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Abstract

Several agents choose positions on the real line (e.g., their levels of conspicuous consumption). Each agent's utility depends on her choice and her "status," which, in turn, is determined by the number of agents with greater choices (the fewer, the better). If the rules for the determination of the status are such that the set of the players is partitioned into just two tiers ("top" and "bottom"), then a strong Nash equilibrium exists, which Pareto dominates every other Nash equilibrium. Moreover, the Cournot tatonnement process started anywhere in the set of strategy profiles inevitably reaches a Nash equilibrium in a finite number of steps. If there are three tiers ("top," "middle," and "bottom"), then the existence of a Nash equilibrium is ensured under an additional assumption; however, there may be no Pareto efficient equilibrium. With more than three possible status levels, there seems to be no reasonably general sufficient conditions for Nash equilibrium existence.

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Key words: status game; strong equilibrium; Nash equilibrium; Cournot tatonnement.

1 Introduction

This paper investigates a class of strategic games with discontinuous utilities. From a purely technical viewpoint, those games are interesting because no general conditions for equilibrium existence, see, e.g., Reny (1999, 2016), McLennan et al. (2011), Prokopovych (2013), or Kukushkin (2018), are applicable to them, even though they display features such as aggregation, monotonicity, and even strict quasiconcavity.

From the viewpoint of economic theory, those games model how concerns for relative social status influence people's decisions. This topic has been present in the literature since, at least, Veblen (1899), and has attracted ever growing attention in recent decades (Frank, 1985a; Akerlof, 1997; Clark and Oswald, 1998; Becker et al., 2005; Arrow and Dasgupta, 2009). Bilancini and Boncinelli (2008) introduced an important distinction between ordinal and cardinal approaches: in the first case, the status of a player is determined by the *comparisons* with other players' choices; in the second case, by the *differences* between them.

The starting point for this paper is the model of Haagsma and von Mouche (2010), henceforth, an "HvM status game," which belongs to the ordinal strand of the literature. In contrast to, say, Frank

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(1985b) or Becker et al. (2005), an HvM status game has a finite number of players; each player's utility depends on her choice and her "status," which is just the order rank of her choice among all choices (simply put, the number of players with the same or lesser choices). The utility function strictly increases in status and is strictly quasiconcave in own choice. The discrete nature of status in such games generates unpleasant discontinuities of the utility functions. Nonetheless, the best responses exist and, typically, exhibit the "keeping up with the Joneses" effect in the literal sense: a player may choose above her intrinsically most preferred alternative in order to become equal with somebody else and obtain a higher status thereby. (This should be contrasted, e.g., with the Status Model of Akerlof (1997, p.1008), where the optimal choice of an agent does not depend on what the others are doing.) Examples show that Nash equilibria may be or not be Pareto efficient.

The weakest point of Haagsma and von Mouche (2010) is the absence of any general result on the existence of Nash equilibrium. The only exception is the two-person case, where it is geometrically obvious that the graphs of the best responses must intersect.

As often happens in mathematics, to make an advance, one has to modify the original posing of the problem. Kukushkin and von Mouche (2018) considered "binary status games," where the number of the players may be arbitrary, but there are only two status levels: a player belongs to the top tier if her choice is the maximal of all, and belongs to the bottom tier otherwise. Every such game possesses a Nash equilibrium; moreover, every best response improvement path, regardless of where it was started and in what order the players act, inevitably reaches a Nash equilibrium after a finite number of improvements.

Here, a step further in the same direction is made. In an "m-consolidated HvM status game," there may be fewer potential status levels (m) than players, so players with different, but close, order ranks may have the same status. For instance, the games of Kukushkin and von Mouche (2018) are 2-consolidated HvM status games. As another example of such a game, the top tier may consist of the players whose choices are greater than or equal to the median choice, while everybody else is relegated to the bottom tier.

The main findings of this paper are as follows. Every 2-consolidated HvM game possesses a strong Nash equilibrium that weakly Pareto dominates all other Nash equilibria (Theorem 1). Moreover, the main result of Kukushkin and von Mouche (2018), about the convergence of all Cournot paths, remains valid in this broader context even with an extension to simultaneous tâtonnement (Proposition 2).

Generally, a 3-consolidated HvM game need not possess an equilibrium. A Nash equilibrium existence result (Theorem 3) is proven under an additional assumption, which could be called "local single crossing." However, there may be no Pareto efficient Nash equilibrium, to say nothing of a strong equilibrium. With more than three possible status levels, there may be no Nash equilibrium at all even under that additional assumption. From the viewpoint of equilibrium existence results of high generality, *tres faciunt collegium* in this context.

Section 2 provides formal descriptions of both HvM status games and our "consolidated" HvM status games. Section 3 contains definitions and auxiliary, technical results concerning the Cournot tâtonnement processes. Our main results are in Section 4 (two possible status levels) and Section 5 (three possible status levels).

