The equivalence of mini-max theorem and existence of Nash equilibrium in asymmetric three-players zero-sum game with two groups

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The equivalence of mini-max theorem and existence of Nash equilibrium in asymmetric three-players zero-sum game with two groups *

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Abstract

We consider the relation between Sion's minimax theorem for a continuous function and a Nash equilibrium in an asymmetric three-players zero-sum game with two groups. Two players are in Group A, and they have the same payoff function and strategy space. One player, Player C, is in Group C. Then,

1. The existence of a Nash equilibrium, which is symmetric in Group A, implies Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A.
2. Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A implies the existence of a Nash equilibrium which is symmetric in Group A.

Thus, they are equivalent.

Keywords: three-players zero-sum game, two groups, Nash equilibrium, Sion's minimax theorem

JEL Classification: C72

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1 Introduction

We consider the relation between Sion's minimax theorem for a continuous function and existence of a Nash equilibrium in an asymmetric three-players zero-sum game with two groups[1]. Two players are in one group (Group A), and they have the same payoff function and strategy space, and so their equilibrium strategies, maximin strategies and minimax strategies are the same. One player, Player C, is in the other group (Group C). We will show the following results.

1. The existence of a Nash equilibrium, which is symmetric in Group A, implies Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A.

2. Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A implies the existence of a Nash equilibrium which is symmetric in Group A.

Thus, they are equivalent.

An example of such a game is a relative profit maximization game in a Cournot oligopoly. Suppose that there are three firms, A, B and C in an oligopolistic industry. Let $\bar{\pi}_A$, $\bar{\pi}_B$ and $\bar{\pi}_C$ be the absolute profits of the firms. Then, their relative profits are

$$\pi_A = \bar{\pi}_A - \frac{1}{2}(\bar{\pi}_B + \bar{\pi}_C),\ \pi_B = \bar{\pi}_B - \frac{1}{2}(\bar{\pi}_A + \bar{\pi}_C),\ \pi_C = \bar{\pi}_C - \frac{1}{2}(\bar{\pi}_B + \bar{\pi}_C).$$

We see

$$\pi_A + \pi_B + \pi_C = \bar{\pi}_A + \bar{\pi}_B + \bar{\pi}_C - (\bar{\pi}_A + \bar{\pi}_B + \bar{\pi}_C) = 0.$$

Thus, the relative profit maximization game in a Cournot oligopoly is a zero-sum game[2]. If the oligopoly is fully asymmetric because the demand function is not symmetric (in a case of differentiated goods) or firms have different cost functions (in both homogeneous and differentiated goods cases), maximin strategies and minimax strategies of firms do not correspond to Nash equilibrium strategies. However, if the oligopoly is symmetric for two firms in one group (Group A) in the sense that demand function is symmetric and two firms have the same cost function, the maximin strategies for those firms with the corresponding minimax strategy of the firm in the other group (Group C) constitute a Nash equilibrium which is symmetric in Group A. In Appendix we present an example of a three-firms relative profit maximizing oligopoly.

[1] In Satoh and Tanaka (2018) we have analyzed a similar problem in a symmetric zero-sum game in which all players are identical.

[2] About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

2 The model and Sion's minimax theorem

Consider a three-players zero-sum game with two groups. There are three players, A, B and C. The strategic variables for Players A, B and C are, respectively, s_A, s_B, s_C, and $(s_A, s_B, s_C) \in S_A \times S_B \times S_C$. S_A, S_B and S_C are convex and compact sets in linear topological spaces. The payoff function of each player is $u_i(s_A, s_B, s_C)$, $i = A, B, C$. They are real valued functions on $S_A \times S_B \times S_C$. We assume

> u_A, u_B and u_C are continuous on $S_A \times S_B \times S_C$, quasi-concave on S_i for each $s_j \in S_j$, $j \neq i$, and quasi-convex on S_j for $j \neq i$ for each $s_i \in S_i$, $i = A, B, C$.

