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Abstract

We introduce the notion of an antichain-convex set to extend Debreu (1954)'s version of the second welfare theorem to economies where either the aggregate production set or preference relations are not convex. We show that—possibly after some redistribution of individuals' wealth—the Pareto optima of some economies which are marked by certain types of non-convexities can be spontaneously obtained as valuation quasi-equilibria and equilibria: both equilibrium notions are to be understood in Debreu (1954)'s sense. From a purely structural point of view, the mathematical contribution of this work is the study of the conditions that guarantee the convexity of the Minkowski sum of finitely many possibly non-convex sets. Such a study allows us to obtain a version of the Minkowski\Hahn-Banach separation theorem which dispenses with the convexity of the sets to be separated and which can be naturally applied in standard proofs of the second welfare theorem; in addition—and equally importantly—the study allows to get a deeper understanding of the conditions on the single production sets of an economy that guarantee the convexity of their aggregate.

JEL: C02; C60; D51; D61

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1 Introduction

The second welfare theorems enunciated in Debreu (1951, 1954) and Arrow (1951) are—more or less explicitly—proved by means of the so-called Minkowski and Hahn-Banach separation theorems. The economic thesis of their welfare theorems is that, possibly after some redistribution of individuals' wealth, the Pareto optima of convex economies[1] can be spontaneously obtained as competitive equilibria of an economy with a finite set of agents where consumers choose optimal affordable consumption vectors and firms maximize own profits. The convexity enables the applications of the mentioned separation theorems but is known to be liable to objection. However, if such a condition were simply dropped then the previous thesis would not hold anymore in general.

Motivated by the need of relaxing the convexity requisites of an economy, in the seventies Guesnerie (1975) extended the second welfare theorem to non-convex preferences and technologies: his extension pertained the *first-order necessary conditions* for consumers' expenditure minimization and firms' profit maximization. In convex economies the necessary conditions are also sufficient to guarantee that a Pareto optimum is the solution to such optimization problems; but the *sufficiency* is not generally guaranteed without convexity assumptions and hence a Pareto optimum of a non-convex economy need not be supportable as a valuation quasiequilibrium. The main results concerning the extension of the second welfare to non-convex economies followed the pioneering approach of Guesnerie (1975): they were devoted to finding "marginal" prices at Pareto optima which—satisfying the first-order necessary conditions—lie in suitably chosen normal cones. Much effort within this literature has been made to seek the right notion of a tangent cone (and of its corresponding normal cone). Among the articles of this strand of the literature we mention in particular Khan and Vohra (1987,1988), Bonnisseau and Cornet (1988), Khan (1999), Mordukhovich (2000), Bonnisseau (2002), Florenzano et al. (2006), Jofré and Rivera (2006), Habte and Mordukhovich (2011).

It is important to observe that Debreu (1954)'s second welfare theorem does not posit the convexity of production sets but only that of their aggregate. As the finite sum of convex sets is convex, that second welfare theorem holds for convex economies: this is undisputed. On the other hand one can easily construct specific examples of economies with a convex aggregate production sets where at least one firm has a non-convex production set. Therefore Debreu (1954)'s economies are not convex stricto sensu and hence the second welfare theorem stated therein holds even for some non-convex economies. However, one is left in the dark when trying to figure out which (general) conditions on firms' production sets can guarantee the convexity of their aggregate in non-convex economies. To the best of our knowledge, the subsequent literature has not illuminated this issue which, from a mathematical viewpoint, boils down to understanding which properties—other than convexity—guarantee that the sum of a finite family of sets is convex.

The previous observation on Debreu (1954)'s assumptions is made more accurate when noting that the condition which, in fact, plays a role in the proof of Debreu (1954)'s second welfare theorem is the convexity of the Pareto

[1] An economy is convex if *all* preference relations and *all* production sets are convex.

improving set Z of scarce resources.[2] The set Z is the sum of the aggregate production set and a certain Pareto improving set of aggregate consumption vectors: thus Z is the Minkowski sum of two Minkowski sums. Its convexity condition is well-known to be met in convex economies. But what can we say as for non-convex economies? Once again one runs into the key issue of seeking conditions ensuring the convexity of the sum of finitely many (possibly non-convex) sets.

In this paper we tackle the issue of extending the second welfare theorem to non-convex economies by applying a reformulation of the Minkowski\Hahn-Banach theorem that dispenses with convexity assumptions on sets separated by a linear continuous functional. In the same spirit of Debreu (1954), we provide sufficient conditions for the supportability of Pareto optima as valuation quasiequilibria and as valuation equilibria. But unlike Debreu (1954), we do not assume the convexity of both the aggregate production set and the preference relations of an economy. Various alternative versions of the second welfare theorem will be presented: one of them—more precisely our Theorem 6—properly generalizes Theorem 2 of Debreu (1954) on the supportability of Pareto optima as valuation quasiequilibria in the case of an economy with locally nonsatiated preferences. Some versions—like for instance our Theorem 7—are not stricto sensu comparable to Theorem 2 of Debreu (1954) but nevertheless explicitly display conditions on (possibly non-convex) production sets which ensure the convexity of their aggregate.

Our reformulation of the Minkowski\Hahn-Banach theorem—more precisely our Theorem 4—relies on a notion of generalized convexity introduced in Ceparano and Quartieri (2017) which is here extended to arbitrary cones and to a possibly infinite-dimensional setting. Such a notion is here called C-antichain-convexity and impose the usual notion of convexity requisites only on the linear span of any two vectors whose differences do not belong to some fixed cone C. To obtain the desired reformulation, we preliminarily address the problem of establishing which conditions can guarantee the convexity of the Minkowski sum of finitely many sets when some summands are not convex. One of the results of this work—more precisely our Theorem 1—displays these conditions proving that the sum of finitely many sets is convex when each summand is C-antichain-convex and at least one of them is C-upward (which is a sort general free-disposability condition).[3] From the pure point of view of the mathematical structure that underlies the economic results of this work, this result is perhaps our key-contribution.

The paper is tacitly organized into two parts. The first part is merely mathematical and consists of Sect. 2–4 and Appendix A. Sect. 2 presents the mathematical definitions of a C-antichain-convex and of a C-upward sets and illustrate some of their general properties. Sect. 3 shows that the sum of finitely many C-antichain-convex sets is convex provided one of the summands is C-upward.

[2] The mentioned set Z is defined at the beginning of page 591 in Debreu (1954). On this observation see also Sect. 8 of Debreu (1951).

[3] As we shall remark, every convex set is $\{0\}$-antichain-convex and $\{0\}$-upward (and so that result implies the well-known fact that the finite sum of convex sets is convex).

Sect. 4 uses this last result to obtain a separation theorem which applies also to non-convex sets. Appendix A contains some mathematical facts. The second part—where the results of the first are applied—is of economic nature and consists of Sect. 5–8 and Appendices B–C. Sect. 5 recalls some classical economic notions and definitions. Sect. 6 provides several second theorems of welfare for possibly non-convex economies. Sect. 5 contains a discussion of the hypotheses posited in the second welfare theorems . Sect. 8 shows some concluding corollaries and some examples of non-convex economies where the economic results of the paper apply. Appendix B contains some economic facts and Appendix C examines the representability of C-antichain-convex preferences by means of C-antichain-quasiconcave (utility) functions.

2 Fundamental mathematical notions

Hereafter a **real vector space**—i.e., a vector space over the reals—is sometimes abbreviated by **RVS** and a **topological real vector space**—i.e., a topological vector space over the reals—by **TRVS**.[4] A subset C of a RVS is a **cone** iff

$$(\lambda, x) \in \mathbb{R}_{++} \times C \Rightarrow \lambda x \in C.$$

Under our definition a cone can be empty; however, a cone need not be convex or contain the zero vector. Given a finite nonempty subset $S = \{s_1, \ldots, s_k\}$ of a RVS, we respectively denote by $\mathrm{co}(S)$ and $\mathrm{coni}(S)$ the **convex hull** of S and the **convex conical hull** of S defined by

$$\mathrm{co}(S) = \{\lambda_1 s_1 + \ldots + \lambda_k s_k : (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k_+ \text{ and } \lambda_1 + \ldots + \lambda_k = 1\}$$

and

$$\mathrm{coni}(S) = \{\lambda_1 s_1 + \ldots + \lambda_k s_k : (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k_+\}.$$

Note that $0 \in \mathrm{coni}(S)$. When V is a RVS and S is a subset of V, we denote by $\overline{S}$ the **complement** of S to V.

Notation 1 *Given two points x and y of a RVS and a real λ, the expression*

$$x \langle\lambda\rangle y$$

will henceforth denote the affine combination $\lambda x + (1 - \lambda)y$.

The notion of "chain-convexity" firstly introduced in Ceparano and Quartieri (2017) is now generalized to arbitrary real vector spaces and cones.[5]

Definition 1 *Let V be a RVS and S be a subset of V. Let C be a cone in V.*

[4] These definitions can be found, e.g., in Kelley and Namioka (1963, pp. 1-2 and 34). We recall that the topology of a TRVS is translation invariant.

[5] The definition of a "chain-convex set" provided in the mentioned paper coincides with that of a C-chain-convex set—in the sense of Definition 1—when $V = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$.

- *S is said to be C-**chain-convex** iff*

$$(x, y, \lambda) \in S \times S \times [0,1] \text{ and } y - x \in C \Rightarrow x \langle\lambda\rangle\, y \in S. \tag{1}$$

- *S is said to be C-**antichain-convex** iff*

$$(x, y, \lambda) \in S \times S \times [0,1] \text{ and } y - x \notin C \cup -C \Rightarrow x \langle\lambda\rangle\, y \in S. \tag{2}$$

Definition 2 *Let V be a RVS and C be a cone in V. A subset S of V is **decomposably C-antichain-convex** iff S can be expressed as the Minkowski sum of finitely many C-antichain-convex subsets of V.*

Example 1 *Let C be a cone in $\mathbb{R}^2$ such that $\mathbb{R}^2_+ \subseteq C$.*

a. The discrete set $\mathbb{N}^2_+$ is not $\mathbb{R}^2_+$-antichain-convex.

b. The discrete sets $\{0\} \times \mathbb{N}_+$ and $\{0\} \times \mathbb{N}_+$ are C-antichain-convex.

c. The discrete set $\mathbb{N}^2_+$ is decomposably C-antichain-convex (as $\mathbb{N}^2_+$ is the sum of the C-antichain-convex sets $\mathbb{N}_+ \times \{0\}$ and $\{0\} \times \mathbb{N}_+$).

In fact C-antichain-convexity can be recovered from C-chain-convexity and vice versa; in a sense, however, the two notions are complementary of one another with respect to convexity. Propositions 1 and 2—proved in Appendix A.2—clarify the point with a precise statement. Proposition 3—whose elementary proof is omitted—highlights some implications of Definition 1 which are worth to be remarked. Proposition 4—proved in Appendix A.2—clarifies that C-antichain-convex decomposability generalizes C-antichain-convexity.

Proposition 1 *Let V be a RVS, S be a subset of V and C be a cone in V.*

1. S is C-antichain-convex if and only if S is $\overline{C \cup -C}$-chain-convex.

2. S is C-chain-convex if and only if S is $\overline{C \cup -C}$-antichain-convex.

Proposition 2 *Let V be a RVS, S be a subset of V and C be a cone in V. Then S is convex if and only if S is C-chain-convex and C-antichain-convex.*

Proposition 3 *Let V be a RVS and S be a subset of V. Suppose $C^\bullet$ and C° are cones in V such that $\emptyset \subseteq C^\bullet \subseteq C^\circ \subseteq V$.*

1. S is $\emptyset$-chain-convex.

2. S is convex if and only if S is V-chain-convex.

3. If S is C°-chain-convex then S is $C^\bullet$-chain-convex.

4. S is V-antichain-convex.

5. S is convex if and only if S is $\emptyset$-antichain-convex.

6. *If S is $C^\bullet$-antichain-convex then S is C°-antichain-convex.*

Remark 1 *Proposition 3 continues to hold true if one replaces "$\emptyset$" with "$\{0\}$" in its parts 1 and 5 and "V" with "$V \setminus \{0\}$" in its parts 2 and 4.*

Proposition 4 *Let V be a RVS, S be a subset of V and C be a cone in V. If S is C-antichain-convex then S is decomposably C-antichain-convex.*

In a same vein we redefine the notions of upward and downward sets.

Definition 3 *Let V be a RVS and S be a subset of V. Let C be a cone in V.*

- *S is said to be C-**upward** iff*

$$(x, y) \in S \times V \text{ and } y - x \in C \Rightarrow y \in S. \tag{3}$$

- *S is said to be C-**downward** iff*

$$(x, y) \in S \times V \text{ and } x - y \in C \Rightarrow y \in S. \tag{4}$$

Proposition 5—proved in Appendix A.3—clarifies that the two notions just introduced are special cases of C-chain-convexity. Proposition 6—whose elementary proof is omitted—highlights some implications of Definition 3 which are worth to be explicitly remarked.

Proposition 5 *Let V be a RVS, S be a subset of V and C be a cone in V.*

1. *If S is C-upward then S is C-chain-convex.*
2. *If S is C-downward then S is C-chain-convex.*

Proposition 6 *Let V be a RVS, S be a subset of V and C be a cone in V. Suppose $C^\bullet$ and C° are cones in V such that $\emptyset \subseteq C^\bullet \subseteq C^\circ \subseteq V$.*

1. *S is $\emptyset$-upward.*
2. *If S is C°-upward then S is $C^\bullet$-upward.*
3. *S is $\emptyset$-downward.*
4. *If S is C°-downward then S is $C^\bullet$-downward.*

Remark 2 *Proposition 6 continues to hold true if one replaces "$\emptyset$" with "$\{0\}$"in its parts 1 and 3.*

Proposition 7—proved in Appendix A.3—shows an important property of C-upward and C-downward sets.

Proposition 7 *Let V be a RVS and C a cone in V. Let X,Y be subsets of V.*

1. *If X is C-upward then $X + Y$ is C-upward.*
2. *If X is C-downward then $X + Y$ is C-downward.*

3 On the convex sum of sets

The sum of two C-antichain-convex sets need not be C-antichain-convex.[6] However, if one of the addends is either C-upward or C-downward then their sum is C-antichain-convex: in fact even convex.