2 Status games

An *HvM status game* is a strategic game with a finite set of players N (we assume $n := \#N \geq 2$), where strategy sets and utility functions satisfy these requirements: (i) There is a closed subset $X \subseteq \mathbb{R}$ ("conceivable strategies") and there are $a_i \leq b_i$ in $\mathbb{R}$ for each $i \in N$ such that $X_i = [a_i, b_i] \cap X$; hence each X_i is compact. (ii) Each player's utility depends on her strategy $x_i \in X_i$ and her *status* $s_i \in S := \{1, \ldots, n\}$, which, in turn, is determined by the *rank mapping* $\rho_i \colon X_N \to S$,

$$\rho_i(x_N) := \#\{j \in N \mid x_j \leq x_i\}. \tag{1}$$

To be more precise, there is a function $U_i \colon X_i \times S \to \mathbb{R}$ such that $u_i(x_N) = U_i(x_i, \rho_i(x_N))$ for all $i \in N$ and $x_N \in X_N$. (iii) Each function $U_i(x_i, s)$ is strictly increasing in s, and upper semicontinuous in x_i; moreover, there are $\hat{x}_i^s \in X_i$ for all $i \in N$ and $s \in S$ such that $U_i(x_i, s)$ strictly increases when $x_i \leq \hat{x}_i^s$ and strictly decreases when $x_i \geq \hat{x}_i^s$.

Given an HvM status game with n players, a *consolidated HvM status game* is defined by a natural number $m \in [2, n]$ and m natural numbers β_s such that $1 = \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m \leq n$. The set of possible status levels is $S := \{1, \ldots, m\}$. The *consolidated status* of player i under strategy profile x_N is

$$\sigma_i(x_N) := \max\{s \in S \mid \rho_i(x_N) \geq \beta_s\}, \tag{2}$$

where ρ_i is defined by (1).

We use the term *m-consolidated HvM status game* when a particular m is essential. "Binary status games" of Kukushkin and von Mouche (2018) are 2-consolidated HvM status games with $\beta_2 = n$. Every HvM status game is simultaneously an n-consolidated HvM status game with $\beta_s = s$ for all $s \in S$; so the class of all consolidated HvM status games includes the original HvM status games. Note also that, given $m' > m$, every m-consolidated HvM status game is simultaneously an m'-consolidated HvM status game (with a sufficient number of equalities between β_s's).

The *best response correspondences* $\mathcal{R}_i \colon X_{-i} \to 2^{X_i}$ ($i \in N$) are defined in the usual way:

$$\mathcal{R}_i(x_{-i}) := \operatorname*{Argmax}_{x_i \in X_i} u_i(x_i, x_{-i}).$$

Since non-strict inequalities are used in (1) and (2), both ρ_i and σ_i are upper semicontinuous; the upper semicontinuity of u_i easily follows. Therefore, $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for every $x_{-i} \in X_{-i}$. A strategy profile $x_N^0 \in X_N$ is a (pure strategy) *Nash equilibrium* if $x_i^0 \in \mathcal{R}_i(x_{-i}^0)$ for all $i \in N$.

Given $x_N \in X_N$ and $I \subseteq N$ ($I \neq \emptyset$), $y_I \in X_I [:= \prod_{i \in I} X_i]$ is called a *strong coalitional improvement* at x_N if $u_i(y_I, x_{-I}) > u_i(x_N)$ for all $i \in I$; $y_I \in X_I$ is a *weak coalitional improvement* at x_N if $u_i(y_I, x_{-I}) \geq u_i(x_N)$ for all $i \in I$ while $u_i(y_I, x_{-I}) > u_i(x_N)$ for at least one $i \in I$. A strategy profile $x_N^0 \in X_N$ is a *strong equilibrium* if there is no strong coalitional improvement at x_N^0; $x_N^0 \in X_N$ is a *very strong equilibrium* if there is no weak coalitional improvement at x_N^0.

Lemma 2.1. *$x_N^0 \in X_N$ is a very strong equilibrium if and only if this condition holds: Whenever $u_i(y_N) > u_i(x_N^0)$ for some $i \in N$ and $y_N \in X_N$, there is $j \in N$ such that $y_j \neq x_j^0$ and $u_j(y_N) < u_j(x_N^0)$.*

A straightforward proof is omitted.

3 Cournot tâtonnement

There are two varieties of Cournot tâtonnement in the literature: consecutive and simultaneous ones. In some contexts, one of them behaves nicely; in some contexts, the other. In the models considered in Section 4, both work nicely; moreover, steps of both kinds can be combined arbitrarily and the convergence to a Nash equilibrium will still be ensured.

A *generalized Cournot path* in a strategic game is a finite or infinite sequence of strategy profiles $\langle x_N^k \rangle_{k=0,1,\dots}$ such that, whenever x_N^{k+1} is defined, each strategy x_i^{k+1} either coincides with x_i^k or is the best response to the strategies of others at x_N^k while x_i^k is not among the best responses. More technically, whenever x_N^{k+1} is defined, there is a nonempty subset $I(k) \subseteq N$ such that $x_{-I(k)}^k = x_{-I(k)}^{k+1}$ and $x_i^k \notin \mathcal{R}_i(x_{-i}^k) \ni x_i^{k+1}$ for each $i \in I(k)$. If $I(k)$ is a singleton at each step, we have a consecutive Cournot path; if $I(k) = \{i \in N \mid x_i^k \notin \mathcal{R}_i(x_{-i}^k)\}$ at each step, we have a simultaneous Cournot path. A *generalized Cournot cycle* is a generalized Cournot path $\langle x_N^k \rangle_{k=0,1,\dots,K}$ such that $K > 0$ and $x_N^K = x_N^0$.