Three players are partitioned into two groups. Group A and Group C. Group A includes Player A and Player B, and Group C includes only Player C. In Group A two players are symmetric, that is, they have the same payoff function, and $S_A = S_B$. Thus, their equilibrium strategies, maximin strategies and minimax strategies are the same.

Since the game is a zero-sum game, we have

$$u_A(s_A, s_B, s_C) + u_B(s_A, s_B, s_C) + u_C(s_A, s_B, s_C) = 0, \tag{1}$$

for given (s_A, s_B, s_C).

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

We follow the description of this theorem in Kindler (2005).

Let s_B be given. Then, $u_A(s_A, s_B, s_C)$ is a function of s_A and s_C. We can apply Lemma 1 to such a situation, and get the following equation.

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B, s_C). \tag{2}$$

Note that we do not require

$$\max_{s_C \in S_C} \min_{s_A \in S_A} u_C(s_A, s_B, s_C) = \min_{s_A \in S_A} \max_{s_C \in S_C} u_C(s_A, s_B, s_C),$$

$$\max_{s_A \in S_A} \min_{s_B \in S_B} u_A(s_A, s_B, s_C) = \min_{s_B \in S_B} \max_{s_A \in S_A} u_A(s_A, s_B, s_C) \text{ given } s_C.$$

We assume that $\arg\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B, s_C)$ and $\arg\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B, s_C)$ are unique, that is, single-valued. By the maximum theorem they are continuous in s_B.

Also, throughout this paper we assume that the maximin strategy and the minimax strategy of players in any situation are unique, and the best responses of players in any situation are unique. Similarly, we obtain

$$\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A, s_B, s_C) = \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A, s_B, s_C), \tag{3}$$

given s_A[3].

Let $s_B = s$. Consider the following function.

$$s \to \arg\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s, s_C).$$

Since u_A is continuous, S_A and S_C are compact and $S_A = S_B$, this function is also continuous. Thus, there exists a fixed point. Denote it by $\tilde{s}$. $\tilde{s}$ satisfies

$$\arg\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, \tilde{s}, s_C) = \tilde{s}.$$

From (2) $\tilde{s}$ satisfies

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, \tilde{s}, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C). \tag{4}$$

From symmetry for Players A and B, $\tilde{s}$ also satisfies

$$\arg\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(\tilde{s}, s_B, s_C) = \tilde{s},$$

and

$$\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(\tilde{s}, s_B, s_C) = \min_{s_C \in S_C} \max_{s_B \in S_B} u_B(\tilde{s}, s_B, s_C).$$

3 The main results

Consider a Nash equilibrium of a three-players zero-sum game. Let s_A^*, s_B^*, s_C^* be the values of $s_A,\ s_B,\ s_C$ which, respectively, maximize u_A given s_B and s_C, maximize u_B given s_A and s_C, maximize u_C given s_A and s_B in $S_A \times S_B \times S_C$. Then,

$$u_A(s_A^*, s_B^*, s_C^*) \geq u_A(s_A, s_B^*, s_C^*) \text{ for all } s_A \in S_A,$$

$$u_B(s_A^*, s_B^*, s_C^*) \geq u_B(s_A^*, s_B, s_C^*) \text{ for all } s_B \in S_B,$$

and

$$u_C(s_A^*, s_B^*, s_C^*) \geq u_C(s_A^*, s_B^*, s_C) \text{ for all } s_C \in S_C.$$

[3]We do not require

$$\max_{s_C \in S_C} \min_{s_B \in S_B} u_C(s_A, s_B, s_C) = \min_{s_B \in S_B} \max_{s_C \in S_C} u_C(s_A, s_B, s_C),$$

$$\max_{s_B \in S_B} \min_{s_A \in S_A} u_B(s_A, s_B, s_C) = \min_{s_A \in S_A} \max_{s_B \in S_B} u_B(s_A, s_B, s_C).$$