Theorem 1 *Let V be a RVS and C be a cone in V. Let X,Y be C-antichain-convex subsets of V.*

1. *If X is C-upward then $X+Y$ is convex and C-upward.*
2. *If X is C-downward then $X+Y$ is convex and C-downward.*

Proof. 1. Suppose X is C-upward. Propositions 2 and 5 ensure that X is convex. By part 1 of Lemma 1, part 2 of Lemma 3 and part 1 of Lemma 4—all contained in Appendix A—we can assume without loss of generality that $0 \in C$. Part 1 of Proposition 7 ensures that $X+Y$ is C-upward. Pick an arbitrary

$$(v^\bullet, v^\circ, \lambda) \in (X+Y) \times (X+Y) \times [0,1].$$

Then there exists $(x^\bullet, x^\circ, y^\bullet, y^\circ) \in X \times X \times Y \times Y$ such that

$$v^\bullet = x^\bullet + y^\bullet \text{ and } v^\circ = x^\circ + y^\circ.$$

As X is convex, $x^\bullet \langle\lambda\rangle\, x^\circ \in X$. Therefore

$$x^\bullet \langle\lambda\rangle\, x^\circ + y^\bullet \in X+Y \tag{5}$$

and

$$x^\bullet \langle\lambda\rangle\, x^\circ + y^\circ \in X+Y. \tag{6}$$

The proof continues distinguishing three exhaustive cases.

Case 1. Suppose $y^\circ - y^\bullet \notin C \cup -C$. Then $y^\bullet \langle\lambda\rangle\, y^\circ \in Y$ by the C-antichain-convexity of Y and hence

$$v^\bullet \langle\lambda\rangle\, v^\circ = (x^\bullet \langle\lambda\rangle\, x^\circ) + (y^\bullet \langle\lambda\rangle\, y^\circ) \in X+Y.$$

Case 2. Suppose $y^\circ - y^\bullet \in C$. Then $(1-\lambda)(y^\circ - y^\bullet) \in C$ as the cone C contains 0. Since $(1-\lambda)(y^\circ - y^\bullet) \in C$ and $X+Y$ is C-upward, from (5) and part 3 of Lemma 5—in Appendix A—we infer that

$$v^\bullet \langle\lambda\rangle\, v^\circ = (x^\bullet \langle\lambda\rangle\, x^\circ + y^\bullet) + (1-\lambda)(y^\circ - y^\bullet) \in X+Y.$$

[6] Even the sum of a C-antichain-convex set and a convex set need not be C-antichain-convex (and a fortiori convex): for instance, putting $C = \mathbb{R}^2_+$, one readily verifies that the subsets $\{(0,0),(0,1)\}$ and $[0,1] \times \{0\}$ of $\mathbb{R}^2$ are both C-antichain-convex and the latter is even convex but their (decomposably C-antichain-convex) sum $[0,1] \times \{0,1\}$ is not C-antichain-convex.

Case 3. Suppose $y^\circ - y^\bullet \in -C$. Then $y^\bullet - y^\circ \in C$ and $\lambda(y^\bullet - y^\circ) \in C$ as the cone C contains 0. Since $\lambda(y^\bullet - y^\circ) \in C$ and $X + Y$ is C-upward, from (6) and part 3 of Lemma 5—in Appendix A—we infer that

$$v^\bullet \langle\lambda\rangle\, v^\circ = (x^\bullet \langle\lambda\rangle\, x^\circ + y^\circ) + \lambda(y^\bullet - y^\circ) \in X + Y.$$

In each of the three cases $v^\bullet \langle\lambda\rangle\, v^\circ \in X + Y$ and so part 1 of Theorem 1 is true.

2. Suppose X is C-downward. Then part 6 of Lemma 3 and part 4 of Lemma 4—both contained in Appendix A—respectively guarantee that the sets $-X$ and $-Y$ are C-antichain-convex and that $-X$ is C-upward. Then part 1 of Theorem 1 implies the convexity of $(-X) + (-Y)$. Therefore also $X + Y$ is convex. Part 2 of Proposition 7 ensures that $X + Y$ is C-downward. ■

The previous result can be conveniently generalized as follows.

Corollary 1 *Let V be a RVS and C be a cone in V. Assume that the subsets $X_1, \ldots, X_m$ of V are C-antichain-convex.*

1. *If X_1 is C-upward then $X_1 + \ldots + X_m$ is convex and C-upward.*
2. *If X_1 is C-downward then $X_1 + \ldots + X_m$ is convex and C-downward.*

Proof. When $m = 1$, Corollary 1 follows from Propositions 2 and 5. A convex subset of V is C-antichain-convex by Proposition 2: noted this fact one readily proves by induction the case $m \geq 2$ using Theorem 1. ■

4 On the separation of sets

4.1 Known results

Let V denote a RVS and V' denote the algebraic dual (i.e., the set of all real-valued linear functions on V). We say that a linear functional $h \in V'$ **separates** two subsets X and Y of V iff $\inf h[X] \geq \sup h[Y]$ and h is non-zero (i.e., $h(v) \neq 0$ for some $v \in V$). So the linear functional h separates X and Y if and only if h separates $-Y$ and $-X$ and the linear functional h separates X and Y if and only if $-h$ separates Y and X. We have adopted the previous definition of separation—where the order of the sets matters—for expositional simplicity. Clarified this, we recall a known geometric form of the Hahn-Banach theorem.[7]

Theorem 2 (Separation Theorem I) *Let V be a TRVS. Assume that X and Y are nonempty convex subsets of V. Suppose X and Y are disjoint.*

1. *If V is finite-dimensional then X and Y can be separated by a continuous linear functional on V.*
2. *If either X or Y has nonempty interior then X and Y can be separated by a continuous linear functional on V.*

[7]For a proof of Theorem 2 see, e.g., Theorem 14.2 in Kelley and Namioka (1963) and Theorem 7.30 in Aliprantis and Border (2006). About part 1—i.e., the Minkowski separation theorem—recall that every linear functional on a finite-dimensional TRVS is continuous.

Observation Let V be a RVS and $h \in V'$ and let X and Y be subsets of V: (i) the sets X and Y are disjoint if and only if so are $X - Y$ and $\{0\}$; (ii) the sets X and Y are separated by h if and only if so are $X - Y$ and $\{0\}$.

In the light of the previous Observation, Theorem 2 can be restated thus.

Theorem 3 (Separation Theorem II) *Let V be a TRVS. Assume that X and Y are nonempty subsets of V such that $X - Y$ is convex. Suppose X and Y are disjoint.*

1. *If V is finite-dimensional then X and Y can be separated by a continuous linear functional on V.*

2. *If $X - Y$ has nonempty interior then X and Y can be separated by a continuous linear functional on V.*

4.2 A reformulation

Theorems 2 and 3 are essentially the restatement of one another. This does not mean that they are perfectly equivalent: Theorem 3 can directly apply when Theorem 2 cannot. Such a direct application, however, is possible only when $X - Y$ is known to be convex. To the best of our knowledge, there do not exist general results that guarantee the convexity of the Minkowski sum of two sets when either of them is not convex. So Theorem 1—and its Corollary 1—can be used to obtain reformulations of Theorem 3 which explicitly dispense with the convexity of either X or Y. One of the possible reformulations is as follows.

Theorem 4 (Separation Theorem III) *Let V be a TRVS and C be a cone in V. Assume that $X_1,\ldots,X_m,Y_1,\ldots,Y_n$ are C-antichain-convex subsets of V and that at least one of such $m+n$ sets is either C-upward or C-downward. Put*

$$X = X_1 + \ldots + X_m \text{ and } Y = Y_1 + \ldots + Y_n.$$

Suppose X and Y are disjoint.

1. *If V is finite-dimensional then X and Y can be separated by a continuous linear functional on V.*

2. *If $X - Y$ has nonempty interior then X and Y can be separated by a continuous linear functional on V.*

Proof. By assumption, one of the $m+n$ sets $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ is either C-upward or C-downward. Thus by parts 3 and 4 of Lemma 4—in Appendix A—one of the $m+n$ sets $X_1, \ldots, X_m, -Y_1, \ldots, -Y_n$ is then either C-upward or C-downward. Each of the $m+n$ sets $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ is C-antichain-convex and by part 6 of Lemma 3—in Appendix A—the set $-Y_l$ is C-antichain-convex for all $l = 1, \ldots, n$. Then Corollary 1 guarantees that the Minkowski sum $X - Y$ of the $m+n$ sets $X_1, \ldots, X_m, -Y_1, \ldots, -Y_n$ is convex. Noted this, Theorem 3 applies and ensures the validity of Theorem 4. ■

Remark 3 *It should be clear from part 5 of Proposition 3, parts 1 and 3 of Proposition 6 and Remarks 1 and 2 that Theorem 4 has the same hypotheses and theses of Theorem 2 if* $m = n = 1$ *and* $C \subseteq \{0\}$ *and if, in its part 2, it is additionally supposed that either* X *or* Y *has nonempty interior (this last condition is stronger than the nonemptiness of the interior of* $X - Y$*). Thus Theorem 4 subsumes Theorem 2.*

4.3 Positivity of the separating functional

Theorem 5 *Let* V *be a RVS and* C *be a cone in* V*. Assume that* X *and* Y *are nonempty subsets of* V *separated by a linear functional* $h : V \to \mathbb{R}$*. If* X *is* C*-upward then* h *is nonnegative on* C*.*

Proof. Suppose X is C-upward. The assumption that X and Y are separated by h entails the existence of a real r such that

$$h(x) \geq r \geq h(y) \text{ for all } (x, y) \in X \times Y. \tag{7}$$

Now, by way of contradiction, suppose there exists $\bar{c} \in C$ such that $h(\bar{c}) < 0$. Pick some $\bar{x} \in X$ and put $\bar{t} = (r - h(\bar{x}))/h(\bar{c})$. As X is C-upward, part 1 of Lemma 5—in Appendix A—ensures that $\bar{x} + C \subseteq X$. Then the assumption that C is a cone containing $\bar{c}$ entails that

$$\bar{x} + t\bar{c} \in X \text{ for all } t \in \mathbb{R}_{++}. \tag{8}$$

But h is linear and negative at $\bar{c}$ and therefore $h(\bar{x} + t\bar{c}) = h(\bar{x}) + th(\bar{c}) < r$ for all positive $t > \bar{t}$: a contradiction with (7) and (8). ∎

Corollary 2 *Let* V *be a TRVS and* C *be a cone in* V*. Assume that* X *and* Y *are nonempty subsets of* V *separated by a linear functional* $h : V \to \mathbb{R}$*. Besides assume that either* X *is* C*-upward or* Y *is* C*-downward. Then* h *is nonnegative on* C *and positive on the interior of* C*.*

Proof. If X is C-upward then h is nonnegative on C by Theorem 5. If Y is C-downward then $-Y$ is C-upward by part 4 of Lemma 4—in Appendix A—and hence $-Y$ and $-X$ are separated by h in that so are X and Y: also in this case Theorem 5 ensures that h is nonnegative on C. Lemma 5.66 in Aliprantis and Border (2006) guarantees that h is positive on the interior of C. ∎

5 Definition of an economy

An **economy** E is a quintuple

$$((V, C), M, N, \{X_i, R_i, \omega_i\}_{i \in M}, \{Y_i\}_{i \in N})$$

with a finite nonempty set $M = \{1, \ldots, m\}$ whose elements are called **consumers** and a finite nonempty set $N = \{1, \ldots, n\}$ whose elements are called

firms. Each consumer $i \in M$ is described by a nonempty **consumption set** X_i, by a **preference relation** $R_i \subseteq X_i \times X_i$ and by an **endowment** $\omega_i \in X_i$. Each firm $l \in N$ is described by a nonempty **production set** Y_l. All consumption and production sets are subsets of a **commodity space** V which is assumed to be a topological real vector space containing a—possibly empty and possibly not convex—cone C called the **relational commodity subspace**. The cone C induces a binary relation $\sqsubset$ on V defined by the double implication $y - x \in C \Leftrightarrow x \sqsubset y$: if C is a convex cone such that $0 \in C$ (resp. such that $C \cap -C = \{0\}$) then $\sqsubset$ is a preorder relation (resp. a partial order relation). As usual, the set of all real-valued linear functions on V is denoted by V'. A functional in V' is sometimes called a **valuation**. A valuation f in V' is **non-zero** if $f(v)$ does not vanish for at least one v in V.