As a modification of Milchtaich's (1996) definition of the FBRP, we say that a strategic game has the *finite generalized best response improvement property* (*FGBRP*) if it admits no infinite generalized Cournot path. Then every generalized Cournot path, if extended whenever possible, ends at a Nash equilibrium. The FGBRP implies the absence of generalized Cournot cycles; for *finite* games, the opposite implication also holds.

Remark. The FBRP is implied by the *finite improvement property* (*FIP*) of Monderer and Shapley (1996); the FGBRP is logically independent of the FIP. Generally, HvM status games do not have the FIP, even finite games with $n = 2$.

Lemma 3.1. *Let* Γ *be a consolidated HvM status game and* $x_N^0 \in X_N$*. Let* $Y_*(x_N^0) := \{\hat{x}_i^s\}_{i \in N, s \in S} \cup \{x_i^0\}_{i \in N} \subseteq X$*. Let* $\langle x_N^k \rangle_{k=0,1,\dots,K}$ *be a finite generalized Cournot path starting at* x_N^0*. Then* $x_N^K \in (Y_*(x_N^0))^N$*.*

Proof. We start with an auxiliary statement: If $x_N \in (Y_*(x_N^0))^N$, $i \in N$, and $y_i \in \mathcal{R}_i(x_{-i})$, then $y_i \in Y_*(x_N^0)$. We denote $s := \sigma_i(y_i, x_{-i})$. If $y_i = \hat{x}_i^s$, then we are home. If $y_i < \hat{x}_i^s$, then $u_i(\hat{x}_i^s, x_{-i}) > u_i(y_i, x_{-i})$, contradicting the optimality of y_i. Finally, if $y_i > \hat{x}_i^s$, then y_i can only be optimal if any decrease of y_i leads to a lower status of player i, but then $y_i = x_j$ for some $j \neq i$ and we are home again.

Now the statement of the lemma is proven with a straightforward recursion. □

Lemma 3.2. *A consolidated HvM status game has the FGBRP if and only if it admits no generalized Cournot cycle.*

Immediately follows from Lemma 3.1 since $Y_*(x_N^0)$ is finite for every $x_N^0 \in X_N$.

Given a consolidated HvM status game Γ, $x_N \in X_N$, and $s \in S$, we define

$$\xi_s(x_N) := \min\{x_i \mid \sigma_i(x_N) \geq s\},$$

the minimal strategy choice that ensures (under strategy profile x_N) the status s or higher.

Lemma 3.3. *Let* $\langle x_N^k \rangle_{k=0,1,\dots,K}$ *be a generalized Cournot cycle in an* m*-consolidated HvM status game. Then* $\xi_m(x_N^k)$ *does not depend on* k*.*

Proof. Supposing the contrary, we may, without restricting generality, assume that $\xi_m(x_N^0) < \xi_m(x_N^1) = \max_k \xi_m(x_N^k)$. Then we must have $i \in I(0)$ for which $x_i^0 < \xi_m(x_N^0) < \xi_m(x_N^1) \leq x_i^1$; hence $\hat{x}_i^m > \xi_m(x_N^0)$, the status of player i at x_N^1 is m, and $x_i^1 = \hat{x}_i^m$. Thus, the utility of player i attains its global maximum at $x_i^1 \geq \xi_m(x_N^1)$; since $\xi_m(x_N^1) = \max_k \xi_m(x_N^k)$, the status of player i remains the highest, m, at all x_N^k, and hence her utility remains maximal all along. Therefore, x_i^1 could not be replaced with $x_i^K = x_i^0$ at any stage. That contradiction proves the lemma. □

4 Two consolidated status levels

To avoid confusion between superscripts related to status levels and to steps in Cournot paths, we redefine $S := \{\mathfrak{b}, \mathfrak{t}\}$ with $\mathfrak{b} < \mathfrak{t}$ (bottom and top) in this section.

Theorem 1. *Every 2-consolidated HvM status game Γ possesses a very strong equilibrium $x_N^0 \in X_N$ which weakly Pareto dominates every Nash equilibrium y_N of Γ except when $u_N(y_N) = u_N(x_N^0)$.*

Proof. We define $\hat{x}_N \in X_N$ by $\hat{x}_i := \hat{x}_i^{\mathfrak{t}}$ for each $i \in N$, and then define $\bar{x} := \xi_{\mathfrak{t}}(\hat{x}_N)$, $N_0 := \{i \in N \mid \hat{x}_i^{\mathfrak{t}} \geq \bar{x}\} = \{i \in N \mid \sigma_i(\hat{x}_N) = \mathfrak{t}\}$, $N_1 := \{i \in N \setminus N_0 \mid U_i(\bar{x}, \mathfrak{t}) > U_i(\hat{x}_i^{\mathfrak{b}}, \mathfrak{b})\}$, and $N_2 := N \setminus (N_0 \cup N_1)$.

