omega^N$ such that Q is a partnership in v and $i \in Q$, we have

$$\frac{\varphi_i^\Lambda(N, v^\lambda)}{\lambda_i} = \text{const.}$$

Marginality$^\Lambda$ (in the coalition function)**, M$^\Lambda$.** For all $N \in \mathcal{N}$, $v^\lambda, w^\lambda \in \mathbb{V}^\Lambda(N)$ and $i \in N$ such that $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i^\Lambda(N, v^\lambda) = \varphi_i^\Lambda(N, w^\lambda)$.

Amalgamating payoffs$^\Lambda$, AP$^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, $Q \in \Omega^N$ a partnership in v, $q := Q$ if Q is regarded as a player and $\big(N^Q, (v^\lambda)^Q\big) \in \mathbb{V}^\Lambda(N^Q)$ an MPW-game to v^λ, we have

$$\sum_{i \in Q} \varphi_i^\Lambda(N, v^\lambda) = \varphi_q^\Lambda\big(N^Q, (v^\lambda)^Q\big).$$

Continuity$^\Lambda$, C$^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $\alpha \in \mathbb{R}$, we have $\varphi^\Lambda(N, \alpha v^\lambda)$ is a continuous function of variable α.

Remark 4.2. *Note that $\boldsymbol{A}^\Lambda$, $\boldsymbol{L}^\Lambda$ and $\boldsymbol{M}^\Lambda$ hold only in the coalition function and therefore only on subdomains. The weight systems must always be equal.*

[7]In Radzik (2012) this game is only mentioned as a reduced game.

4.3. Additional axioms for WTU-values

Besner (2017a) introduced a corresponding splitted player game for TU-games where a fixed player is splitted in two new players. In analogy we define a splitted player game for WTU-games. In difference to the MPW-game, in the splitted player game the new players are completely independent from the "splitted" player in the original game apart from the given properties in the following definition.

Definition 4.3. *Let $N, N^j \in \mathcal{N}$, $j \in N$, $N^j := (N\backslash\{j\}) \cup \{k,\ell\}$, $k,\ell \in \mathfrak{U}$, $k,\ell \notin N$, $(N,v^\lambda) \in \mathbb{V}^\Lambda(N)$, $\big(N^j,(v^\lambda)^j\big) \in \mathbb{V}^\Lambda(N^j)$, $(v^\lambda)^j := (\lambda^j, v^j)$. The game $\big(N^j,(v^\lambda)^j\big)$ is called a **splitted player WTU-game (SPW-game)** to (N,v^λ) if for all $S \subseteq N\backslash\{j\}$*

- $\lambda^j_k + \lambda^j_\ell = \lambda_j$ *and* $\lambda^j_i = \lambda_i$ *if* $i \in N\backslash\{j\}$,
- $v^j(S \cup \{m\}) = v(S)$ *if* $m \in \{k,\ell\}$,
- $v^j(S \cup \{k,l\}) = v(S \cup \{j\})$ *and*
- $v^j(S) = v(S)$.

Remark 4.4. *Regard that the worths of $v^j(k)$ and $v^j(\ell)$ are zero and the players k,ℓ are dependent in the game v^j. Like as the null player property can be strengthened to the dummy player property we can strengthen definition 4.3 to a definition where the players k,ℓ are weakly dependent in the game v^j and so $v^j(S \cup \{m\}) = v(S) + v^j(\{m\})$ as in example 3.1.2, situation 2, for the profit game.*

In the following we present additional axioms for WTU-values:

Equal weighted symmetry$^\Lambda$, EWS$^\Lambda$ (Shapley, 1981). For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, $i,j \in N$ such that i and j are symmetric in v and $\lambda_i = \lambda_j$, we have $\varphi^\Lambda_i(N,v^\lambda) = \varphi^\Lambda_j(N,v^\lambda)$.

Null player out$^\Lambda$[8], NO$^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $j \in N$ such that j is a null player in v, we have $\varphi^\Lambda_i(N,v^\lambda) = \varphi^\Lambda_i(N\backslash\{j\},v^\lambda)$ for all $i \in N\backslash\{j\}$.

Weighted proportionality$^\Lambda$, WP$^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $i,j \in N$ such that i and j are dependent in v, we have

$$\frac{\varphi^\Lambda_i(N,v^\lambda)}{\lambda_i} = \frac{\varphi^\Lambda_j(N,v^\lambda)}{\lambda_j}.$$

Players merging$^\Lambda$, PM$^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, $k,\ell \in N$ two dependent players in v and $\big(N^{\{k,\ell\}},(v^\lambda)^{\{k,\ell\}}\big) \in \mathbb{V}^\Lambda(N^{\{k,\ell\}})$ an MPW-game to (N,v^λ), we have

$$\varphi^\Lambda_i(N,v^\lambda) = \varphi^\Lambda_i\big(N^{\{k,\ell\}},(v^\lambda)^{\{k,\ell\}}\big) \text{ for all } i \in N\backslash\{k,\ell\}.$$

Player splitting$^\Lambda$, PS$^\Lambda$. For all $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, $j \in N$ and $\big(N^j,(v^\lambda)^j\big) \in \mathbb{V}^\Lambda(N^j)$ an SPW-game to (N,v^λ), we have

$$\varphi^\Lambda_i(N,v^\lambda) = \varphi^\Lambda_i\big(N^j,(v^\lambda)^j\big) \text{ for all } i \in N\backslash\{j\}.$$

[8]This axiom is part of the dummy elimination axiom in (Shapley, 1981) and comes as a TU-axiom from (Derks and Haller, 1999).

Weighted standardness$^\Lambda$, WS$^\Lambda$. For all $N \in \mathcal{N}$, $|N| = 2$, $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $i, j \in N, i \neq j$, we have

$$\varphi_i^\Lambda(N, v^\lambda) = v(\{i\}) + \frac{\lambda_i}{\lambda_i + \lambda_j}\big[v(\{i,j\}) - v(\{i\}) - v(\{j\})\big].$$

Remark 4.5. *It is easy to see that* $\boldsymbol{WP}_Q^\Lambda$ *is equivalent to* $\boldsymbol{WP}^\Lambda$*. Hereafter, we use only the shortcut* $\boldsymbol{WP}^\Lambda$*.*

The players merging and the player splitting property both pick up the idea in Banker (1981) that splitting up a cost center or merging cost centers should not change the allocation of costs to the remaining cost centers. The procedure of amalgamating players in the amalgamating payoffs axiom can be interpreted as a form of adapting the idea in Lehrer (1988) that the new players get together the same payoff as the old splitted player. Here Lehrer's idea is satisfied too if the WTU-value is efficient.

