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Abstract

This paper presents new approach to financial modeling and forecasting that is based on economic space notion. Economic space is defined as generalization of risk ratings and allows boost methods and description of financial processes. Risk ratings of economic agents are treated as coordinates of economic agents on economic space. Economic and financial variables of separate economic agents determine macroeconomic and financial variables as functions of time and coordinates on economic space. That permits describe financial relations similar to mathematical physics equations. Financial models can be described on discreet and continuous economic spaces with dimension determined by number of major risks measured simultaneously. To show advantages of economic space usage to financial modeling we present extension of Black-Scholes-Merton equation on *n*-dimensional economic space; develop macroeconomic models on economic space in a way similar to hydrodynamics and derive financial wave equations.

Keywords: risk ratings, economic space, option pricing, financial wave equations.

JEL Classifications: C500, C520, C530, C600, G110, G130

Introduction

This paper presents new approach to economic and financial modeling that is based on economic space notion. Economic space is defined as generalization of risk ratings and allows boost methods and description of financial processes. Risk ratings of economic agents are treated as coordinates of economic agents on economic space. Agent based economic modeling (Judd & Tesfatsion (2005)) has a long history. The term "economic space" was used in economics at least since Perroux (1950), and mostly relates to spatial econometrics (Asada & Ishikawa, (2007) and Fujita (2010)). Out treatment of economic agents and economic space is completely different. Let regard economic agents as economic "particles" on economic space and develop economic models alike to description of many particles systems in physics. Each economic agent is described by set of extensive economic and financial variables as Supply and Demand, Production Function and Capital, Consumption and Value and etc. Aggregation of extensive variables of separate economic agents at point $\boldsymbol{x}$ on economic space permit introduce macroeconomic and financial variables as functions of coordinates on economic space. Such approach allows establish parallels with physical kinetics and hydrodynamics and permit derive wave equations for financial variables. Wave processes play fundamental role for most physical phenomena's. It seems important develop and study models that describe wave generation, propagation and interaction of economic and financial variables.

The paper is organized as follows. In Section 2 we argue economic space notion. In Section 3 we discuss option pricing on n-dimensional economic space and derive extension for Black-Scholes-Merton equation. In Section 4 we present macroeconomic models similar to hydrodynamics and derive financial wave equations on economic space. The conclusions are in Section 5.

On economic space notion

Any attempt to develop a theory on certain space requires procedure that measure coordinates on a given space. In physics such problem is solved by measurements of coordinates of physical particles and bodies on a space-time.

We propose that current risk management and risk ratings in particular form basis for economic space definition. Let note BIS (2011, 2014) as brief references for economic and financial risk management studies and risk ratings that are provided by international rating agencies as Fitch

(2010), Moody's (2007), S&P (2012) and DBRS (2015). Let assume that risk ratings can be established for any economic agent.

Current risk ratings procedures distribute economic agents like companies, corporations and banks over finite number of risk grades, like *AAA, BB, CCC* and so on. Each risk grade can be treated as a point of discreet space. Risk ratings procedures provided by rating agencies can be treated as measurements of coordinates of economic agents on discreet space. Risk grades of a single risk can be treated as coordinates on one-dimensional discreet space. The simultaneous estimations of risk grades for n different risks are similar to measuring coordinates on n-dimensional discreet space. Existing risk ratings practices can be treated as procedures that distribute economic agents on discreet space. Let mention such space as economic space and state that positive direction along each axis is treated as risk growth direction and negative direction points to small risks values.

Current risk ratings methodologies presented by Fitch, Moody's, S&P and DRBS utilize finite number of risk grades that can be treated as discreet space points. Let propose that these methodologies can be extended from discreet to continuous risk space representation. We suggest study n different risk ratings on Euclidian space R^n. Description of economic agents and their economic variables as functions on economic space allows define and describe macroeconomic and finance variables. For example, aggregate Money Demand of separate economic agents that have coordinates near point $\boldsymbol{x}$ determine macroeconomic Money Demand at point $\boldsymbol{x}$. Money Demand becomes a function of economic space coordinates and integral of Money Demand function over economic space defines Money Demand of entire Economics. Thus financial variables of economic agents form basis for definition of macroeconomic variables as functions on e-space. Let refer Euclidian space R^n or discreet space, or any other mathematical space that is used for mapping risk grades of economic agents as economic space or e-space. Below we develop economic modeling on Euclidian R^n economic space.

Definition of economic space as an extension of risk grades and ability to describe global financial variables as functions on economic space allows apply modern methods of mathematical physics and enrich financial modeling. On the other hand introduction of economic space arises new problems that outline internal complexity of global financial modeling and forecasting.

To describe financial processes on economic space R^n it is necessary determine n risks that disturb finance system. It seems impossible to take into account all existing risks that affect current financial evolution. Risk ratings procedures contain internal uncertainty and that uncertainty will grow up with the number of simultaneously measured risks. If one takes into account too many different risks then the simultaneous measurements of all these risk ratings will have too high variability and hence the model description can be too uncertain. To determine a reasonable economic space one should estimate current risks and select two, three, four most important risks as main factors affecting contemporary economics and financial system. Then it is possible to define economic space with two or three dimensions and derive appropriate initial distributions of economic variables as functions of most powerful risks. To select most valuable risks one should establish procedures that allow compare the influence of different risks on financial processes. That permits determine the initial state of economic space R^n.