Now we define our strategy profile $x_N^0 \in X_N$:

$$x_i^0 := \begin{cases} \hat{x}_i^{\mathfrak{t}}, & i \in N_0; \\ \bar{x}, & i \in N_1; \\ \hat{x}_i^{\mathfrak{b}}, & i \in N_2. \end{cases}$$

Let $y_N \in X_N$, $i \in N$, and $u_i(y_N) > u_i(x_N^0)$. First, we note that $u_i(x_N^0) = \max_{x_N \in X_N} u_i(x_N)$ for $i \in N_0$; hence $i \notin N_0$. Second, $u_i(x_N^0) \geq \max_{x_i \in X_i} U_i(x_i, \mathfrak{b})$ for all $i \in N$ (strict inequality if $i \in N_0 \cup N_1$ and an equality if $i \in N_2$); therefore, $\sigma_i(y_N) = \mathfrak{t}$. Thirdly, $U_i(x_i, \mathfrak{t}) \leq U_i(\bar{x}, \mathfrak{t}) \leq u_i(x_N^0)$ whenever $i \in N \setminus N_0$ and $x_i \geq \bar{x}$; hence $\xi_{\mathfrak{t}}(y_N) < \bar{x}$. The latter inequality is only possible if there is $j \in N_0$ for which

$$y_j < \bar{x} \leq \hat{x}_j^{\mathfrak{t}} = x_j^0. \tag{3}$$

Thus, we must have $u_j(y_N) < u_j(x_N^0)$ and $y_j \neq x_j^0$; hence, x_N^0 is a very strong equilibrium by Lemma 2.1. Finally, (3) for $j \in N_0$ implies that y_N is not a Nash equilibrium since player j can improve by replacing y_j with $\hat{x}_j^{\mathfrak{t}}$. □

In addition to Theorem 1, we prove a generalization of the main result of Kukushkin and von Mouche (2018); but first an auxiliary statement.

Lemma 4.1. *Let Γ be a 2-consolidated HvM status game, $x_N \in X_N$, $i \in N$, and $\bar{x} := \xi_{\mathfrak{t}}(x_N)$. Then $x_i \in \mathcal{R}_i(x_{-i})$ if and only if one of the following conditions holds:*

$$x_i = \hat{x}_i^{\mathfrak{t}} > \bar{x};$$

$$x_i = \bar{x} \geq \hat{x}_i^{\mathfrak{t}} \ \&\ U_i(\hat{x}_i^{\mathfrak{b}}, \mathfrak{b}) \leq U_i(\bar{x}, \mathfrak{t});$$

$$x_i = \hat{x}_i^{\mathfrak{b}} < \bar{x} \ \&\ U_i(\hat{x}_i^{\mathfrak{b}}, \mathfrak{b}) \geq U_i(\bar{x}, \mathfrak{t}).$$

The proof is essentially the same as in the case of two-person HvM status games (Haagsma and von Mouche, 2010, Section 3.1) and is omitted.

Proposition 2. *Every 2-consolidated HvM status game has the FGBRP.*

Proof. In light of Lemma 3.2, it is enough to show the impossibility of generalized Cournot cycles. Supposing, to the contrary, $\langle x_N^k \rangle_{k=0,1,\ldots,K}$ to be a generalized Cournot cycle, we denote $\bar{x} := \xi_{\mathfrak{t}}(x_N^k)$; by Lemma 3.3, it does not depend on k. By definition, we have $x_i^0 \neq x_i^1 \in \mathcal{R}_i(x_{-i}^0)$ for each $i \in I(0)$. Since the necessary and sufficient conditions in Lemma 4.1 only refer to x_i and $\bar{x}$, we see that $x_i^1 \in \mathcal{R}_i(x_{-i}^k)$ for all $k \geq 1$, and hence $x_i^k = x_i^1$ for all k with the same contradiction as in the proof of Lemma 3.3. $\square$

Even a two-person HvM status game may possess a continuum of inefficient Nash equilibria (Haagsma and von Mouche, 2010, Figure 4), so one cannot expect that an arbitrary generalized Cournot path will lead to a very strong equilibrium.

Remark. In contrast to Proposition 2, Rauscher (1992) showed the possibility of chaotic simultaneous Cournot dynamics in a two-player status game; however, the status there was modeled as cardinal.

5 Three consolidated status levels

In this section, we make a morally dubious, but helpful notational convention: In Examples 5.1, 5.2, 5.3, and 5.4, the status levels are literally defined by (2), i.e., $S = \{1, 2, 3\}$ or $S = \{1, 2, 3, 4\}$ as the case may be. In the formulation and proof of Theorem 3, however, we, similar to Section 4, assume $S := \{\mathfrak{b}, \mathfrak{m}, \mathfrak{t}\}$ with $\mathfrak{b} < \mathfrak{m} < \mathfrak{t}$ (bottom, middle, top).

To start with, a Nash equilibrium may fail to exist when there are more than two possible status levels, even just in a three-person HvM status game.