Remark 4.6. *Let* $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, $j \in N$ *and* $\big(N^j, (v^\lambda)^j\big) \in \mathbb{V}^\Lambda(N^j)$ *a SPW-game to* (N, v^λ)*. If* φ^Λ *is a WTU-value that satisfies* $\boldsymbol{E}^\Lambda$ *and* $\boldsymbol{PS}^\Lambda$*, then we have*

$$\varphi_k^\Lambda\big(N^j, (v^\lambda)^j\big) + \varphi_\ell^\Lambda\big(N^j, (v^\lambda)^j\big) = \varphi_j^\Lambda(N, v^\lambda) \text{ for } k, \ell \in N^j,\ k, \ell \notin N.$$

4.4. Subdomains

Any TU-value ignores weights which potentially are allocated to the players. Thus we can interpret each TU-value also as a WTU-value:

Remark 4.7. *Let* $N \in \mathcal{N}$ *and* $v^\lambda \in \mathbb{V}^\Lambda(N)$*. Each TU-value* φ *coincides with a WTU-value* φ^Λ *by* $\varphi_i^\Lambda(N, v^\lambda) := \varphi_i(N, v)$ *for all* $i \in N$*, in particular we have for a weighted Shapley value* Sh^{λ^k}, $\lambda^k \in \Lambda$, $(Sh_i^{\lambda^k})^\Lambda(N, v^\lambda) := Sh_i^{\lambda^k}(N, v)$ *for all* $i \in N$*.*

At first glance the Shapley set value looks very similar to a weighted Shapley value. But for one thing, we have only one value for all weight systems λ, whereas the weight system used in a weighted Shapley value is fixed, and secondly, for some λ the weights can depend on the coalition function and are therefore not endogenous given as required for a weighted Shapley value. For different subdomains this value coincides with different values.

Remark 4.8. *Let* $\lambda' \in \Lambda$ *and* $\mathbb{V}^{\Lambda_{\lambda'}}(N)$ *the set of all WTU-games* $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$ *with* $\lambda := \lambda'$*.* Sh^Λ *coincides on* $\mathbb{V}^{\Lambda_{\lambda'}}(N)$ *with* $Sh^{\lambda'}$*.*

But the Shapley set value coincides on some subdomains also with non-linear values like the proportional Shapley value.

Remark 4.9. *For all* $v \in \mathbb{V}_0(N)$ *let* $\lambda^v \in \Lambda$ *such that* $\lambda_i^v := v(\{i\})$ *for all* $i \in N$ *and let* $\mathbb{V}^{\Lambda_{\lambda^v}}(N)$ *the set of all WTU-games* $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$ *with* $\lambda := \lambda^v$*.* Sh^Λ *coincides on* $\mathbb{V}^{\Lambda_{\lambda^v}}(N)$ *with* Sh^p*.*

Casajus (2017) presented a huge class of solutions φ^ω, $\omega \in \overline{\Omega}$, $\overline{\Omega} := \{f : \mathbb{R} \times \mathfrak{U} \to \mathbb{R}_{++}\}$, given by

$$\varphi_i^\omega(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\omega(v(\{i\}), i)}{\sum_{j \in S} \omega(v(\{j\}), j)} \Delta_v(S) \quad \text{for all } i \in N \text{ and } v \in \mathbb{V}(N).$$

With the weight functions $\omega \in \overline{\Omega}$ we can generalize the previous two remarks.

Remark 4.10. *Let $\omega \in \overline{\Omega}$, $\lambda^{\omega_v} \in \Lambda$ such that $\lambda_i^{\omega_v} := \omega(v(\{i\}), i)$ for all $i \in N$ and all $v \in \mathbb{V}(N)$ such that $v(\{i\})$ is in the domain of ω for all $i \in N$ and let $\mathbb{V}^{\Lambda_{\lambda^{\omega_v}}}(N)$ the set of all WTU-games $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$ with $\lambda := \lambda^{\omega_v}$. Sh^Λ coincides on $\mathbb{V}^{\Lambda_{\lambda^{\omega_v}}}(N)$ with φ^ω.*

4.5. Additivity can replace linearity

Among other axiomatizations Radzik presented the following one.

Theorem 4.11. *(Radzik, 2012) Sh^Λ is the unique WTU-value that satisfies $\boldsymbol{E}^\Lambda$, $\boldsymbol{L}^\Lambda$, $\boldsymbol{N}^\Lambda$ and $\boldsymbol{WP}^\Lambda$.*

In remark 4.6 in Radzik (2012) is pointed out that $\mathbf{L}^\Lambda$ can be weakened to $\mathbf{A}^\Lambda$ if we replace $\mathbf{L}^\Lambda$ by $\mathbf{A}^\Lambda$ and add $\mathbf{C}^\Lambda$. But this is not necessary what the following theorem and the next corollary are showing.

Theorem 4.12. *(Besner, 2017b) Sh^λ is the unique TU-value that satisfies $\boldsymbol{E}$, $\boldsymbol{N}$, $\boldsymbol{WP}$ and $\boldsymbol{A}$.*

This axiomatization is a special case of theorem 6.4 in Besner (2017b). Since there the proof is oriented on level structures we give a short proof in the appendix 7.1.

By theorem 4.12 and remark 4.8 we have the following corollary.

Corollary 4.13. *Sh^Λ is the unique WTU-value that satisfies $\boldsymbol{E}^\Lambda$, $\boldsymbol{N}^\Lambda$, $\boldsymbol{WP}^\Lambda$ and $\boldsymbol{A}^\Lambda$.*

4.6. Insertion

We introduce a special case of the weighted proportionality property.

Dependency, D. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$ and $i, j \in N$ such that i and j are dependent in v, we have $\varphi_i(N, v) = \varphi_j(N, v)$.

By theorem 4.12 we obtain the following corollary for the Shapley value.

Corollary 4.14. *Sh is the unique TU-value that satisfies $\boldsymbol{E}$, $\boldsymbol{N}$, $\boldsymbol{D}$ and $\boldsymbol{A}$.*

This axiomatization is weaker then the well-known axiomatization in Shapley (1953b) by efficiency, symmetry, additivity and the null player property because symmetry implies dependency but not vice versa.

4.7. Player splitting

In our example, subsection 3.1.2, situation 2, the game $\tilde{q}$ is an SPW-game to the game q in situation 1. If we use the stronger version of an SPW-game as mentioned in remark 4.4 this holds also for the profit game v. The payoff to unchanged players calculated by the Shapley set value Sh^Λ is in situation 2 the same as in situation 1 as required in $\mathbf{PS}^\Lambda$. Using the weaker version of an SPW-game (definition 4.3) we show that Sh^Λ satisfies $\mathbf{PS}^\Lambda$ in general what could with less effort also be shown for the stronger version.