Let E be an economy and $i \in M$ be a consumer. The set

$$\{(x_i^\bullet, x_i^\circ) \in R_i : (x_i^\circ, x_i^\bullet) \notin R_i\}$$

is called the **strict preference relation** for consumer i and denoted by P_i. Henceforth $R_i(x_i)$ will denote the set $\{v \in X_i : (v, x_i) \in R_i\}$ and $P_i(x_i)$ will denote the set $\{v \in X_i : (v, x_i) \in P_i\}$. A real-valued (**utility**) function u_i on X_i **represents the preference relation** R_i iff for all $x_i \in X_i$ the upper level set $\{v \in X_i : u_i(v) \geq u_i(x_i)\}$ at height $u_i(x_i)$ of u_i equals $R_i(x_i)$. The product $X_1 \times \ldots \times X_m$ is called the **joint consumption set** and denoted by $\mathcal{X}$. The product $Y_1 \times \ldots \times Y_n$ is called the **joint production set** and denoted by $\mathcal{Y}$. The sum $\omega_1 + \ldots + \omega_m$ is called the **aggregate endowment** and denoted by ω. The Minkowski sum $Y_1 + \ldots + Y_n$ is called the **aggregate production set** and denoted by Y. The set

$$\mathcal{A} = \{(x, y) \in \mathcal{X} \times \mathcal{Y} : x_1 + \ldots + x_m = \omega + y_1 + \ldots + y_n\}$$

is the **set of attainable allocations** and the set

$$\mathcal{X}_\mathcal{A} = \{x \in \mathcal{X} : (x, y) \in \mathcal{A} \text{ for some } y \in \mathcal{Y}\}$$

is the **set of attainable consumption allocations**. The **set of attainable consumption allocations Pareto dominating** $\bar{x} \in \mathcal{X}_\mathcal{A}$ is

$$\mathcal{D}(\bar{x}) = \{v \in \mathcal{X}_\mathcal{A} : v_i \in R_i(\bar{x}_i) \text{ for all } i \in M \text{ and } v_j \in P_j(\bar{x}_j) \text{ for some } j \in M\}.$$

The **set of consumption allocations weakly improving** $x \in \mathcal{X}$ is

$$\mathcal{R}(x) = R_1(x_1) \times \ldots \times R_m(x_m).$$

Definition 4 *Let E be an economy. A strict preference relation P_i is:*

- *C-**antichain-convex** iff $P_i(x_i)$ is C-antichain-convex for all $x_i \in X_i$;*
- *C-**upward** iff $P_i(x_i)$ is C-upward for all $x_i \in X_i$;*
- *C-**monotone** iff $(x_i^\bullet, x_i^\circ) \in X_i \times V$ and $x_i^\circ - x_i^\bullet \in C \backslash \{0\} \Rightarrow x_i^\circ \in P_i(x_i^\bullet)$.*

Definition 5 *Let E be an economy. A strict preference relation P_i is **convex** iff $P_i(x_i)$ is convex for all $x_i \in X_i$.*

Definition 6 *Let E be an economy. A preference relation R_i is:*

- *C-**antichain-convex** iff $R_i(x_i)$ is C-antichain-convex for all $x_i \in X_i$;*
- *C-**upward** iff $R_i(x_i)$ is C-upward for all $x_i \in X_i$;*
- *C-**monotone** iff $(x_i^\bullet, x_i^\circ) \in X_i \times V$ and $x_i^\circ - x_i^\bullet \in C \backslash \{0\} \Rightarrow x_i^\circ \in R_i(x_i^\bullet)$.*

Definition 7 *Let E be an economy. A strict preference relation P_i is:*

- ***wide** iff $\operatorname{int}(P_i(x_i)) \neq \emptyset$ for all $x_i \in X_i$ such that $P_i(x_i) \neq \emptyset$ (the interior $\operatorname{int}(P_i(x_i))$ of $P_i(x_i)$ is understood w.r.t. the topology of V);*
- ***locally nonsatiated** iff $P_i(x_i) \cap U \neq \emptyset$ for all neighborhoods U of x_i and for all $x_i \in X_i$ (neighborhoods are understood w.r.t. the topology of V);*
- ***D-lower semicontinuous** iff $\{t \in [0,1] : x_i^\bullet \langle t \rangle x_i^\circ \in P_i(x_i)\}$ is open in $[0,1]$ for all $(x_i, x_i^\bullet, x_i^\circ) \in X_i \times X_i \times X_i$ (the real interval $[0,1]$ is endowed with the relative topology from $\mathbb{R}$).*

Definition 8 *An economy E is **regular** iff*

i) each consumption set X_i is convex;

ii) each preference relation R_i is a preorder;

iii) each strict preference relation P_i is locally nonsatiated;

iv) at least one strict preference relation P_i is wide when both $\operatorname{int}(Y)$ is empty and V is not finite-dimensional.

Definition 9 *A regular economy E is a **strictly regular economy** iff*

i) each X_i and each Y_l contain 0;

ii) each P_i is D-lower semicontinuous.

Definition 10 *Let E be an economy. A pair $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}$ is a **Pareto optimum** for E iff (i) $(\bar{x}, \bar{y}) \in \mathcal{A}$ and (ii) $\mathcal{D}(\bar{x}) = \emptyset$.*

Definition 11 *Let E be an economy. A triple $(\hat{x}, \hat{y}, f) \in \mathcal{X} \times \mathcal{Y} \times V'$ is a **valuation equilibrium** (resp. **valuation quasiequilibrium**) for E iff (i) $(\hat{x}, \hat{y}) \in \mathcal{A}$, (ii) f is non-zero and (iii) the implications*

$$x_i \in P_i(\hat{x}_i) \Rightarrow f(x_i) > f(\hat{x}_i) \tag{9}$$

$$\text{(resp. } x_i \in R_i(\hat{x}_i) \Rightarrow f(x_i) \geq f(\hat{x}_i)\text{)} \tag{10}$$

and

$$y_l \in Y_l \Rightarrow f(\hat{y}_l) \geq f(y_l) \tag{11}$$

hold true for every $(i, l) \in M \times N$.

6 Second welfare theorems

6.1 From Pareto optima to quasiequilibria

Theorems 6 and 7 guarantee the supportability of Pareto optima as valuation quasiequilibria: both dispense with some usual convexity conditions. If attention is restricted to an economy with locally nonsatiated preference relations then Theorem 6 properly subsumes Theorem 2 in Debreu (1954). Examples of economies where Theorem 6 applies will be shown in Sect. 8.

Theorem 6 *Let E be a regular economy. Assume that:*

1. *P_i is C-antichain-convex for all $i \in M$;*
2. *P_i is C-upward for at least one $i \in M$;*
3. *Y is decomposably C-antichain-convex.*

If $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}$ is a Pareto optimum then $(\bar{x}, \bar{y}, f)$ is a valuation quasiequilibrium for a continuous $f \in V'$ that is nonnegative on C and positive on $\mathrm{int}(C)$.

Proof. Suppose $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}$ is a Pareto optimum. Put

$$A = \textstyle\sum_{i \in M} R_i(\bar{x}_i),\ \hat{A} = \sum_{i \in M} P_i(\bar{x}_i) \text{ and } B = \omega + Y.$$

The nonemptiness of $\hat{A}$ follows from the local nonsatiation of strict preference relations while that of B from the nonemptiness of productions sets. By the regularity of E, when V is not finite-dimensional either $\hat{A}$ or B has nonempty interior;[8] therefore $\hat{A} - B$ has nonempty interior when V is not finite-dimensional. By the Pareto optimality of $(\bar{x}, \bar{y})$, the sets $\hat{A}$ and B are disjoint. Thus[9] Theorem 4 and Corollary 2 ensure the existence of a continuous linear functional

$$f \in V'$$

separating $\hat{A}$ and B which is nonnegative on C and positive on $\mathrm{int}(C)$. As $\hat{A}$ and B are separated by $f \in V'$, there must exist $\alpha \in \mathbb{R}$ and two intervals $I^\bullet = \{r \in \mathbb{R} : r \leq \alpha\}$ and $I^\circ = \{r \in \mathbb{R} : r \geq \alpha\}$ such that $\hat{A} \subseteq f^{-1}[I^\circ]$ and $B \subseteq f^{-1}[I^\bullet]$. Put

$$K = f^{-1}[I^\circ].$$

As f is continuous, the preimage K through f of the closed (in $\mathbb{R}$) interval I° is closed (in V). As K is closed (in V), Lemma 15 in Appendix B entails that A and B are separated by f. So $f(a) \geq f(b)$ for all $(a, b) \in A \times B$ and hence

$$f(x_1 + \ldots + x_m) \geq f(\omega + y_1 + \ldots + y_n) \text{ for all } (x, y) \in \mathcal{R}(\bar{x}) \times \mathcal{Y}. \tag{12}$$

[8] When V is not finite-dimensional, we have that: if $\mathrm{int}(Y) \neq \emptyset$ then $\mathrm{int}(B) \neq \emptyset$; if $\mathrm{int}(Y) = \emptyset$ then the regularity of E ensures that $\mathrm{int}(\hat{A}) \neq \emptyset$.

[9] Clearly, the set $\hat{A}$ is convex and C-upward by Corollary 1. The set $B = \omega + Y$ can be written as the finite sum of C-antichain-convex sets because $\{\omega\}$ is trivially C-antichain-convex and Y is decomposably C-antichain-convex and hence Y can be written as the sum $Y_1^*, \ldots, Y_\nu^*$ of ν sets that are C-antichain-convex (needless to say that this assumption does not require the C-antichain-convexity of the n production sets $Y_1, \ldots, Y_n$).

As $(\bar{x}, \bar{y}) \in \mathcal{A}$ by the definition of a Pareto optimum, we have

$$f(\bar{x}_1 + \ldots + \bar{x}_m) = f(\omega + \bar{y}_1 + \ldots + \bar{y}_n). \tag{13}$$

As f is linear, from (12) and (13) we obtain

$$f(x_1 - \bar{x}_1) + \ldots + f(x_m - \bar{x}_m) \geq f(y_1 - \bar{y}_1) + \ldots + f(y_n - \bar{y}_n) \tag{14}$$

for all $(x, y) \in \mathcal{R}(\bar{x}) \times \mathcal{Y}$: therefore the two implications

$$i \in M \text{ and } x_i \in R_i(\bar{x}_i) \Rightarrow f(x_i) \geq f(\bar{x}_i)$$

and

$$l \in N \text{ and } y_l \in Y_l \Rightarrow f(\bar{y}_l) \geq f(y_l)$$

hold true[10] as the Pareto optimum $(\bar{x}, \bar{y})$ lies in $\mathcal{R}(\bar{x}) \times \mathcal{Y}$ by the reflexivity of R_i for all $i \in M$. We conclude that $(\bar{x}, \bar{y}, f)$ is a valuation quasiequilibrium. ■

Conditions 1 and 2 of Theorem 6 entail that at least one consumer has a convex and C-upward strict preference relation. Condition 3 of Theorem 6 is evidently met when all production sets are decomposably C-antichain-convex and so one obtains the following immediate Corollary 3, whose proof is omitted.

Corollary 3 *Let E be a regular economy. Assume that:*

1. *P_i is C-antichain-convex for all $i \in M$;*
2. *P_i is convex and C-upward for at least one $i \in M$;*
3. *Y_l is decomposably C-antichain-convex for all $l \in N$.*

If $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}$ is a Pareto optimum then $(\bar{x}, \bar{y}, f)$ is a valuation quasiequilibrium for a continuous $f \in V'$ that is nonnegative on C and positive on $\mathrm{int}(C)$.

We show a variant of Theorem 6 which neither implies nor is implied by it. By Corollary 1, the last two assumptions of Theorem 7 entail the convexity of the aggregate production set Y (though not necessarily that of each production set Y_l). The convexity of preference relations will not be assumed. Examples of economies where Theorem 7 applies will be shown in Sect. 8.

Theorem 7 *Let E be a regular economy. Assume that:*

1. *P_i is C-antichain-convex for all $i \in M$;*
2. *Y_l is convex and C-downward for at least one $l \in N$;*
3. *Y_l is decomposably C-antichain-convex for all $l \in N$.*

If $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}$ is a Pareto optimum then $(\bar{x}, \bar{y}, f)$ is a valuation quasiequilibrium for a continuous $f \in V'$ that is nonnegative on C and positive on $\mathrm{int}(C)$.

Proof. Exactly[11] the same proof as that of Theorem 6. ■

[10] Each R_i is reflexive: so to obtain the former (resp. latter) of the implications, reconsider (14) first fixing $i \in M$ (resp. $l \in N$) and $x_i \in R_i(\bar{x}_i)$ (resp. $y_l \in Y_l$) and then putting $y = \bar{y}$ and $x_j = \bar{x}_j$ for all $j \in M \setminus \{i\}$ (resp. putting $x = \bar{x}$ and $y_j = \bar{y}_j$ for all $j \in N \setminus \{l\}$).

[11] Clearly, now B is convex and C-downward by Corollary 1 and the set $\hat{A}$ is the finite sum of C-antichain-convex sets.

6.2 From quasiequilibria to equilibria

We show sufficient conditions for a quasiequilibrium to be an equilibrium.

Proposition 8 *Suppose E is an economy. Let $f \in V'$ and $i \in M$ be a consumer with a convex consumption set X_i containing a consumption vector $\overline{x}_i$. Put $\phi_i = \inf f[X_i]$. Suppose P_i is D-lower semicontinuous and consider the implications*

$$x_i \in R_i(\overline{x}_i) \Rightarrow f(x_i) \geq f(\overline{x}_i) \tag{15}$$

and

$$x_i \in P_i(\overline{x}_i) \Rightarrow f(x_i) > f(\overline{x}_i). \tag{16}$$

1. *If $f(\overline{x}_i) > \phi_i$ then the validity of (15) implies the validity of (16).*
2. *If $f(\overline{x}_i) = \phi_i$ and $P_i(\overline{x}_i) \subseteq \{x_i \in X_i : f(x_i) > \phi_i\}$ then (16) is true.*

Proof. 1. Assume that $f(\overline{x}_i) > \phi_i$ and that (15) is true. Then

$$f(\overline{x}_i) > f(\underline{x}_i) \tag{17}$$

for some $\underline{x}_i \in X_i$. The contrapositive of (15) entails that

$$x_i \in X_i \text{ and } f(\overline{x}_i) > f(x_i) \Rightarrow x_i \notin R_i(\overline{x}_i)$$

and a fortiori that

$$x_i \in X_i \text{ and } f(\overline{x}_i) > f(x_i) \Rightarrow x_i \notin P_i(\overline{x}_i). \tag{18}$$

Suppose for a moment that x_i is an element of X_i such that $f(\overline{x}_i) \geq f(x_i)$: the convexity of X_i, the linearity of f and inequality (17) imply that

$$x_i \langle t \rangle \underline{x}_i \in X_i \text{ and } f(\overline{x}_i) > f(x_i \langle t \rangle \underline{x}_i) \text{ for all } t \in [0, 1[. \tag{19}$$

From (18) and (19) we infer that $\{t \in [0,1] : x_i \langle t \rangle \underline{x}_i \in P_i(\overline{x}_i)\} \subseteq \{1\}$. Thus

$$x_i \in X_i \text{ and } f(\overline{x}_i) \geq f(x_i) \Rightarrow \{t \in [0,1] : x_i \langle t \rangle \underline{x}_i \in P_i(\overline{x}_i)\} \subseteq \{1\}$$

and so, by virtue of the D-lower semicontinuity of P_i, we infer that

$$x_i \in X_i \text{ and } f(\overline{x}_i) \geq f(x_i) \Rightarrow x_i \notin P_i(\overline{x}_i). \tag{20}$$

The contrapositive of (20) entails that (16) is true.
2. The immediate proof is omitted. ∎

Corollary 4 *Suppose E is a strictly regular economy with a valuation quasi-equilibrium $(\bar{x}, \bar{y}, f)$.*

1. *$(\bar{x}, \bar{y}, f)$ is a valuation equilibrium if f is positive at $\bar{x}_i$ for all $i \in M$.*
2. *$(\bar{x}, \bar{y}, f)$ is a valuation equilibrium if f is positive on $X_i \backslash \{0\}$ for all $i \in M$.*

Proof. 1. A consequence of Proposition 8 and of the fact that f is vanishing at $0 \in X_i$ for all $i \in M$.