They mean

$$\arg\max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*) = s_A^*,$$

$$\arg\max_{s_B \in S_B} u_B(s_A^*, s_B, s_C^*) = s_B^*,$$

and

$$\arg\max_{s_C \in S_C} u_C(s_A^*, s_B^*, s_C) = s_C^*, .$$

We assume that the Nash equilibrium is symmetric in Group A, that is, it is symmetric for Player A and Player B. Then, $s_B^* = s_A^*$, and $u_A(s_A^*, s_B^*, s_C^*) = u_B(s_A^*, s_B^*, s_C^*)$. Also we have

$$u_A(s_A^*, s_B^*, s_C) = u_B(s_A^*, s_B^*, s_C).$$

Since the game is zero-sum,

$$u_A(s_A^*, s_B^*, s_C) + u_B(s_A^*, s_B^*, s_C) = 2u_A(s_A^*, s_B^*, s_C) = 2u_B(s_A^*, s_B^*, s_C) = -u_C(s_A^*, s_B^*, s_C).$$

Thus,

$$\arg\min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C) = \arg\max_{s_C \in S_C} u_C(s_A^*, s_B^*, s_C) = s_C^*,$$

and

$$\arg\min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C) = \arg\max_{s_C \in S_C} u_C(s_A^*, s_B^*, s_C) = s_C^*.$$

They imply

$$\min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C) = u_A(s_A^*, s_B^*, s_C^*) = \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*),$$

and

$$\min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C) = u_B(s_A^*, s_B^*, s_C^*) = \max_{s_B \in S_B} u_B(s_A, s_B^*, s_C^*).$$

First we show the following theorem.

Theorem 1. *The existence of a Nash equilibrium, which is symmetric in Group A, implies Sion's minimax theorem for pairs of a player in Group A and Player C with symmetry in Group A.*

Proof. 1. Let (s_A^*, s_B^*, s_C^*) be a Nash equilibrium of a three-players zero-sum game. This means

$$\begin{aligned}\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) &\leq \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*) \\ = \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C) &\leq \max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C),\end{aligned} \tag{5a}$$

for Player A.

$$\begin{aligned}\min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C) &\leq \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C^*) \\ = \min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C) &\leq \max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C),\end{aligned} \tag{5b}$$

for Player B.

On the other hand, since

$$\min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) \le u_A(s_A, s_B^*, s_C),$$

we have

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) \le \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C).$$

This inequality holds for any s_C. Thus,

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) \le \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C).$$

With (5a), we obtain

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C). \tag{6a}$$

Similarly, for Player B we can show

$$\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C) = \min_{s_C \in S_C} \max_{s_B \in S_B} u_A(s_A^*, s_B, s_C). \tag{6b}$$

(5a), (5b), (6a) and (6b) imply

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*),$$

$$\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) = \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C),$$

$$\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C) = \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C^*),$$

$$\min_{s_c \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C) = \min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C).$$

From

$$\min_{s_c \in S_C} u_A(s_A, s_B^*, s_C) \le u_A(s_A, s_B^*, s_C^*),$$

and

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*),$$

we have

$$\arg\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \arg\max_{s_A \in S_A} u_A(s_A, s_B^*, s_C^*) = s_A^*.$$

Also, from

$$\max_{s_A \in S_A} u_A(s_A, s_B, s_C) \ge u_A(s_A^*, s_B, s_C^*),$$

and

$$\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) = \min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C),$$

we get

$$\arg\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) = \arg\min_{s_C \in S_C} u_A(s_A^*, s_B^*, s_C) = s_C^*.$$

Similarly, we can show

$$\arg\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C) = \arg\max_{s_B \in S_B} u_B(s_A^*, s_B, s_C^*) = s_B^* = s_A^*,$$

and

$$\arg\min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C) = \arg\min_{s_C \in S_C} u_B(s_A^*, s_B^*, s_C) = s_C^*.$$

Therefore,

$$\arg\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, s_B^*, s_C) = \arg\max_{s_B \in S_B} \min_{s_C \in S_C} u_B(s_A^*, s_B, s_C),$$

and

$$\arg\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, s_B^*, s_C) = \arg\min_{s_C \in S_C} \max_{s_B \in S_B} u_B(s_A^*, s_B, s_C).$$

□

Next we show the following theorem.