Lemma 4.15. *Sh^Λ satisfies $\boldsymbol{PS}^\Lambda$.*

For the proof see appendix 7.2. The following lemma shows dependence on $\mathbf{EWS}^{\Lambda}$ for efficient values which satisfy $\mathbf{PS}^{\Lambda}$.

Lemma 4.16. *Let $N \in \mathcal{N}$ and $v^{\lambda} \in \mathbb{V}^{\Lambda}(N)$. If a WTU-value φ^{Λ} satisfies $\boldsymbol{E}^{\Lambda}$ and $\boldsymbol{PS}^{\Lambda}$ then φ^{Λ} satisfies also $\boldsymbol{EWS}^{\Lambda}$.*

For the proof see appendix 7.3. The next lemma makes use of lemma 4.16 in the proof.

Lemma 4.17. *Let $N \in \mathcal{N}$ and $v^{\lambda} \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$. If a WTU-value φ^{Λ} satisfies $\boldsymbol{E}^{\Lambda}$ and $\boldsymbol{PS}^{\Lambda}$ then φ^{Λ} satisfies also $\boldsymbol{WP}^{\Lambda}$.*

For the proof see appendix 7.4. We obtain by lemma 4.15, lemma 4.17 and corollary 4.13 the following corollary.

Corollary 4.18. *Let $N \in \mathcal{N}$ and $v^{\lambda} \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$. Sh^{Λ} is the unique WTU-value that satisfies $\boldsymbol{E}^{\Lambda}$, $\boldsymbol{N}^{\Lambda}$, $\boldsymbol{PS}^{\Lambda}$ and $\boldsymbol{A}^{\Lambda}$.*

In Radzik (2012) is also presented the following axiomatization.

Theorem 4.19. *(Radzik, 2012) Sh^{Λ} is the unique WTU-value that satisfies $\boldsymbol{E}^{\Lambda}$, $\boldsymbol{M}^{\Lambda}$ and $\boldsymbol{WP}^{\Lambda}$.*

By lemma 4.17 and theorem 4.19 we get another corollary.

Corollary 4.20. *Let $N \in \mathcal{N}$ and $v^{\lambda} \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$. Sh^{Λ} is the unique WTU-value that satisfies $\boldsymbol{E}^{\Lambda}$, $\boldsymbol{PS}^{\Lambda}$ and $\boldsymbol{M}^{\Lambda}$.*

Remark 4.21. *Lemma 4.17 holds for $v^{\lambda} \in \mathbb{V}^{\Lambda}(N)$ if we require continuity of φ^{Λ} in λ for all $\lambda \in \Lambda^{N}$ in an additional axiom. Thus also corollary 4.18 and corollary 4.20 are valid for $v^{\lambda} \in \mathbb{V}^{\Lambda}(N)$ if there is is in each case an additional continuity axiom.*

4.8. Players merging

The Shapley set value and the players merging property fit together well.

Lemma 4.22. *Sh^{Λ} satisfies $\boldsymbol{PM}^{\Lambda}$.*

Since it is easy to adapt the proof from lemma 4.15 the proof is omitted. It follows an axiomatization which uses the players merging property.

Theorem 4.23. *Sh^{Λ} is the unique WTU-value that satisfies $\boldsymbol{NO}^{\Lambda}$, $\boldsymbol{PM}^{\Lambda}$, $\boldsymbol{WS}^{\Lambda}$ and $\boldsymbol{A}^{\Lambda}$.*

For the proof see appendix 7.5.

5. The Harsanyi set value

The Harsanyi payoffs H^{γ} need a sharing system that is exogenously given. If the sharing system depends on the coalition function we can use the Harsanyi set value H^{Γ}.

5.1. Axioms for STU-values

To transfer axioms for WTU-values into axioms for STU-values we need two definitions we already know as definitions in the version for WTU-values.

Definition 5.1. *Let $N, N^Q \in \mathcal{N}$, $Q \in \Omega^N$, $q := Q$ if Q is regarded as a player, $N^Q := (N \backslash Q) \cup \{q\}$, $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$, $\left(N^Q, (v^\gamma)^Q\right) \in \mathbb{V}^\Gamma(N^Q)$, $(v^\gamma)^Q := \left(\gamma^Q, v^Q\right)$ such that $v^Q(S) := v(S)$ and $v^Q(S \cup \{q\}) := v(S \cup Q)$ for all $S \subseteq N \backslash Q$ and*

$$\gamma^Q_{S,k} := \begin{cases} \gamma_{S,k}, & \text{if } S \in N \backslash Q \text{ and } k \in S, \\ \gamma_{T \cup Q,k}, & \text{if } S = T \cup \{q\},\ T \subseteq N \backslash Q \text{ and } k \in T, \\ \sum_{i \in Q} \gamma_{T \cup Q,i} & \text{if } S = T \cup \{q\},\ T \subseteq N \backslash Q \text{ and } k = q. \end{cases}$$

*Then $\left(N^Q, (v^\gamma)^Q\right)$ is called an **merged players STU-game(MPS-game)** to (N, v^γ).*

Definition 5.2. *Let $N, N^j \in \mathcal{N}$, $j \in N$, $N^j := (N \backslash \{j\}) \cup \{k, \ell\}$, $k, \ell \in \mathfrak{U}$, $k, \ell \notin N$, $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$, $\left(N^j, (v^\gamma)^j\right) \in \mathbb{V}^\Gamma(N^j)$, $(v^\gamma)^j := \left(\gamma^j, v^j\right)$. The game $\left(N^j, (v^\gamma)^j\right)$ is called a **splitted player STU-game (SPS-game)** to (N, v^γ) if for all $S \subseteq N \backslash \{j\}$, $i \in S$, $m \in \{k, \ell\}$,*

- $\gamma^j_{S,i} = \gamma_{S,i}$, $\gamma^j_{S \cup \{k,\ell\},k} + \gamma^j_{S \cup \{k,\ell\},\ell} = \gamma_{S \cup \{j\},j}$ *and* $\gamma^j_{S \cup \{k,\ell\},i} = \gamma_{S \cup \{j\},i}$,
- $v^j(S \cup \{m\}) = v(S)$,
- $v^j(S \cup \{k, l\}) = v(S \cup \{j\})$ *and*
- $v^j(S) = v(S)$.

Remark 5.3. *Note that the shares of all players from a coalition $(S \cup \{m\})$ that contains only one splitted player are arbitrary within the domain of a sharing system.*

Remark 5.4. *Regard that the worths of $v^j(k)$ and $v^j(\ell)$ are zero and k, ℓ are dependent in v^j. Like as the null player property can be strengthened to the dummy property we can strengthen definition 5.2 to a definition where the players k, ℓ are weakly dependent in the game v^j and so $v^j(S \cup \{m\}) = v(S) + v^j(\{m\})$ as in example 3.1.2, situation 2.*

Our axioms for STU-values are simple adaptions of the related axioms for WTU-values.