To describe financial evolution in a time term T it is necessary to forecast m main risks that will play major role in a particular time term and to define economic space R^m. The set of m risks can be the same as for the initial state, or different one. This set of m risks defines the target state of economic space R^m. Then it is necessary to define the transition dynamics that describes the move from initial set of n main risk on economic space R^n to the target set of m main risk on economic space R^m. Such transition dynamics from initial set of n main risk to the target set of m risks describes the evolution of initial representation R^n of to the target one R^m.

That arises a lot of difficult problems. The selection of main risk factors simplifies description and allows neglect "small risks". On the other hand the selection process becomes a part of validation procedure. As one can select and measure main risk factors, then it is possible to validate the initial and target set of risk and to prove or disprove initial model assumptions. The procedure and criteria's to measure and to compare the power of different risks on financial processes should be determined and that is a separate tough problem.

The financial modeling and forecasting on economic space is splitting into a set of verification procedures. It gives a chance to make financial modeling more measurable and it's forecasting more faithful. In the next two Sections we demonstrate advantages of economic space usage for financial modeling: we present treatment of option pricing on economic space. Then we develop macroeconomic models similar to hydrodynamics and derive financial wave equations.

Option pricing on economic space

Option pricing theory is based on the Black-Scholes-Merton equation (BSM) (Black & Scholes, 1973; Merton, 1973; Merton, 1998) and that is one of the most recognized equations in financial theory. We present an extension of BSM equation on the n-dimensional economic space. Further we shall mention economic space as e-space.

The BSM equation for the price V of the option on the underlying asset with price a has the form:

$$\frac{\partial V}{\partial t} + ra\frac{\partial V}{\partial a} + \frac{1}{2}\sigma^2 a^2 \frac{\partial^2}{\partial a^2} V = rV \quad (1)$$

Here, r is the risk-free interest rate. A simple way to derive BSM equation (Merton, 1998; Hull, 2009) is based on the assumption that the asset price a obeys Brownian motion $dW(t)$

$$da = a\, c\, dt + a\, \sigma\, dW(t) \quad (2)$$

$$< dW(t) > = 0; < dW(t) dW(t+T) > = \delta(T) dt$$

c – is the instantaneous rate of return on the security, and σ^2 – is the instantaneous variance rate. The option price $V = V(t,a)$ is a function of time t and security price a. Operator <…> denotes the averaging procedure.

Let argue option pricing on n-dimensional e-space and derive extension for BSM equation. Let regard options on shares of corporations and banks and mention these economic agents as economic particles (e-particles). Shares price of individual e-particle is determined by market value of e-particle and depends on its e-space coordinates or risk grades associated with selected e-particle. Market value of e-particle that is under the influence of n risks becomes function of time and coordinates of e-particle on n-dimensional e-space R^n. Thus, for fixed outstanding shares, shares price a of e-particle becomes a function time t and coordinates $\boldsymbol{x}=(x_1,\ldots x_n)$ on n-dimensional e-space R^n and $a=a(t,\boldsymbol{x})$. Hence option price V also should be a function of time t, coordinates $\boldsymbol{x}=(x_1,\ldots x_n)$ on the n-dimensional e-space R^n and stocks price a and takes the form $V=V(t,\boldsymbol{x},a)$. To derive extension for BSM equation on e-space R^n let suggest two assumptions.

First, let extend (2) and assume that shares price has linear dependence on $d\boldsymbol{x}$:

$$da = a\, c\, d + a\, \sigma\, dW(t) + a\, \boldsymbol{k} \cdot d\boldsymbol{x} \quad (3)$$

Vector $\boldsymbol{k}$ describes the input of the e-space coordinates variation $d\boldsymbol{x}$ on the value of e-particle and thus on shares price; $\boldsymbol{k}\cdot d\boldsymbol{x}$ denotes the scalar product.

Second, let assume that the coordinates $\boldsymbol{x}$ of the e-particle also obey Brownian walk $d\boldsymbol{Z}(t)$ on the n-dimensional e-space

$$d\boldsymbol{x} = \boldsymbol{v} dt + d\boldsymbol{Z}(t) \quad (4)$$

Vector $\boldsymbol{v}$ defines the regular speed of the e-particle on the e-space. Brownian motion $dZ_i(t)$ along each axis of the n-dimensional e-space follows

$$< dZ_i(t) > = 0; \; < dZ_i(t) dZ_j(t+T) > = \eta_i^2\, \delta_{i,j} \delta(T) dt \quad (5)$$

Vector $\boldsymbol{\eta}=(\eta_1,\ldots\eta_n)$ determines the instantaneous variance rate along each axis in the e-space R^n. For simplicity, we assume that there are no correlations between $dW(t)$ and $dZ_i(t)$

$$< dW(t_1)\, dZ_i(t_2) > = 0$$

Assumptions (3-5) allow derive the equation for the option price $V=V(t,\boldsymbol{x},a)$ as an extension of the BSM equation (3) on the n-dimensional e-space R^n.

