A convexity result for the range of vector measures with applications to large economies

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Abstract

On a Boolean algebra we consider the topology u induced by a finitely additive measure μ with values in a locally convex space and formulate a condition on u that is sufficient to guarantee the convexity and weak compactness of the range of μ. This result à la Lyapunov extends those obtained in (Khan, Sagara 2013) to the finitely additive setting through a more direct and less involved proof. We will then give an economical interpretation of the topology u in the framework of coalitional large economies to tackle the problem of measuring the bargaining power of coalitions when the commodity space is infinite dimensional and locally convex. We will show that our condition on u plays the role of the "many more agents than commodities" condition introduced by Rustichini and Yannelis in (1991). As a consequence of the convexity theorem, we will obtain two straight generalizations of Schmeidler's and Vind's Theorems on the veto power of coalitions of arbitrary economic weight.

Keywords: Lyapunov's Theorem, finitely additive measures, correspondences, coalitional economies

JEL Classification: C02, C71, D51

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1 Introduction

Being one of the main assumptions of Lyapunov's Theorem on the range of vector measures, the notion of non-atomicity of measure spaces is of great importance in a wide variety of applications. Many significant results, like Aumann's or Richter's Theorems on the range and integrals of correspondences, depend on the fact that, given a non-atomic measure space (Σ, λ), every $\mathbb{R}^N$-valued measure on Σ absolutely continuous with respect to λ has a convex and weakly compact range[1]. As it is known, the validity of this statement depends directly on the dimension of $\mathbb{R}^N$ and, in general, it does not hold if we consider measures with values in infinite dimensional spaces, suggesting that the non-atomicity of λ should be replaced by a stronger property.

Progresses in this direction have been made in the last years since Khan and Sagara presented in [26] a version of Lyapunov's Theorem for Banach-space valued σ-additive measures in which non-atomicity is substituted by Maharam-type homogeneity[2]. This result can be formally stated as follows:

> **Theorem:** Given a Banach space E, a σ-algebra Σ and a homogeneous σ-additive measure $\mu\colon \Sigma \to E$, every measure $\nu\colon \Sigma \to E$ absolutely continuous with respect to μ has a convex and weakly compact range whenever the Maharam-type of μ is strictly greater than the density of E.

This idea was then sharpened by Greinecker and Podczeck in [20] and applied to economic models of exchange economies. As observed in [27], this convexity result still holds under the milder assumption that E is a locally convex space, provided that the measure μ admits a real valued control measure, a condition that is always satisfied by Banach-space valued measures.

In their recent work [28], Khan and Sagara came back on this problem removing the hypothesis on the existence of the control measure by means of a Theorem by Knowles ([29, Theorem V.1.1]). Although their approach follows from a close range the previous one, the tools involved are founded on deep concepts of measure theory and functional analysis that were not required in the Banach-space valued case and which are hard to be used in several applications.

In this paper we give an alternative, but yet equivalent, formulation of the Theorem above using the so-called Fréchet Nikodym approach, in which a measure is studied via the topological structure it induces on its Boolean algebra of definition. By doing so we will extend the mentioned results to include the case of a finitely additive measure μ that takes values in a locally convex space and that is not necessarily controlled. To overcome the difficulties faced in [28], we will decomposed μ as $\mu = \sum_i \mu_i$, where each μ_i admits a control measure. This way we will be able to reduce the proof to the simpler case of controlled measures and hence to Greinecker and Podzeck's approach. Our contribution

[1] Aumann explicitly refers to this formulation of Lyapunov's Theorem in [6]. For a short proof of Richter's Theorem that makes clear the connections with Lyapunov's see [23, Theorem 3, page 62].

[2] The Maharam-type of a measure μ on a Boolean algebra Σ is the least among the cardinalities of sub-algebras of the quotient $\Sigma/\mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the ideal of μ-null sets, whose order closure is the whole $\Sigma/\mathcal{N}(\mu)$. μ is *homogeneous* if the restriction of μ to every non-null principal ideal of Σ has the same Maharam-type.

is therefore twofold: while on the one hand we extend the convexity result to the finitely additive setting, on the other hand we present an alternative approach that shortcuts some technicalities of [28] and opens to a wide range of applications.

This part of our work is organized as follows: In Section 2 we recall some definitions and basic properties of topological Boolean algebras, and we introduce the key notion of *degree of saturation*, a cardinal invariant similar to the Maharam-type. In Section 3 we consider a topological Boolean algebra (Σ, u), a locally convex space E and find a condition under which every u-continuous measure $\mu\colon \Sigma \to E$ has a convex and weakly compact range. This result is obtained first under the assumption that u is metrizable (or, equivalently, that the measures we consider are controlled) and then in the general case. Moreover, in the spirit of [26, Theorem 4.2], we will find a partial converse to this result in the special case in which E is separable and metrizable.

In the second part of the paper, we provide applications of our main theorem to models of exchange economies. We first prove in Section 4.1 a convexity result for finitely additive correspondences and then focus on the study of coalitional finitely additive exchange economies with a locally convex space of commodities. Within this general model we study the problem of representing the influence that coalitions have on the economic activity. Specifically, we will show how in every exchange economy the set of all coalitions can be represented as a topological Boolean algebra so that coalitions with "small" economic power correspond to "topologically small" elements of the algebra. What will emerge is that the topological approach proposed in the paper is not only a natural consequence of the commodity-price duality, but also a necessary tool to study economies with a locally convex space of commodities without imposing significant (and apparently unjustified) restrictions on the model. As a corollary of our main theorem, we shall derive in Section 4 a condition ensuring the convexity of values of demand correspondences and then two characterizations of core allocations in the spirit of Schmeidler and Vind's Theorems ([33],[38]).

Notation

Throughout, E will denote a complete, Hausdorff locally convex topological vector space with continuous dual E^*. For $x^* \in E^*$, $x \in E$ we will also write $\langle x^*, x\rangle$ meaning $x^*(x)$. As usual, $\sigma(E, E^*)$ will denote the week topology on E induced by E^*.

We agree to denote by Σ a Boolean algebra, to use the symbols $\triangle$, $\wedge$, $\vee$, $\setminus$ and $\leq$ respectively for the Boolean operations of symmetric difference (sum), infimum (multiplication), supremum, difference and for the natural order, and to call 0_Σ (or simply 0) and 1_Σ respectively the null and unit element in Σ. For any $x \in \Sigma$, we will write x^c for $1_\Sigma \setminus x$ and $\Sigma \wedge x$ for the principal ideal generated by x, i.e. the set $\{y \in \Sigma : \ y \leq x\}$. For algebras of sets (i.e. sub-algebras of the power set of a non-empty set) we will also use the standard set notation.

By *measure* we will always mean a finitely additive function on a Boolean algebra. We will say that a measure μ on Σ is *exhaustive* if $\mu(x_n) \to 0$ whenever x_n, $n \in \mathbb{N}$, is a sequence of pairwise disjoint elements of Σ. When μ is a measure on Σ, we will refer to the set $\mathcal{N}(\mu) := \{x \in \Sigma : \ \mu(y) = 0 \ \forall y \leq x\}$ as the ideal of *μ-null elements* and denote by

$\hat{\Sigma}_\mu$ the quotient algebra $\Sigma/\mathcal{N}(\mu)$ so that the elements of $\hat{\Sigma}_\mu$ are the classes of equivalence determined by the relation $x \sim_\mu y \iff x\triangle y \in \mathcal{N}(\mu)$, for $x, y \in \Sigma$.

Other notation conventions will be introduced below. As our main references, we cite [30, 39] for the theory of finitely additive measures (charges) and topological Boolean algebras, [14, 2, 18] for elements of vector measures, integration and functional analysis.

2 Measures and topologies

We start this section by considering a bounded, positive scalar measure λ defined on a Boolean algebra Σ: as it is well known, the function $d_\lambda\colon (x, y) \mapsto \lambda(x\triangle y)$, for $x, y \in \Sigma$ defines on Σ an invariant pseudo-metric and hence a ring-topology on Σ that we will denote by $\tau(\lambda)$. Such a topology, whose 0-neighborhood system is generated by the sets $\{x \in \Sigma : \ \lambda(x) \leq 2^{-n}\}$ with n ranging in $\mathbb{N}$, will result to be the coarsest one making λ a uniformly continuous function. In this way, absolute continuity of a measure with respect to λ is translated in uniform continuity with respect to the uniform structure induced on Σ by λ. This means that a measure $\mu\colon \Sigma \to \mathbb{R}^N$ will be absolutely continuous with respect to λ[3] if and only if it is continuous with respect to the topology $\tau(\lambda)$. For an arbitrary measure $\mu\colon \Sigma \to E$ we follow a similar idea and define on Σ the ring-topology $\tau(\mu)$ as the one whose 0-neighborhood system is generated by the sets $\{x \in \Sigma : \ y \in U, \ y \leq x\}$ with U ranging over the 0-neighborhoods in E. Just like in the case of scalar measures, $\tau(\mu)$ will result to be the coarsest group-topology on $(\Sigma, \triangle)$ making μ a continuous function.

All these topologies on Σ belong to the family of the so-called *Fréchet-Nikodym topologies* (or simply FN-topologies) which are all the ring-topologies on Σ that make the ring-operations $\triangle$ and $\wedge$ uniformly continuous ([39, Proposition 1.6]. When u is a FN-topology on Σ we refer to (Σ, u) by calling it a *topological Boolean algebra.*

We will say that the topological Boolean algebra (Σ, u) is *exhaustive* if every sequence of pairwise disjoint elements of Σ converges to 0 or, equivalently, if every monotone net in Σ is Cauchy ([39, Proposition 3.4]). With this definition, a measure is exhaustive if and only if it induces an exhaustive FN-topology. On the other hand, if every monotone net in Σ order converging to some $x \in \Sigma$ is also topologically convergent to x we will call u an *order continuous* topology. These two classes of measures, which link the algebraic and the uniform nature of topological Boolean algebras, are related by the following property.

Proposition 2.1 (Proposition 4.2 in [39])**.** *Let (Σ, u) be an exhaustive, Hausdorff topological Boolean algebra that is complete (as a uniform space). Then Σ is a complete Boolean algebra and u is order continuous.*

In view of Proposition 2.1, we give a special importance to those measures inducing a complete FN-topology on Σ, namely the *closed measures.*

Definition 2.2. *A measure μ on Σ is* closed *if $(\Sigma, \tau(\mu))$ is a (uniformly) complete topological Boolean algebra.*

[3]We refer to the $\epsilon - \delta$ notion of absolute continuity as defined in [30, Definition 6.1.1]: i.e. μ is absolutely continuous with respect to λ if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\mu(y)| \leq \epsilon$ whenever $\lambda(x) \leq \delta$ for all $y \leq x \in \Sigma$.