Theorem 2. *Sion's minimax theorem with symmetry in Group A implies the existence of a Nash equilibrium which is symmetric in Group A.*

Proof. Let $\tilde{s}$ be a value of s_B such that

$$\tilde{s} = \arg\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, \tilde{s}, s_C).$$

Then, we have

$$\max_{s_A \in S_A} \min_{s_C \in S_C} u_A(s_A, \tilde{s}, s_C) = \min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C). \tag{7}$$

Since

$$u_A(\tilde{s}, \tilde{s}, s_C) \le \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C),$$

and

$$\min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C),$$

we get

$$\arg\min_{s_C \in S_C} u_A(\tilde{s}, \tilde{s}, s_C) = \arg\min_{s_C \in S_C} \max_{s_A \in S_A} u_A(s_A, \tilde{s}, s_C). \tag{8}$$

Since the game is zero-sum,

$$u_A(\tilde{s}, \tilde{s}, s_C) + u_B(\tilde{s}, \tilde{s}, s_C) = 2u_A(\tilde{s}, \tilde{s}, s_C) = -u_C(\tilde{s}, \tilde{s}, s_C).$$

Therefore,

$$\arg\min_{s_C\in S_C} u_A(\tilde{s},\tilde{s},s_C) = \arg\max_{s_C\in S_C} u_C(\tilde{s},\tilde{s},s_C).$$

Let

$$\hat{s}_C = \arg\min_{s_C\in S_C} u_A(\tilde{s},\tilde{s},s_C) = \arg\max_{s_C\in S_C} u_C(\tilde{s},\tilde{s},s_C). \tag{9}$$

Then, from (7) and (8)

$$\min_{s_C\in S_C}\max_{s_A\in S_A} u_A(s_A,\tilde{s},s_C) = \max_{s_A\in S_A} u_A(s_A,\tilde{s},\hat{s}_C) = \min_{s_C\in S_C} u_A(\tilde{s},\tilde{s},s_C) = u_A(\tilde{s},\tilde{s},\hat{s}_C). \tag{10}$$

Similarly, we can show

$$\max_{s_B\in S_B} u_B(\tilde{s},s_B,\hat{s}_C) = u_B(\tilde{s},\tilde{s},\hat{s}_C). \tag{11}$$

(9), (10) and (11) mean that $(s_A,s_B,s_C) = (\tilde{s},\tilde{s},\hat{s}_C)$ is a Nash equilibrium which is symmetric in Group A. □

4 Concluding Remark

In this paper we have examined the relation between Sion's minimax theorem for a continuous function and a Nash equilibrium in an asymmetric three-players zero-sum game with two groups. We want to extend this result to more general multi-players zero-sum game.

Appendix: Example of relative profit maximizing three-firms oligopoly

Consider a three-players game. Suppose that the payoff functions of players are

$$\pi_A = (a-x_A-x_B-x_C)x_A-c_Ax_A-\frac{1}{2}[(a-x_B-x_A-x_C)x_B-c_Bx_B+(a-x_C-x_B-x_A)x_C-c_Cx_C],$$

$$\pi_B = (a-x_B-x_A-x_C)x_B-c_Bx_B-\frac{1}{2}[(a-x_A-x_B-x_C)x_A-c_Ax_A+(a-x_C-x_B-x_A)x_C-c_Cx_C],$$

and

$$\pi_C = (a-x_C-x_B-x_A)x_C-c_Cx_C-\frac{1}{2}[(a-x_A-x_B-x_C)x_A-c_Ax_A+(a-x_B-x_A-x_C)x_B-c_Bx_B].$$