$$\frac{\partial V}{\partial t} + ra\frac{\partial V}{\partial a} + rx_i\frac{\partial V}{\partial x_i} + \frac{1}{2}a^2q^2\frac{\partial^2 V}{\partial a^2} + k_i\eta_i^2\frac{\partial^2 V}{\partial a \partial x_i} + \frac{1}{2}\eta_i^2\frac{\partial^2 V}{\partial x_i^2} = rV \tag{6}$$

$$q^2 = \left(\sigma^2 + \,k_i{}^2 \cdot \eta_i{}^2\right);\ i = 1,..n$$

The derivation of (6) based on Ito formula and similar to Hull (2009) and we omit it here. Here r is same risk-free interest rate as in (1).

Equation (6) has the same structure as (1). It belongs to diffusion like equations and it's solutions are well known.

Let outline some issues concern option pricing on n-dimensional e-space. Equations (1,6) are valid if initial set of n risks that define n-dimensional e-space is constant. Due to considerations presented in Section 2, initial selection of n risks that determines initial e-space R^n can differ from final set of m risks and final e-space R^m. If during the time to expiration the forecast predicts that new risks can affect the e-particle value, then the option pricing should be corrected. For such a case the option pricing should depends on description of the transition from the initial e-space R^n to the final e-space R^m. Thus equation (6) has sense if initial set of risks remains constant. If some new risks grow up during the term to expiration then assumptions (3-5) and equation (6) should be modified. Thus, original assumption (2) and BSM equations (1) seems fail to describe the option price V for the case with different sets of initial and final risks that affect the value a of the underling assets. The modification of (1,2) and (3-6) should describe the transformation from initial e-space to the final one and that requires additional studies and considerations.

Further let regard macroeconomic models on e-space.

Macroeconomic models and financial wave equations

Let study economics as ensemble of economic agents like banks and corporations, consumers and personal investors, householders and labor and so on. Let assume that each economic agent is described by a set of l extensive economic and financial variables that form the vector $(u_1,\ldots u_l)$. Introduction of economic space allows develop models that have parallels with physical kinetics and hydrodynamics. Let refer further economic agents as economic particles (e-particles). Let denote economic kinetics as approximations that describe a system of e-particles on e-space and economic hydrodynamics as approximations that model behavior of macroeconomic and financial variables that are treated as economic fluids (e-fluids) on e-space.

Economic Kinetics

Let assume that each e-particle (each economic agent) on n-dimensional e-space R^n at moment t is described by coordinates $\boldsymbol{x}=(x_1,...x_n)$, velocity $\boldsymbol{v}=(v_1,...\ v_n)$, and l extensive economic and financial variables $(u_1,...u_l)$. Let regard extensive variables because it is possible to average them within probability distribution. Intensive variables like prices or interest rates cannot be averaged directly. Enormous number of extensive variables like Value and Capital, Demand and Supply, Profits and Production Function and so on describe each e-particle and that increase complexity of description to compare with physical kinetics. Contrary to physics, economic and financial variables do not obey conservation laws and can change their values due to economic processes and their motion on e-space.

Let assume that there are N e-particles on e-space and number of e-particles that are observed at point $\boldsymbol{x}$ equals $N(\boldsymbol{x})$. Let state that velocities of e-particles at point $\boldsymbol{x}$ equal $\boldsymbol{v}=(\boldsymbol{v}_1,...\ \boldsymbol{v}_{N(x)})$. Each e-particle has l economic variables $(u_1,...u_l)$. Let assume that values of economic variables equal $(u_{1i},...u_{li})$, $i=1,..N(x)$. Each economic variable u_j at point $\boldsymbol{x}$ defines macroeconomic variable U_j as sum of economic variables u_j of $N(x)$ e-particles at point $\boldsymbol{x}$

$$U_j = \sum\nolimits_i u_{ji}\,; \quad j=1,..l; \quad i=1,...N(x)$$

For each macroeconomic variable U_j let define analogy of impulses $\boldsymbol{P}_j$ as

$$\boldsymbol{P}_j = \sum\nolimits_i u_{ji}\boldsymbol{v}_i\,; \quad j=1,..l; \quad i=1,...N(x)$$

Let follow Landau and Lifshitz (1981) and introduce economic analogy of Boltzmann's distribution function $f=f(t,x;\ U_1,..U_l,\ \boldsymbol{P}_1,..\boldsymbol{P}_l)$ on n-dimensional e-space that determine probability to observe macroeconomic variables U_j and impulses $\boldsymbol{P}_j$ at point $\boldsymbol{x}$ at time t. Let define macroeconomic density function $U_j(t,\boldsymbol{x})$

$$U_j(t,\boldsymbol{x}) = \int U_j\, f(t,\boldsymbol{x},U_1,...U_l,\boldsymbol{P}_1,..\boldsymbol{P}_l)\, dU_1..dU_l d\boldsymbol{P}_1..d\boldsymbol{P}_l\,;\ j=1,...l \quad (7.1)$$

and impulse density $P_j(t,x)$ as

$$\boldsymbol{P}_j(t,\boldsymbol{x}) = \int \boldsymbol{P}_j\, f(t,\boldsymbol{x},U_1,...U_l,\boldsymbol{P}_1,..\boldsymbol{P}_l)\, dU_1..dU_l d\boldsymbol{P}_1..d\boldsymbol{P}_l\,;\ j=1,...l \quad (7.2)$$