We stress that if E is metrizable, Σ is a σ-algebra and $\mu\colon \Sigma \to E$ is σ-additive then μ is automatically closed ([39, Corollary 3.7]).

Let us call $\mathcal{N}(u)$ the closure of $\{0\}$ in (Σ, u). One sees that $\mathcal{N}(u)$ is a closed ideal in (Σ, u) (that coincides with $\mathcal{N}(\mu)$ if u is the FN-topology induced by a measure μ) so that the quotient $(\hat{\Sigma}, \hat{u}) := (\Sigma, u)/\mathcal{N}(u)$ results to be a Hausdorff topological Boolean algebra which is exhaustive or complete whenever u is so. Furthermore, if μ is a u-continuous measure on Σ then $\hat{\mu}\colon \hat{x} \mapsto \mu(x)$, for $x \in \hat{x} \in \hat{\Sigma}$, defines on $\hat{\Sigma}$ a $\hat{u}$-continuous measure. This, together with Proposition 2.1, gives us the following key result.

Proposition 2.3. *Let (Σ, u) be a complete and exhaustive topological Boolean algebra and let $(\hat{\Sigma}, \hat{u})$ be the quotient $(\Sigma, u)/\mathcal{N}(u)$. Then $\hat{\Sigma}$ is a complete Boolean algebra and $\hat{u}$ is order continuous.*

In addition, if $\mu\colon \Sigma \to E$ is a u-continuous measure and $\hat{\mu}\colon \hat{\Sigma} \to E$ is the function defined by $\hat{\mu}(\hat{x}) = \mu(x)$ for $x \in \hat{x} \in \hat{\Sigma}$ then $\hat{\mu}$ is a completely additive measure[4] *and $\mu(\Sigma) = \hat{\mu}(\hat{\Sigma})$.*

Finally, we introduce the notion of *absolute continuity*. Given two measures μ and ν over Σ, ν is *absolutely continuous with respect to* μ if $\tau(\nu)$ is coarser than $\tau(\mu)$ (in this case we write $\nu \ll \mu$). From this definition, which can be shown to be coherent with the usual ϵ-δ definition for real valued measures, it follows that for closed and exhaustive measures μ and ν over a complete algebra, $\nu \ll \mu$ if and only if $\mathcal{N}(\mu) \subseteq \mathcal{N}(\nu)$. This follows from the fact that for any two order continuous topologies u and v over a complete Boolean algebra Σ, $u \subseteq v$ if and only if $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ ([39, Theorem 4.8]).

2.1 The degree of saturation of a topological Boolean algebra

Recall that the *density* of a topological group G, denoted by $dens(G)$, is the least among the cardinalities of all dense subsets of G. It is straightforward to see that, if H is the τ-closure of the identity in G, then the Hausdorff quotient G/H has the same density as G.

In general, when H is a subset of G, it is not necessarily true that $dens(H) = dens(G)$. Consequently, for a given topological Boolean algebra (Σ, u) we could have that the subspace $\Sigma \wedge x$, considered with the topology induced by u, has density strictly smaller than $dens(\Sigma)$. This observation brings us to the following definition.

Definition 2.4. *The* degree of saturation *of a topological Boolean algebra (Σ, u), denoted by $sat(u)$, is the least among the densities of all $\Sigma \wedge x$, with $x \in \Sigma \setminus \mathcal{N}(u)$, each one considered as a topological subspace of (Σ, u).*

If μ is a measure over Σ, we also write $sat(\mu)$ to denote $sat(\tau(\mu))$ and call it degree of saturation *of the measure μ.*

Just like the density character, we note that the degree of saturation of a topological Boolean algebra (Σ, u) is the same as the one of the correspondent Hausdorff quotient. In fact, if one calls $(\hat{\Sigma}, \hat{u})$ the quotient $(\Sigma, u)/\mathcal{N}(u)$, then $dens(\Sigma \wedge x) = dens(\hat{\Sigma} \wedge \hat{x})$ for every $x \in \hat{x} \in \hat{\Sigma}$.

[4] i.e. for every net $(x_i)_{i\in\mathcal{I}}$ of pairwise disjoint elements of $\hat{\Sigma}$, $(\hat{\mu}(x_i))_{i\in\mathcal{I}}$ is summable and $\sum_i \hat{\mu}(x_i) = \hat{\mu}(\sup_i x_i)$.

Proposition 2.5. *Let Σ be a complete Boolean algebra and u, v two order continuous FN-topologies over Σ such that $v \subseteq u$. Then $sat(u) \leq sat(v)$.*

Proof. We prove that for every $x \in \Sigma \setminus \mathcal{N}(v)$ there is a continuous function $f\colon (\Sigma, v) \to (\Sigma, u)$ such that $f(\Sigma \wedge x)$ is of the form $\Sigma \wedge y$ for some $y \in \Sigma \setminus \mathcal{N}(u)$. This way, for every v-dense subset D of $\Sigma \wedge x$, $f(D)$ is a u-dense subset of $\Sigma \wedge y$ with cardinality smaller or equal than $|D|$. We do it only for $x := 1_\Sigma$, as the proof strategy remains the same for a generic $x \in \Sigma \setminus \mathcal{N}(v)$.

The ideal $\mathcal{N}(v)$ can be seen as a monotone net in Σ, hence convergent to $a := \sup \mathcal{N}(v) \in \Sigma$ (which exists by the completeness of Σ) by the order continuity assumption. This, being $\mathcal{N}(v)$ closed, implies that $\mathcal{N}(v)$ can be written as the principal ideal $\Sigma \wedge a$.

Let $b := a^c$ and call u_b and v_b the subspace topologies induced on $\Sigma \wedge b$ by u and v respectively. v_b and u_b are order-continuous topologies defined on a complete Boolean algebra and, moreover, by the choice of b, $\mathcal{N}(v_b) = \mathcal{N}(v) \wedge b = \{0\} = \mathcal{N}(u) \wedge b = \mathcal{N}(u_b)$. But then, a glance at [39, Theorem 4.8] gives us $v_b = u_b$. Let $f\colon \Sigma \to \Sigma \wedge b$ be the function that assigns $x \wedge b$ to each $x \in \Sigma$. Of course, f is surjective and continuous with respect to v and v_b. Since $v_b = u_b$, f is the desired function. □

Corollary 2.6. *Let (Σ, u) be a complete and exhaustive topological Boolean algebra and μ a u-continuous measure on Σ. Then $sat(u) \leq sat(\mu)$.*

Proof. Let $(\hat{\Sigma}, \hat{u})$, $\hat{\mu}$ as in Proposition 2.3 so that Σ is a complete Boolean algebra and u (and hence $\tau(\mu)$) are order-continuous topologies on Σ. Since $\hat{u}$ and $\tau(\hat{\mu})$ satisfy the assumptions of Proposition 2.5, $sat(\hat{u}) \leq sat(\tau(\hat{\mu})) = sat(\hat{\mu})$. The thesis follows from the fact that $sat(u) = sat(\hat{u})$ and $sat(\mu) = sat(\hat{\mu})$.

□

Remark 2.7. Let μ be a measure on Σ and $\pi\colon \Sigma \to \hat{\Sigma}_\mu$ the quotient map. By the argument above, $sat(\mu) < \infty$ if and only if $\pi(\Sigma \wedge x)$ is finite for some $x \in \Sigma \setminus \mathcal{N}(\mu)$. But the latter is equivalent with saying that x is the join of μ-atoms[5] and, consequently, $sat(\mu) = 1$.

In other words, μ is non-atomic if and only if $sat(\mu)$ is infinite if and only if $sat(\mu) > 1$.

Remark 2.8. The notion of *saturation* of a measure space has been widely employed in different applications of measure and probability theory in the last decades (see [19, 25] and their references for a survey). However, it is in [28] that we find this notion adapted to topological Boolean algebras with the following definition: a measure μ on Σ is *saturated* if there is no $x \in \Sigma \setminus \mathcal{N}(\mu)$ such that $\Sigma \wedge x$, endowed with the topology $\tau(\mu)$, is separable. The definition of *degree of saturation* we gave in 2.4 can be seen as a natural extension of this concept: in fact a measure μ is saturated in the sense given by Khan and Sagara if and only if $sat(\mu)$ is uncountable.

[5] $a \in \Sigma$ is a *μ-atom* if for all $b \leq a$, $b \in \mathcal{N}(\mu)$ or $a \setminus b \in \mathcal{N}(\mu)$.

3 A Convexity result for the range of vector measures

A significant consequence of Lyapunov's Theorem is that if Σ is a σ-algebra and $\lambda\colon \Sigma \to [0,+\infty[$ a σ-additive, non-atomic measure, then the range of every $\mathbb{R}^N$-valued measure absolutely continuous with respect to λ is convex and compact. By adapting the terminology used in [26], we say that the measure λ has the *Lyapunov property with respect* to any finite dimensional space. This brings us to the following definition.

Definition 3.1. *We say that a topological Boolean algebra* (Σ, u) *has the* Lyapunov property *with respect to the space* E *if every* u*-continuous measure* $\nu\colon \Sigma \to E$ *has a convex and weakly compact range.*

Similarly, a measure μ *on* Σ *has the* Lyapunov property *with respect to* E *if* $\tau(\mu)$ *has the Lyapunov property with respect to* E*.*

In other words, a measure $\mu\colon \Sigma \to E$ has the Lyapunov property with respect to E if every E-valued measure absolutely continuous with respect to μ has a convex and weakly compact range. Our main problem can then be written in the following way:

> **Problem:** Given the locally convex space E, which topological Boolean algebras have the Lyapunov property with respect to E?

We divide our analysis in two steps: first we consider only topological Boolean algebras whose uniform structure is induced by a scalar measure, then we tackle the problem in the general case.

3.1 The case of measures admitting a control

We say that a vector measure $\mu\colon \Sigma \to E$ has a *control measure* $\lambda\colon \Sigma \to [0,+\infty[$ if $\mu \ll \lambda$, i.e. if $\lim_n \mu(x_n) = 0$ whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in Σ such that $\lim_n \lambda(x_n) = 0$. In this case, μ is continuous with respect to the λ-topology and therefore it is exhaustive and bounded and it is σ-additive when λ is σ-additive. Moreover, both μ and λ will be closed whenever Σ is a σ-algebra and λ is σ-additive (see [39, Corollary 3.7]).

In general, not all vector measures are controlled, however, when E is metrizable, a slight generalization of Bartle-Dunford-Schwartz's Theorem ensures that if μ is a E-valued exhaustive measure then it admits a control measure λ which can be taken σ-additive if μ is so ([39, Corollary 7.5]).