Efficiency$^\Gamma$, E$^\Gamma$. For all $N \in \mathcal{N}$, $v^\gamma \in \mathbb{V}^\Gamma(N)$, we have $\sum_{i \in N} \varphi^\Gamma_i(N, v^\gamma) = v(N)$.

Null player out$^\Gamma$, NO$^\Gamma$. For all $N \in \mathcal{N}$, $v^\gamma \in \mathbb{V}^\Gamma(N)$ and $j \in N$ such that j is a null player in v, we have $\varphi^\Gamma_i(N, v^\gamma) = \varphi^\Gamma_i(N \backslash \{j\}, v^\gamma)$ for all $i \in N \backslash \{j\}$.

Additivity$^\Gamma$(in the coalition function), **A$^\Gamma$.** For all $N \in \mathcal{N}$, $v^\gamma, w^\gamma \in \mathbb{V}^\Gamma(N)$, we have $\varphi^\Gamma(N, v^\gamma) + \varphi^\Gamma(N, w^\gamma) = \varphi^\Gamma(N, (v + w)^\gamma)$.

Players merging$^\Gamma$, PM$^\Gamma$. For all $N \in \mathcal{N}$, $v^\gamma \in \mathbb{V}^\Gamma(N)$, $k, \ell \in N$ two dependent players in v and $\left(N^{\{k,\ell\}}, (v^\gamma)^{\{k,\ell\}}\right) \in \mathbb{V}^\Gamma(N^{\{k,\ell\}})$ an MPS-game to (N, v^γ), we have $\varphi^\Gamma_i(N, v^\gamma) = \varphi^\Gamma_i\left(N^{\{k,\ell\}}, (v^\gamma)^{\{k,\ell\}}\right)$ for all $i \in N \backslash \{k, \ell\}$.

Player splitting$^\Gamma$, PS$^\Gamma$. For all $N \in \mathcal{N}$, $v^\gamma \in \mathbb{V}^\Gamma(N)$, $j \in N$ and $\left(N^j, (v^\gamma)^j\right) \in \mathbb{V}^\Gamma(N^j)$ an SPS-game to (N, v^γ), we have $\varphi^\Gamma_i(N, v^\gamma) = \varphi^\Gamma_i\left(N^j, (v^\gamma)^j\right)$ for all $i \in N \backslash \{j\}$.

Weighted standardness$^\Gamma$, WS$^\Gamma$. For all $N \in \mathcal{N}$, $|N| = 2$, $v^\gamma \in \mathbb{V}^\Gamma(N)$ and $i, j \in N, i \neq j$, we have

$$\varphi_i^\Gamma(N, v^\gamma) = v(\{i\}) + \frac{\gamma_{N,i}}{\gamma_{N,i} + \gamma_{N,j}}\big[v(\{i,j\}) - v(\{i\}) - v(\{j\})\big].$$

5.2. Subdomains

Similar to the WTU-values we can interpret each TU-value also as an STU-value:

Remark 5.5. *Let $N \in \mathcal{N}$ and $v^\gamma \in \mathbb{V}^\Gamma(N)$. Each TU-value φ coincides with an STU-value φ^Γ by $\varphi_i^\Gamma(N, v^\gamma) := \varphi_i(N, v)$ for all $i \in N$, in particular we have for a Harsanyi payoff H^{γ^k}, $\gamma^k \in \Gamma$, $(H_i^{\gamma^k})^\Gamma(N, v^\gamma) := H_i^{\gamma^k}(N, v)$ for all $i \in N$.*

Also the Harsanyi set value looks very similar to a Harsanyi payoff. But for one thing, we have only one value for all sharing systems γ, whereas the sharing system used in a Harsanyi payoff is fixed, and secondly, for some γ the sharing systems can also depend on the coalition function and are therefore not endogenous given as required for a Harsanyi payoff again. For different subdomains this value coincides with different values.

Remark 5.6. *Let $\gamma' \in \Gamma^N$ and $\mathbb{V}^{\Gamma_{\gamma'}}(N)$ the set of all STU-games $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$ with $\gamma := \gamma'$. H^Γ coincides on $\mathbb{V}^{\Gamma_{\gamma'}}(N)$ with the Harsanyi payoff $H^{\gamma'}$.*

Remark 5.7. *Let $\lambda \in \Lambda^N$ and $\gamma' \in \Gamma^N$ such that*

$$\gamma'_{T,i} := \frac{\lambda_i}{\sum_{j \in T} \lambda_j}, T \in \Omega^N,\ i \in T,$$

and let $\mathbb{V}^{\Gamma_{\gamma'}}(N)$ the set of all STU-games $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$ with $\gamma := \gamma'$. H^Γ coincides on $\mathbb{V}^{\Gamma_{\gamma'}}(N)$ with the weighted Shapley value Sh^λ.

Remark 5.8. *For all $\lambda \in \Lambda^N$ let $\gamma^\lambda \in \Gamma^N$ such that*

$$\gamma^\lambda_{T,i} := \frac{\lambda_i}{\sum_{j \in T} \lambda_j}, T \in \Omega^N,\ i \in T,$$

$\Gamma_\Lambda(N) := \{\gamma^\lambda : \lambda \in \Lambda^N\}$ and let $\mathbb{V}^{\Gamma_\Lambda}(N)$ the set of all STU-games $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$ with $\gamma \in \Gamma_\Lambda(N)$. H^Γ coincides on $\mathbb{V}^{\Gamma_\Lambda}(N)$ with the Shapley set value Sh^Λ.

Also the Harsanyi set value coincides with non-linear values on some subdomains[9].

Remark 5.9. *Let $v \in \mathbb{V}_0(N)$ and $\gamma^v \in \Gamma^N$ such that*

$$\gamma^v_{T,i} := \frac{v(\{i\})}{\sum_{j \in T} v(\{j\})}, T \in \Omega^N,\ i \in T,$$

and let $\mathbb{V}_0^{\Gamma_{\gamma^v}}(N)$ the set of all STU-games $(N, v^\gamma) \in \mathbb{V}_0^\Gamma(N)$ with $\gamma := \gamma^v$. H^Γ coincides on $\mathbb{V}_0^{\Gamma_{\gamma^v}}(N)$ with the proportional Shapley value Sh^p.

[9] Easily can be shown that the Harsanyi set value also coincides on certain subdomains with non-linear values for level structures presented in Besner (2018), with the proportional Shapley support levels value, the proportional Shapley alliance levels value and the proportional Shapley collaboration levels value.