Example 5.1. Let us consider an HvM status game Γ where $N := \{1, 2, 3\}$, $X := X_i := [0, 4] \subset \mathbb{R}$ (for all $i \in N$), and utility functions are:

$$U_1(x, s) := s + 2x + 1;$$

$$U_2(x, s) := \begin{cases} \min\{4s + 2x - 1, 4s - 3x + 9\}, & s = 3 \text{ or } s = 2; \\ \min\{3 + 2x, 11 - 2x\}, & s = 1; \end{cases}$$

$$U_3(x, s) := \begin{cases} \min\{s + x + 9, s - 10x + 42\}, & s = 3 \text{ or } s = 2; \\ \min\{10 + x, 14 - 3x\}, & s = 1. \end{cases}$$

It is easily seen that $\hat{x}_1^s = 4$ and $\hat{x}_2^s = 2$ for all s, while $\hat{x}_3^3 = \hat{x}_3^2 = 3$ and $\hat{x}_3^1 = 1$. An important observation is that $U_2(2, 2) = 11 > 9 = U_2(4, 3) > 7 = U_2(2, 1)$ and $U_3(3, 2) = 14 > 11 = U_3(1, 1) > 5 = U_3(4, 3)$.

Assuming x_N^0 to be a Nash equilibrium, we immediately see that $x_1^0 = 4$. Now we consider three alternatives, computing the best response of one player (2 or 3) to the assumed strategy of the other.

First, let $x_2^0 \leq 3$. Then $x_3^0 = 3$; hence $x_2^0 = 4$: a contradiction. Now let $3 < x_2^0 < 4$. Then either $x_3^0 = x_2^0$ or $x_3^0 = 1$; hence, either $x_2^0 = 4$ or $x_2^0 = 2$: a contradiction again. Finally, let $x_2^0 = 4$. Then $x_3^0 = 1$; hence $x_2^0 = 2$: a final contradiction. Thus, there is no Nash equilibrium in the game.

The existence of a Nash equilibrium in a 3-consolidated HvM status game is ensured under an additional assumption: $\hat{x}_i^{\mathfrak{b}} \geq \hat{x}_i^{\mathfrak{m}}$ for all $i \in N$. The assumption can be reformulated in a way resembling the single crossing condition of Milgrom and Shannon (1994): Let $x_N, y_N \in X_N$ and $i \in N$ be such that $y_i > x_i$, $\sigma_i(x_N) = \mathfrak{m} = \sigma_i(y_i, x_{-i})$, $y_{-i} \geq x_{-i}$, and $u_i(y_i, x_{-i}) > u_i(x_N)$; then $u_i(y_N) > u_i(x_i, y_{-i})$.

Theorem 3. *If a 3-consolidated HvM status game has the property that $\hat{x}_i^{\mathfrak{b}} \geq \hat{x}_i^{\mathfrak{m}}$ for each $i \in N$, then the game possesses a Nash equilibrium.*

Proof. We start with some auxiliary notations. Similarly to the proof of Lemma 3.1, we denote $Y_* := \{\hat{x}_i^s\}_{i \in N, s \in S} \subseteq X$. Arguing similarly to Lemma 4.1, we see that $\mathcal{R}_i(x_{-i})$ is finite for all $i \in N$ and $x_{-i} \in X_{-i}$; we define $r_i(x_{-i}) := \min \mathcal{R}_i(x_{-i})$ and $r_N \colon X_N \to X_N$ by $r_N(x_N) := \langle r_i(x_{-i}) \rangle_{i \in N}$. Every fixed point of r_N is a Nash equilibrium.

Similarly to the proof of Theorem 1, we define $\hat{x}_N \in X_N$ by $\hat{x}_i := \hat{x}_i^{\mathfrak{t}}$ for each $i \in N$, and then define $\bar{x} := \xi_{\mathfrak{t}}(\hat{x}_N)$, $N_0 := \{i \in N \mid \hat{x}_i^{\mathfrak{t}} \geq \bar{x}\} = \{i \in N \mid \sigma_i(\hat{x}_N) = \mathfrak{t}\}$, $N_1 := \{i \in N \setminus N_0 \mid U_i(\bar{x}, \mathfrak{t}) > U_i(\hat{x}_i^{\mathfrak{m}}, \mathfrak{m})\}$, $N_2 := N \setminus (N_0 \cup N_1)$, and

$$\hat{X} := \{x_N \in (Y_*)^N \mid \forall i \in N_0\, [x_i = \hat{x}_i^{\mathfrak{t}}]\ \&\ \forall i \in N_1\, [x_i = \bar{x}]\ \&\ \forall i \in N_2\, [x_i \leq \bar{x}]\}.$$

Claim 3.1. *Let $x_N \in \hat{X}$ and $y_N = r_N(x_N)$; then $y_i = x_i$ for each $i \in N_0 \cup N_1$, while $y_i \leq \bar{x}$ for each $i \in N_2$. In other words, $y_N \in \hat{X}$.*

Proof of Claim 3.1. The utility of each player from N_0 attains its global maximum at $\hat{x}_i^{\mathfrak{t}}$. For each $i \in N \setminus N_0$, we have $\hat{x}_i^{\mathfrak{t}} < \bar{x}$ and hence $y_i = \bar{x}$ is strictly preferable to any $y_i > \bar{x}$; for $i \in N_1$, it is actually the best option. □

Now we define a strategy profile $y_N^0 \in \hat{X}$:

$$y_i^0 := \begin{cases} \hat{x}_i^{\mathfrak{t}}, & i \in N_0; \\ \bar{x}, & i \in N_1; \\ \hat{x}_i^{\mathfrak{m}}, & i \in N_2. \end{cases}$$

For each $k = 1, 2, \ldots$, we recursively define $y_N^{k+1} := r_N(y_N^k)$; by Claim 3.1, we have $y_N^k \in \hat{X}$ for all k. If $y_N^{k+1} = y_N^k$ at some stage, then y_N^k is a Nash equilibrium and we are home. In particular, y_N^0 is a Nash equilibrium if $N_2 = \emptyset$.