2. Suppose f is positive on $X_i \backslash \{0\}$ for all $i \in M$. Then $f_i(0) = 0 = \inf f[X_i]$ for all $i \in M$. Part 2 of Proposition 8 ensures that (16) is true whenever $i \in M$ and $\bar{x}_i = 0$ by the irreflexivity of P_i. When $i \in M$ and $\bar{x}_i \neq 0$, the implication in (15) holds true because $(\bar{x}, \bar{y}, f)$ is a quasiequilibrium: therefore (16) is true by virtue of part 1 of Proposition 8 (and by the positivity of f on $X_i \backslash \{0\}$). ■

Proposition 9 provides sufficient conditions only on the primitives of an economy E for a valuation quasiequilibrium to be a valuation equilibrium.

Proposition 9 *Let E be a strictly regular economy. Suppose $(\bar{x}, \bar{y}, f)$ is a valuation quasiequilibrium such that $f \in V'$ is nonnegative on C.*

1. *$(\bar{x}, \bar{y}, f)$ is a valuation equilibrium if $X_i \backslash \{0\} \subseteq \operatorname{int}(C)$ for all $i \in M$.*

2. *$(\bar{x}, \bar{y}, f)$ is a valuation equilibrium if $X_i \subseteq C$ for all $i \in M$ and $P(x_i) \subseteq \operatorname{int}(X_i)$ for all $x_i \in X_i \backslash \operatorname{int}(X_i)$ and for all $i \in M$.*

3. *$(\bar{x}, \bar{y}, f)$ is a valuation equilibrium if $X_i = C$ for all $i \in M$, P_i is C-monotone for all $i \in M$, and $\omega \in \operatorname{int}(C)$.*

Proof. 1. Lemma 5.66 in Aliprantis and Border (2006) ensures that f is positive on $\operatorname{int}(C)$. So part 1 of Proposition 9 is a direct consequence of part 2 of Corollary 4.

2. Suppose that $X_i \subseteq C$ for all $i \in M$ and that $P(x_i) \subseteq \operatorname{int}(X_i)$ for all $x_i \in X_i \backslash \operatorname{int}(X_i)$ and for all $i \in M$. Lemma 5.66 in Aliprantis and Border (2006) ensures that f is positive on $\operatorname{int}(C) \supseteq \operatorname{int}(X_i)$ for all $i \in M$. As f is vanishing at $0 \in X_i$ and nonnegative on $C \supseteq X_i$ for all $i \in M$, we have that $\inf f[X_i] = 0$. By assumption, (15) is true for all $i \in M$. If $i \in M$ and $f(\overline{x}_i) > 0$ then (16) is true by part 1 of Proposition 8 as (15) is true for all $i \in M$. If instead $i \in M$ and $f(\overline{x}_i) = 0$ then $\overline{x}_i \notin \operatorname{int}(X_i)$ and using part 2 of Proposition 8 we can infer that (16) is true. In conclusion, (16) is true for all $i \in M$ and hence $(\bar{x}, \bar{y}, f)$ is a valuation equilibrium.

3. Suppose $X_i = C$ for all $i \in M$, P_i is C-monotone for all $i \in M$, and $\omega \in \operatorname{int}(C)$. Lemma 5.66 in Aliprantis and Border (2006) ensures that f is positive on $\operatorname{int}(C)$. So

$$f(\omega) > 0. \tag{21}$$

By the strict regularity of E each firm can be inactive and hence

$$f(\overline{y}_l) \geq 0 \text{ for all } l \in N \tag{22}$$

since in quasiequilibrium firms maximize own profits (namely, the implication in (11) must hold for all $l \in N$). As $(\bar{x}, \bar{y}) \in \mathcal{A}$, we have

$$\overline{x}_1 + \ldots + \overline{x}_m = \omega + \overline{y}_1 + \ldots + \overline{y}_n$$

and hence the inequalities in (21) and (22) imply

$$f(\overline{x}_1) + \ldots + f(\overline{x}_m) = f(\omega) + f(\overline{y}_1) + \ldots + f(\overline{y}_n) > 0$$

by the linearity of f. Thus there exists $i^* \in M$ such that $f(\overline{x}_{i^*}) > 0$; as E is strictly regular, $0 \in X_{i^*}$ and so $\inf f[X_{i^*}] \leq 0$. Then

$$f(\overline{x}_{i^*}) > \inf f[X_{i^*}]. \tag{23}$$

As P_{i^*} is C-monotone and $\overline{x}_{i^*} \in X_{i^*}$, we have $\overline{x}_{i^*} + c \in X_{i^*}$ for all $c \in C$ and

$$\overline{x}_{i^*} + c \in P_i(\overline{x}_{i^*}) \text{ for all } c \in C\backslash\{0\}. \tag{24}$$

As $(\bar{x}, \bar{y}, f)$ is a valuation quasiequilibrium, the implication in (15) holds true for $i = i^*$ and so from (23) and part 1 of Proposition 8 we infer that (16) holds true for $i = i^*$. This last fact, together with (24), entails that

$$f(\overline{x}_{i^*} + c) > f(\overline{x}_{i^*}) \text{ for all } c \in C\backslash\{0\}. \tag{25}$$

As f is linear, (25) implies $f(c) > 0$ for all $c \in C\backslash\{0\}$. Thus f is positive on $X_i\backslash\{0\} = C\backslash\{0\}$ for all $i \in M$ and part 2 of Corollary 4 ensures that $(\bar{x}, \bar{y}, f)$ is a valuation equilibrium. ■

7 Discussion of some assumptions

7.1 On antichain-convex preferences

Convexity of preferences is at times understood as an expression of the inclination of a consumer to diversification\variety: just to provide two references, see Mas-Colell et al. (1995, p. 44) and Villar (2000, p. 20). This interpretation, however, might oversimplify the actual implications of convexity. Consider, for instance, the case of a preordered preference relation R defined on $\mathbb{R}^2_+$ and—for simplicity—suppose that R can be represented by a utility function u. If R is convex then the equality $u(x^\bullet) = u(x^\circ)$ implies $u(x^\bullet/2 + x^\circ/2) \geq u(x^\bullet) = u(x^\circ)$ for *all* consumption vectors $x^\bullet$ and x° in $\mathbb{R}^2_+$. When $x^\bullet$ and x° cannot be compared under the usual product order relation for $\mathbb{R}^2_+$—e.g., when $x^\bullet = (4, 0)$ and $x^\circ = (0, 4)$—the consumption vector $x^\bullet/2 + x^\circ/2$ might be legitimately interpreted as a diversifying consumption vector. However, there is some doubt that $x^\bullet/2 + x^\circ/2$ can be legitimately considered a diversifying vector when $x^\bullet$ and x° can be compared. For instance, one might consider the consumption vectors $x^\bullet = (10, 10)$ and $x^\circ = (30, 30)$ and wonder whether $(20, 20)$ can be properly considered a diversifying consumption vector. The interpretation of the convexity of a preference relation as the consumer's inclination to diversification appears as an oversimplification: the condition of convexity is in fact more demanding. The introduction of the notion of a C-antichain-convex preference allows to express the inclination of a consumer to diversification in a more precise and circumstantial form. For instance, if in the previous example

R is assumed to be $\mathbb{R}^2_+$-antichain-convex then the inclination of a consumer to diversification is effectively restricted to mixtures of consumption vectors that cannot be compared under the usual product order of $\mathbb{R}^2_+$.

So far we have considered a preordered preference relation, the interpretation of the C-antichain-convexity of the associated strict preference is analogous. Proposition 10—proved in Appendix B.1—shows that in the case of totally preordered relation the condition of C-antichain-convexity is even equivalent to that of the associated strict preference relation.

Proposition 10 *Let E be an economy and $i \in M$ be a consumer with a C-antichain-convex consumption set X_i. Suppose R_i is totally preordered. Then P_i is C-antichain-convex if and only if R_i is C-antichain-convex.*

7.2 On upward preferences and downward production sets

Like C-antichain-convex preference relations represent a circumstantial formulation of the notion of convexity, also the definitions of C-monotone and C-upward preference relations enunciated above allow circumstantial formulations of various notions of monotonicity. Propositions 11 and 12—proved in Appendix B.1—show how these definitions precisely relate one to each other.

Proposition 11 *Let E be an economy and $i \in M$ be a consumer. If the preference relation R_i is preordered then*

$$P_i \text{ is } C\text{-monotone} \Rightarrow R_i \text{ is } C\text{-monotone} \Leftrightarrow R_i \text{ is } C\text{-upward} \Rightarrow P_i \text{ is } C\text{-upward.}$$

Proposition 12 *Let E be an economy and $i \in M$ be a consumer. If the preference relation R_i is totally preordered then*

$$P_i \text{ is } C\text{-monotone} \Rightarrow R_i \text{ is } C\text{-monotone} \Leftrightarrow R_i \text{ is } C\text{-upward} \Leftrightarrow P_i \text{ is } C\text{-upward.}$$

When C is degenerate—i.e., when $C \subseteq \{0\}$—the condition that a production set Y_l is C-downward does not impose any actual requisite on the production set. When C is not degenerate, the condition that a production set Y_l is C-downward cannot be understood as a circumstantial formulation of the usual notion of free-disposability, at least in general.[12] Proposition 13—proved in Appendix B.1—shows a decomposition that allows us to have a clear economic interpretation of such a condition.

Proposition 13 *Let E be an economy and $l \in N$ be a firm: (i) $C^\star = -C \cup \{0\}$ is a cone; (ii) the production set Y_l is C-downward if and only if $Y_l = Y_l + C^\star$.*

Economically, Proposition 13 says that the assumption that Y_l is C-downward is equivalent to the assumption that the production Y_l is invariant under the addition of the constant returns to scale technology $C^\star = -C \cup \{0\}$.

[12] To see why, consider a two dimensional Euclidean space and suppose C is a cone properly including the nonnegative orthant.

7.3 On antichain-convex production sets

When inaction is possible, a known implication of convexity are non-increasing returns to scale. While the possibility of inaction can seem reasonable in several contexts, non-increasing returns to scale need not in many cases: just to provide two references, see Kreps (1990, pp. 235-236) and Villar (2000, Sect. 3.4). The C-antichain-convexity of a production set does not generally imply non-increasing returns to scale even when inaction is possible. Being a circumstantial form of convexity, C-antichain-convexity requires the production feasibility only of some mixtures of two feasible activities (in fact C-antichain-convexity is compatible even with discrete production sets and hence with indivisibilities).

7.4 On decomposably antichain-convex production sets

A **decomposition** $\{Y_l^1, \ldots, Y_l^k\}$ **of a production set** Y_l is a finite collection of subsets of V such that

$$Y_l = Y_l^1 + \ldots + Y_l^k.$$

Following Debreu (1951, pp. 277–278), we can suppose that the aggregate production set is the sum of the **activity possibility sets**—each activity possibility set formally defined as a subset of V—of the **production units** of the economy: a production unit does not necessarily coincide with a firm. The assumption that Y_l is decomposably C-antichain-convex can be interpreted as the assumption that firm l is made up of k production units—e.g., its plants—with C-antichain-convex activity possibility sets whose sum $Y_l^1 + \ldots + Y_l^k$ equals Y_l. The interpretation of the C-antichain-convexity of an activity possibility set is analogous to that of a C-antichain-convex production set.

7.5 On lower semicontinuous preferences

The notion of a lower semicontinuous preference—paralleling that of a lower semicontinuous (utility) function—requires the openness of strictly preferred sets. The literature, however, has employed also some nonequivalent variants.

Definition 12 *Let E be an economy and $i \in M$ be a consumer. Put*

$$I(x_i^\bullet, x_i^\circ) = \{t \in \mathbb{R} : x_i^\bullet \langle t \rangle x_i^\circ \in X_i\}$$

for all $(x_i^\bullet, x_i^\circ) \in X_i \times X_i$ and endow $I(x_i^\bullet, x_i^\circ)$ and X_i with, respectively, the relative topology from $\mathbb{R}$ and V. The strict preference relation P_i is:

- ***D-lower* semicontinuous** iff $\{t \in I(x_i^\bullet, x_i^\circ) : x_i^\bullet \langle t \rangle x_i^\circ \in P_i(x_i)\}$ is open in $I(x_i^\bullet, x_i^\circ)$ for all $(x_i, x_i^\bullet, x_i^\circ) \in X_i \times X_i \times X_i$.*
- ***lower semicontinuous** iff $P_i(x_i)$ is open in X_i for all $x_i \in X_i$.*

Proposition 14—proved in Appendix B.1—relates the various definitions of semicontinuity introduced sofar.

Proposition 14 *Let E be an economy and $i \in M$ be a consumer.*

1. *If P_i is lower semicontinuous then P_i is D-lower* semicontinuous.*
2. *If P_i is D-lower* semicontinuous then P_i is D-lower semicontinuous.*

Condition III in Debreu (1954) implies D-lower* semicontinuity (and so even D-lower semicontinuity by virtue of Proposition 14): this can be readily checked recalling that in Debreu (1954) preference relations are total preorders.

8 Numerical examples

We show seven economies satisfying the assumptions of either Theorem 7 or Corollary 3 (and hence of Theorem 6) whose set of valuation quasiequilibria coincides with that of valuation equilibria by virtue of Proposition 9.

Remark 4 *We affirm now, once for all, that in Examples 2–8* every *P_i is C-antichain-convex for* any *cone C such that $\mathbb{R}_+ \subseteq C \subseteq \mathbb{R}$ by virtue of Proposition 16 and either Examples 9 or 11: see Appendix C. Also, in Examples 2–8 a direction is indicated along which utility functions are strictly increasing: the local nonsatiation of P_i is immediately verified considering such a direction.*

Before introducing the seven economies, it is worth to remark that for each of them the existence of a Pareto optimum—which, however, is not the object of our inquiry—obtains from known results of the literature.[13]

8.1 "Specialized" economies with non-convexities

Examples of "specialized" economies are shown where each commodity can be produced by exactly one firm and where each firm can produce exactly one commodity. The examples are of interest because—despite the convexity of the aggregate production set—one of the (two) commodities of the economy can be produced only by means of a non-convex technology.