5.3. Axiomatizations

Lemma 5.10. *H^Γ satisfies $\boldsymbol{E}^\Gamma$, $\boldsymbol{NO}^\Gamma$, $\boldsymbol{PS}^\Gamma$, $\boldsymbol{PM}^\Gamma$, $\boldsymbol{WS}^\Gamma$ and $\boldsymbol{A}^\Gamma$.*

For the proof see appendix 7.6. In difference to corollary 4.18, the next theorem needs the null player out property instead of the null player property.

Theorem 5.11. *Let $N \in \mathcal{N}$ and $v^\gamma \in \mathbb{V}^{\Gamma_\mathbb{Q}}(N)$. H^Γ is the unique WTU-value that satisfies $\boldsymbol{E}^\Gamma$, $\boldsymbol{NO}^\Gamma$, $\boldsymbol{PS}^\Gamma$ and $\boldsymbol{A}^\Gamma$.*

For the proof see appendix 7.7.

Remark 5.12. *Similar to remark 4.21 theorem 5.11 holds for $v^\gamma \in \mathbb{V}^\Gamma(N)$ if we require continuity of φ^Γ in γ for all $\gamma \in \Gamma^N$ in an additional axiom.*

Our last axiomatization can be transferred from theorem 4.23 one to one.

Theorem 5.13. *Let $N \in \mathcal{N}$ and $v^\gamma \in \mathbb{V}^\Gamma(N)$. H^Γ is the unique WTU-value that satisfies $\boldsymbol{NO}^\Gamma$, $\boldsymbol{PM}^\Gamma$, $\boldsymbol{WS}^\Gamma$ and $\boldsymbol{A}^\Gamma$.*

The proof is omitted because it can also be transmitted one to one from theorem 4.23.

6. Conclusion

In this paper we have carved out the differences between the weighted Shapley values and the Shapley set value and the differences between the Harsanyi payoffs and the Harsanyi set value. The introduced player splitting and players melting properties lead to convincing axiomatizations, particularly in the context of profit and cost allocation. Especially the axiomatizations of the Harsanyi set value, not using any dividends, astonish through their practicable presentation.

It is immediate, besides the given axiomatizations in Radzik (2012) and Shapley (1981)/Dehez (2011), that the Shapley set value also satisfies adaptions for WTU-values from the well-known axiomatizations of the weighted Shapley values given in Myerson (1980)\Hart and Mas-Colell (1989) by efficiency and weighted balanced contributions and Hart and Mas-Colell (1989) by consistency and weighted standardness.

Also the amalgamating payoffs property allows interesting axiomatizations. Whereas in the axiomatization in Radzik (2012, theorem 2.1) $\mathbf{AP}^\Lambda$ is redundant, it is clear, applying ideas from the proof of theorem 4.23, that $\mathbf{Sh}^\Lambda$ ($\mathbf{H}^\Gamma$) can be axiomatized by $\mathbf{E}^\Lambda$, $\mathbf{NO}^\Lambda$, $\mathbf{AP}^\Lambda$, $\mathbf{WS}^\Lambda$ and $\mathbf{A}^\Lambda$ ($\mathbf{E}^\Gamma$, $\mathbf{NO}^\Gamma$, $\mathbf{AP}^\Gamma$, $\mathbf{WS}^\Gamma$ and $\mathbf{A}^\Gamma$). Here $\mathbf{AP}^\Lambda$ is not redundant. Look for this, e. g., to the WTU-value φ, defined by

$$\varphi_i(N, v^\lambda) = \sum_{\substack{S \subseteq N,\\ S \ni i, |S| \le 2}} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \Delta_v(S) + \sum_{\substack{S \subseteq N,\\ S \ni i, |S| \ge 3}} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N,$$

that satisfies $\mathbf{E}^\Lambda$, $\mathbf{NO}^\Lambda$, $\mathbf{WS}^\Lambda$ and $\mathbf{A}^\Lambda$ and doesn't match $\mathbf{AP}^\Lambda$. But it is still an open question if all axioms are logically independent.

We have desisted from possibly null weights for the Shapley set value. If one wants to allow null weights we recommend to use the Harsanyi set value. There one has to specify the weights for coalitions where in the Shapley set value all players have a weight of zero how to share in this case the dividend of these coalitions.

7. Appendix

Lemma 7.1. *(Besner, 2017b) Players $i, j \in N$, $i \neq j$, are dependent in $v \in \mathbb{V}(N)$, iff $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, for all $S \subseteq N \backslash \{i, j\}$.*

7.1. Proof of theorem 4.12

Let $v^\lambda \in \mathbb{V}^\Lambda(N)$. It is well-known that Sh^λ satisfies all axioms from theorem 4.12. So, due to property (1) and **A**, it is sufficient to show that φ is uniquely defined on games $v_S := \Delta_v(S) \cdot u_S$, $S \in \Omega^N$.

Let $S \in \Omega^N$ arbitrary and φ a value that satisfies all axioms from theorem 4.12. All players $j \in N \backslash S$ are null players in v_S and so φ is unique on v_S for all $j \in N \backslash S$ by **N**. Thus, if S is a singleton, φ is unique on v_S by **E**. Let now $|S| \geq 2$. By lemma 7.1 all $i \in S$ are dependent in v_S and therefore, by **E** and **WP**, φ is also unique on v_S for all $i \in N$. $\square$

7.2. Proof of lemma 4.15

Let $v^\lambda \in \mathbb{V}^\Lambda(N)$, $j \in N$ and $\big(N^j, (v^\lambda)^j\big)$ an SPW-game to (N, v^λ). We point out that we have for all $S \in \Omega^{N\backslash\{j\}}$, $\Delta_{v^j}(S) = \Delta_v(S)$, $\Delta_{v^j}(S \cup \{k, l\}) = \Delta_v(S \cup \{j\})$ and, by lemma 7.1, $\Delta_{v^j}(S \cup \{k\}) = \Delta_{v^j}(S \cup \{\ell\}) = 0$. Then we get for all $i \in N \backslash \{j\}$

$$
\begin{aligned}
Sh_i^\Lambda(N, v^\lambda) &\underset{(3)}{=} \sum_{R \subseteq N,\, R \ni i} \frac{\lambda_i}{\sum_{m \in R} \lambda_m} \Delta_v(R) \\
&= \sum_{S \subseteq N \backslash \{j\},\, S \ni i} \frac{\lambda_i}{\sum_{m \in S} \lambda_m} \Delta_v(S) + \sum_{S \subseteq N \backslash \{j\},\, S \ni i} \frac{\lambda_i}{\sum_{m \in S \cup \{j\}} \lambda_m} \Delta_v(S \cup \{j\}) \\
&= \sum_{\substack{S \subseteq N^j \backslash \{k, \ell\}, \\ S \ni i}} \frac{\lambda_i^j}{\sum_{m \in S} \lambda_m^j} \Delta_{v^j}(S) + \sum_{\substack{S \subseteq N^j \backslash \{k, \ell\}, \\ S \ni i}} \frac{\lambda_i^j}{\sum_{m \in S \cup \{k, \ell\}} \lambda_m^j} \Delta_{v^j}(S \cup \{k, \ell\}) \\
&= \sum_{R \subseteq N^j,\, R \ni i} \frac{\lambda_i^j}{\sum_{m \in R} \lambda_m^j} \Delta_{v^j}(R) = Sh_i^\Lambda\big(N^j, (v^\lambda)^j\big).
\end{aligned}
$$