Thus, let $N_2 \neq \emptyset$. We define $\bar{y}^k := \xi_{\mathfrak{m}}(y_N^k)$. The main point of the rest of the proof is that $\bar{y}^k$ increases in k. Since Y_* is finite, the sequence must stabilize at some stage, and then the equality $y_N^{k+1} = y_N^k$ will be achieved.

To have a base for induction, let us compare y_N^0 and y_N^1. If $y_i^0 \geq \bar{x}$, then $i \in N_0 \cup N_1$ and hence $y_i^1 = y_i^0$. If $\bar{y}^0 \leq y_i^0 < \bar{x}$, then $y_i^1 = y_i^0$ again since player i has the middle status $\mathfrak{m}$, makes the best choice for this status, and is not interested in obtaining the top status because $i \in N_2$. Thus, if $y_i^0 \geq \bar{y}^0$ for all $i \in N$, then $y_N^1 = y_N^0$ and we are home. Actually, in this case, y_N^0 is a very strong equilibrium for the same reason as in the proof of Theorem 1. If $y_i^0 < \bar{y}^0$, then three choices may be optimal for player i: $y_i^1 = \hat{y}_N^{\mathfrak{b}}$, $y_i^1 = \bar{y}^0$, or $y_i^1 = \bar{x}$; note that $y_i^1 \geq y_i^0$ in each case, which immediately implies $\bar{y}^1 \geq \bar{y}^0$. Moreover, if $\bar{y}^1 = \bar{y}^0$, then $y_N^2 = y_N^1$ since $y_i^1 = \hat{y}_i^{\mathfrak{b}}$ whenever $y_i^1 < \bar{y}^0 = \bar{y}^1$ and the choice of $y_i^2 = \bar{y}^0$ or $y_i^2 = \bar{x}$ has already been ruled out.

Now, we proceed with the induction step.

Claim 3.2. *Let $k \geq 1$ and $\bar{y}^{h+1} > \bar{y}^h$ for each $h < k$. Then these statements hold for each $i \in N$:*

$$y_i^k > \bar{x} \Rightarrow y_i^k = \hat{x}_i^{\mathfrak{t}}; \tag{4a}$$

$$[y_i^k = \bar{x} \;\&\; i \in N_2] \Rightarrow [\hat{x}_i^{\mathfrak{m}} < \bar{y}^{k-1} \;\&\; U_i(\bar{x}, \mathfrak{t}) > \max\{U_i(\bar{y}^{k-1}, \mathfrak{m}), U_i(\hat{x}_i^{\mathfrak{b}}, \mathfrak{b})\}]; \tag{4b}$$

$$\bar{y}^{k-1} < y_i^k < \bar{x} \Rightarrow y_i^k = \hat{x}_i^{\mathfrak{m}}; \tag{4c}$$

$$y_i^k = \bar{y}^{k-1} \Rightarrow [\hat{x}_i^{\mathfrak{m}} \leq \bar{y}^{k-1} \;\&\; U_i(\bar{x}, \mathfrak{t}) \leq U_i(\bar{y}^{k-1}, \mathfrak{m}) > U_i(\hat{x}_i^{\mathfrak{b}}, \mathfrak{b})]; \tag{4d}$$

$$y_i^k < \bar{y}^{k-1} \Rightarrow [y_i^k = \hat{x}_i^{\mathfrak{b}} \;\&\; U_i(\hat{x}_i^{\mathfrak{b}}, \mathfrak{b}) \geq \max\{U_i(\bar{x}, \mathfrak{t}), U_i(\bar{y}^{k-1}, \mathfrak{m})\}]; \tag{4e}$$

$$\bar{y}^{k+1} \geq \bar{y}^k. \tag{5}$$

Proof of Claim 3.2. Assuming all conditions (4) to hold for k, we have to show that they still hold for $k+1$. As a first step, we check these statements:

$$y_i^k \geq \bar{y}^k \Rightarrow y_i^{k+1} = y_i^k; \tag{6a}$$

$$\bar{y}^{k-1} \leq y_i^k < \bar{y}^k \Rightarrow y_i^{k+1} \in \{\hat{x}_i^{\mathfrak{b}}, \bar{y}^k, \bar{x}\}; \tag{6b}$$