Example 2 *Let E be an economy with two consumers and two firms. Let the commodity space V coincide with $\mathbb{R}^2$, let each consumption set X_i equal $\mathbb{R}^2_+$ and let each preference relation R_i be represented by a continuous utility function u_i strictly increasing at all $v \in \mathbb{R}^2_+$ along $(1,1)$.*[14] *In particular, suppose that*

$$C = \operatorname{coni}(T^\bullet) \text{ with } T^\bullet = \{(-1,1),(1,0)\},$$

that $u_1(v) = u_2(v) = v_1 v_2$ and that $Y_1 = -C$ and

$$Y_2 = A + \mathbb{R}^2_- \text{ with } A = \{(-a,a) : a = 0,1,2,3,4\}.$$

[13] In all our examples the boundedness of the attainable set $\mathcal{A}$ is a consequence of Theorem 12.3 in Villar (2000). Noted this, one readily verifies the nonemptiness and the compactness of the attainable set; the existence of a Pareto optimum is then guaranteed by representability of preference relations by means of continuous utility functions.

[14] I.e., for all $v \in \mathbb{R}^2_+$ the map $\mathbb{R}_+ \to \mathbb{R}$ defined by $\lambda \mapsto u_i(v + \lambda(1,1))$ is strictly increasing.

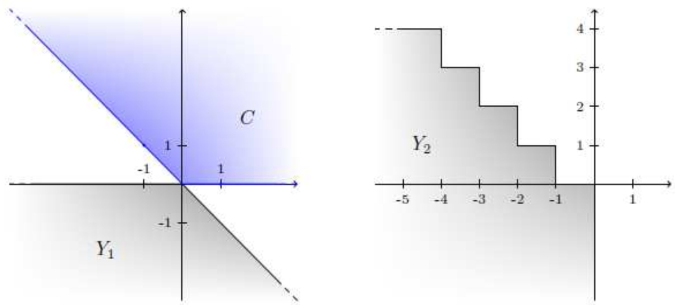

Fig. 1 The sets C, Y_1 and Y_2 of Examples 2 and 3.

It is not difficult to verify that A is coni$(T^{\bullet})$-antichain-convex and it is immediate that $\mathbb{R}^2_-$ is convex (and hence a fortiori coni$(T^{\bullet})$-antichain-convex): we can conclude that Y_2 is decomposably coni$(T^{\bullet})$-antichain-convex. Clearly, Y_1 is coni$(T^{\bullet})$-downward and convex (and hence a fortiori decomposably coni$(T^{\bullet})$-antichain-convex). Noted this, one can readily verify that all conditions of Theorem 7 are satisfied. Also, one can readily verify that also the conditions of part 2 of Proposition 9 are satisfied.

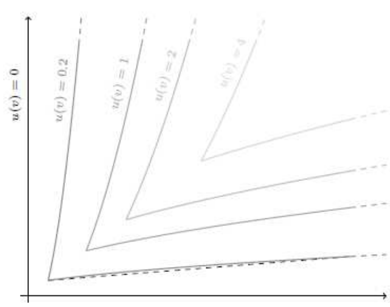

Fig. 2 Upper level sets of u_1 and u_2 in Examples 3 and 8.

Example 3 *Exactly the same economy illustrated in Example 2, but now let preference relations be representable by*

$$u_1(v) = u_2(v) = \min \left\{ \frac{v_1^2}{v_2 + 1}, \frac{v_2^2}{v_1 + 1} \right\}.$$

The continuous utility functions u_1 and u_2 keep on being strictly increasing at all $v \in \mathbb{R}^2_+$ along $(1,1)$ but they are not quasiconcave anymore: see Remark 6. Even though preferences now are non-convex, the economy continues to satisfy all conditions of Theorem 7. One can readily verify that also the conditions of part 2 of Proposition 9 continue to be satisfied.

Example 4 *Exactly the same economy illustrated in Example 3, but now put $Y_2 = A$ (where A is the discrete set defined in Example 2).*

8.2 "Unspecialized" economies with non-convexities

Here we show two examples of "unspecialized" economies where any commodity can be produced by any firm.

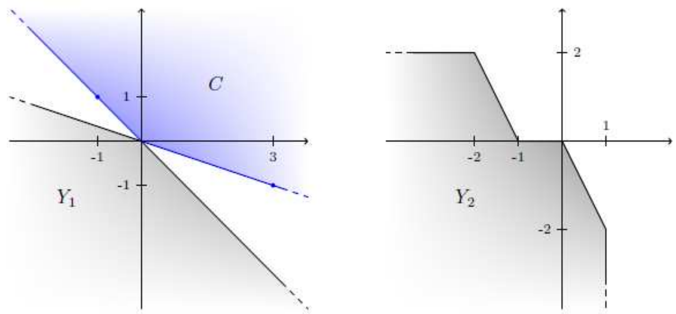

Fig. 3 The sets C, Y_1 and Y_2 of Examples 5 and 6.

Example 5 *Let E be an economy with two consumers and two firms. Let the commodity space V coincide with $\mathbb{R}^2$, let each consumption set X_i equal $\mathbb{R}^2_+$ and let each preference relation R_i be represented by a continuous utility function u_i strictly increasing in the second argument. In particular, suppose that*

$$C = \operatorname{coni}(T^\circ) \text{ with } T^\circ = \{(-1, 1), (3, -1)\},$$

that $u_1(v) = u_2(v) = v_1 + v_2$ and that $Y_1 = -C$ and

$$Y_2 = A + \operatorname{co}(B) + \mathbb{R}^2_- \text{ with } A = \{(0, 0), (2, -2)\} \text{ and } B = \{(-2, 2), (-1, 0)\}.$$

It is not difficult to verify that A is $\operatorname{coni}(T^\circ)$-antichain-convex and it is immediate that $\operatorname{co}(B)$ and $\mathbb{R}^2_-$ are convex (and hence a fortiori $\operatorname{coni}(T^\circ)$-antichain-convex): we can conclude that Y_2 is decomposably $\operatorname{coni}(T^\circ)$-antichain-convex. Clearly, Y_1 is $\operatorname{coni}(T^\circ)$-downward and convex (and hence a fortiori decomposably $\operatorname{coni}(T^\circ)$-antichain-convex). Noted this, one can readily verify that all conditions of Theorem 7 are satisfied. Also, one can readily verify that also the conditions of part 1 of Proposition 9 are satisfied in Example 5.

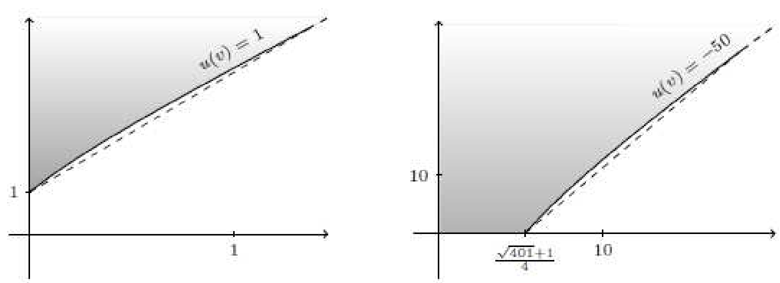

Fig. 4 Upper level sets of u_1 and u_2 in Example 6.

Example 6 *Exactly the same economy illustrated in Example 5, but now let preference relations be representable by*

$$u_1(v) = \frac{v_1 v_2}{v_1 + 1} - 5v_1 + v_2 \text{ and } u_2(v) = -2v_1^2 + v_1 + v_1 v_2 + v_2.$$

The continuous utility functions u_1 and u_2 keep on being strictly increasing in the second argument but they are not quasiconcave anymore: see Remark 5. Even though preferences now are non-convex, the economy continues to satisfy all conditions of Theorem 7. One can readily verify that also the conditions of part 1 of Proposition 9 continue to be satisfied.

8.3 Economies with non-convex aggregate production sets

In all previous examples the aggregate production set was convex. Here we show two examples of economies where aggregate production set is non-convex.

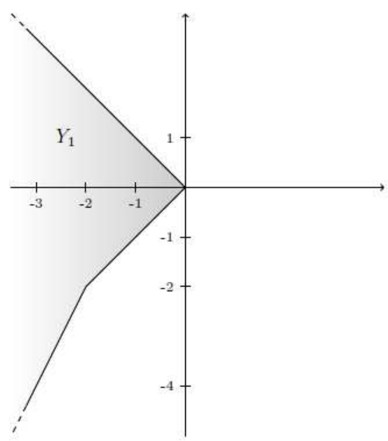

Fig. 5 The production set Y_1 of Examples 7 and 8 (where $Y_1 = Y_2 = Y$).

Example 7 *Let E be an economy with four consumers and two firms. Let the commodity space V coincide with $\mathbb{R}^2$, let each consumption set X_i equal $\mathbb{R}^2_+$ and let each preference relation R_i be represented by a continuous utility function u_i strictly increasing at all $v \in \mathbb{R}^2_+$ along $(1, 1)$. In particular, suppose that*

$$C = \mathbb{R}^2_+,$$

that $u_1(v) = u_2(v) = u_3(v) = u_4(v) = v_1 + v_2$ and that

$$Y = Y_1 = Y_2 = A + \text{coni}(B)$$

with

$$A = \{v \in \mathbb{R}^2_- : 2v_1 + 2 = v_2\} \cup \{(0, 0)\} \text{ and } B = \{(-1, 1), (-1, -1)\}.$$

Besides assume $\omega_1 = \omega_2 = (3, 0)$ and $\omega_3 = \omega_4 = (0, 3)$.

It is not difficult to verify that A is $\mathbb{R}^2_+$-antichain-convex and it is immediate that coni(B) is convex (and hence a fortiori $\mathbb{R}^2_+$-antichain-convex): we can conclude that Y_1 is decomposably $\mathbb{R}^2_+$-antichain-convex (and that so are also Y_2 and Y as $Y = Y_1 = Y_2$ in this example). It is readily seen that all strict preference relations are $\mathbb{R}^2_+$-monotone (and hence they are all $\mathbb{R}^2_+$-upward by virtue of Proposition 11). Noted this, one can readily verify that all conditions of Corollary 3—and hence those of Theorem 6—are satisfied. One can readily verify that also the conditions of part 3 of Proposition 9 are satisfied.

Example 8 *Exactly the same economy illustrated in Example 7 but now remove the assumption that* $\omega_1 = \omega_2 = (3, 0)$ *and* $\omega_3 = \omega_4 = (0, 3)$ *and suppose that*

$$u_1(v) = u_2(v) = \min\left\{\frac{v_1^2}{v_2+1}, \frac{v_2^2}{v_1+1}\right\}$$

and that $u_3(v) = v_1 v_2$ *and* $u_4(v) = \min\{v_1, v_2\}$. *(See again Fig. 2 for a graphical representation of* u_1 *and* u_2*).*

The continuous utility functions u_1,u_2, u_3 and u_4 keep on being strictly increasing at all $v \in \mathbb{R}^2_+$ along $(1, 1)$ but they are not quasiconcave anymore: see Remark 6. Even though some preferences now are non-convex, the economy continues to satisfy all conditions of Theorem 6 (note that P_3 and P_4 continue to be $\mathbb{R}^2_+$-upward). One can readily verify that now the conditions of part 2 of Proposition 9 are satisfied.

Appendix

A Some mathematical facts

This Appendix contains some mathematical facts. Parts 1 and 5 of Lemma 3 are presented only for expositional completeness.

A.1 On cones

Lemma 1 *Let* V *be a RVS and* C *be a subset of* V*.*

1. C *is a cone in* V *if and only if* $C \cup \{0\}$ *is a cone in* V*.*
2. C *is a cone in* V *if and only if* $-C$ *is a cone in* V*.*
3. C *is a cone in* V *if and only if* $\overline{C}$ *is a cone in* V*.*
4. *If* C *is a cone in* V *then* $C \cup -C$ *is a cone in* V*.*
5. *If* C *is a cone in* V *then* $\overline{C \cup -C}$ *is a cone in* V*.*

Proof. *If part.* Suppose $C \cup \{0\}$ is a cone. If $0 \in C$ then $C = C \cup \{0\}$. If $0 \notin C$ then $(x, \alpha) \in C \times \mathbb{R}_{++}$ implies $0 \neq \alpha x \in C$. In both cases C is a cone.

Only if part. Suppose C is a cone. Then $(x, \alpha) \in (C \cup \{0\}) \times \mathbb{R}_{++}$ implies $\alpha x \in C \cup \{0\}$ (when $\alpha x \in C$ because C is a cone and when $x = 0$ because $\alpha x = 0$). So $C \cup \{0\}$ is a cone.

2. It suffices to prove the *if part* since $C = -(-C)$. Such a proof is as follows. Suppose $-C$ is a cone. Then $(x, \alpha) \in C \times \mathbb{R}_{++}$ implies $(-x, \alpha) \in -C \times \mathbb{R}_{++}$ and hence $-\alpha x$ belongs to the cone $-C$: this implies $\alpha x \in C$. So C is a cone.

3. It suffices to prove the *if part* since $C = \overline{(\overline{C})}$. Such a proof is as follows. Suppose $\overline{C}$ is a cone. Then $(x, \alpha) \in C \times \mathbb{R}_{++}$ implies $x \notin \overline{C}$ and hence αx does not belong[15] to the cone $\overline{C}$: this implies $\alpha x \in C$. So C is a cone.

4. Suppose C is a cone. The set $-C$ is a cone by part 2 of Lemma 1. Let α be an arbitrary positive real. If $x \in C \cup -C$ then x belongs to either the cone C or the cone $-C$ and hence $\alpha x \in C \cup -C$. Thus $C \cup -C$ is a cone.