Throughout all this section, we assume that $\mathcal{A}$ is a σ-algebra of subsets of a non-empty Ω and that $\lambda\colon \mathcal{A} \to [0,+\infty[$ is a σ-additive measure. For all $A \in \mathcal{A}$, we will use χ_A to denote the *characteristic function* of A. If we identify the functions that are equal λ-almost everywhere we can define a unique operator $T_\mu\colon L^\infty(\lambda) \to E$ with the property that $T_\mu(\chi_A) = \mu(A)$, $A \in \mathcal{A}$, and prove that T_μ is continuous with respect to the weak* topology on $L^\infty(\lambda)$ and the weak topology on E ([14, IX.1.4]). We call T_μ the *integral operator associated to* μ and also write $\int f\, d\mu$ for $T_\mu(f)$ (see [29] for references on this integration procedure). Observe that, for a continuity argument, for all $x^* \in E^*$ and $f \in L^\infty(\lambda)$ $x^* \circ T_\mu(f) = x^*\left(\int f\, d\mu\right) = \int f\, d(x^* \circ \mu) T_{x^*\circ\mu}(f)$.

The following Theorem shows the relation between the non-injectiveness of the operator T_μ and the convexity of the range of μ. Its proof can be found in [14, IX.1.4] for Banach-space valued measures and in [36, Proposition 2.3] for the locally convex case.

Proposition 3.2. *Let $\mu\colon \mathcal{A} \to E$ be a vector measure over a σ-algebra of sets and $\lambda\colon \mathcal{A} \to [0, +\infty[$ a σ-additive control for μ. For every $A \in \mathcal{A} \setminus N(\lambda)$, assume that the restriction of the operator T_μ to the space $L^\infty(\lambda_A)$, consisting of functions in $L^\infty(\lambda)$ vanishing off A, is non-injective. Then $\mu(\mathcal{A} \cap A)$ is weakly compact and convex for all $A \in \mathcal{A}$.*

In view of the above, our next aim is to find conditions on the measures μ and λ ensuring that each of the operators $T_\mu\colon L^\infty(\lambda_A) \to E$, $A \in \mathcal{A} \setminus \mathcal{A}$, is non-injective. For example, we could ask that $dim\ (L^\infty(\lambda_A)) > dim\ E$[6] for every $A \in \mathcal{A} \setminus N(\mu)$, a condition studied by Rustichini and Yannelis in [32]. The approach below closely follows the line of Greinecker and Podzeck ([20]) and it is included here for the sake of completeness. First we will need the following lemma.

Lemma 3.3. *Let $\lambda\colon \mathcal{A} \to [0, +\infty[$ be a σ-additive measure over a σ-algebra of sets. Then $dens(\mathcal{A}, \tau(\lambda)) \le dens(L^1(\lambda), \|\cdot\|_1)$.*

Proof. Take a dense set $\mathcal{F} \subset L^1(\lambda)$ and for $f \in \mathcal{F}$ define $B_f := \{x :\ |1 - f(x)| \le \frac{1}{2}\} \in \mathcal{A}$. Our goal is to prove that for any $A \in \mathcal{A}$ and $\epsilon > 0$ we can take $f \in \mathcal{F}$ such that $\lambda(A \triangle B_f) < 2\epsilon$. This way $\{B_f :\ f \in \mathcal{F}\}$ is dense in $(\mathcal{A}, \tau(\lambda))$ and so the thesis will follow from the generality of $\mathcal{F}$.

Choose $A \in \mathcal{A}$, $\epsilon > 0$ and take $f \in \mathcal{F}$ such that $\|\chi_A - f\|_1 < \epsilon$. We have that:

$$\epsilon > \|\chi_A - f\|_1 = \int |\chi_A - f(x)|\ d\lambda(x) \ge \int_{A \setminus B_f} |1 - f(x)|\ d\lambda(x) + \int_{B_f \setminus A} |f(x)|\ d\lambda(x)$$

By construction, $|f(x)| \ge \frac{1}{2}$ for $x \in B_f$ while $|1 - f(x)| \ge \frac{1}{2}$ for $x \notin B_f$ so from the previous equation follows that:

$$\epsilon > \|\chi_A - f\|_1 \ge \int_{A \setminus B_f} \frac{1}{2}\ d\lambda + \int_{B_f \setminus A} \frac{1}{2}\ d\lambda = \frac{1}{2}\lambda(A \setminus B_f) + \frac{1}{2}\lambda(B_f \setminus A) = \frac{1}{2}\lambda(A \triangle B_f)$$

as claimed. ☐

It is quite easy to prove that in Lemma 3.3 the equality $dens(\mathcal{A}, \tau(\lambda)) = dens(L^1(\lambda), \|\cdot\|_1)$ holds true (see [31]).

In [20], the authors consider a σ-algebra of sets $\mathcal{A}$ and for every infinite cardinal number κ they define a class of *κ-atomless* measures. The latter consists of σ-additive measure $\lambda\colon \mathcal{A} \to [0, 1]$ such that an equivalent to Lemma 3.3 holds (i.e. $densL^1(\lambda) \ge \kappa$, with κ infinite cardinal number). By doing so, they were able to prove in [20, Section 3] that if $\lambda\colon \mathcal{A} \to [0, 1]$ is a σ-additive, κ-atomless measure and E is a Banach space separated by a family $\mathcal{F} \subset E^*$ with $|\mathcal{F}| < \kappa$ then every measure $\mu\colon \mathcal{A} \to E$ absolutely continuous with respect to λ has a convex and weakly compact range.

Next Theorem can be seen as an extension of Greinecker and Podczeck's main result.

[6]Here $dim\ E$ stands for the algebraic dimension of the linear space E.

Theorem 3.4. *Let $\mu\colon \mathcal{A} \to E$ be a measure on a σ-algebra of sets and $\lambda\colon \mathcal{A} \to [0, +\infty[$ be a σ-additive control measure for μ with infinite degree of saturation. Assume that there exists a family $\mathcal{F} \subset E^*$ that separates the points of $\overline{span}\mu(\mathcal{A})$ with $|\mathcal{F}| < sat(\lambda)$. Then $\mu(\mathcal{A})$ is convex and weakly compact.*

Proof. We identify functions which are λ-almost everywhere equal.

By Proposition 3.2 it will be sufficient to prove that for any $A \in \mathcal{A} \setminus N(\lambda)$, the restriction of the operator $T_\mu\colon f \mapsto \int f\, d\mu$ to $L^\infty(\lambda_A)$ is non-injective. We will do this for $A = \Omega$, since the proof remains the same for the general case.

If $\mathcal{F}$ is finite then $\overline{span}\mu(\Sigma)$ must be finite dimensional. At the same time, being $sat(\lambda)$ infinite, the space $L^\infty(\lambda)$ has infinite dimension and so the operator $T_\mu\colon L^\infty(\lambda) \to E$ cannot be injective. Therefore we can assume that $\mathcal{F}$ is infinite.

By the Radon-Nikodym Theorem, to every $x^* \in \mathcal{F}$ we can associate a function $g_{x^*} \in L^1(\lambda)$ so that the measure $x^* \circ \mu$ is described by the relation $A \mapsto \int_A g_{x^*}\, d\lambda$ for $A \in \mathcal{A}$. Put $Y := \overline{span}\{g_{x^*} :\ x^* \in \mathcal{F}\}$. Since the set of finite linear combinations of the g_{x^*}'s with rational coefficients is a dense subset of Y with cardinality $|\mathcal{F}|$ and the latter is strictly smaller than $sat(\lambda)$ by hypothesis, we have that $dens(Y) = |\mathcal{F}| < dens(\mathcal{A}, \tau(\lambda))$. Consequently, Y cannot be the whole $L^1(\lambda)$, since $L^1(\lambda)$ has density greater or equal to $(\mathcal{A}, \tau(\lambda))$ by Lemma 3.3.

Now, because Y is a closed proper subspace of $L^1(\lambda) = L^\infty(\lambda)^*$, as a consequence of the Hahn-Banach Theorem there must be a $f \in L^\infty(\lambda) \setminus \{0\}$ such that for all $x^* \in \mathcal{F}$, $\int f\, d(x^* \circ \mu) = 0$ and so, by a continuity argument, $x^* \circ T_\mu(f) = 0$. But $T_\mu(f)$ belongs to $\overline{span}\mu(\mathcal{A})$, so $x^* \circ T_\mu(f) = 0$ for all $x^* \in \mathcal{F}$ implies that $T_\mu(f) = 0$ and hence that T_μ is non-injective as claimed. □

In the assumptions of Theorem 3.4, λ is a σ-additive real valued measure defined on a σ-algebra and as such it is closed. The corollary that follows shows that this property alone is enough to guarantee the validity of the results.

Corollary 3.5. *Let $\lambda\colon \Sigma \to [0, +\infty[$ be a closed control measure for $\mu\colon \Sigma \to E$ with infinite degree of saturation and assume that there is a family $\mathcal{F} \subseteq E^*$ separating the points of $\overline{span}\mu(\Sigma)$ such that $|\mathcal{F}| < sat(\lambda)$. Then $\mu(\Sigma)$ is convex and weakly compact.*

Proof. Let $(\hat{\Sigma}_\lambda, \hat{u})$ be the quotient $(\Sigma, \tau(\lambda))/N(\lambda)$ and $\hat{\mu}\colon \hat{\Sigma}_\lambda \to E$, $\hat{\lambda}\colon \hat{\Sigma}_\lambda \to [0, +\infty[$ be the measures defined by $\hat{\mu}(\hat{x}) = \mu(x)$, $\hat{\lambda}(\hat{x}) = \lambda(x)$ for $x \in \hat{x} \in \hat{\Sigma}_\lambda$. By Proposition 2.3, being (Σ, u) complete and exhaustive, $\hat{\Sigma}_\lambda$ is complete and $\hat{\lambda}$ is a completely additive control measure for $\hat{\mu}$.

By the Loomis-Sikorski representation Theorem ([34, 29.1]), there exists a σ-algebra of sets $\mathcal{A}$ and a surjective homomorphism $\pi\colon \mathcal{A} \to \hat{\Sigma}_\lambda$ such that $Ker\pi$ is a σ-ideal in $\mathcal{A}$ or, equivalently, $\mathcal{A}/Ker\pi$ is isomorphic to $\hat{\Sigma}_\lambda$. Let us define the measures $\lambda_\pi := \hat{\lambda} \circ \pi\colon \mathcal{A} \to [0, +\infty[$ and $\mu_\pi := \hat{\mu} \circ \pi\colon \mathcal{A} \to E$. By construction, λ_π is a σ-additive control measure for μ_π, $sat(\lambda_\pi) = sat(\hat{\lambda}) = sat(\lambda)$ and $\mu_\pi(\mathcal{A}) = \mu(\Sigma)$. In other words, $\overline{span}\mu_\pi(\mathcal{A})$ is separated by the family $\mathcal{F} \subset E^*$ with $|\mathcal{F}| < sat(\lambda_\pi)$ and so, being satisfied the condition of Theorem 3.4, $\mu_\pi(\mathcal{A})$, and therefore $\mu(\Sigma)$, is convex and weakly compact as claimed. □

3.2 The general case

In the absence of a control measure for $\mu\colon \Sigma \to E$, it is much harder to obtain a result close to Theorem 3.4 with a similar approach. This is mainly due to the difficulties that can arise in generalizing some of the functional analytic tools used throughout the proofs of Proposition 3.2, Lemma 3.3 and Theorem 3.4, in which the properties of $L^\infty(\lambda)$ and $L^1(\lambda)$ were intensively employed.