$\square$

7.3. Proof of lemma 4.16

Let $N = \{1, 2, ..., n\} \in \mathcal{N}$, $n \geq 2$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, φ^Λ a WTU-value that satisfies $\mathbf{E}^\Lambda$ and $\mathbf{PS}^\Lambda$ and, w.l.o.g., player 1 and player 2 symmetric in v with $\lambda_1 = \lambda_2$. We split player 1, according to $\mathbf{PS}^\Lambda$, into two new players, player $n+1$ and player $n+2$, $N^1 := \{2, 3, ..., n, n+1, n+2\}$ and obtain

$$
\varphi_2^\Lambda\big(N^1, (v^\lambda)^1\big) = \varphi_2^\Lambda(N, v^\lambda), \tag{10}
$$

and, if we split player 2, according to $\mathbf{PS}^\Lambda$, into the same players as before, player $n+1$ and player $n+2$, instead, $N^2 := \{1, 3, 4, ..., n, n+1, n+2\}$, we have

$$
\varphi_1\big(N^2, (v^\lambda)^2\big) = \varphi_1(N, v^\lambda), \tag{11}
$$

where we choose $\lambda^2_{n+1} := \lambda^1_{n+1} := \lambda^2_{n+2} := \lambda^1_{n+2}$.

In the same manner we split now in the game $\left(N^1, (v^\lambda)^1\right)$ player 2 into two new players, player $n+3$ and player $n+4$, and analogous in the game $\left(N^2, (v^\lambda)^2\right)$ player 1 into the same players as before, player $n+3$ and player $n+4$, and choose $\lambda^{21}_{n+3} := \lambda^{12}_{n+3} := \lambda^{21}_{n+4} := \lambda^{12}_{n+4}$.

We have $N^{12} = N^{21} = \{3, 4, ..., n, n+1, n+2, n+3, n+4\}$ and $(v^\lambda)^{12} = (v^\lambda)^{21}$ and get by $\mathbf{E}^\Lambda$, according to remark 4.6,

$$\varphi_{n+3}\left(N^{12}, (v^\lambda)^{12}\right) + \varphi_{n+4}\left(N^{12}, (v^\lambda)^{12}\right) = \varphi_2\left(N^1, (v^\lambda)^1\right) \underset{(10)}{=} \varphi_2(N, v^\lambda),$$
$$\varphi_{n+3}\left(N^{21}, (v^\lambda)^{21}\right) + \varphi_{n+4}\left(N^{21}, (v^\lambda)^{21}\right) = \varphi_1\left(N^2, (v^\lambda)^2\right) \underset{(11)}{=} \varphi_1(N, v^\lambda)$$

and hence $\varphi_1(N, v^\lambda) = \varphi_2(N, v^\lambda)$ and $\mathbf{EWS}^\Lambda$ is shown. □

7.4. Proof of lemma 4.17

Let $N \in \mathcal{N}$, $|N| \geq 2$, $v^\lambda \in \mathbb{V}^{\Lambda_\mathbb{Q}}(N)$, φ^Λ a WTU-value that satisfies $\mathbf{E}^\Lambda$ and $\mathbf{PS}^\Lambda$ and therefore, by lemma 4.16, also $\mathbf{EWS}^\Lambda$ and $i, j \in N$ such that i and j are dependent in v. Due to $\lambda_i, \lambda_j \in \mathbb{Q}_{++}$ the weights λ_k, $k \in \{i, j\}$, can be written as a fraction

$$\lambda_k = \frac{p_k}{q_k} \quad \text{with} \quad p_k, q_k \in \mathbb{N}.$$

We choose a main denominator q of these two fractions by $q := q_i q_j$. With $z_i := p_i q_j$ and $z_j := p_j q_i$ we get

$$\lambda_i = \frac{z_i}{q} \quad \text{and} \quad \lambda_j = \frac{z_j}{q}. \tag{12}$$

Applying $\mathbf{PS}^\Lambda$ (repeatedly) to (N, v^λ) and the two players i, j we can get the WTU-game $(N', (v^\lambda)')$ where each player k, $k \in \{i, j\}$, is splitted in z_k players k_1 to $k_{(z_k)}$, such that $N' = (N\backslash\{i, j\}) \cup \{i_m : 1 \leq m \leq z_i\} \cup \{j_m : 1 \leq m \leq z_j\}$ and each player $k_m \in N'\backslash(N\backslash\{i, j\})$, $1 \leq m \leq z_k$, get a singleton worth $v'(\{k_m\}) := 0$ for $k \in \{i, j\}$, with a weight $\lambda_{k_m} := \frac{1}{q}$ where $\left|N'\backslash(N\backslash\{i, j\})\right| = z_1 + z_2$.

All players $\ell \in N'\backslash(N\backslash\{i, j\})$ are symmetric in v' and have the same weights. Hence follows by $\mathbf{EWS}^\Lambda$ and $\mathbf{E}^\Lambda$

$$\varphi_\ell(N', (v^\lambda)') = \frac{\varphi_i(N, v^\lambda) + \varphi_j(N, v^\lambda)}{z_i + z_j} \text{ for } \ell \in N'\backslash(N\backslash\{i, j\}). \tag{13}$$

We get

$$\varphi_k(N, v^\lambda) \underset{(13)}{=} \sum_{1 \leq m \leq z_k} \varphi_{k_m}(N', (v^\lambda)') = \frac{z_k}{z_i + z_j}\left[\varphi_i(N, v^\lambda) + \varphi_j(N, v^\lambda)\right] \text{ for } k \in \{i, j\}.$$

It follows

$$\varphi_i(N, v) = \frac{z_i}{z_j}\varphi_j(N, v^\lambda) \underset{(12)}{=} \frac{\lambda_i}{\lambda_j}\varphi_j(N, v)$$

and $\mathbf{WP}^\Lambda$ is shown. □

7.5. Proof of theorem 4.23

Let $v^\lambda \in \mathbb{V}^\Lambda(N)$. By (3), lemma 4.22 and Radzik (2012), it is clear that Sh^Λ satisfies all axioms from theorem 4.23. Thus we have only to show uniqueness.