5. A consequence of parts 3–4 of Lemma 1. ■

A.2 On C-chain-convex sets

Lemma 2 *Let V be a RVS, C be a cone in V and S be a subset of V.*

1. *S is C-chain-convex if and only if*

$$(x, y) \in S \times S \text{ and } y - x \in C \cup -C \Rightarrow \{x \langle \lambda \rangle y : \lambda \in [0, 1]\} \subseteq S. \quad (26)$$

2. *S is C-antichain-convex if and only if*

$$(x, y) \in S \times S \text{ and } y - x \notin C \cup -C \Rightarrow \{x \langle \lambda \rangle y : \lambda \in [0, 1]\} \subseteq S. \quad (27)$$

Proof. 1. The *if part* is evidently true as (26) implies (1). The proof of the *only if part* is as follows. Assume that S is C-chain-convex. Suppose $(x, y) \in S \times S$ and $y - x \in C \cup -C$. Then either $y - x \in C$ or $y - x \in -C$. If $y - x \in C$ then $\{x \langle \lambda \rangle y : \lambda \in [0, 1]\} \subseteq S$ by the C-chain-convexity of S. If $y - x \in -C$ then $x - y \in C$ and hence $\{x \langle \lambda \rangle y : \lambda \in [0, 1]\} = \{y \langle \lambda \rangle x : \lambda \in [0, 1]\} \subseteq S$ by the C-chain-convexity of S. In both cases implication (26) is true.

2. Implication (2) is readily seen to be equivalent to the implication

$$(x, y, \lambda) \in S \times S \times [0, 1] \text{ and } y - x \notin C \cup -C \Rightarrow x \langle \lambda \rangle y \in S$$

and hence also to implication (27). ■

Lemma 3 *Let V be a RVS, C be a cone in V and S be a subset of V.*

1. *S is C-chain-convex if and only if S is $C \cup \{0\}$-chain-convex.*

2. *S is C-antichain-convex if and only if S is $C \cup \{0\}$-antichain-convex.*

[15] If $\alpha x \in \overline{C}$ then $x = \alpha^{-1}(\alpha x) \in \overline{C}$ as $\overline{C}$ is a cone: but this is impossible as $x \in C$.

3. *S is C-chain-convex if and only if S is $C \cup -C$-chain-convex.*

4. *S is C-antichain-convex if and only if S is $C \cup -C$-antichain-convex.*

5. *S is C-chain-convex if and only if $-S$ is C-chain-convex.*

6. *S is C-antichain-convex if and only if $-S$ is C-antichain-convex.*

Proof. 1. A consequence of part 1 of Lemma 1 and of the (obvious) implication

$$(x, y, \lambda) \in S \times S \times [0, 1] \text{ and } y - x = 0 \Rightarrow x \langle\lambda\rangle y \in S. \tag{28}$$

2. A consequence of part 1 of Lemma 1 and of implication (28).
3. A consequence of part 4 of Lemma 1 and of part 1 of Lemma 2.
4. Note that $C \cup -C$ equals $(C \cup -C) \cup -(C \cup -C)$ and is a cone by part 4 of Lemma 1. Noted this, part 4 of Lemma 3 readily follows from Definition 1.
5. The *if part* is an immediate consequence of the *only if part* and of the equality $-(-S) = S$. We prove just the *only if part*, as follows. Assume that S is C-chain-convex. By way of contradiction, suppose $-S$ is not C-chain-convex; then part 1 of Lemma 2 implies the existence of a triple (x, y, λ) in $-S \times -S \times [0, 1]$ such that $y - x \in C \cup -C$ and $x \langle\lambda\rangle y \notin -S$. So, putting $\hat{x} = -x$ and $\hat{y} = -y$, we equivalently have that $(\hat{x}, \hat{y}, \lambda) \in S \times S \times [0, 1]$, $\hat{y} - \hat{x} \in C \cup -C$ and $\hat{x} \langle\lambda\rangle \hat{y} \notin S$: a contradiction with the assumption that S is C-chain-convex and with part 1 of Lemma 2.
6. Essentially the same proof of part 5: just replace "C-chain-convex" with "C-antichain-convex", "part 1" with "part 2", "$y - x \in C \cup -C$" with "$y - x \notin C \cup -C$" and "$\hat{y} - \hat{x} \in C \cup -C$" with "$\hat{y} - \hat{x} \notin C \cup -C$". ■

Proof of Proposition 1. 1. Part 5 of Lemma 1 ensures that $\overline{C \cup -C}$ is a cone. By part 3 of Lemma 3, S is C-chain-convex if and only if

$$(x, y, \lambda) \in S \times S \times [0, 1] \text{ and } y - x \in C \cup -C \Rightarrow x \langle\lambda\rangle y \in S.$$

Equivalently, S is C-chain-convex if and only if

$$(x, y, \lambda) \in S \times S \times [0, 1] \text{ and } y - x \notin \overline{C \cup -C} \Rightarrow x \langle\lambda\rangle y \in S.$$

But[16]

$$\overline{C \cup -C} = (\overline{C \cup -C}) \cup -(\overline{C \cup -C}). \tag{29}$$

Thus S is C-chain-convex if and only if

$$(x, y, \lambda) \in S \times S \times [0, 1] \text{ and } y - x \notin (\overline{C \cup -C}) \cup -(\overline{C \cup -C}) \Rightarrow x \langle\lambda\rangle y \in S$$

and therefore—by virtue of Definition 1—it readily follows that S is C-chain-convex if and only if S is $\overline{C \cup -C}$-antichain-convex.

2. Part 5 of Lemma 1 ensures that $\overline{C \cup -C}$ is a cone. Put

$$C^* = \overline{C \cup -C} \text{ and } C^{**} = \overline{C^* \cup -C^*}.$$

[16] As $\overline{C \cup -C} = \overline{C} \cap \overline{-C} = \overline{C} \cap -\overline{C} = -(\overline{C} \cap -\overline{C}) = -(\overline{C} \cap \overline{-C}) = -(\overline{C \cup -C})$.

The C^*-chain-convexity of S is equivalent to the C^{**}-antichain-convexity of S by part 1 of Proposition 1 and hence also to the $C \cup -C$-antichain-convexity of S because $C^{**} = C \cup -C$ by (29). The $C \cup -C$-antichain-convexity of S is equivalent to the C-antichain-convexity of S by part 4 of Lemma 3. ■

Proof of Proposition 2. A consequence of parts 1–2 of Lemma 2. ■

Proof of Proposition 4. Suppose S is C-antichain-convex. Note that $\{0\}$ is C-antichain-convex and $\{0\} + S = S$. Conclude that S is decomposably C-antichain-convex. ■

A.3 On C-upward sets

Lemma 4 *Let V be a RVS, C be a cone in V and S be a subset of V.*

1. *S is C-upward if and only if S is $C \cup \{0\}$-upward.*
2. *S is C-downward if and only if S is $C \cup \{0\}$-downward.*
3. *S is C-upward if and only if $-S$ is C-downward.*
4. *S is C-downward if and only if $-S$ is C-upward.*

Proof. 1. Part 1 of Lemma 1 ensures that $C \cup \{0\}$ is a cone. The *if part* is immediate. The proof of the *only if* part is as follows. Suppose S is C-upward. Then $(x, y) \in S \times V$ and $y - x \in C$ imply $y \in S$. As $(x, y) \in S \times V$ and $y - x = 0$ imply $y \in S$, we infer that S is $C \cup \{0\}$-upward.

2. Analogous to the proof of part 1 of Lemma 4.

3. We prove the *if part*: the proof of the *only if part* is analogous and omitted. Assume that $-S$ is C-downward. Suppose $(x, y) \in S \times V$ and $y - x \in C$. Then $(-x, -y) \in -S \times V$ and $-x - (-y) \in C$ and the assumption that $-S$ is C-downward implies $-y \in -S$ and hence $y \in S$. Therefore S is C-upward.

4. A consequence of part 3 of Lemma 4 and of the equality $-(-S) = S$. ■

Lemma 5 *Let V be a RVS, C be a cone in V and S be a subset of V.*

1. *S is C-upward if and only if $S + C \subseteq S$.*
2. *S is C-downward if and only if $S - C \subseteq S$.*
3. *Suppose $0 \in C$. Then S is C-upward if and only if $S + C = S$.*
4. *Suppose $0 \in C$. Then S is C-downward if and only if $S - C = S$.*

Proof. 1. *If part.* Suppose $S + C \subseteq S$. If $(x, y) \in S \times V$ and $y - x \in C$ then $y \in x + C \subseteq S + C \subseteq S$. We conclude that S is C-upward.

Only if part. Suppose S is C-upward. If $y \in S + C$ then there exists $(x, c) \in S \times C$ such that $y = x + c$ and hence that $y - x = c \in C$: the assumption that

S is C-upward implies $y \in S$. Thus $y \in S + C$ implies $y \in S$ and we conclude that $S + C \subseteq S$.

2. By part 4 of Lemma 4, S is C-downward if and only if $-S$ is C-upward. By part 1 of Lemma 5, $-S$ is C-upward if and only if $-S + C \subseteq -S$. As the inclusion $-S + C \subseteq -S$ is true if and only if so is the inclusion $S - C \subseteq S$, we conclude that S is C-downward if and only if $S - C \subseteq S$.

3. Note that $S + C \supseteq S$ as $0 \in C$. Noted this, part 3 of Lemma 5 is an immediate consequence of part 1 of Lemma 5.

4. Note that $0 \in -C$ and that $S - C \supseteq S$ as $0 \in -C$. Noted this, part 4 of Lemma 5 is an immediate consequence of part 2 of Lemma 5. ■

Proof of Proposition 5. 1. Suppose S is C-upward. Let $(x, y, \lambda) \in S \times S \times [0, 1]$ and $y - x \in C$. When $\lambda = 1$ we have that $x \langle\lambda\rangle y = y \in S$ by assumption. Henceforth suppose $\lambda \neq 1$. Then $(1 - \lambda)(y - x) \in C$ because C is a cone. As S is a C-upward set containing x, we infer that $x + (1 - \lambda)(y - x) \in S$. Noting that $x \langle\lambda\rangle y = x + (1 - \lambda)(y - x)$, we conclude that $x \langle\lambda\rangle y \in S$ and hence that S is C-chain-convex.

2. Analogous to the proof of part 1 of Proposition 5. ■

Proof of Proposition 7. 1. Suppose X is C-upward. Then $X + C \subseteq X$ by part 1 of Lemma 5. So $(X+Y)+C = (X+C)+Y \subseteq X+Y$ by basic properties of Minkowski addition and part 1 of Lemma 5 ensures that $X + Y$ is C-upward.

2. Suppose X is C-downward. Then $-X$ is C-upward by part 4 of Lemma 4. As also $-Y$ is a subset of V, part 1 of Proposition 7 ensures that $-X - Y$ is C-upward. So $X + Y$ is C-downward by part 4 of Lemma 4. ■

B Some economic facts

B.1 On preferences and production sets

Lemma 6 *Let E be an economy and $i \in M$ be a consumer. Assume that R_i is transitive and $x_i^\bullet \in X_i$.*

1. *If $x_i^\circ \in R_i(x_i^\bullet)$ then $P_i(x_i^\circ) \subseteq P_i(x_i^\bullet)$.*
2. *If $x_i^\circ \in R_i(x_i^\bullet)$ and $x_i^\bullet \in R_i(x_i^\circ)$ then $P_i(x_i^\circ) = P_i(x_i^\bullet)$.*
3. *If $x_i^\circ \in R_i(x_i^\bullet)$ and $x_i \in R_i(x_i^\circ)$ and in addition either $x_i \in P_i(x_i^\circ)$ or $x_i^\circ \in P_i(x_i^\bullet)$ then $x_i \in P_i(x_i^\bullet)$.*

Proof. 1. Suppose $x_i^\circ \in R_i(x_i^\bullet)$. If $P_i(x_i^\circ) = \emptyset$ then $P_i(x_i^\circ) \subseteq P_i(x_i^\bullet)$. Suppose $P_i(x_i^\circ) \neq \emptyset$ and pick an arbitrary $v \in P_i(x_i^\circ)$. Then $v \in R_i(x_i^\circ)$ and

$$x_i^\circ \notin R_i(v). \tag{30}$$

As $v \in R_i(x_i^\circ)$ and $x_i^\circ \in R_i(x_i^\bullet)$, the transitivity of R_i implies $v \in R_i(x_i^\bullet)$. If $x_i^\bullet \in R_i(v)$ then the transitivity of R_i and $x_i^\circ \in R_i(x_i^\bullet)$ implies $x_i^\circ \in R_i(v)$ in contradiction with (30): thus $x_i^\bullet \notin R_i(v)$. As $v \in R_i(x_i^\bullet)$ and $x_i^\bullet \notin R_i(v)$, we infer that $v \in P_i(x_i^\bullet)$ and hence $P_i(x_i^\circ) \subseteq P_i(x_i^\bullet)$ as v is arbitrary in $P_i(x_i^\circ)$.

2. A consequence of part 1 of Lemma 6.

3. Assume that $x_i \in R_i(x_i^\circ)$ and $x_i^\circ \in R_i(x_i^\bullet)$ and that either $x_i \in P_i(x_i^\circ)$ or $x_i^\circ \in P_i(x_i^\bullet)$. Then at least one of the following two exhaustive cases is true.
Case $x_i^\circ \in P_i(x_i^\bullet)$. As $x_i \in R_i(x_i^\circ)$ and $x_i^\circ \in R_i(x_i^\bullet)$, the transitivity of R_i implies $x_i \in R_i(x_i^\bullet)$. As R_i is transitive and $x_i \in R_i(x_i^\circ)$ by assumption, if $x_i^\bullet \in R_i(x_i)$ then $x_i^\bullet \in R_i(x_i^\circ)$ in contradiction with $x_i^\circ \in P_i(x_i^\bullet)$. So $x_i^\bullet \notin R_i(x_i)$ and hence $x_i \in P_i(x_i^\bullet)$ as $x_i \in R_i(x_i^\bullet)$.
Case $x_i \in P_i(x_i^\circ)$ *and* $x_i^\bullet \in R_i(x_i^\circ)$. Part 2 of Lemma 6 ensures the validity of the equality $P_i(x_i^\circ) = P_i(x_i^\bullet)$. Therefore $x_i \in P_i(x_i^\bullet)$. ∎

Proof of Proposition 10. *If part.* Suppose R_i is C-antichain-convex and $x_i \in X_i$. Assume that

$$(x_i^\bullet, x_i^\circ, \lambda) \in P_i(x_i) \times P_i(x_i) \times [0,1] \text{ and } x_i^\bullet - x_i^\circ \notin C \cup -C.$$

The totality of R_i entails that either $x_i^\circ \in R_i(x_i^\bullet)$ or $x_i^\bullet \in R_i(x_i^\circ)$: without loss of generality suppose $x_i^\circ \in R_i(x_i^\bullet)$. The reflexivity of R_i entails that $x_i^\bullet \in R_i(x_i^\bullet)$. As $x_i^\circ \in R_i(x_i^\bullet)$ and $x_i^\bullet \in R_i(x_i^\bullet)$, the C-antichain-convexity of R_i implies $x_i^\bullet \langle\lambda\rangle\, x_i^\circ \in R_i(x_i^\bullet)$. Then part 3 of Lemma 6 implies $x_i^\bullet \langle\lambda\rangle\, x_i^\circ \in P_i(x_i)$ as $x_i^\bullet \in P(x_i)$ by assumption. We conclude that $P_i(x_i)$ is C-antichain-convex.