Our main goal in this Section is to prove the following:

Theorem 3.6. *Let $\mu\colon \Sigma \to E$ be a closed and exhaustive measure with infinite degree of saturation and suppose that there is a family $\mathcal{F} \subseteq E^*$ that separates the points of $\overline{span}\mu(\Sigma)$ with $|\mathcal{F}| < sat(\mu)$. Then $\mu(\Sigma)$ is convex and weakly compact.*

The idea behind the proof is simple and what has been done in [36]. It consists in decomposing μ as a sum $\mu = \sum_{i\in\mathcal{I}} \mu_i$ in which each one of the μ_i's is a measure satisfying the hypothesis of Corollary 3.5 and then showing that $\mu(\Sigma) = \sum_i \mu_i(\Sigma)$. However, if on one hand the writing $\sum_{i\in\mathcal{I}} \mu(\Sigma)$ has a clear meaning when $\mathcal{I}$ is finite, in the infinite case things must be handled much more carefully and a little additional terminology is needed. For convenience, in the following we recall some of the definitions and results used in [36, Section 3] to study infinite sums and uniform summability.

We recall that a family x_i, $i \in \mathcal{I}$, in E is *summable* if the net of partial sums $\sum_{i\in F} x_i$, $F \subset \mathcal{I}$ finite, converges to some element $x_0 \in E$. In this case, we write $\sum_{i\in\mathcal{I}} x_i := x_0$. With this definitions, the set $\ell^1(\mathcal{I}, E)$ of all summable families of elements of E indexed by $\mathcal{I}$ will form a vector subspace of $E^{\mathcal{I}}$.

We say that a system $A := \prod_{i\in\mathcal{I}} A_i \subset E^{\mathcal{I}}$ of summable families is *uniformly summable* if the nets of partial sums of the families in A converge uniformly, i.e. if for every 0-neighborhood U in E there is a finite subset $F \subset \mathcal{I}$ such that $\sum_{i\in F_0} x_i \in U$ for every finite $F_0 \subset \mathcal{I} \setminus F$ and every $(x_i)_{i\in\mathcal{I}} \in A$. In this case, we write $\sum_{i\in\mathcal{I}} A_i$ for the set $\{\sum_{i\in\mathcal{I}} x_i :\ (x_i)_i \in A\}$.

We will need the following two results. The first one is a consequence of lemmas [36, 3.6, 3.8, 3.9].

Lemma 3.7. *Let $A_i \subset E$, $i \in \mathcal{I}$ be a family of non-empty convex and weakly compact subsets of E such that $A := \prod_{i\in\mathcal{I}} A_i$ is uniformly summable. Then $\sum_{i\in\mathcal{I}} A_i$ is convex and weakly compact too.*

Proof. Let us denote by w the subspace topology on $\ell^1(\mathcal{I}, E)$ induced by the product topology on $(E, \sigma(E, E^*))^{\mathcal{I}}$. Since each of the A_i's is convex and weakly compact by assumption, $A = \prod_{i\in\mathcal{I}} A_i$ is convex and compact with respect to the topology w by Tychonoff's Theorem.

Our aim is to show that the relation $S\colon (x_i)_{i\in\mathcal{I}} \mapsto \sum_{i\in\mathcal{I}} x_i$ defines a linear function $S\colon \ell^1(\mathcal{I}, E) \to E$ whose restriction to A is continuous with respect to the topology w and the weak topology on E. As a consequence $S(A) = \sum_{i\in\mathcal{I}} A_i$ will be a convex and weakly compact subset of E.

The linearity of S is immediate so we focus on the continuity of its restriction to A. Let U be a closed and symmetric 0-neighborhood of $(E, \sigma(E, E^*))$. By the uniform summability of $\prod_{i\in\mathcal{I}} A$, there is a finite subset F of $\mathcal{I}$ such that $\sum_{i\in F_0} x_i \in U$ for all finite subsets F_0 of $\mathcal{I}\setminus F$ and $(x_i)_{i\in\mathcal{I}} \in A$. Let V be a 0-neighborhood of $(E,\sigma(E,E^*))$ such that $\sum_{i\in F} V \subset U$, then consider the set $W := \{(x_i)_{i\in\mathcal{I}} : \ x_i \in V \ \forall \ i \in F\}$ which is a 0-neighborhood in $(\ell^1(\mathcal{I},E), w)$.

If we choose $x := (x_i)_{i\in\mathcal{I}}$ and $y := (y_i)_{i\in\mathcal{I}}$ in A so that $x - y \in W$, it will follow that

$$S(x-y) = \sum_{i\in F}(x_i - y_i) + \sum_{i\notin F} x_i - \sum_{i\notin F} y_i \in \sum_{i\in F} V + U - U \subseteq U + U + U$$

proving that the restriction of S to A is continuous as claimed. □

Proposition 3.8 (Theorem 4.5 in [36]). *Let $\mu\colon \Sigma \to E$ be a closed and exhaustive measure. Then there is a system $a_i \in \Sigma$ of almost disjoint*[7] *elements and $x_i^* \in E^*$, $i \in \mathcal{I}$, such that the measures $\mu_i\colon \Sigma \to E$ defined by $\mu_i(x) = \mu(x \wedge a_i)$, $x \in \Sigma$, satisfy the following conditions:*

1. *for each $i \in \mathcal{I}$ the measure μ_i is absolutely continuous with respect to $|x_i^* \circ \mu|$;*
2. *for all $x \in \Sigma$, $(\mu_i(x))_{i\in\mathcal{I}}$ is summable and $\mu(x) = \sum_{i\in\mathcal{I}} \mu_i(x)$;*
3. *$\prod_{i\in\mathcal{I}} \mu_i(\Sigma)$ is uniformly summable and $\sum_{i\in\mathcal{I}} \mu_i(\Sigma) = \mu(\Sigma)$.*

What Proposition 3.8 ensures is that whenever we have a closed and exhaustive measure $\mu\colon \Sigma \to E$, we can always write it as the infinite sum of some controlled measures μ_i's that can be chosen so that $(\mu_i(\Sigma))$ is uniformly summable. We now have all the ingredients to prove our main theorem.

Theorem 3.9. *Let $\mu\colon \Sigma \to E$ be a closed and exhaustive measure with infinite degree of saturation and suppose that there is a family $\mathcal{F} \subseteq E^*$ that separate the points of $\overline{span}\mu(\Sigma)$ with $|\mathcal{F}| < sat(\mu)$. Then $\mu(\Sigma)$ is convex and weakly compact.*

Proof. Let $x_i^* \in E^*$, $a_i \in \Sigma$ and $\mu_i\colon \Sigma \to E$, $i \in \mathcal{I}$, be as in Proposition 3.8 so that $\prod_{i\in\mathcal{I}} \mu_i(\Sigma)$ is uniformly summable and $\mu(\Sigma) = \sum_{i\in\mathcal{I}} \mu_i(\Sigma)$. If we prove that each of the μ_i's satisfies the assumptions of Corollary 3.5, the thesis will follow from Lemma 3.7.

Fix a $i \in \mathcal{I}$ and call λ the measure $|x_i^* \circ \mu|\colon \Sigma \to [0,+\infty[$, which is a control measure for μ_i by point (1) in 3.8. By construction, λ is absolutely continuous with respect to μ and therefore, beside being closed, it has a degree of saturation greater or equal than $sat(\mu)$ so that $|\mathcal{F}| < sat(\lambda)$.

Moreover, since $\mu_i(\Sigma) \subseteq \mu(\Sigma)$, the family $\mathcal{F}$ separates the points of $\overline{span}\mu_i(\Sigma)$ too. But then all the assumptions on Corollary 3.5 are satisfied and $\mu_i(\Sigma)$ is convex and weakly compact as claimed. □

Corollary 3.10. *Let $\mu\colon \Sigma \to E$ be a closed and exhuaustive measure with infinite degree of saturation and suppose that there is a family $\mathcal{F} \subset E^*$ that separates the points of E with $|\mathcal{F}| < sat(\mu)$. Then μ has the Lyapunov property with respect to E.*

[7]i.e. such that $a_i \wedge a_j \in \mathcal{N}(\mu)$ for all $i, j \in \mathcal{I}$ distinct.

Proof. Let $\nu\colon \Sigma \to E$ be a measure absolutely continuous with respect to μ. Then ν is closed, exhaustive and has degree of saturation greater or equal to $sat(\mu)$, where the latter is strictly greater than $|\mathcal{F}|$ by assumption. Since $\mathcal{F}$ separates the points of E, and consequently of $\overline{span}\nu(\Sigma)$, ν satisfies all the assumptions of Theorem 3.9 and as such it has a convex and weakly compact range. $\square$

Corollary 3.11. *Let (Σ, u) be a complete and exhaustive topological Boolean algebra such that $sat(u)$ is infinite. Furthermore, assume that there is a family $\mathcal{F} \subset E^*$ that separates the points with $|\mathcal{F}| < sat(u)$. Then (Σ, u) has the Lyapunov property with respect to E.*

Proof. It follows directly from Corollary 2.6 that every u-continuous measure $\mu\colon \Sigma \to E$ satisfies the assumptions of Corollary 3.10 and therefore it has a convex and wekly compact range. $\square$

Remark 3.12 (Remarks on the main theorem)**.** As mentioned before, what makes it quite easier to work with a vector measure $\mu\colon \mathcal{R} \to E$ admitting a control $\lambda\colon \mathcal{R} \to [0, +\infty[$ is the possibility of employing many fine properties of the spaces $L^1(\lambda)$ and $L^\infty(\lambda)$. When such a λ does not exist, it is necessary to study other function spaces in order to replace $L^1(\lambda)$ and $L^\infty(\lambda)$. This is done, for example, in [29] where a generalization of Proposition 3.2 is given. Following this line of investigation, in [28] Khan and Sagara proved that a closed, σ-additive measure over a σ-algebra $\mu\colon \mathcal{A} \to E$ has convex and weakly compact range if it is homogeneous of type strictly greater than the topological dimension of E, generalizing a previous result contained in [26]. The problem with this approach is mainly due to the very deep analytical tools employed which seem to be a very high price to be payed in this framework.