That allows define e-space velocity $\boldsymbol{v}_j(t,\boldsymbol{x})$ of density $U_j(t,\boldsymbol{x})$ as

$$U_j(t,\boldsymbol{x})\boldsymbol{v}_j(t,\boldsymbol{x}) = \boldsymbol{P}_j(t,\boldsymbol{x})$$

Densities $U_j(t,\boldsymbol{x})$ and impulses $\boldsymbol{P}_j(t,\boldsymbol{x})$ are determined as aggregates of corresponding economic and financial variables of separate e-particles. Functions $U_j(t,\boldsymbol{x})$ can describe e-space density of

Demand and Supply, Assets and Debts, Production Function and Value Added and so on. E-space densities $U_j(t,\boldsymbol{x})$ as Value and Capital, Supply and Demand play the role similar to mass density distribution $\rho(t,\boldsymbol{x})$ in physical kinetics (Landau & Lifshitz, 1981). We use (7.1-7.2) as tool to establish transition from description of economic and financial variables $(u_1,\dots u_l)$ of separate economic agents to description of macroeconomic densities $(U_1,\dots U_l)$, $U_j=U_j(t,\boldsymbol{x})$ on e-space and develop economic and financial models alike hydrodynamics.

Economic Hydrodynamics

In parallels to physical mass densities $\rho(t,\boldsymbol{x})$ of physical fluids, let treat e-space densities $U_1(t,\boldsymbol{x}),\dots$ $U_l(t,\boldsymbol{x})$ like Value and Capital, Production Function and Investments, Demand and Supply as economic fluids. Macroeconomics describes interaction of economic and financial densities $U_1(t,\boldsymbol{x}),\dots U_l(t,\boldsymbol{x})$ similar to multi-fluids hydrodynamics and appears to be extremely complex. Parallels between physical and economic densities permit obtain e-fluids equations alike to Continuity Equation and Equation of Motion for physical fluids (Landau & Lifshitz, 1987). Let present phenomenological derivation of e-fluid equations.

Continuity Equation for density $U_i(t,x)$, $i=1,..l$ takes form

$$\frac{\partial U_i}{\partial t} + div(\boldsymbol{v}_i U_i) = Q_1 \qquad (8)$$

$\boldsymbol{v}_i(t,\boldsymbol{x})$ - is the velocity of e-fluid U_i on e-space. Left side describes the flux of density $U_i(t,\boldsymbol{x})$ through the unit volume surface on e-space and factor Q_1 describes transformation of $U_i(t,\boldsymbol{x})$. Contrary to physics and mass conservation law, Continuity Equations on densities don't conform the value of densities and $U_i(t,\boldsymbol{x})$ can increase or decrease in time and during the motion of the selected volume on e-space due to economic reasons. For example, Value in e-space unit volume can increase in time due to economic activity and can decrease if unit volume moves in the direction of risk growth. We state that there are no conservation laws for economic and financial densities.

Equation of Motion for density $U_i(t,\boldsymbol{x})$ takes form

$$U_i\left[\frac{\partial \boldsymbol{v}_i}{\partial t} + (\boldsymbol{v}_i \cdot \nabla)\boldsymbol{v}_i\right] = Q_2 \qquad (9)$$

Left side describes the flux of $U_i(t,\boldsymbol{x})\boldsymbol{v}_i(t,\boldsymbol{x})$ through the surface of unit volume on e-space, taking into account Continuity Equation. Q_2 describes factors that change the flux. We state that in the first approximation extensive economic variables of different economic agents (different e-particles) do not interact and do not depend on the same economic variables of separate e-particles. For example, Production Function of e-particle does not depend on the Production

Function of other e-particles, but depends on other financial and economic variables like Capital, Labor, Market Demand, Investments and so on. As well Consumption of e-particle does not depend on Consumption of other e-particles, but is determined by Income, Savings, Inflation and etc. In the first approximation we neglect any interactions between same economic variables of different e-particles and state that there are no economic analogies for such physical factors as pressure or viscosity. We state that Q_1 and Q_2 in equations on $U_i(t,\boldsymbol{x})$ depend on economic densities $U_j(t,\boldsymbol{x})$ or their velocities those different from $U_i(t,\boldsymbol{x})$. We denote such densities $U_j(t,\boldsymbol{x})$ or their velocities $\boldsymbol{v}_j(t,\boldsymbol{x})$ that define Q_1 and Q_2 and induce changes of particular variable $U_i(t,\boldsymbol{x})$ as conjugate densities to $U_i(t,\boldsymbol{x})$. For example, Production Function density may have conjugate densities like Capital and Labor. Conjugate variables define Q_1 and Q_2 in the right hand side of Continuity Equation and Motion Equations (8,9). The simplest model (8,9) describes two conjugate e-fluids and for that case we derive economic wave equations on e-space.