Let φ a value that satisfies all axioms from theorem 4.23. If $|N| = 2$, φ is unique on v^λ by $\mathbf{WS}^\Lambda$. Let now one player $i \in N$ a null player. Then, by $\mathbf{NO}^\Lambda$, φ is unique on $(\{j\}, v^\lambda)$ for the other player $j \in N$, but then also for $|N| = 1$ in general.

Let now $|N| \geq 3$. Due to property (1) and $\mathbf{A}^\Lambda$, it is sufficient to show that φ is uniquely defined on games v_S^λ with $v_S := \Delta_v(S) \cdot u_S$, $S \in \Omega^N$.

Let $S \in \Omega^N, S \neq N$. By lemma 7.1, all $j \in S$ are dependent in v_S and all $k \in N \backslash S$ are null players in v_S. If we delete, by $\mathbf{NO}^\Lambda$, all but one null player $k \in N \backslash S$ and merge all players $j \in S$ by $\mathbf{PM}^\Lambda$, step by step, we get, by $\mathbf{WS}^\Lambda$, that φ is unique on v_S^λ for the null player k and since k was arbitrary, that φ is unique on v_S^λ for all null players $k \in N \backslash S$.

Now we delete all null players $k \in N \backslash S$ or we have $S = N$. By $\mathbf{NO}^\Lambda$, we have $Sh_j^\Lambda(S, v^\lambda) = Sh_j^\Lambda(N, v^\lambda)$ for all $j \in S$. If $|S| = 1$, φ is unique on v_S^λ as shown above. Let now $|S| \geq 2$ and $i \in S$. We merge all players $j \in S \backslash \{i\}$ by $\mathbf{PM}^\Lambda$, step by step, and obtain $\varphi_i^\Lambda(N, v_S^\lambda) = \varphi_i^\Lambda\big(S^{S \backslash \{i\}}, (v_S^\lambda)^{S \backslash \{i\}}\big)$. By $\mathbf{WS}^\Lambda$, $\varphi_i^\Lambda\big(S^{S \backslash \{i\}}, (v_S^\lambda)^{S \backslash \{i\}}\big)$ is unique on $(v_S^\lambda)^{S \backslash \{i\}}$. Therefore $\varphi_i^\Lambda(N, v_S^\lambda)$ is unique on v_S^λ. Since $i \in S$ was arbitrary, φ^Λ is unique for all $i \in S$ and theorem 4.23 is shown. □

7.6. Proof of lemma 5.10

Obviously, by (4), H^Γ satisfies $\mathbf{E}^\Gamma$, $\mathbf{NO}^\Gamma$, $\mathbf{WS}^\Gamma$ and $\mathbf{A}^\Gamma$.

We show that H^Γ meets $\mathbf{PS}^\Gamma$. The proof is similar to the proof of lemma 4.15. Let $v^\gamma \in \mathbb{V}^\Gamma(N)$, $j \in N$ and $\big(N^j, (v^\gamma)^j\big)$ an SPS-game to (N, v^γ). We point out that we have for all $S \in \Omega^{N \backslash \{j\}}$, $\Delta_{v^j}(S) = \Delta_v(S)$, $\Delta_{v^j}(S \cup \{k, l\}) = \Delta_v(S \cup \{j\})$ and, by lemma 7.1, $\Delta_{v^j}(S \cup \{k\}) = \Delta_{v^j}(S \cup \{\ell\}) = 0$. Then we get for all $i \in N \backslash \{j\}$

$$\begin{aligned} H_i^\Gamma(N, v^\gamma) &\underset{(4)}{=} \sum_{R \subseteq N,\, R \ni i} \gamma_{R,i} \Delta_v(R) = \sum_{S \subseteq N \backslash \{j\},\, S \ni i} \gamma_{S,i} \Delta_v(S) + \sum_{S \subseteq N \backslash \{j\},\, S \ni i} \gamma_{S,i} \Delta_v(S \cup \{j\}) \\ &= \sum_{S \subseteq N^j \backslash \{k,\ell\},\, S \ni i} \gamma_{S,i}^j \Delta_{v^j}(S) + \sum_{S \subseteq N^j \backslash \{k,\ell\},\, S \ni i} \gamma_{S \cup \{k,\ell\},i}^j \Delta_{v^j}(S \cup \{k, \ell\}) \\ &= \sum_{R \subseteq N^j,\, R \ni i} \gamma_{R,i}^j \Delta_{v^j}(R) = H_i^\Gamma\big(N^j, (v^\gamma)^j\big). \end{aligned}$$

Since it is easy to adapt this part of the proof to show also that H^Γ meets $\mathbf{PM}^\Gamma$ we have omitted this adaption. □

7.7. Proof of theorem 5.11

Let $v^\gamma \in \mathbb{V}^{\Gamma_\mathbb{Q}}(N)$. By lemma 5.10, due to property (1) and $\mathbf{A}$, it is sufficient to show that φ is uniquely defined on games $v_S := \Delta_v(S) \cdot u_S$, $S \in \Omega^N$.

Let $S \in \Omega^N$ arbitrary and φ a value that satisfies all axioms from theorem 4.12. It is obvious that $\mathbf{E}^\Gamma$ and $\mathbf{NO}^\Gamma$ imply together $\mathbf{N}^\Gamma$. Thus φ also satisfies $\mathbf{N}^\Gamma$. All players

$j \in N \backslash S$ are null players in v_S and so φ is unique on v_S for all $j \in N \backslash S$ by **N**. So, by $\mathbf{NO}^\Gamma$, it is sufficient to show that φ is unique on games $\bar{v}_S := (S, v_S)$.

Let $\gamma' \in \Gamma_{\mathbb{Q}}^N$ such that

$$\gamma'_{T,i} := \frac{\gamma_{S,i}}{\sum_{j \in T} \gamma_{S,j}}, T \in \Omega^S, i \in T.$$

We show that

$$\varphi_i^\Gamma(S, v^{\gamma'}) = \varphi_i^\Gamma(S, v^\gamma) \text{ for all } i \in S. \tag{14}$$

If $|S| \leq 2$, we have $\gamma' = \gamma$ and (14) is satisfied. Let now $|S| \geq 3$, $i \in S$ and $Q \in \mathcal{N}$, $Q := \{i, q\}$, $q \notin N$. Accordingly to $\mathbf{PS}^\Gamma$, we split (possibly repeatedly), the player q into all other players $j \in S \backslash \{i\}$. By remark 5.3, we finally can obtain both games $(S, v^{\gamma'})$ and (S, v^γ) and get, by $\mathbf{PS}^\Gamma$, $\varphi_i^\Gamma(S, v^{\gamma'}) = \varphi_i^\Gamma(S, v^\gamma)$ for all $i \in S$ since i was arbitrary.