To prove Theorem 3.9, which can be seen as a general case of the above mentioned result of Khan and Sagara, we decided to follow a completely different path inspired by [36]. Theorem 3.9 improves previous results in two respects: the less restrictive hypothesis, in which neither σ-additiveness of the measures nor the σ-completeness of the algebra are required, and the proof strategy itself which seems to be more flexible to further developments.

3.3 A refinement of the main result

As it was first proved in [35, Theorem 3], a finitely additive measure taking values in a locally convex space has a relatively weakly compact range if and only if it is exhaustive (see also or [14, Corollary 18.1.I] for the case of Banach-space valued measures). This implies that whenever $\mu\colon \Sigma \to E$ is exhaustive, the space $\overline{span}\mu(\Sigma)$ belongs to the class of *weakly compactly generated spaces*, where a linear subspace Y of E is *weakly compactly generated* if it is the closed linear span of a weakly compact subset of E. In the light of this remark, we might agree with saying that much of the results on the range of E-valued exhaustive measures can be reformulated in terms of weakly compactly generated subsets of E. The following Theorem is a way to do this.

Theorem 3.13. *Let $\mu\colon \Sigma \to E$ be a closed and exhaustive measure with infinite degree of saturation and assume that for every weakly compactly generated subspace Y of E there* *is a family $\mathcal{F} \subset E^*$ separating the points of Y such that $|\mathcal{F}| < sat(\mu)$. Then μ has the Lyapunov property with respect to E.*

Proof. Let $\nu\colon \Sigma \to E$ be a measure absolutely continuous with respect to μ and call $Y := \overline{span}\nu(\Sigma)$. Our goal is to prove that ν has a convex and weakly compact range by showing that is satisfies the assumptions of Theorem 3.9.

Being ν exhaustive, Y is a weakly compactly generated subspace of E and so, by hypothesis, its points are separated by a family $\mathcal{F} \subset E^*$ with $|\mathcal{F}| < sat(\mu)$. The fact that ν is closed, so that $sat(\mu) \leq sat(\nu)$ (Corollary 2.6), concludes the proof. □

We stress that Theorem 3.13 is a significant improvement of Corollary 3.10 as it allows us to consider a much wider class of measures. In the following example, we describe a locally convex, infinite dimensional space whose weakly compactly generated subspaces are finite dimensional.

Example 3.14. Consider the infinite dimensional space $X = c_{00}$ consisting of all real sequences with finite support (i.e. sequences $(x_n)_n \subset \mathbb{R}$ such that $x_n = 0$ for all but a finite number of indexes $n \in \mathbb{N}$). On E we take the topology τ_B generated by the base:

$$\left\{ E \cap \left(\prod_{n\in\mathbb{N}} U_n \right) : \ U_n \text{ is an open set of } \mathbb{R} \text{ for all } n \in \mathbb{N} \right\}.$$

Such τ_B is commonly known as *box topology* and, by [24, section 6.6], it makes (X, τ_B) a complete locally convex space. Furthermore, one observes that bounded sets in E must lie in finite dimensional subspaces of X (see [41, Theorem 4]).

Consider now the algebra $\mathcal{B}$ of measurable subsets of the real unit interval $[0, 1]$ with the Lebesgue measure λ. Since the range of every measure $\mu\colon \mathcal{B} \to X$ absolutely continuous with respect to λ lies in a finite dimensional subspace of X, by the classical Lyapunov's Theorem $\mu(\mathcal{B})$ must be compact and convex. This implies that $(\mathcal{B}, \tau(\lambda))$ has the Lyapunov's property with respect to X even though there is no family of functionals $\mathcal{F} \subset X^*$ that separates the points of X with $|\mathcal{F}| < sat(\lambda)$.

Similarly with what is done in [26, 27] and [20, Corollary 1], one might want to find a relation between the density of the space E and the degree of saturation of a E-valued measure with the Lyapunov property with respect to E. In order to do this, we will recall the following preliminary result, due to Amir and Lindenstrauss ([3]), whose proof can be found in [18, Theorem 13.3] for Banach spaces and in [10, Theorem 13] for a general class of spaces that includes locally convex metrizable spaces.

Proposition 3.15. *If E is a metrizable weakly compactly generated locally convex space then $dens(E) = dens(E^*)$.*

Proposition 3.15 allows us reformulate the conditions in 3.13 in terms of the density of weakly compactly generated subspaces of E.

Proposition 3.16. *Let $\mu\colon \Sigma \to E$ be a closed and exhaustive measure and assume that every weakly compactly generated subspace of E is linearly homeomorphic to some metrizable space with density strictly smaller than $sat(\mu)$. Then μ has the Lyapunov property with respect to E.*

Proof. Let $\nu \colon \Sigma \to E$ be a measure absolutely continuous with respect to μ. We need to prove that $\nu(\Sigma)$ is convex and weakly compact. Since ν is closed and exhaustive, it is sufficient to show that the points of $Y := \overline{span}\nu(\Sigma)$ are separated by an infinite family $\mathcal{F} \subset E^*$ with $|\mathcal{F}| < sat(\nu)$, then apply Theorem 3.9.

Being ν exhaustive, Y is weakly compactly generated subspace of E and therefore it is metrizable by assumption. Thus, by applying Proposition 3.15, we can take a family $\mathcal{F} \subset Y^*$ with cardinality $dens(Y)$ that is dense in Y^* with respect to the weak* topology. The family $\mathcal{F}$ has therefore cardinality strictly smaller than $sat(\mu)$ by assumption and it separates the points of Y as a consequence of the Hahn-Banach Theorem ([18, Proposition 3.39]).

This, together with the fact that $sat(\mu) \leq sat(\nu)$ (Proposition 2.6), implies that $|\mathcal{F}| < sat(\nu)$ as desired. $\square$

Remark 3.17. In the setting of Proposition 3.16, the measure μ takes values in a subspace of E whose topology can be induced by a metric. Thus, by the Theorem of Bartle-Dunford-Schwartz (as formulated in [39, Corollary 7.5]) the measure μ is equivalent with respect to a scalar measure $\lambda \colon \Sigma \to [0, +\infty[$ (i.e. $\tau(\mu) = \tau(\lambda)$).

This makes possible to prove Proposition 3.16 via Corollary 3.5 without recurring to Theorem 3.9.

3.4 A necessary condition for a measure to have convex range

It is known that Lyapunov's Theorem also characterizes finite dimensional spaces. In fact, if E is a F-space[8] such that every E-valued non-atomic σ-additive measure on σ-algebras has compact or convex range, E cannot have infinite dimension (see [14, Corollary 6 on pg 265] for the case E is a Banach space, for the general result see [40]).

We wonder whether a similar statement can be generalized to spaces with higher dimension, proving that those conditions that in Theorem 3.9 were shown to be sufficient for the convexity result are also necessary. In other words, we ask if the following question can be answered positively:

> **Question:** Let $\mu \colon \Sigma \to E$ be a closed and exhaustive measure with the Lyapunov property with respect to E. Is it true that $sat(\mu)$ must be strictly greater then the cardinality of the minimum family $\mathcal{F} \subset E^*$ that separates the points of $\overline{span}\ \mu(\Sigma)$?

Following an idea of Wnuk ([40]), we use the existence of a topologically independent sequence in E to provide a partial answer to the previous question.

Recall that $(e_n)_{n\in\mathbb{N}}$ is a *topologically linearly independent sequence* in E if for every $f \in \ell^\infty(\mathbb{N})$ (i.e. the space of bounded functions $f \colon \mathbb{N} \to \mathbb{R}$) $\sum_{n\in\mathbb{N}} f(n)e_n = 0$ implies $f = 0$. In [16] it is proved that every infinite dimensional metrizable vector space (X, τ) contains a topologically linearly independent sequence $(e_n)_{n\in\mathbb{N}}$.

Proposition 3.18. *Suppose that E is metrizable and infinite dimensional. Let $\mu \colon \Sigma \to E$ be an exhaustive measure such that every measure $\nu \colon \Sigma \to E$ absolutely continuous with respect to μ has a convex range. Then $sat(\mu)$ is uncountable.*

[8] i.e. a complete metrizable topological vector space.

Proof. Since E is metrizable and μ exhaustive, by Bartle-Dunford-Schwartz's Theorem ([39, Corollary 7.5]) there is a measure $\lambda\colon \Sigma \to [0, +\infty[$ which is equivalent to μ, i.e. such that $\tau(\lambda) = \tau(\mu)$. By contradiction, suppose that $sat(\lambda) \leq |\mathbb{N}|$. Then there exists a $x \in \Sigma \setminus \mathcal{N}(\lambda)$ such that $\Sigma \wedge x$ is a separable topological subspace of $(\Sigma, \tau(\lambda))$. Without loss of generality we can assume that $x = 1_\Sigma$ and take a sequence b_n, $n \in \mathbb{N}$ dense in $(\Sigma, \tau(\lambda))$.

On Σ we define the family of scalar measures $\lambda_n\colon x \mapsto \lambda(x \wedge b_n)$, $n \in \mathbb{N}$, and observe that $x \triangle y \notin \mathcal{N}(\lambda)$ implies that $\lambda_n(x) \neq \lambda_n(y)$ for at least one $n \in \mathbb{N}$.

Since E is metrizable, we can select a topologically linearly independent sequence e_n, $n \in \mathbb{N}$, in E and choose a sequence of non-zero $t_n \in \mathbb{R}$, $n \in \mathbb{N}$, so that $(t_n e_n)_{n\in\mathbb{N}}$ is summable in E. Then, we can define the measure $\nu\colon \Sigma \to E$ by setting $\nu(x) := \sum_{n\in\mathbb{N}} \lambda_n(x) t_n e_n$ for $x \in \Sigma$. Since ν is absolutely continuous with respect to μ, $\nu(\Sigma)$ must be a convex subset of E, meaning that there is a $a \in \Sigma$ such that $\nu(a) = \nu(a^c) = \nu(1_\Sigma)/2$. But then:

$$0 = \nu(a) - \nu(a^c) = \sum_{n\in\mathbb{N}} t_n \left(\lambda_n(a) - \lambda_n(a^c)\right) e_n$$

and so, having taken e_n, $n \in \mathbb{N}$, topologically linearly independent and t_n non-zero, it must be $\lambda_n(a) = \lambda_n(a^c)$ for each $n \in \mathbb{N}$.