Only if part. Suppose P_i is C-antichain-convex and $x_i \in X_i$. Assume that

$$(x_i^\bullet, x_i^\circ, \lambda) \in R_i(x_i) \times R_i(x_i) \times [0,1] \text{ and } x_i^\bullet - x_i^\circ \notin C \cup -C.$$

By way of contradiction, suppose $x_i^\bullet \langle\lambda\rangle\, x_i^\circ \notin R_i(x_i)$: the totality of R_i implies $x_i \in P_i(x_i^\bullet \langle\lambda\rangle\, x_i^\circ)$. As $x_i^\bullet \in R_i(x_i)$ and $x_i \in P_i(x_i^\bullet \langle\lambda\rangle\, x_i^\circ)$, part 3 of Lemma 6 ensures that $x_i^\bullet \in P_i(x_i^\bullet \langle\lambda\rangle\, x_i^\circ)$; as $x_i^\circ \in R_i(x_i)$ and $x_i \in P_i(x_i^\bullet \langle\lambda\rangle\, x_i^\circ)$, part 3 of Lemma 6 ensures that $x_i^\circ \in P_i(x_i^\bullet \langle\lambda\rangle\, x_i^\circ)$. So $x_i^\bullet \in P_i(x_i^\bullet \langle\lambda\rangle\, x_i^\circ)$ and $x_i^\circ \in P_i(x_i^\bullet \langle\lambda\rangle\, x_i^\circ)$ and hence $x_i^\bullet \langle\lambda\rangle\, x_i^\circ \in P_i(x_i^\bullet \langle\lambda\rangle\, x_i^\circ)$ by the C-antichain-convexity of P_i: a contradiction with the irreflexivity of P_i. ∎

Lemma 7 *Let E be an economy and $i \in M$ be a consumer. If P_i is C-monotone then R_i is C-monotone.*

Proof. Suppose P_i is C-monotone. Then the implication

$$(x_i^\bullet, x_i^\circ) \in X_i \times V \text{ and } x_i^\circ - x_i^\bullet \in C\backslash\{0\} \Rightarrow x_i^\circ \in P_i(x_i^\bullet)$$

is true. As $P_i(x_i^\bullet) \subseteq R_i(x_i^\bullet)$, we conclude that R_i is C-monotone. ∎

Lemma 8 *Let E be an economy and $i \in M$ be a consumer. Suppose R_i is transitive. If R_i is C-monotone then P_i is C-upward.*

Proof. Suppose R_i is C-monotone. Assume that $x_i \in X_i$, that

$$x_i^\bullet \in P_i(x_i) \tag{31}$$

and that x_i° is an element of V such that $x_i^\circ - x_i^\bullet \in C$: we are done if we show

$$x_i^\circ \in P_i(x_i). \tag{32}$$

If $x_i^\circ = x_i^\bullet$ then $x_i^\circ \in P_i(x_i)$ by (31). Henceforth suppose $x_i^\circ \neq x_i^\bullet$. Then $x_i^\circ - x_i^\bullet \in C\backslash\{0\}$ and the C-monotonicity of R_i entails that

$$x_i^\circ \in R_i(x_i^\bullet). \tag{33}$$

Then (33) and (31) imply (32) by virtue of part 3 of Lemma 6. ∎

Lemma 9 *Let E be an economy and $i \in M$ be a consumer. Suppose R_i is reflexive. If R_i is C-upward then R_i is C-monotone.*

Proof. Suppose R_i is C-upward. Assume that $(x_i^\bullet, x_i^\circ) \in X_i \times V$ and that $x_i^\circ - x_i^\bullet \in C\backslash\{0\}$: we are done if we show

$$x_i^\circ \in R_i(x_i^\bullet).$$

By assumption, $R_i(x_i^\bullet)$ is C-upward. As R_i is reflexive, $x_i^\bullet \in R_i(x_i^\bullet)$. Then $x_i^\circ \in R_i(x_i^\bullet)$ as $R_i(x_i^\bullet)$ is a C-upward set containing $x_i^\bullet$ and $x_i^\circ - x_i^\bullet \in C\backslash\{0\}$. ∎

Lemma 10 *Let E be an economy and $i \in M$ be a consumer. Suppose R_i is transitive. If R_i is C-monotone then R_i is C-upward.*

Proof. Suppose R_i is C-monotone. Assume that $x_i \in X_i$, that

$$x_i^\bullet \in R_i(x_i) \tag{34}$$

and that x_i° is an element of V such that $x_i^\circ - x_i^\bullet \in C$: we are done if we show

$$x_i^\circ \in R_i(x_i).$$

If $x_i^\circ = x_i^\bullet$ then $x_i^\circ \in R_i(x_i)$ by (34). Henceforth suppose $x_i^\circ \neq x_i^\bullet$. Then $x_i^\circ - x_i^\bullet \in C\backslash\{0\}$ and $x_i^\circ \in R_i(x_i^\bullet)$ by the C-monotonicity of R_i. As $x_i^\circ \in R_i(x_i^\bullet)$, the transitivity of R_i and (34) imply $x_i^\circ \in R_i(x_i)$. ∎

Lemma 11 *Let E be an economy and $i \in M$ be a consumer. Suppose R_i is total. If P_i is C-upward then R_i is C-monotone.*

Proof. Suppose P_i is C-upward. Assume that $(x_i^\bullet, x_i^\circ) \in X_i \times V$ and that $x_i^\circ - x_i^\bullet \in C\backslash\{0\}$: we are done if we show

$$x_i^\circ \in R_i(x_i^\bullet). \tag{35}$$

If $x_i^\circ \notin R_i(x_i^\bullet)$ then $x_i^\bullet \in P_i(x_i^\circ)$ by the totality of R_i: but this yields a contradiction with the irreflexivity of P_i as $(x_i^\bullet, x_i^\circ) \in X_i \times V$, $x_i^\bullet \in P_i(x_i^\circ)$, $x_i^\circ - x_i^\bullet \in C \backslash \{0\}$ and the assumption that P_i is C-upward imply $x_i^\circ \in P_i(x_i^\circ)$. So (35) is true. ■

Proof of Proposition 11. Suppose R_i is preordered. The C-monotonicity of P_i implies that of R_i by Lemma 7 and so the first one-way implication is true. The validity of the double implication is guaranteed by Lemmas 9 and 10. Furthermore, such a double implication and Lemma 8 guarantees the validity of the remaining one-way implication. ■

Proof of Proposition 12. An immediate consequence of Lemma 11 and Proposition 11. ■

Proof of Proposition 13. Part 2 of Lemma 1 ensures that $-C$ is a cone. So $C^\star = -C \cup \{0\}$ is a cone by part 1 of Lemma 1 and hence $-C^\star$ is cone again by part 2 Lemma 1. As $-C^\star = C \cup \{0\}$, part 2 of Lemma 4 ensures that S is C-downward if and only if S is $-C^\star$-downward; part 4 of Lemma 5 ensures that S is $-C^\star$-downward if and only if $S = S - (-C^\star)$. As $S - (-C^\star) = S + C^\star$, we conclude that S is C-downward if and only if $S = S + C^\star$. ■

Proof of Proposition 14. The proof of part 2 is a direct consequence of the inclusion $\{t \in [0,1] : x_i^\bullet \langle t \rangle x_i^\circ \in X_i\} \subseteq I(x_i^\bullet, x_i^\circ)$ and of basic facts about relative topologies. The proof of part 1 is as follows. Suppose P_i is lower semicontinuous. Fix and arbitrary i and an arbitrary triple $(x_i, x_i^\bullet, x_i^\circ) \in X_i \times X_i \times X_i$. As V is a topological real vector space, the function $g : \mathbb{R} \to V$ defined by

$$g(\lambda) = x_i^\bullet + \lambda(x_i^\circ - x_i^\bullet)$$

is continuous. The preimage through a continuous function of an open set is open and hence

$$g^{-1}[P_i(x_i)] = \{t \in \mathbb{R} : x_i^\bullet \langle t \rangle x_i^\circ \in P_i(x_i)\}$$

is open in $\mathbb{R}$; so, as $I(x_i^\bullet, x_i^\circ) \subseteq \mathbb{R}$, basic facts about relative topologies guarantee that $\{t \in I(x_i^\bullet, x_i^\circ) : x_i^\bullet \langle t \rangle x_i^\circ \in P_i(x_i)\}$ is open in $I(x_i^\bullet, x_i^\circ)$. ■

B.2 On half-space inclusion of aggregate preferred sets

Lemma 12 *Let E be an economy and $i \in M$ be a consumer. Assume that P_i is locally nonsatiated and that K is a closed subset of V. If $x_i \in X_i$ and $P_i(x_i) \subseteq K$ then $x_i \in K$.*

Proof. Suppose $x_i \in X_i$ and $P_i(x_i) \subseteq K$. As P_i is locally nonsatiated, the intersection $P_i(x_i) \cap U$ is nonempty for every neighborhood U of x_i. Pick x_i^U in $P_i(x_i) \cap U$ for every neighborhood U of x_i. As $P_i(x_i) \subseteq K$ and $x_i^U \in P_i(x_i) \cap U$ for every neighborhood U of x_i, we have that $x_i^U \in K \cap U$ for every neighborhood U of x_i. We conclude that x_i is a limit point for K. So $x_i \in K$ as every closed set contains its limit points. ■

Lemma 13 *Let E be an economy and $i \in M$ be a consumer. Assume that P_i is locally nonsatiated and transitive and that K is a closed subset of V. If $x_i \in X_i$ and $P_i(x_i) \subseteq K$ then $R_i(x_i) \subseteq K$.*

Proof. Suppose $x_i \in X_i$ and $P_i(x_i) \subseteq K$. As $P_i(x_i) \subseteq R_i(x_i)$, the local nonsatiation of P_i ensures that $R_i(x_i) \neq \emptyset$. Pick $x_i^\bullet \in R_i(x_i)$. As $P_i(x_i) \subseteq K$,

$$x_i^\bullet \in P_i(x_i) \Rightarrow x_i^\bullet \in K. \tag{36}$$

Suppose for a moment that $x_i^\bullet \notin P_i(x_i)$. As $x_i^\bullet \notin P_i(x_i)$ and $x_i^\bullet \in R_i(x_i)$, we have $x_i \in R_i(x_i^\bullet)$. So $x_i^\bullet \in R_i(x_i)$ and $x_i \in R_i(x_i^\bullet)$ and part 2 of Lemma 6 implies $P_i(x_i^\bullet) = P_i(x_i)$. Therefore $P_i(x_i^\bullet) \subseteq K$ and from Lemma 12 we infer that $x_i^\bullet \in K$. In conclusion,

$$x_i^\bullet \notin P_i(x_i) \Rightarrow x_i^\bullet \in K. \tag{37}$$

From (36) and (37) we conclude that the arbitrary element $x_i^\bullet$ of $R_i(x_i)$ is also an element of K. So $R_i(x_i) \subseteq K$. ■

Lemma 14 *Let E be an economy and $i \in M$ be a consumer. Assume that R_i transitive, that P_i is locally nonsatiated and that K is a closed subset of V. Suppose $Q \subseteq V$ is an arbitrary subset of the commodity space and x_i is an element of X_i. If $P_i(x_i) + Q \subseteq K$ then $R_i(x_i) + Q \subseteq K$.*

Proof. Suppose $P_i(x_i) + Q \subseteq K$. If $Q = \emptyset$ then $\emptyset = R_i(x_i) + Q \subseteq K$ and there is nothing to prove. Assume that $Q \neq \emptyset$ and pick an arbitrary $q \in Q$. Then $P_i(x_i) + q \subseteq K$ and hence

$$P_i(x_i) \subseteq -q + K. \tag{38}$$

As K is closed in V, then so is $-q + K$. As $-q + K$ is closed, inclusion (38) implies $R_i(x_i) \subseteq -q + K$ by Lemma 13. Consequently $R_i(x_i) + q \subseteq K$. Being q arbitrary in Q, we conclude that $R_i(x_i) + Q \subseteq K$ also when $Q \neq \emptyset$. ■

Lemma 15 *Let E be an economy. Assume that $R_1,\ldots,R_m$ are transitive, that $P_1,\ldots,P_m$ are locally nonsatiated and that K is a closed subset of V and that $x \in \mathcal{X}$. Then*

$$P_1(x_1) + \ldots + P_m(x_m) \subseteq K \Rightarrow R_1(x_1) + \ldots + R_m(x_m) \subseteq K.$$

Proof. If $m = 1$ then the validity of Lemma 15 is ensured by Lemma 13. Henceforth assume that $m > 1$ and that

$$P_1(x_1) + \ldots + P_m(x_m) \subseteq K. \tag{39}$$

Case $m = 2$. Inclusion (39) and Lemma 14 imply that $R_1(x_1) + P_2(x_2) \subseteq K$ (to check the implication identify $P_2(x_2)$ with Q). The previous inclusion and Lemma 14 imply that $R_1(x_1) + R_2(x_2) \subseteq K$ (to check the implication identify $R_1(x_1)$ with Q).