Financial wave equations

Wave processes describe enormous amount of physical phenomena's and play the core role in current understanding of Nature. Certain parallels should exist in economics and finance. Up now terms "waves" are used for Kondratieff waves, Inflation waves, Crisis waves, Demographic waves and so on. All these issues describe time oscillations only but not wave propagation. To describe financial wave propagation one requires determine certain space. Introduction of economic space allows study financial and economic wave processes that can be extremely important for financial analysis and crisis modeling, forecasting and managing.

Let show that economic equations (8,9) can be origin for various wave processes and derive wave equations on e-space for the simplest model of two conjugate e-fluids. Let study relations between Money Demand and Interest Rate: the rise in Money Demand lead to Interest Rate growth; as well Interest Rates growth decline Demand for Money. The similar relations exist between Investments and Interest Rates; between Commodity Demand and Commodity Price etc. Relations between these variables reflect positive-negative response. Let describe interaction for two conjugate e-fluids model.

Model: Money Demand – Interest Rate.

Let argue the interaction between Money Demand and Interest Rate. Money Demand is extensive variables and thus Money Demand $u_d\ (t,\boldsymbol{x})$ of separate e-particles, separate economic agents in the neighborhood of point $\boldsymbol{x}$ on e-space construe Money Demand density function $U_D(t,\boldsymbol{x})$ (7.1). Interest Rate $r(t,\boldsymbol{x})$ is intensive economic variable and for fixed time term describes Cost of Money supply $U_S(t,\boldsymbol{x})$ available to e-particles at point $(t,\boldsymbol{x})$. For given Money Supply $U_S(t,\boldsymbol{x})$ and fixed time term Cost of Money $U_C(t,\boldsymbol{x})$:

$U_C(t,\boldsymbol{x}) = r(t,\boldsymbol{x})\ U_S(t,\boldsymbol{x})$

Thus for constant Money Supply $U_S(t,\boldsymbol{x})$, the Cost of Money supply $U_C(t,\boldsymbol{x})$ depends on Interest Rate only. Rise in Money Demand $U_D(t,\boldsymbol{x})$ lead to growth of Interest Rate $r(t,\boldsymbol{x})$ and hence growth of Money supply Cost $U_C(t,\boldsymbol{x})$. As well growth of Money supply Cost $U_C(t,\boldsymbol{x})$ being induced by growth of Interest Rate $r(t,\boldsymbol{x})$ imply decline of Money Demand $U_D(t,\boldsymbol{x})$. Thus let replace Interest Rate $r(t,\boldsymbol{x})$ as intensive variable by Money supply Cost $U_C(t,\boldsymbol{x})$ as extensive variable taking into account that $U_C(t,\boldsymbol{x})$ depends on Interest Rate only with $U_S(t,\boldsymbol{x})$ being constant. Hence we obtain two conjugate e-fluids model Money Demand $U_D(t,\boldsymbol{x})$ - $U_C(t,\boldsymbol{x})$ Money supply Cost and can argue forms of right hand side factors for equations (8,9).

Let assume that Q_1 factor in the right hand side of Continuity equation (8) on Money Demand density function U_D is proportional to time derivative of $U_C(t,\boldsymbol{x})$ Money supply Cost density function with negative factor $\alpha_C<0$:

$Q_1 \sim \alpha_C \partial U_C(t,\boldsymbol{x})/\partial t$

Q_1 factor for Continuity equation on Money supply Cost density function U_C is proportional to time derivative of Money Demand density function $U_D(t,\boldsymbol{x})$ with positive factor $\alpha_D>0$:

$Q_1 \sim \alpha_D \partial U_D(t,\boldsymbol{x})/\partial t$

Thus positive growth in time of Money Demand density $U_D(t,\boldsymbol{x})$ induce growth of Money supply Cost density function $U_C(t,x)$ due to rise of interest rate $r(t,\boldsymbol{x})$. As well positive growth in time of Money supply Cost density $U_C(t,x)$ reduce Money Demand density function $U_D(t,\boldsymbol{x})$. Let state that Q_2 factor for Equation of Motion (9) on Money Demand velocity $\boldsymbol{v}_D$ is proportional to gradient ∇U_C of Money supply Cost density function U_C with negative factor $\beta_C<0$:

$Q_2 \sim \beta_C\ \nabla U_C$

Let state that Q_2 factor for Equation of Motion on Money supply Cost velocity $\boldsymbol{v}_C$ is proportional to gradient ∇U_D of Money Demand density function U_D with positive factor $\beta_D>0$:

$Q_2 \sim \beta_D\ \nabla U_D$

Our assumptions means that Money Demand velocity $\boldsymbol{v}_D$ decrease in the direction of positive gradient of Money supply Cost density function U_C and Money supply Cost velocity $\boldsymbol{v}_C$ increase in the direction with positive gradient of Money Demand density function U_D. Our assumptions present the simplest model of possible mutual dependence of two conjugate e-fluids like Money