However, since $\lambda(a \triangle a^c) > 0$ by construction, there must be a $n \in \mathbb{N}$ with $\lambda_n(a) \neq \lambda_n(a^c)$. □

We stress that in the settings of Proposition 3.18 the assumption of metrizability of the space E cannot be directly dropped. As seen in example 3.14, if E is infinite dimensional but not metrizable it is possible to find a measure $\mu\colon \Sigma \to E$ with the Lyapunov proeperty with respect to E such that $sat(\mu)$ is countable.

Corollary 3.19. *Let E be separable, infinite dimensional and metrizable and let $\mu\colon \Sigma \to E$ be a closed and exhaustive measure. Then the following are equivalent:*

1. *μ has the Lyapunov property with respect to E;*
2. *every $\nu\colon \Sigma \to E$ with $\nu \ll \mu$ has convex range.*
3. *$sat(\mu)$ is uncountable;*

Proof. The implication $(1 \Rightarrow 2)$ is obvious while $(2 \Rightarrow 3)$ is a consequence of Proposition 3.18. Finally, $(1 \Rightarrow 2)$ can be seen as a special case of Theorem 3.16. □

Remark 3.20. Corollary 3.19 still holds if we replaced the hypothesis on the metrizability and separability of E with the less restrictive hypothesis that every weakly compactly generated subset of E is linearly homeomorphic to a separable and metrizable space. In fact, under this milder assumptions, the proofs remains identical.

Remark 3.21. Using [14, Corollary 6, pg. 265], Khan and Sagara proved in [26, Section 4.2] that if E is an infinite dimensional separable Banach space and $\mu\colon \mathcal{A} \to E$ is a σ-additive homogeneous measure over a σ-algebra then μ is saturated[9] if and only if every $\nu\colon \mathcal{A} \to E$ absolutely continuous with respect to μ has convex and weakly compact range.

[9] i.e. $sat(\mu)$ is uncountable.

Their result is extended in this section via Proposition 3.18 and Corollary 3.19 to include finitely additive measures that are not necessarily homogeneous and that can take values in locally convex metrizable spaces. Moreover, the proof is significantly simplified by avoiding the necessity of recurring to Maharam's Theorem of classification of homogeneous measure algebras.

4 Applications

4.1 The range of finitely additive correspondences

Let us denote by $\mathcal{P}(E)$ the family of all subsets of E and by $\mathcal{P}_0(E)$ the non-empty parts of E. By *fintely additive correspondence* with values in E we mean a function $\Phi\colon \Sigma \to \mathcal{P}_0(E)$ that is finitely additive in the sense that $\Phi(0_\Sigma) = \{0\}$ and $\Phi(x) + \Phi(y) = \Phi(x \vee y)$ for all $x, y \in \Sigma$ disjoint.

Given a finitely additive correspondence $\Phi\colon \Sigma \to \mathcal{P}_0(E)$, we say that a measure $\mu\colon \Sigma \to E$ is a *selection* of Φ if $\mu(x) \in \Phi(x)$ for every $x \in \Sigma$. We denote by $S(\Phi)$ the set of all selections of Φ and we say that Φ is *rich in selections* if for all $x \in \Sigma$, $y \in \Phi(x)$ there is a $\mu \in S(\Phi)$ such that $\mu(x) = y$. In particular, if u is a FN-topology on Σ, we say that Φ on Σ is rich in u-continuous selections if for for all $x \in \Sigma$, $y \in \Phi(x)$ we can take a u-continuous $\mu \in S(\Phi)$ such that $\mu(x) = y$. To prove our main result in Theorem 4.2 we will first need the following technical Lemma.

Lemma 4.1. *Let μ and E be as in Theorem 3.13. Let $\nu_1, \nu_2\colon \Sigma \to E$ be two measures absolutely continuous with respect to μ and let $a \in \Sigma$, $t \in [0,1]$. Then there is an element $b \in \Sigma \wedge a$ such that $t\nu_1(a) = \nu_1(b)$ and $(1-t)\nu_2(a) = \nu_2(a \setminus b)$.*

Proof. Let us define the measure $\eta\colon \Sigma \to E \times E$ by setting $\eta(x) = (\nu_1(x \wedge a), \nu_2(x \wedge a))$. On the one hand, η is absolutely continuous with respect to μ, thus it is exhaustive, closed and it has degree of saturation greater than or equal to $sat(\mu)$. On the other $\overline{span}\eta(\Sigma) \subseteq \overline{span}\nu_1(\Sigma) \times \overline{span}\nu_2(\Sigma)$ and hence there is a family $\mathcal{F} \subset (E \times E)^*$ that separates its the points such that $|\mathcal{F}| < sat(\eta)$. We conclude that η satisfies the assumptions of Theorem 3.13 and as such it has a convex range.

Let $y \in \Sigma$ be such that $\eta(y) = t\eta(1_\Sigma)$, then set $b := y \wedge a$. By construction we have that:

$$(\nu_1(b), \nu_2(b)) = (\nu_1(y \wedge a), \nu_2(y \wedge a) = \eta(y) = t\eta(1_\Sigma) = t(\nu_1(a), \nu_2(a))$$

and so $\nu_1(b) = t\nu_1(a)$ and $\nu_2(a \setminus b) = \nu_2(a) - \nu_2(b) = (1-t)\nu_2(b)$ as claimed. □

Theorem 4.2. *Let μ and E be as in Theorem 3.13 and let $\Phi\colon \Sigma \to \mathcal{P}_0(E)$ be a finitely additive correspondence rich in $\tau(\mu)$-continuous selections.*

1. *Then $\Phi(a)$ is convex for every $a \in \Sigma$;*
2. *$\bigcup_{x\in\Sigma} \Phi(x)$ is convex.*

Proof. 1. Let $a \in \Sigma$, $x_1, x_2 \in \Phi(a)$ and $t \in [0,1]$. We claim that $tx_1 + (1-t)x_2 \in \Phi(a)$. Being Φ rich in $\tau(\mu)$-continuous selections, we can take $\nu_1, \nu_2 \in S(\Phi)$ such that $\nu_i \ll \mu$ and $\nu_i(a) = x_i$ for $i = 1, 2$.

   Since ν_1, ν_2 satisfy the assumptions of Lemma 4.1 we can take $b \in \Sigma \wedge a$ such that $\nu_1(b) = t\nu_1(a)$ and $\nu_2(a \setminus b) = (1-t)\nu_2(a)$. But then $tx_1 = \nu_1(b) \in \Phi(b)$, $(1-t)x_2 = \nu_2(a \setminus b) \in \Phi(a \setminus b)$ and by the finite additivity of Φ we conclude that $tx_1 + (1-t)x_2 \in \Phi(b \vee (a \setminus b)) = \Phi(a)$ as claimed.

2. Let $x_1, x_2 \in \bigcup_{x \in \Sigma} \Phi(x)$, $t \in [0,1]$ and take $a_1, a_2 \in \Sigma$ such that $x_i \in \Phi(a_i)$ for $i = 1, 2$. We claim that $tx_1 + (1-t)x_2 \in \Phi(d)$ for some $d \in \Sigma$. Being Φ rich in $\tau(\mu)$-continuous selections, we can take $\nu_1, \nu_2 \in S(\Phi)$ such that $\nu_i \ll \mu$ and $\nu_i(a_i) = x_i$ for $i = 1, 2$.

   Call $b_1 := a_1 \setminus a_2$, $b_2 := a_2 \setminus a_1$ and $b_3 := a_1 \wedge a_2$. By point (1), we can find $c_1 \leq b_1$, $c_2 \leq b_2$ such that $\nu_1(c_1) = t\nu_1(b_1)$ and $\nu_2(c_2) = (1-t)\nu_2(b_2)$. Apply Lemma 4.1 to b_3 to find $c_3 \leq b_3$ such that $\nu_1(c_3) = t\nu_1(b_3)$ and $\nu_2(b_3 \setminus c_3) = (1-t)\nu_2(b_3)$. Then call $d_1 := c_1 \vee c_3$ and $d_2 := c_2 \vee (b_3 \setminus c_3)$ and observe that they are disjoint elements of Σ. We have:

$$\begin{aligned} tx_1 + (1-t)x_2 =& t\nu_1(a_1) + (1-t)\nu_2(a_2) = (\nu_1(b_1 \vee b_3)) + (1-t)(\nu_2(b_2 \vee b_3)) = \\ =& t\nu_1(b_1) + t\nu_1(b_3) + (1-t)\nu_2(b_2) + (1-t)\nu_2(b_3) = \\ =& \nu_1(c_1) + \nu_1(c_3) + \nu_2(c_2) + \nu_2(b_3 \setminus c_3) = \nu_1(d_1) + \nu_2(d_2) \end{aligned}$$

   and the latter is an vector in $\Phi(d_1 \vee d_2)$.

□

Remark 4.3. On the set $\mathcal{P}_0(E)$ we can define a topology, sometimes called the Hausdorff topology, defined by the semi-metrics $d_p(X,Y) := \sup_{x \in X} \inf_{y \in Y} p(x-y)$ with p ranging over continuous semi-norms on E. Thanks to this topological structure on $\mathcal{P}_0(E)$, one could define the *FN-topology induced by Φ on Σ* as the coarsest FN-topology on Σ making Φ a continuous function and denote it by $\tau(\Phi)$. This approach is followed, for example in [7], [9].

If Φ is continuous with respect to some FN-topology u on Σ, then every selection of Φ is so.

4.2 Coalitional representations of exchange economies with many commodities

This section is devoted to the description of an exchange economy $\mathcal{E}$ with infinitely many agents and commodities. We mainly adapt the coalitional approach described in [4] to obtain a finitely additive economy with an infinite dimensional locally convex space of commodities. The main idea behind coalitional representations of economies is to take coalitions, instead of agents, as the main actors of the model so as to ignore all entities unable to influence the economic activity. We take [37, 4, 8] and [11, 15] for classical references on coalitional representations both in finite dimensional and infinite dimensional economies.

In our model, we will consider an exchange economy $\mathcal{E}$ in which:

- the commodity-price duality is represented via the dual pair (E, E^*) of infinite dimensional ordered locally convex spaces spaces, where the positive orthant of E, denoted by E_+, stands for the spaces of commodity bundles while E_+^* for the set of prices.

- *Coalitions* are taken as the primitive entity of the economy, formally represented as the elements of an abstract Boolean algebra Σ. Even though Σ is a purely algebraic object, for the sake of simplicity we will think of Σ as the algebra of sub-sets of a given set $\Omega \neq \emptyset$ representing the totality of the agents. This identification is made possible by the Stone's representation theorem (see [34, pg. 117]).

- *Assignments* are (finitely additive) vector measures $\alpha\colon \Sigma \to E_+$ with the idea that $\alpha(F) \in E_+$ represents the consumption bundle assigned by α to the coalition F. Since we want the total amount of resources available in the economy to be bounded, we require that every assignment $\alpha\colon \Sigma \to E_+$ has a totally bounded range $\alpha(\Sigma)$ which, in our situation, is equivalent with asking that α is an *exhaustive* measure ([35, Theorem 3]).