Case $m > 2$. Put

$$Q_1 = P_2(x_2) + \ldots + P_m(x_m) \text{ and } Q_m = R_1(x_1) + \ldots + R_{m-1}(x_{m-1})$$

and for every integer $k \in \{2, \ldots, m-1\}$ put

$$Q_k = \sum_{i=1}^{k-1} R_i(x_i) + \sum_{i=k+1}^{m} P_i(x_i).$$

Inclusion (39) and Lemma 14 imply $Q_1 + R_1(x_1) \subseteq K$. The previous inclusion and Lemma 14 imply $Q_2 + R_2(x_2) \subseteq K$ and reiterating this type of reasoning we get $Q_m + R_m(x_m) \subseteq K$. As

$$Q_m + R_m(x_m) = R_1(x_1) + \ldots + R_m(x_m),$$

we have $R_1(x_1) + \ldots + R_m(x_m) \subseteq K$. ■

C On antichain-quasiconcavity

Constructing C-antichain-convex and decomposably C-antichain-convex sets is quite simple. Constructing examples of strict preference relations which are C-antichain-convex might at first appear not so elementary when we require them to be also non-convex. The end of this Sect. 5 is dedicated to the illustration of two simple methods of constructing non-trivial non-convex C-antichain-convex preference. To this end, we introduce some definitions which generalize to arbitrary real vector spaces and cones analogous notions of generalized convexity already introduced in Ceparano and Quartieri (2017).

Definition 13 *Let V be a RVS, C be a cone in V, S be a C-chain-convex subset of V and $f : S \to \mathbb{R}$. The function f is **C-chain-quasiconcave** iff its upper level sets are C-chain-convex. The function f is **C-chain-concave** iff*

$$(x, y, \lambda) \in S \times S \times [0, 1] \text{ and } y - x \in C \Rightarrow f(x \langle\lambda\rangle y) \geq f(x) \langle\lambda\rangle f(y).$$

Definition 14 *Let V be a RVS, C be a cone in V, S be a C-antichain-convex subset of V and $f : S \to \mathbb{R}$. The function f is **C-antichain-quasiconcave** iff its upper level sets are C-antichain-convex. The function f is **C-antichain-concave** iff*

$$(x, y, \lambda) \in S \times S \times [0, 1],\ y - x \notin C \cup -C \Rightarrow f(x \langle\lambda\rangle y) \geq f(x) \langle\lambda\rangle f(y).$$

Proposition 15—whose elementary proof is omitted—highlights some implications which are worth to be explicitly remarked.

Proposition 15 *Let V be a RVS, C be a cone in V, S be a C-antichain-convex (resp. C-chain-convex) subset of V and $f : S \to \mathbb{R}$. The function f is C-antichain-quasiconcave (resp. C-chain-quasiconcave) if f is C-antichain-concave (resp. C-chain-concave). Suppose C is even convex: the function f is C-antichain-quasiconcave (resp. C-chain-quasiconcave) if f is quasiconcave; the function f is C-antichain-concave (resp. C-chain-concave) if f is concave.*

Proposition 16 contains a simple but useful observation.

Proposition 16 *Let E be an economy and i be a consumer. Let $C^\bullet$ and C° be cones in V such that $C^\bullet \subseteq C^\circ$. Suppose R_i can be represented by means of a utility function u_i. If u_i is $C^\bullet$-antichain-quasiconcave then P_i is C°-antichain-convex.*

Proof. Suppose u_i is $C^\bullet$-antichain-quasiconcave. By virtue of part 6 of Proposition 3 it suffices to show that P_i is $C^\bullet$-antichain-convex. As R_i is representable by means of utility function[17], the $C^\bullet$-antichain-convexity of P_i is equivalent to that of R_i by virtue of Proposition 10. The fact that R_i is $C^\bullet$-antichain-convex follows immediately from the fact that R_i is representable by means of a $C^\bullet$-antichain-quasiconcave utility function. ■

Providing examples of C-antichain-(quasi)concave functions which are not quasiconcave might not be simple because a mathematical analysis of the former has not been developed yet. Without trying to systematically start such an analysis, we show some simple-to-check conditions guaranteeing the C-antichain-concavity—and hence the C-antichain-quasiconcavity—of a real-valued function defined on a subset of a two-dimensional Euclidean space when $\mathbb{R}^2_+ \subseteq C$.

Proposition 17 *Let $Z \subseteq \mathbb{R}^2$ be nonempty and open and let $f : Z \to \mathbb{R}$ be twice continuously differentiable with Hessian matrix at $z \in Z$ denoted by*

$$\mathrm{H}(z) = \begin{bmatrix} \mathrm{H}_{1,1} & \mathrm{H}_{1,2} \\ \mathrm{H}_{2,1} & \mathrm{H}_{2,2} \end{bmatrix}.$$

Assume that S is a convex subset of Z and that for all $s \in S$ the matrix $\mathrm{H}(s)$ has nonpositive diagonal entries $\mathrm{H}_{1,1}$ and $\mathrm{H}_{2,2}$ and nonnegative off-diagonal entries $\mathrm{H}_{2,1}$ and $\mathrm{H}_{1,2}$. Let C be a cone in $\mathbb{R}^2$ such that $\mathbb{R}^2_+ \subseteq C \subseteq \mathbb{R}^2$. Then:

1. *f is $\mathbb{R}^2_+$-antichain-concave on S;*
2. *f is C-antichain-concave on S;*
3. *f is C-antichain-quasiconcave on S;*
4. *f need not be quasiconcave.*

Proof. 1. Put $Y = \{(z_1, -z_2) : z \in Z\}$ and $X = \{(s_1, -s_2) : s \in S\}$. Then Y is open and X is a convex subset of Y. Let $f^\circ : Y \to \mathbb{R}$ be defined by

$$f^\circ(y) = f(y_1, -y_2).$$

Note that f° is twice continuously differentiable on Y and that we are done if we show that f° is $\mathbb{R}^2_+$-chain-concave on X. The assumptions on the Hessian

[17] Recall that the representability of a preference relation R_i by means of a utility function implies that R_i is totally preordered.

matrix of f imply that the Hessian matrix of f° has nonpositive entries at all points in X. Corollary 1 in Ceparano and Quartieri (2017) ensures that f° is $\mathbb{R}^2_+$-chain-concave on X.

2. A consequence of Definition 14 and part 1 of Proposition 17.
3. A consequence of Proposition 15 and part 2 of Proposition 17.
4. See Example 9 and Remark 5. ■

Example 9 *Suppose C is a cone in $\mathbb{R}^2$ such that $\mathbb{R}^2_+ \subseteq C \subseteq \mathbb{R}^2$. Let $(\alpha, \delta) \in \mathbb{R}^2_+$ and $(\beta, \gamma) \in \mathbb{R}^2$. The functions $f^\bullet : \mathbb{R}^2_+ \to \mathbb{R}$ and $f^\circ : \mathbb{R}^2_+ \to \mathbb{R}$ defined by*

$$f^\bullet(v) = \alpha\frac{v_1v_2}{v_1+1}+\beta v_1+\gamma v_2+\delta\frac{v_1v_2}{v_2+1} \text{ and } f^\circ(v) = -\alpha v_1^2+\beta v_1+v_1v_2+\gamma v_2-\delta v_2^2$$

are C-antichain-quasiconcave by virtue of part 3 of Proposition 17 above.[18]

Remark 5 *The functions $f^\bullet$ and f° in Example 9 are not quasiconcave for some of their specifications: for instance the specifications*

$$f^\bullet(v) = \frac{v_1v_2}{v_1+1} - 5v_1 + v_2 \text{ and } f^\circ(v) = -2v_1^2 + v_1 + v_1v_2 + v_2$$

are not quasiconcave (the first has an upper level set at height 1 which is not convex and the second has an upper level set at height -50 which is not convex).

Proposition 18 *Let C be a cone in $\mathbb{R}^2$ such that*

$$\mathbb{R}^2_+ \subseteq C \subseteq \mathbb{R}^2$$

and f_1 and f_2 be real-valued functions on $\mathbb{R}^2_+$. Assume that f_1 is increasing in one of the two arguments and decreasing in the other. Also, assume that f_2 is increasing in one of the two arguments and decreasing in the other. Then

1. *f_1 and f_2 are $\mathbb{R}^2_+$-antichain-quasiconcave;*
2. *$\min\{f_1, f_2\}$ is $\mathbb{R}^2_+$-antichain-quasiconcave;*
3. *f_1, f_2 and $\min\{f_1, f_2\}$ are C-antichain-quasiconcave;*
4. *f_1, f_2 and $\min\{f_1, f_2\}$ need not be quasiconcave.*

Proof. 1. We show that f_1 is $\mathbb{R}^2_+$-antichain-quasiconcave: the proof for f_2 is the same. Without loss of generality suppose f_1 is increasing in the first argument and decreasing in the second: the proof for the other case is analogous. Pick an arbitrary quadruple $(x, y, r, \lambda) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \times \mathbb{R} \times [0, 1]$ such that $x - y \notin \mathbb{R}^2_+ \cup \mathbb{R}^2_-$ and suppose x and y are in the upper level set of f_1 at height r. Then

$$r \leq f_1(x) \text{ and } r \leq f_1(y).$$

[18] Both $f^\bullet$ and f° extends to $Z =]-1, +\infty[\times]-1, +\infty[$ as twice continuously differentiable functions: putting $S = \mathbb{R}^2_+$ apply the Hessian condition in Proposition 17 to such extensions.

As $x-y \notin \mathbb{R}^2_+ \cup \mathbb{R}^2_-$ we have two exhaustive cases: either $x_1 \leq y_1$ and $x_2 \geq y_2$ or $x_1 \geq y_1$ and $x_2 \leq y_2$. If $x_1 \leq y_1$ and $x_2 \geq y_2$ then

$$r \leq f_1(x) \leq f(x_1 \langle\lambda\rangle\, y_1, x_2) \leq f(x_1 \langle\lambda\rangle\, y_1, x_2 \langle\lambda\rangle\, y_2) = f(x \langle\lambda\rangle\, y)$$

(the second inequality holds because f_1 is increasing in the first argument and the third inequality holds because f_1 is decreasing in the second argument); if $x_1 \geq y_1$ and $x_2 \leq y_2$ then

$$r \leq f_1(y) \leq f(y_1, x_2 \langle\lambda\rangle\, y_2) \leq f(x_1 \langle\lambda\rangle\, y_1, x_2 \langle\lambda\rangle\, y_2) = f(x \langle\lambda\rangle\, y)$$

(the second inequality holds because f_1 is decreasing in the second argument and the third inequality holds because f_1 is increasing in the first argument). Therefore $f(x \langle\lambda\rangle\, y)$ is in the upper level set of f_1 at height r. We conclude that f_1 is $\mathbb{R}^2_+$-antichain-quasiconcave.

2. Pick an arbitrary quadruple $(x, y, r, \lambda) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \times \mathbb{R} \times [0,1]$ such that $x-y \notin \mathbb{R}^2_+ \cup \mathbb{R}^2_-$ and suppose $r \leq \min\{f_1(x), f_2(x)\}$ and $r \leq \min\{f_1(y), f_2(y)\}$. Then x and y are in the upper level sets at height r of both f_1 and f_2 and hence their $\mathbb{R}^2_+$-antichain-quasiconcavity entails that

$$r \leq f_1(x \langle\lambda\rangle\, y) \text{ and } r \leq f_2(x \langle\lambda\rangle\, y).$$

Thus $\min\{f_1(x \langle\lambda\rangle\, y), f_2(x \langle\lambda\rangle\, y)\} \geq r$ and we can conclude that $\min\{f_1, f_2\}$ is $\mathbb{R}^2_+$-antichain-quasiconcave.

3. A consequence of part 6 of Proposition 3, of Definition 14 and of parts 1 and 2 of Proposition 18.

4. See Examples 10 and 11 and Remark 6. ■

Example 10 *The real-valued functions f defined on $\mathbb{R}^2_+$ by $f(v) = 1 + v_1^2 - v_2$ is $\mathbb{R}^2_+$-antichain-quasiconcave (by virtue of Part 1 of Proposition 18) but f is not quasiconcave (as $f(0,0) = f(1,1) = 1$ and $f(1/2, 1/2) < 1$).*

Example 11 *Suppose C is a cone in $\mathbb{R}^2$ such that $\mathbb{R}^2_+ \subseteq C \subseteq \mathbb{R}^2$. Let $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4_+$ and let $f_1 : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $f_2 : \mathbb{R}^2_+ \to \mathbb{R}_+$ be defined by*

$$f_1(v) = \frac{v_1^\alpha}{(v_2+1)^\beta} \text{ and } f_2(v) = \frac{v_2^\gamma}{(v_1+1)^\delta}.$$

Let $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ be the function defined by $f = \min\{f_1, f_2\}$: so

$$f(v) = \min\left\{\frac{v_1^\alpha}{(v_2+1)^\beta}, \frac{v_2^\gamma}{(v_1+1)^\delta}\right\}.$$

Note that f_1 is increasing in the first argument and decreasing in the other and that f_2 is decreasing in the first argument and increasing in the other: then f is C-antichain-quasiconcave by virtue of Part 3 of Proposition 18.

Remark 6 *The function $f = \min\{f_1, f_2\}$ in Example 11 is not quasiconcave for some of its specifications: for instance the specification of*

$$f(v) = \min\{f_1, f_2\} \text{ with } f_1(v) = \frac{v_1^2}{v_2 + 1} \text{ and } f_2(v) = \frac{v_2^2}{v_1 + 1}$$

is not quasiconcave (such a function is positive on $\mathbb{R}^2_{++}$ and any of its upper level sets at positive height is not convex). Putting

$$F_1 = \{v \in \mathbb{R}^2_+ : v_2 \geq v_1\} \text{ and } F_2 = \{v \in \mathbb{R}^2_+ : v_1 \geq v_2\}$$

and noting that the previous specification can be equivalently defined by

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in F_1 \\ f_2(v) & \text{if } v \in F_2 \end{cases}$$

we can infer that f is strictly increasing along $(1, 1)$ in that so are f_1 and f_2 on, respectively, the sets F_1 and F_2.

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