$$y_i^k < \bar{y}^{k-1} \Rightarrow y_i^{k+1} = y_i^k. \tag{6c}$$

(6c) follows from (4e): since $\bar{y}^k > \bar{y}^{k-1} > y_i^k = \hat{x}_i^{\mathfrak{b}} \geq \hat{x}_i^{\mathfrak{m}}$, we have $U_i(\bar{y}^{k-1}, \mathfrak{m}) > U_i(\bar{y}^k, \mathfrak{m})$ and hence $U_i(\hat{x}_i^{\mathfrak{b}}, \mathfrak{b}) > \max\{U_i(\bar{x}, \mathfrak{t}), U_i(\bar{y}^k, \mathfrak{m})\}$. (6b) is straightforward; (6a) for $i \in N_0 \cup N_1$ immediately follows from Claim 3.1. We only have to check (6a) for $i \in N_2$. If $\bar{y}^{k-1} < \bar{y}^k \leq y_i^k < \bar{x}$, then invoking (4c) is enough. When $y_i^k = \bar{x}$, a more sophisticated argument is needed.

Let us develop it. First, $\bar{y}^{k-1} = y_j^{k-1} = \hat{x}_j^{\mathfrak{m}}$ for some $j \in N_2$. When computing $y_j^k = r_j(y_{-j}^{k-1})$, we have to compare $U_j(\bar{x}, \mathfrak{t})$ and $U_j(\hat{x}_j^{\mathfrak{m}}, \mathfrak{m})$; since $j \in N_2$, the latter wins. Therefore, $\bar{y}^{k-1} = y_j^k = \hat{x}_j^{\mathfrak{m}} < \bar{y}^k$. By the definition of $\bar{y}^k$, we have $\#\{\nu \in N \mid y_\nu^k \leq \bar{y}^{k-1}\} < \beta_{\mathfrak{m}}$; for every $y < \bar{y}^{k-1}$, we have $\#\{\nu \in N \mid y_\nu^k \leq y\} < \beta_{\mathfrak{m}} - 1$. Therefore, $y_i \geq \bar{y}^{k-1}$ whenever $\sigma_i(y_i, y_{-i}^k) = \mathfrak{m}$, and hence $U_i(y_i, \mathfrak{m}) \leq U_i(\bar{y}^{k-1}, \mathfrak{m})$. Now $y_i^{k+1} = r_j(y_{-i}^k) = \bar{x} = y_i^k$ by (4b).

Now (5) is straightforward: by (6), $y_i^{k+1} \geq \bar{y}^k$ whenever $y_i^k \geq \bar{y}^k$. It is as easily checked that (6) imply all conditions (4) for $k+1$. □

Claim 3.3. *If $\bar{y}^{k+1} = \bar{y}^k$, then $y_N^{k+2} = y_N^{k+1}$.*

Immediately follows from (6) since (6b) becomes inapplicable.

Thus, we showed that the sequence of strategy profiles $y_N^0, y_N^1, \ldots, y_N^k$ must stabilize at some stage, and such a stable point must be an equilibrium. The proof of Theorem 3 is finished. □

The equilibrium built in the proof of Theorem 3 need not be strong; actually there may be no Pareto efficient Nash equilibrium, to say nothing of a strong equilibrium, in the three-tier case. There may also be no Nash equilibrium Pareto dominating all other equilibria.

Example 5.2. Let us consider an HvM status game Γ where $N := \{1, 2, 3\}$, $X := X_i := [0, 3] \subset \mathbb{R}$ (for all $i \in N$), and utility functions are:

$$U_1(x, s) := \min\{3s + x + 1, 3s - 2x + 4\};$$

$$U_2(x, s) := \min\{3s + x + 1, 3s - 5x + 13\};$$

$$U_3(x, s) := s + x + 2;$$

It is easily seen that $\hat{x}_i^s = i$ for all $i \in N$ and $s \in S$. An important observation is that $U_1(1, 2) = 8 > 7 = U_1(3, 3) > 6 = U_2(2, 2) > 5 = U_2(1, 1)$ and $U_2(1, 2) = 8 > 7 = U_2(3, 3) > 6 = U_2(2, 1)$. Whenever x_N^0 is a Nash equilibrium, we must have $x_3^0 = 3$. Fixing this choice by player 3, we easily find the best responses of players 1 and 2 to the strategies of each other: $\mathcal{R}_1(x_2 = 1) = \{1\}$, $\mathcal{R}_1(x_2 = 2) = \mathcal{R}_1(x_2 = 3) = \{3\}$; $\mathcal{R}_2(x_1 = 1) = \mathcal{R}_2(x_1 = 2) = \{2\}$, $\mathcal{R}_2(x_1 = 3) = \{3\}$. Thus, $x_1^0 = x_2^0 = x_3^0 = 3$ is the unique Nash equilibrium of the game; the utility vector is $\langle 7, 7, 8\rangle$. Defining $y_1 := y_2 := 1$ and $y_3 := 3$, we see that the utility vector at y_N is $\langle 8, 8, 8\rangle$; in other words, (y_1, y_2) is a strong coalitional improvement at x_N^0, so x_N^0 is not a strong equilibrium, and not even a Pareto efficient Nash equilibrium.