But then coincides φ^Γ on (S, v^γ) also with an WTU-value φ^Λ on $(S, v^\lambda) \in \mathbb{V}^\Lambda(S)$ with $\lambda_i := \gamma_{S,i}$ for all $i \in S$ that satisfies $\mathbf{E}^\Lambda$, $\mathbf{N}^\Lambda$, $\mathbf{PS}^\Lambda$ and $\mathbf{A}^\Lambda$. Therefore, by corollary 4.18, φ^Γ is unique on v_S and theorem 5.11 is shown. □

7.8. Logical independence

Finally, we want to show the independence of the axioms used in the axiomatizations.

Remark 7.2. *Let $v^\lambda \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$, $N \in \mathcal{N}$. The axioms in corollary 4.20 are logically independent:*

- $\boldsymbol{E}^\Lambda$*: The WTU-value φ, defined by $\varphi_i(N, v^\lambda) := 0$ for all $i \in N$, satisfies* $\boldsymbol{PS}^\Lambda$ *and* $\boldsymbol{M}^\Lambda$ *but not* $\boldsymbol{E}^\Lambda$*.*
- $\boldsymbol{PS}^\Lambda$*: The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) := \sum_{S \subseteq N, S \ni i} \frac{1}{|S|} \Delta_v(S) \text{ for all } i \in N,$$

  satisfies $\boldsymbol{E}^\Lambda$ *and* $\boldsymbol{M}^\Lambda$ *but not* $\boldsymbol{PS}^\Lambda$*.*
- $\boldsymbol{M}^\Lambda$*: The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) := \frac{\lambda_i}{\sum_{j \in N} \lambda_j} v(N) \text{ for all } i \in N,$$

  satisfies $\boldsymbol{E}^\Lambda$ *and* $\boldsymbol{PS}^\Lambda$ *but not* $\boldsymbol{M}^\Lambda$*.*

Remark 7.3. *Let $v^\lambda \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$, $N \in \mathcal{N}$. The axioms in corollary 4.18 are logically independent:*

- $\boldsymbol{E}^\Lambda$*: The WTU-value φ, defined by $\varphi_i(N, v^\lambda) := 0$ for all $i \in N$, satisfies* $\boldsymbol{N}^\Lambda \backslash \boldsymbol{NO}^\Lambda$*,* $\boldsymbol{PS}^\Lambda$ *and* $\boldsymbol{A}^\Lambda$ *but not* $\boldsymbol{E}^\Lambda$*.*
- $\boldsymbol{N}^\Lambda \backslash \boldsymbol{NO}^\Lambda$*: The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) = \frac{\lambda_i}{\sum_{j \in N} \lambda_j} v(N) \text{ for all } i \in N,$$

  satisfies $\boldsymbol{E}^\Lambda$*,* $\boldsymbol{PS}^\Lambda$ *and* $\boldsymbol{A}^\Lambda$ *but not* $\boldsymbol{N}^\Lambda \backslash \boldsymbol{NO}^\Lambda$*.*

- $\boldsymbol{PS}^\Lambda$: *The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) = \sum_{S\subseteq N,\, S\ni i} \frac{1}{|S|}\Delta_v(S) \text{ for all } i \in N,$$

satisfies $\boldsymbol{E}^\Lambda$, $\boldsymbol{N}^\Lambda\backslash\boldsymbol{NO}^\Lambda$ and $\boldsymbol{A}^\Lambda$ but not $\boldsymbol{PS}^\Lambda$.

- $\boldsymbol{A}^\Lambda$: *The WTU-value φ, defined for all $i \in N$ by*

$$\varphi_i(N, v^\lambda) = \begin{cases} 0, & \text{if } i \text{ is a null player in } v, \\ \dfrac{\lambda_i}{\sum_{\substack{j\in N,\\ j \text{ is no null player in } v}} \lambda_j} v(N), & \text{else,} \end{cases}$$

satisfies $\boldsymbol{E}^\Lambda$, $\boldsymbol{N}^\Lambda\backslash\boldsymbol{NO}^\Lambda$ and $\boldsymbol{PS}^\Lambda$ but not $\boldsymbol{A}^\Lambda$.

Remark 7.4. *Let $v^\lambda \in \mathbb{V}^\Lambda(N)$, $N \in \mathcal{N}$. The axioms in theorem 4.23 are logically independent:*

- $\boldsymbol{NO}^\Lambda$: *The WTU-value φ, defined for all $i \in N$ by*

$$\varphi_i(N, v^\lambda) := \begin{cases} 0, & \text{if } |N| = 1, \\ \sum_{\substack{S\subseteq N,\\ S\ni i}} \dfrac{\lambda_i}{\sum_{j\in S}\lambda_j}\Delta_v(S), & \text{else,} \end{cases}$$

satisfies $\boldsymbol{PM}^\Lambda$, $\boldsymbol{WS}^\Lambda$ and $\boldsymbol{A}^\Lambda$ but not $\boldsymbol{NO}^\Lambda$.

- $\boldsymbol{PM}^\Lambda$: *The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) = \sum_{\substack{S\subseteq N,\\ S\ni i,\, |S|\le 2}} \frac{\lambda_i}{\sum_{j\in S}\lambda_j}\Delta_v(S) \text{ for all } i \in N,$$

satisfies $\boldsymbol{NO}^\Lambda$, $\boldsymbol{WS}^\Lambda$ and $\boldsymbol{A}^\Lambda$ but not $\boldsymbol{PM}^\Lambda$.

- $\boldsymbol{WS}^\Lambda$: *The WTU-value φ, defined by $\varphi_i(N, v^\lambda) := 0$ for all $i \in N$, satisfies $\boldsymbol{NO}^\Lambda$, $\boldsymbol{PM}^\Lambda$ and $\boldsymbol{A}^\Lambda$ but not $\boldsymbol{WS}^\Lambda$.*

- $\boldsymbol{A}^\Lambda$: *Let $R := \{i \in N : i \text{ is no null player in } v\}$. The WTU-value φ, defined for all $i \in N$ by*

$$\varphi_i(N, v^\lambda) = \begin{cases} v(\{i\}), & \text{if } i \in N\backslash R \text{ or } |N| = 1 \text{ or } |R| = 1, \\ v(\{i\}) + \dfrac{\lambda_i}{\sum_{j\in R}\lambda_j}\Delta_v(R), & \text{else,} \end{cases}$$

satisfies $\boldsymbol{NO}^\Lambda$, $\boldsymbol{PM}^\Lambda$ and $\boldsymbol{WS}^\Lambda$ but not $\boldsymbol{A}^\Lambda$.

It is easy to transfer our considerations for logical independence above to STU-games and STU-values. We obtain the following remark.

Remark 7.5. *The axioms in theorem 5.11 and in theorem 5.13 are logically independent.*

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