  In the set $\mathcal{A}$ of all assignments, which we consider endowed with the uniform-convergence topology, we give a special importance to the *initial endowment* $\omega\colon \Sigma \to E_+$ which describes how the wealth is initially distributed among coalitions.

- A binary relation $\succ_F$ on $\mathcal{A}$ is associated to every coalition $F \in \Sigma$ to represent the *preference* of coalition F following the intuition that $\alpha \succ_F \beta$ means that every agent in F prefers what she obtains from α with respect to β. Once $\succ_F$ is defined we can introduce a weak preference relation $\succcurlyeq_F$ on $\mathcal{A}$ by setting $\alpha \succcurlyeq_F \beta$ if and only if there is no non-null coalition $F' \subset F$ such that $\beta \succ_{F'} \alpha$.

In conclusion, a *finitely additive coalitional economy* $\mathcal{E}$ will be fully described by the tupla:

$$\mathcal{E} := (\Sigma, (E, E^*), \omega\colon \Sigma \to E_+, \{\succ_F\colon\ F \in \Sigma\}) .$$

For any assignment $\mu \in \mathcal{A}$ and coalition $F \in \Sigma$, $\mu|_F$ will stand for the assignment that associates $\mu(G \cap F)$ to each $G \in \Sigma$. In addition to the assumptions made above, following [4] we will require that preference relations satisfy some standard assumptions:

Assumption 4.4. **(i)** for all $\alpha, \beta \in \mathcal{A}$, the set $\{G \in \Sigma :\ \alpha \succ_G \beta\}$ is an ideal in Σ;

(s) for $\alpha, \beta \in \mathcal{A}$, $\alpha \succ_F \beta$ if and only if $\alpha|_F \succ_F \beta$ if and only if $\alpha \succ_F \beta|_F$;

(m) if $\alpha, \beta \in \mathcal{A}$ are such that $\alpha|_F \geq \beta|_F$ and $\alpha|_F \neq \beta|_F$ then $\alpha \succcurlyeq_F \beta$;

(c) the set $\{(\alpha, \beta) \in \mathcal{A}^2 :\ \alpha \succcurlyeq_F \beta\}$ is closed in the product topology.

The condition (i), where i stands for *ideal*, reflects the idea that all members of a coalition F must agree on the preference $\succ_F$ so that the formation of coalitions is totally voluntary. On the other hand, condition (s), that stands for *selfishness*, is needed to exclude the presence of externalities of consumption. Last, (m) and (c) are simply conditions of *monotonicity* and *continuity* of preferences.

Remark 4.5. The description of the commodity-price duality in terms of a dual pair of ordered linear spaces is a common expedient we owe to Debreu ([12]). Thanks to this idea, the topological structure of the commodity space, and hence the continuity of price-evaluation functions, is justified as a natural consequence of the algebraic properties determined by the commodity-price duality and it is not imposed for technical necessities. In this perspective, locally convex topologies are the most natural ones to be considered on a space of commodities. For more references on this issue see [1, Chapter 8.2].

Remark 4.6. Starting from an *individual representation* of the economy $\mathcal{E}$, i.e. a model in which agents are represented as the points of a measure space and assignments as integrable functions, it is always possible to derive an equivalent coalitional representation of $\mathcal{E}$ via a suitable integration procedure. Thus, in some sense, the coalitional approach can be seen as a generalization of the individual one which short circuits some technical aspects of mathematical integration that are intrinsic in Aumann's model for competitive economies. In [13], Debreu showed that , as long as we consider countably additive economies with finitely many commodities, the differences between the two approaches are not too significant: in fact, for every coalitional economy with finitely many commodities, a σ-algebra of coalitions and σ-additive assignments, it is possible to construct an equivalent individual representation with the aid of a specific version of Radon-Nikodym Theorem for preference relations.

It is only with [4] that Armstrong and Richter provided a class of examples of coalitional representations of economies that cannot be derived from individual models.

4.3 A qualitative measurement of the power of coalitions

In the study of large economies we often make considerations on the *economic weight* of coalitions intended as the capacity of a group of agents to influence trades. The problem of understanding how the actors in the economy and their economic weight should be represented, is a classical and significant issue. This is especially true dealing with competitive economies, where the notion of economic negligibility of individual traders plays a crucial role.

Let us consider the exchange economy $\mathcal{E}$ as defined in 4.2. Loosely speaking, we expect that coalitions with "better" initial endowment will more likely play a significant role in the economic activity and therefore have a larger economic weight. Following this intuition, we should measure how powerful a coalition $F \in \Sigma$ is in terms of what it will be able to attain if she decides to deviate from the rest of economy and act independently. In other words, we need to focus on the set $\omega(\Sigma \cap F) = \{\omega(G) : \Sigma \ni G \subseteq F\}$ which is the collection of all bundles initially owned by F and its sub-coalitions. In the light of this, the smaller the set $\omega(\Sigma \cap F)$ is, the "weaker" we expect F to be.

With the observations above we focus on the the uniform structure induced on Σ by the correspondence $F \mapsto \omega(\Sigma \cap F)$ and say that $F \in \Sigma$ is U-small if $\omega(\Sigma \cap F) \subseteq U$, where U is a 0-neighborhood of E. All this brings us to the following definition:

Definition 4.7. *We call* distribution of the economic weight *the ω-topology $\tau(\omega)$ in Σ and denote it by the letter u.*

Moving from Definition 4.7, we shall refer to the *space of coalitions* in the economy $\mathcal{E}$ as the topological Boolean algebra (Σ, u). Finally, we can define *allocations* all the assignments that are consistent with the topological structure of the space of coalitions.

Definition 4.8. *An* allocation *is a u-continuous assignment* $\alpha\colon (\Sigma, u) \to E_+$.

We will denote by $\mathcal{M}$ the set of all allocations and observe that it is a closed linear subspace of $\mathcal{A}$. An allocation α will be *feasible* if $\alpha(\Omega) = \omega(\Omega)$.

A possible alternative way of defining the economic weight of coalitions, more closely related to the commodity-price duality, is to measure the economic potential of each coalitions under all possible price-systems that can emerge. Formally, one could associate to every $p \in E_+^*$ the positive measure

$$\nu_p\colon F \mapsto \sup\{\langle p, \omega(G)\rangle : \ G \in \Sigma \cap F\}$$

which assign to every coalition $F \in \Sigma$ the maximum possible income she can attain at price p if she deviates from the rest of the economy. With this idea we are brought to say that the economic power of a coalition $F \in \Sigma$ should be "small"whenever $\nu_p(F)$ is small for some $p \in E_+^*$. Quite surprisingly, this approach can be shown to be equivalent with the one showed above thanks to the following proposition:

Proposition 4.9 (Corollary 7.3 in [39])**.** *For any net of coalitions F_i, $i \in \mathcal{I}$, and $F_0 \in \Sigma$ it is equivalent to say that:*

1. *the net F_i converges to F_0 in (Σ, u),*
2. *for all $p \in E_+^*$ the net $\nu_p(F_i)$ converges to $\nu_p(F_0)$.*

In other words, the topology induced on Σ by the collection ν_p, $p \in E_+^*$, coincides with $\tau(\omega)$ and hence with the *distribution of economic weight* as defined in Definition 4.7, showing that our notion of economic weight can be derived directly from the commodity-price duality.

Remark 4.10. A common solution adopted in most of the literature on coalitional economies is to include in the description of the model a *numerical estimation* of the economic weight by assuming the existence of a control measure $\lambda\colon \Sigma \to [0, +\infty[$ for ω. This way $\omega(\Sigma \cap F)$ "gets smaller with $\lambda(F)$"in the sense that for every 0-neighborhood U of E there is a $\epsilon > 0$ such that $\lambda(F) < \epsilon \Rightarrow \omega(\Sigma \cap F) \subseteq U$ and we can think of $\lambda(F)$ as a numerical expression of the economic weight of $F \in \Sigma$.

Despite its intuitiveness, this approach as it forces to limit the analysis only to controlled assignments, a limitation which does not seem to have any economic justification. By moving the attention from the measure λ to the uniform structure it induces on Σ, we move from a quantitative to a qualitative measurement of the economic weight. We will see next that this weakening does not affect important features we are interested in.

Remark 4.11. When the space of commodities E is equipped with a norm $\|\cdot\|$, it is common to call *diameter* the function $F \mapsto |F|_\omega := \sup\{\|\omega(G)\| : \ \Sigma \ni G \subseteq F\}$, $F \in \Sigma$ (see for example [21] or [22, 17] for the case in which the dimension of E is infinite). Since every $p \in E^*_+$ defines on E the semi-norm $x \mapsto |\langle p, x\rangle|$, we could see the function ν_p as a special case of a diameter function.

It is also worth stressing that, when E is normed, the diameter function $|\cdot|_\omega$ provides a sort of numerical description of the economic weight in the sense described in remark 4.10.

4.4 A condition for competitive markets

When the commodity space E has a finite dimension, Aumann's notion of perfect competitiveness of the market is stated in terms of non-atomicity of the initial endowment and, consequently, of all allocations (see [5]). We follow his idea to extend this notion to the case of infinite dimensional spaces by requiring that ω satisfies the following conditions.

Assumption 4.12. $sat(\omega)$ is infinite and there is a family $\mathcal{F} \subset E^*$ separating the points of E such that $|\mathcal{F}| < sat(\omega)$.

Once again we stress that in the finite dimensional settings Assumption 4.12 is equivalent to the condition of non-atomicity of allocations and therefore to Aumann's notion of perfect competitive market. Also, in line with Aumann, as a direct consequence of Theorem 3.9 we have that, under Assumption 4.12, every closed allocation has convex and weakly compact range. It is in view of this that throughout we will also assume the following:

Assumption 4.13. $\omega\colon \mathcal{R} \to E_+$ is a closed measure.

Assumptions 4.12, 4.13 give us important results on convexity of preferences. Precisely, if we fix a coalition $F \in \Sigma$ and an allocation $\alpha \in \mathcal{M}$ we can represent the set of bundles preferred to $\alpha(F)$ by means of the set $\{\beta(F) : \ \beta \in \mathcal{M}, \ \beta \succ_F \alpha\}$ which we claim to be convex.

Proposition 4.14. *Under assumptions 4.12 and 4.13 the correspondence $P_\alpha\colon F \mapsto \{\beta(F) : \beta \in \mathcal{M}, \ \beta \succ_F \alpha\}$ has convex values.*

Proof. Fix $\alpha \in \mathcal{M}$ and $F \in \mathcal{M} \setminus \mathcal{N}(\omega)$.