This is a model of relative profit maximization in a three firms Cournot oligopoly with constant marginal costs and zero fixed cost producing a homogeneous good. x_i's,$i = A,B,C$, are the outputs of the firms. The conditions for maximization of $\pi_i,\ i = A,B,C$, are

$$\frac{\partial \pi_A}{\partial x_A} = a - 2x_A - (x_B + x_C) - c_A + \frac{1}{2}(x_B + x_C) = 0,$$

$$\frac{\partial \pi_B}{\partial x_B} = a - 2x_B - (x_A + x_C) - c_B + \frac{1}{2}(x_A + x_C) = 0,$$

and

$$\frac{\partial \pi_C}{\partial x_C} = a - 2x_C - (x_B + x_A) - c_C + \frac{1}{2}(x_B + x_A) = 0.$$

The Nash equilibrium strategies are

$$x_A = \frac{3a - 5c_A + c_B + c_C}{9},\ x_B = \frac{3a - 5c_B + c_A + c_C}{9},\ x_C = \frac{3a - 5c_C + c_B + c_A}{9}. \quad (12)$$

Next consider maximin and minimax strategies about Player A and Player C. The condition for minimization of π_A with respect to x_C is $\frac{\partial \pi_A}{\partial x_C} = 0$. Denote x_C which satisfies this condition by $x_C(x_A, x_B)$, and substitute it into π_A. Then, the condition for maximization of π_A with respect to x_A given $x_C(x_A, x_B)$ and x_B is

$$\frac{\partial \pi_A}{\partial x_A} + \frac{\partial \pi_A}{\partial x_C}\frac{\partial x_C}{\partial x_A} = 0.$$

It is denoted by $\arg\max_{x_A}\min_{x_C}\pi_A$. The condition for maximization of π_A with respect to x_A is $\frac{\partial \pi_A}{\partial x_A} = 0$. Denote x_A which satisfies this condition by $x_A(x_B, x_C)$, and substitute it into π_A. Then, the condition for minimization of π_A with respect to x_C given $x_A(x_B, x_C)$ is

$$\frac{\partial \pi_A}{\partial x_C} + \frac{\partial \pi_A}{\partial x_A}\frac{\partial x_A}{\partial x_C} = 0.$$

It is denoted by $\arg\min_{x_C}\max_{x_A}\pi_A$. In our example we obtain

$$\arg\max_{x_A}\min_{x_C}\pi_A = \frac{3a - 4c_A + c_C}{9},\ \arg\min_{x_C}\max_{x_A}\pi_A = \frac{6a - 9x_B - 2c_A - 4c_C}{9}.$$

Similarly, we get the following results.

$$\arg\max_{x_B}\min_{x_C}\pi_B = \frac{3a - 4c_B + c_C}{9},\ \arg\min_{x_C}\max_{x_B}\pi_B = \frac{6a - 9x_A - 2c_B - 4c_C}{9}.$$

If $c_A \neq c_B$, $\arg\max_{x_A}\min_{x_C}\pi_A \neq \arg\max_{x_B}\min_{x_C}\pi_B$, and they are not equal to the Nash equilibrium strategies for Players A and B. However, if $c_B = c_A$, we have

$$\arg\max_{x_A}\min_{x_C}\pi_A = \arg\max_{x_B}\min_{x_C}\pi_B,$$

and those strategies and the Nash equilibrium strategies for Players A and B are equal. Further, when $c_B = c_A$ and

$$x_A = x_B = \frac{3a - 4c_A + c_C}{9},$$

we get

$$\arg\min_{x_C}\max_{x_A}\pi_A = \arg\min_{x_C}\max_{x_B}\pi_B = \frac{3a - 5c_C + 2c_A}{9}.$$

This is equal to the Nash equilibrium strategy for Player C when $c_B = c_A$.

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