Demand density function U_D and Money supply Cost density function U_C on e-space. For simplicity we use money supply cost density function $U_C(t,\boldsymbol{x})$ notion to define dependence of Money Demand density $U_D(t,\boldsymbol{x})$ on Interest Rate $r(t,\boldsymbol{x})$. Equations (8,9) describe two conjugate e-fluids model $U_D(t,\boldsymbol{x})$ - $U_C(t,\boldsymbol{x})$ of Money Demand - Cost of Money supply. Continuity Equations take form:

$$\frac{\partial U_D}{\partial t} + \nabla \cdot (\boldsymbol{v}_D U_D) = \alpha_C \frac{\partial U_C}{\partial t} \quad ; \quad \frac{\partial U_C}{\partial t} + \nabla \cdot (\boldsymbol{v}_C U_C) = \alpha_D \frac{\partial U_D}{\partial t}$$

Equations of Motion take form:

$$U_D \left[\frac{\partial \boldsymbol{v}_D}{\partial t} + (\boldsymbol{v}_D \cdot \nabla)\boldsymbol{v}_D\right] = \beta_C \nabla U_C \quad ; \quad U_C \left[\frac{\partial \boldsymbol{v}_C}{\partial t} + (\boldsymbol{v}_C \cdot \nabla)\boldsymbol{v}_C\right] = \beta_D \nabla U_D$$

$$\alpha_D > 0 \; ; \; \alpha_C < 0 \; ; \; \beta_D > 0 \; ; \; \beta_C < 0 \; ; \qquad (10)$$

To derive wave equations let's regard linear approximation on small disturbances $q_{D,C}$ for constant Money Demand U_D and Money supply Cost densities U_C and assume that velocities $\boldsymbol{v}_{D,C}$ are small. In physics similar approximations are used for derivation of acoustic wave equations (Landau & Lifshitz, 1987)

$$U_D = U_{D0} + q_D \; ; \; U_{D0} = const \quad ; \quad U_C = U_{C0} + q_C \; ; \; U_{C0} = const$$

Continuity Equations on small disturbances $q_{D,C}$ and $\boldsymbol{v}_{D,C}$ takes the form

$$\frac{\partial q_D}{\partial t} + U_{D0} \nabla \cdot \boldsymbol{v}_D = \alpha_C \frac{\partial q_C}{\partial t} \quad ; \quad \frac{\partial q_C}{\partial t} + U_{C0} \nabla \cdot \boldsymbol{v}_C = \alpha_D \frac{\partial q_D}{\partial t}$$

Equations of Motion on small velocities take the form:

$$U_{D0} \frac{\partial \boldsymbol{v}_D}{\partial t} = \beta_C \nabla q_C \quad ; \quad U_{C0} \frac{\partial \boldsymbol{v}_C}{\partial t} = \beta_D \nabla q_D$$

The second derivation by time of Continuity Equation implies:

$$\frac{\partial^2 q_D}{\partial t^2} + U_{D0} \nabla \cdot \frac{\partial}{\partial t} \boldsymbol{v}_D = \alpha_C \frac{\partial^2 q_C}{\partial t^2} \quad ; \quad \frac{\partial^2 q_D}{\partial t^2} = \alpha_C (\alpha_D \frac{\partial^2 q_D}{\partial t^2} - \beta_D \Delta q_D) - \beta_C \Delta q_C$$

$$\frac{\partial^2}{\partial t^2} [(1 - \alpha_C \alpha_D) \frac{\partial^2 q_D}{\partial t^2} + \alpha_C \beta_D \Delta q_D] = - \beta_C \Delta (\alpha_D \frac{\partial^2 q_D}{\partial t^2} - \beta_D \Delta q_D)$$

Equations on disturbances $q_{D,C}$ take form

$$\left[(1 - \alpha_C \alpha_D) \frac{\partial^4}{\partial t^4} + (\alpha_C \beta_D + \beta_C \alpha_D) \Delta \frac{\partial^2}{\partial t^2} - \beta_C \beta_D \Delta^2 \right] q_{D,C} = 0 \qquad (11)$$

To show that equations (11) admit wave solutions let regard $q_{D,C} = q(x-ct)$ or take Fourier transforms for $exp(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$ and define speed c as $c=\omega/k$. For positive $c^2_{1,2}>0$ equations (11) on disturbances $q_{D,C}$ take form of bi-wave equation:

$$(\frac{\partial^2}{\partial t^2} - c_1^2 \Delta) (\frac{\partial^2}{\partial t^2} - c_2^2 \Delta) q_{D,C} = 0 \qquad (12)$$

$$c_{1,2}^2 = \frac{-(\alpha_D \beta_C + \alpha_C \beta_D) +/- \sqrt{(\alpha_D \beta_C - \alpha_C \beta_D)^2 + 4 \beta_D \beta_D}}{2(1 - \alpha_D \alpha_C)} \qquad (13)$$

For (10) $c^2_{1,2}>0$ if $(\alpha_D \beta_C - \alpha_C \beta_D)^2 + 4 \beta_D \beta_D > 0$

Bi-wave equations (12) describe propagation of waves $q=q(\boldsymbol{x}-\boldsymbol{c}t)$ with speed $\boldsymbol{c}$ equals $\boldsymbol{c}_1$ or $\boldsymbol{c}_2$ as in the direction of risks growth as in the direction of small risks values. Existence of wave processes on economic space allows describe economic and financial wave generation, propagation and interaction, permit modeling possible wave response on economic and financial shocks and study wave phenomena's of financial crisis evolution. Bi-wave equations (12) reflect wave processes that are more complex to compare with second order wave equations. Green functions for bi-wave equations take form of convolution of two Green functions for wave equations.