Example 5.3. Let us consider a 3-consolidated HvM status game Γ where $N := \{1, 2, 3, 4\}$, $X := X_i := [0, 4] \subset \mathbb{R}$ (for all $i \in N$), $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 4$, and utility functions are:

$$U_1(x, s) := \min\{3s + x + 2, 3s - 2x + 5\};$$

$$U_2(x, s) := \min\{2s + x + 4, 2s - 3x + 12\};$$

$$U_3(x, s) := U_4(x, s) := \min\{2s + x + 3, 2s - x + 9\}.$$

It is easily seen that, for each $s \in S$, $\hat{x}_i^s = i$ for $i < 4$ while $\hat{x}_4^s = 3$.

The proof of Theorem 3 constructs the equilibrium x_N^0 where $x_i^0 = 3$ for all $i \in N$; the utility vector is $\langle 8, 9, 12, 12\rangle$. Meanwhile, for every $w \in [3, 4]$, there is a Nash equilibrium $y_N(w)$ with $y_3(w) = y_4(w) = w$:

$$y_1(w) = \begin{cases} 2, & w \geq 3.5; \\ w, & w \leq 3.5; \end{cases} \quad y_2(w) = \begin{cases} 2, & w \geq 3\frac{1}{3}; \\ w, & w \leq 3\frac{1}{3} \end{cases}$$

(when $w = 3.5$ or $w = 3\frac{1}{3}$, both variants are possible). The utility vector at $y_N(w)$ is $\langle 14 - 2w, 18 - 3w, 15 - w, 15 - w\rangle$ if $w \leq 3 + 1/3$, $\langle 14 - 2w, 8, 15 - w, 15 - w\rangle$ if $3 + 1/3 \leq w \leq 3.5$, and $\langle 10, 7, 15 - w, 15 - w\rangle$ if $w \geq 3.5$. We see that x_N^0 and $y_N(w)$ for $w \geq 3.5$ are Pareto incomparable.

Remark. A conjecture that the assumptions of Theorem 3 imply the FGBRP of the game does not seem implausible, but there is no proof in sight. It is not difficult to show that the sequence $\langle y_N^k\rangle_k$ in the proof is a generalized Cournot path, i.e., if $y_i^k \neq r_i(y_{-i}^k)$, then $y_i^k \notin \mathcal{R}_i(y_{-i}^k)$; however, this would not be enough even for the weak version of the property because the specific choice of the initial strategy profile y_N^0 was important.

When there are more than three possible status levels, e.g., in a four-person HvM status game, a Nash equilibrium may fail to exist even if the most favorite choice of each player does not depend on her status.

Example 5.4. Let us consider an HvM status game Γ where $N := \{1, 2, 3, 4\}$, $X := X_i := [0, 4] \subset \mathbb{R}$ (for all $i \in N$), and utility functions are:

$$U_1(x, s) := \begin{cases} \min\{7 + s + x, 9 + s - x, 18 + s - 4x\}, & s \in \{3, 4\}; \\ \min\{5 + s + x, 8 + s - 2x\}, & s \in \{1, 2\}; \end{cases}$$

$$U_2(x, s) := \begin{cases} \min\{10 + x, 16 - 2x\}, & s = 4; \\ \min\{8 + x, 16 - 3x\}, & s = 3; \\ \min\{7 + x, 15 - 3x\}, & s = 2; \\ \min\{5 + x, 13 - 3x\}, & s = 1; \end{cases}$$

$$U_3(x, s) := \begin{cases} \min\{9 + x, 18 - 2x\}, & s = 4; \\ \min\{8 + x, 17 - 2x\}, & s = 3; \\ \min\{6 + x, 15 - 2x\}, & s = 2; \\ \min\{5 + x, 14 - 2x\}, & s = 1; \end{cases}$$

$$U_4(x, s) := 2s + 2x.$$

It is easily seen that $\hat{x}_i^s = i$ for all s.

Is is important to note that: $U_1(3, 3) = 9 > 7 = U_1(1, 1) > U_1(2, 2) = 6 = U_1(4, 4) > U_1(3, 2) = 4$; $U_2(2, 2) = 9 > 8 = U_2(4, 4) > 7 = U_2(2, 1) = U_2(3, 3)$; $U_3(3, 3) = 11 > 10 = U_3(4, 4) > 9 = U_3(3, 2)$.

The analysis is somewhat similar to that of Example 5.1. Assuming x_N^0 to be a Nash equilibrium, we immediately see that $x_4^0 = 4$. Now we consider three alternatives, computing the best responses of some players to the assumed strategies of others.

First, let $x_1^0 \leq 3 \geq x_2^0$. Then $x_3^0 = 3$; hence $x_1^0 = 3$; hence $x_2^0 = 4$. Thus, we have a contradiction. Now let $x_1^0 > 3$. Then either $x_3^0 = \max\{x_1^0, x_2^0\} > 3$ or $x_3^0 = 4$. In either case, $x_2^0 = 4$; hence $x_1^0 = 1$: a contradiction again. Finally, let $x_2^0 > 3$. Again, either $x_3^0 = \max\{x_1^0, x_2^0\} > 3$ or $x_3^0 = 4$. In either case, $x_1^0 = 1$; hence $x_2^0 = 2$: a final contradiction. Thus, there is no Nash equilibrium in the game.

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