If $P_\alpha(F)$ is empty there is nothing to prove. So we can assume that there exists at list a $\beta \in \mathcal{M}$ with $\beta \succ_F \alpha$. By the ideal property of preferences, $\beta \succ_G \alpha$ for all $G \in \Sigma \cap F$ thus P_α restricted to $\Sigma \cap F$ is a non-empty valued correspondence. We claim that the restriction of P_α to $\Sigma \cap F$ is rich in u-continuous selections so that the thesis follows from Theorem 4.2.

Let $G \in \Sigma \cap F$ and take $v \in P_\alpha(G)$. By construction, there is a $\gamma \in \mathcal{M}$ such that $\gamma \succ_G \alpha$ and $\gamma(G) = v$. By the selfishness property of preferences, $\gamma \succ_G \alpha$ implies $\gamma|_G \succ_G \alpha$. Call $\eta := \gamma|_G + \beta|_{F\setminus G}$ and observe that, by the ideal property, $\eta \succ_F \alpha$. But then η is a u-continuous selection of the restriction of P_α to $\Sigma \cap F$ such that $\eta(G) = v$ as claimed. $\square$

4.5 On the veto power of coalitions

We now move to the problem of determining under which condition on its economic weight a coalition is capable of improving upon a given allocation.

Precisely, we say that an allocation α is *dominated* by an allocation β if there is a coalition $F \in \Sigma \setminus \mathcal{N}(\omega)$ such that $\beta(F) = \omega(F)$ and $\beta \succ_F \alpha$. In this case, we also say that F *blocks* α *via* β and call F a *blocking coalition for* α. Feasible allocations which are not dominated are called *core allocations*.

In general, the larger the economy, the harder is to check whether a given distribution of resources is a core allocation or not. Here, we will study how we can narrow the area in which we have to look for blocking coalitions from the whole Σ to significantly smaller subsets. We start by extending a theorem due to Schmeidler ([33]), who proved that in perfectly competitive markets, any non-competitive allocation can be blocked by arbitrarily small coalitions. In our framework, this result can be formalized as follows.

Theorem 4.15. *Let α be an allocation that can be blocked by a coalition F via a given $\beta \in \mathcal{M}$. Then for every 0-neighborhood U of (Σ, u) there is a $G \subset F$ in U that blocks α via β.*

Proof. Let U be a 0-neighborhood in (Σ, u). As a consequence of Proposition 4.9 there is a finite number of u-continuous measures $\lambda_i \colon \Sigma \to [0,1]$, $i \leq n$, and an $\epsilon > 0$ such that $G \in U$ whenever $\lambda_i(G) \leq \epsilon\lambda(\Omega)$ for all $i \leq n$.

Define the vector measure $\eta \colon \Sigma \to E \times E \times \mathbb{R}^n$ by setting:

$$\eta(G) := (\omega(G) - \beta(G), \lambda_i(G), \dots, \lambda_n(G))$$

for all $G \in \Sigma$. The measure η is u-continuous and takes values in a space that is separated by a family of functionals $\mathcal{F} \subset D$ with $|\mathcal{F}| < sat(\Sigma, u)$. But then, by Theorem 3.13, $\eta(\Sigma \cap F)$ is convex and as such there must be a $G \subset F$ such that $\epsilon\eta(F) = \eta(G)$. This means that $\omega(G) - \beta(G) = 0$ and that $G \in U$ (since $\lambda_i(G) \leq \epsilon\lambda(F) \leq \epsilon\lambda_i(\Omega)$ for all $i \leq n$). Moreover, since $G \subset F$, by the ideal assumption on preferences, $\beta \succ_F \alpha$ implies that $\beta \succ_G \alpha$ proving that G blocks α via β as claimed. □

In the spirit of the work of Schmeidler, Vind describes in ([38]) sufficient conditions for an allocation outside the core to be blocked by arbitrarily big coalitions. However, despite being symmetrical to the previous situation, the problem introduced by Vind requires us to make additional assumptions on preferences.

Assumption 4.16. Let $\alpha \in \mathcal{M}$ and $v \in E_+$ a non zero commodity bundle. Then for every $F \in \Sigma$ there is an allocation $\beta \in \mathcal{M}$ such that $\beta \succ_F \alpha$ and $\beta(F) = \alpha(F) + v$.

Lemma 4.17. *Let $\alpha, \beta \in \mathcal{M}$ and $F \in \Sigma$ be such that $\beta \succ_F \alpha$. Then there is a $\gamma \in \mathcal{M}$ such that $\gamma \succ_F \alpha$ and $v := \beta(F) - \gamma(F) \geq 0$, $v \neq 0$.*

Proof. Call $\mathcal{C}$ the set $\{(\eta, \zeta) \in \mathcal{M}^2 : \ \eta \succcurlyeq_F \zeta\}$ and observe that, since $\beta \succ_F \alpha$, it cannot be $\alpha \succcurlyeq_F \beta$ and hence $(\alpha, \beta) \notin \mathcal{C}$. For the assumption of continuity of preferences, $\mathcal{C}$ must be a closed set and as such it must have an open complement in $\mathcal{M}$. All this implies that there is a $t \in (0,1)$ such that $(\alpha, t\beta)$ does not belong to $\mathcal{C}$ neither and, consequently, such that $t\beta \succ_{F'} \alpha$ for some non-null coalition $F' \subset F$. In particular, it must be then $\beta(F') \geq 0$, $\beta(F') \neq 0$.

Call G the complementary of F' and set $\gamma = \beta|_G + t\beta|_{F'}$. Then $\gamma \succ_F \alpha$ by the assumption of selfishness on preferences, and $v := \beta(F) - \gamma(F) = (1-t)\beta(F') \geq 0$ and $v \neq 0$ as claimed. $\square$

Theorem 4.18. *Let $\alpha \in \mathcal{M}$ be a feasible non-core allocation. Under assumption 4.16, for every 0-neighborhood U in (Σ, u) there is a coalition D that blocks α and such that $D^c \in U$.*

Proof. Let U be a 0-neighborhood in (Σ, u) and suppose that $A \in \Sigma$ is a non-null coalition that can block α via a certain $\beta \in \mathcal{M}$, i.e. $\beta(A) = \omega(A)$ and $\beta \succ_A \alpha$. By Proposition 4.9, there exists an $\epsilon > 0$ and a finite number of u-continuous measures $\lambda_i \colon \Sigma \to [0,1]$, $i \leq n$, such that $F \in U$ whenever $F \in \Sigma$ is such that $\lambda(F) \leq 1 - \epsilon$ for all $i \leq n$.

We can define the measure $\eta \colon \Sigma \to E \times E \times \mathbb{R}^n$ that associates to $F \in \Sigma$ the vector:

$$\eta(F) := (\omega(F), \alpha(F), \lambda_1(F), \ldots, \lambda_n(F))$$

and observe that it satisfies the assumptions of Theorem 3.13. Therefore, we can take a $B \subseteq A$ such that $\eta(B) = \epsilon\eta(A)$ and a $C \subseteq B^c$ such that $\eta(C) = \epsilon\eta(B^c)$ so that for every $i \leq n$ $\lambda_i(C \cup B) = \lambda_i(C) + \lambda_i(B) = \epsilon(\lambda_i(B^c) + \lambda_i(A)) \geq \epsilon\lambda_i(\Omega)$. Since $D^c \in U$, it is enough to prove that $D := B \cup C$ can block the allocation α.

Let us focus on the coalition B. Since $B \subset A$, by the ideal assumption on preferences $\beta \succ_B \alpha$. This also means, by Lemma 4.17, that there is a $\gamma \in \mathcal{M}$ such that $\gamma \succ_B \alpha$ and $\beta(B) - \gamma(B) > 0$. Now, since the set $\omega(B) + P_\alpha(B)$ is convex, the vector $\epsilon\alpha(B) + (1-\epsilon)\gamma(B)$ still belongs to $\omega(A) + P_\alpha(B)$ thus there must be an allocation $\zeta_1 \in \mathcal{M}$ such that $\zeta_1 \succ_B \alpha$ and $\zeta_1(B) = \epsilon\alpha(B) + (1-\epsilon)\gamma(B)$.

We now move our attention to C. The vector $(1-\epsilon)(\beta(B) - \gamma(B))$ is strictly greater than 0, thus by the assumption 4.16 there is a $\zeta_2 \in \mathcal{M}$ such that $\zeta_2(C) = \alpha(C) + (1-\epsilon)(\beta(B) - \gamma(B))$ and $\zeta_2 \succ_C \alpha$. Let us define $\zeta \in \mathcal{M}$ as the sum $\zeta_1|_B + \zeta_2|_C$ so that, by the property of selfishness of preferences, $\zeta \succ_D \alpha$. We claim that D, which we defined as $B \cup C$, blocks α via ζ. To see this, observe that:

$$\begin{aligned}
\zeta(D) =& \zeta_1(B) + \zeta_2(C) = \\
&= [\epsilon\alpha(B) + (1-\epsilon)\gamma(B)] + [\alpha(C) + (1-\epsilon)(\beta(B) - \gamma(B))] = \\
&= \epsilon\alpha(B) + \alpha(C) + (1-\epsilon)\beta(B) = \\
&= \epsilon\alpha(B) + \epsilon\alpha(B^c) + \beta(B) - \epsilon\beta(B) = \\
&= \epsilon(\alpha(\Omega) - \beta(B)) + \beta(B) = \\
&= \epsilon(\omega(\Omega) - \omega(B)) + \omega(B) = \\
&= \epsilon(\omega(B^c)) + \omega(B) = \\
&= \omega(C) + \omega(B) = \omega(D)
\end{aligned} \tag{1}$$

as claimed. $\square$

The importance of these results can be informally explained as follows: taken any feasible allocation α and 0-neighborhood U in (Σ, u), by Theorem 4.15 if no coalitions in

U can block an allocation α then no coalition in Σ will and α is therefore a core allocation. On the other hand, by 4.18, under the additional assumption 4.16 α is a core allocation if it cannot be blocked by any $F \in U^c$. This holds regardless how small or big we take U with respect to Σ.

Remark 4.19. As mentioned in remark 4.11, when E is a Banach space endowed with a norm $\|\cdot\|$ it is common to call *diameter* the function $F \mapsto \sup\{\|\omega(G)\| : G \in \Sigma \cap F\}$, $F \in \Sigma$. What we have is that for all $\epsilon > 0$ the set of coalitions with diameter smaller than ε forms a 0-neighborhood in $(\Sigma, \tau(\omega))$ showing that our definition of *distribution of economic power* is very closely related to the idea of diameter. In view of this, Theorem 4.15 can be viewed as an infinite dimensional interpretation of Grodal's Theorem ([21]).

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