Positive-negative response is not necessary condition to derive wave equations for two conjugate e-fluids interacting model. For example positive response coefficients like $\alpha_D=2$; $\alpha_C=\beta_D=\beta_C=1$ delivers $c^2_{1,2}>0$ and thus provide bi-wave equation regime.

Let outline that equations (11) admit wave propagation with exponential growth of amplitude. For example small disturbances q_D of money demand can grow as exponent (Appendix) in time as:

$$q_D = \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\exp(\gamma t)$$

$$\gamma^2 = \frac{k^2}{4a}\left(b + \sqrt{-4ad}\ \right) > 0 \ \ ; \ \ \omega^2 = \frac{k^2}{4a}\left(b + \sqrt{-4ad}\ \right) - \frac{bk^2}{2a}$$

For $\gamma>0$ the solution will grow up and for $\gamma<0$ will dissipate.

Equations (11) admit solutions (Appendix) that grow up in direction, defined by vector $\boldsymbol{p}$:

$$q_D = \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\exp(\boldsymbol{p}\cdot\boldsymbol{x})$$

$$\omega^2 = \frac{2d}{b}(k^2 - p^2) \ \ ; \ \ \boldsymbol{p} = \boldsymbol{k}\left(\sqrt{\varepsilon^2+1} - \varepsilon\right)$$

These samples demonstrate possible existence of exponential amplification of small financial density disturbances in the simplest models of two conjugate interacting e-fluids and confirm that modeling on e-space can describe a wide range of wave processes.

Conclusion

Complexity of finance processes requires relevant methods and models. Most methods of theoretical physics are based on the space-time notion. To boost methods of financial modeling we introduce economic space notion as generalization of risk ratings.

Risk ratings can be treated as coordinates of economic agents on discreet economic space. We propose that generalization of risk ratings methodologies can establish risk ratings measurements of n different risks on continuous economic space R^n. Financial and economic risks should be treated as important and necessary drivers of financial and economic evolution.

It appears reasonable that suitable economic space should follow current financial conditions that are determined by set of most valuable risks and define financial dynamics. Economic space can have different representations for different financial conditions and that add complexity to description of financial dynamics. There are no ways to establish determined macroeconomic and financial forecasts as random nature of risks growth and decline insert permanent uncertainty into financial evolution. Long terms financial forecasts require development of evolution model on initial economic space and assumptions on future valuable risk configurations that will define final economic space representation in projected time term.

Definition of economic space requires selection of main risks. Risk benchmarking is a separate tough problem. Possibility to measure and select most valuable risks should establish procedure to validate the initial and target set of risks and to prove or disprove initial model assumptions. It will allow compare predictions of economic and financial models with observations and will help outline causes of disagreement between theoretical predictions and observed reality.

Economic space notion outlines complexity of financial processes and gives base for usage of theoretical physics methods and models. Differences between finance and physics leave no hopes for direct application of mathematical physics methods and models. We believe that finance modeling on economic space may help establish suitable forecasting and improve crises predictions and management.

Our treatments of option pricing of economic agents on n-dimensional e-space R^n permit derive extension for BSM equation and outline additional difficulties for option pricing modeling. These difficulties concern possible changes of main risks that determine the current economic space axes. Random dynamics of main risks during time to expiration means that BSM equations (1,6) should be transformed into other ones on economic space with different axes. That effect might explain differences between predicted and observed option price dynamics. Correct description of these effects might rise up the accuracy for the option pricing.

Nevertheless finance and physics are completely different systems it is reasonable study economic and financial models similar to physical approximations like kinetics and hydrodynamics. Derivation of economic and financial wave equations for the simplest models of interaction between Money Demand and Interest Rate opens a way for further investigation of financial wave processes. Wave theories play core role in physics and might help for financial modeling as well.

Economic space notion outlines the internal complexity of financial and economic systems and requires appropriate econometric foundations. The main problems concern the observation and the choice of most valuable risks, measurement of economic agents distributions on economic space. At present, there are no risk ratings methodologies that allow distribute economic agents on Rn. To establish economic space with one or two dimensions, one needs cooperative efforts of Central Banks, Rating Agencies, Economic and Finance Research Communities, Regulators, Statistical Bureaus, Businesses, etc., etc. Our work is only first step on that long way.

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Appendix

Let take the q_D in the form:

$$q_D = \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\exp(\gamma t + \boldsymbol{p}\cdot\boldsymbol{x}) \tag{A1}$$

Here $\boldsymbol{k}\cdot\boldsymbol{x}$ denote scalar product of two vectors $\boldsymbol{k}$ and $\boldsymbol{x}$. Let denote

$$a=1-\alpha_D\alpha_C>1\ ;\ b=\alpha_C\beta_D+\alpha_D\beta_C<0\ ;\ d=\beta_D\beta_C<0$$

For q_D as (A1) equation (11) becomes a system of two equations:

$$a[(\gamma^2-\omega^2)^2-4\gamma^2\omega^2]+b\,[(p^2-k^2)(\gamma^2-\omega^2)+4\gamma\omega\,\boldsymbol{k}\cdot\boldsymbol{p}]-d[(p^2-k^2)^2-4(\,\boldsymbol{k}\cdot\boldsymbol{p}\,)^2]=0 \tag{A2}$$

$$4a\omega\gamma(\gamma^2-\omega^2)+b[\,2\omega\gamma\,(p^2-k^2)-2(\gamma^2-\omega^2)\,\boldsymbol{k}\cdot\boldsymbol{p}\,]+4d(p^2-k^2)\,\boldsymbol{k}\cdot\boldsymbol{p}\ =0$$

For simplicity let study two cases.

Case 1. Let $\boldsymbol{p}=0$

Then system (A2) takes form:

$$a[(\gamma^2-\omega^2)^2-4\gamma^2\omega^2]\ -bk^2(\gamma^2-\omega^2)-dk^4\ =0$$

$$4a\omega\gamma(\gamma^2-\omega^2)-\ 2b\omega\gamma k^2\ \ =0$$

Hence, from the second equation, a>1, b<0 :

$$\omega^2=\gamma^2-\frac{bk^2}{2a}$$

and first equation takes form

$$\frac{b^2k^4}{4a}\ -4a\gamma^2\omega^2\ -\frac{b^2k^4}{2a}-dk^4\ =0$$

Thus

$$\frac{b^2k^4}{4a}\ -4a\gamma^2\omega^2=\ -\frac{b^2k^4}{2a}-dk^4\ =0\quad;\quad\gamma^2\omega^2=\ -\frac{k^4}{4a}\left(\frac{b^2}{4a}+d\right)>0$$

$$b^2+4ad<0\ \text{ or }(\alpha_D\beta_C-\alpha_C\beta_D)^2+4\beta_D\beta_C<0 \tag{A3}$$

Condition (A3) means that $c^2_{1,2}$ (13) of equations (11) become complex numbers.

$$\gamma^4-\frac{bk^2}{2a}\gamma^2+\frac{k^4(b^2+4ad)}{16a^2}=0\quad:\quad{\gamma^2}_{1,2}=\frac{k^2}{4a}\left(b+/-\sqrt{-4ad}\ \right)$$

Thus for condition (15) obtain

$$\gamma^2=\frac{k^2}{4a}\left(b+\sqrt{-4ad}\ \right)>0\quad;\quad\omega^2=\frac{k^2}{4a}\left(b+\sqrt{-4ad}\ \right)-\frac{bk^2}{2a}$$

For $\gamma>0$ amplitude of solution of equation (11) in form (A1) grow up as *exp(γt)*. Thus wave propagation of small disturbances of economic spatial densities for two conjugate e-fluids model can go along with exponential growth of amplitude of disturbances in time. Exponential growth of money demand perturbations may reflect crisis trends.

Case 2. $\gamma = 0$.

Then system (A2) takes form:

$$a\omega^4 - b\,\omega^2(p^2 - k^2) - d[(p^2 - k^2)^2 \; - 4(\,\boldsymbol{k}\cdot\boldsymbol{p}\,)^2] = 0$$

$$2b\omega^2\,\boldsymbol{k}\cdot\boldsymbol{p} \; + 4d(p^2 - k^2)\,\boldsymbol{k}\cdot\boldsymbol{p} \; = 0$$

Hence from the second equation:

$$(k^2 - p^2) = \frac{b}{2d}\,\omega^2 > 0 \;;\; k^2 > p^2$$

The first equation takes form:

$$a\omega^4 + \frac{b^2}{4d}\,\omega^4 + 4\,d\,(\,\boldsymbol{k}\cdot\boldsymbol{p}\,)^2 = 0 \quad ; \quad \left(a + \frac{b^2}{4d}\right)\omega^4 = -4\,d\,(\,\boldsymbol{k}\cdot\boldsymbol{p}\,)^2 > 0$$

For condition (A3) obtain

$$\omega^4 = -\frac{16\,d^2}{4ad + b^2}\,(\,\boldsymbol{k}\cdot\boldsymbol{p}\,)^2 > 0$$

Let define θ as angle between vectors $\boldsymbol{k}$ and $\boldsymbol{p}$

$$\boldsymbol{k}\cdot\boldsymbol{p} = kpcos(\theta) \;;\;\; \varepsilon = \frac{|b|\,|\cos(\theta)|}{\sqrt{-(4ad+b^2)}}$$

Absolute values of vector p :

$$p = k\,(\sqrt{\varepsilon^2 + 1} - \varepsilon)$$

Amplitude of the wave (A1) can exponentially grow up as exp($\boldsymbol{p}\cdot\boldsymbol{x}$). Vector $\boldsymbol{p}$ belongs to cone that has angle θ with wave vector $\boldsymbol{k}$ that define direction of wave propagation.

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