Two classes of weighted values for coalition structures with extensions to level structures

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Abstract

In this paper we introduce two new classes of weighted values for coalition structures with related extensions to level structures. The values of both classes coincide on given player sets with Harsanyi payoffs and match therefore adapted standard axioms for TU-values which are satisfied by these values.

Characterizing elements of the values from the new classes are a new weighted proportionality within components property and a null player out property, but on different reduced games for each class. The values from the first class, we call them weighted Shapley alliance coalition structure values (weighted Shapley alliance levels values), satisfy the null player out property on usual reduced games. By contrast, the values from the second class, named as weighted Shapley collaboration coalition structure values (weighted Shapley collaboration levels values) have this property on new reduced games where a component decomposes in components of lower levels (these are singletons in a coalition structure) if one player of this component is removed from the game. The first class contains the Owen value (Shapley levels value) and the second class includes a new extension of the Shapley value to coalition structures (level structures) as a special case.

Keywords Cooperative game · Weighted Shapley coalition structure values · Weighted Shapley levels values · Weighted proportionality within components · Dividends

1 Introduction

Whereas e. g. companies, governments or political organizations are mostly structured strong hierarchical and in a statical manner supply chains or electricity and other networks have often a more dynamical and not so strong top down frame work. For hierarchical organized structures Winter (1989) developed a model, called level structure, that is an extension of a coalition structure (Aumann and Drèze, 1974). Therefore Winter (1989) introduced his value, we name it Shapley levels value, that is an extension of the Owen value (Owen, 1977), itselves an extension of the Shapley value (Shapley, 1953b). A level structure consists of ordered partitions (the levels) of the original player set. Each partition consists of disjoint coalitions of players (the components) such that each component

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of a level higher that contains a player from a component of the lower level must contain all players of this component. So in a level structure the lowest level contains all the singletons as components and the highest level contains the grand coalition as the only component.

Vidal-Puga (2012) introduced an interesting weighted value for coalition structures, extended by Gómez-Rúa and Vidal-Puga (2011) to level structures[1]. There the weights are given by the size of the coalitions. This value does not satisfy the null player property.

The size of coalitions may be a considerable magnitude that is to take into account by sharing the payoff in a cooperative game. But often also other factors, e. g. in the context of cost allocation, play a decisive role: the size of firms or departments, political influence, fixed costs of the units, spent time of working members and so on. Levy and McLean (1989) and McLean (1991) extended the weighted Shapley values (Shapley, 1953a) to coalition structures for such arbitrary weights which don't depend on the coalition function. The main class of these values, in Dragan (1992) called McLean weighted coalition structure values, is extended to level structures in Besner (2018), called weighted Shapley support levels values. There the hierarchy of the level structure is treated more statically. If a subcoalition of a component deals with other coalitions outside of this component the subcoalition is supported by the weight of the whole component. In general, without further statement, it is not clear how to go on if a player is removed from the player set meaning what are the weights of the new components in the new game. In this respect, the Shapley levels value forms an exception within this class of values. It can be easily be treated also dynamically since in a game where some players are removed its weights are the same as before, they are always equal.

To handle also weighted values for level structures in the same manner we introduce the class of weighted Shapley alliance levels values (weighted Shapley alliance coalition structure values). There the weights are not only assigned to components, each subset of a component owns a weight too. If a player is removed from a component the remaining players build a new component with the weight of the coalition of these players. Thus here the coalitions within a component can act more independently in the corset of a level structure. They can act with players outside of the component without support of the whole component but all involved players of a component form always an alliance. The weighted Shapley alliance levels values coincide with payoff vectors from the Harsanyi set (Hammer, 1977; Vasil'ev, 1978), called Harsanyi payoffs, and inherit, for fixed player sets, all properties of these payoff vectors adapted to level structures. Interestingly, the Shapley levels value (Owen value) is a special case of these values too. So our axiomatization gives also deeper insight into this value. A characterizing element of the weighted Shapley alliance levels values is a null player out property. If we delete a player of a component this doesn't influence the payoff to the other players.

In some situations it is possible that if a player leaves a component the component loses its cohesion. For such situations the values from our second new class, called weighted Shapley collaboration levels values (weighted Shapley collaboration coalition structure values) are recommended. Especially if a null player is removed from a component and the coalition of the remaining players of this component is smashed in the next smaller components. Here the payoff to all players is the same as before in the game with the complete player set. If some players of a component B are involved in a bargaining situa-

[1]In this context it is also worth mentioning the paper of Gómez-Rúa and Vidal-Puga (2010).

tion with players from outside of B all players which form the next largest subcomponent of B collaborate together. So the players of a component can act more independently too. Here, similar as by the weighted Shapley support levels values, only the components need a weight. As a special case of these values we single out a new extension of the Shapley value to level structures, named Shapley collaboration levels value (Shapley collaboration caolition structure value).

Whereas the Shapley levels value and the value in Gómez-Rúa and Vidal-Puga (2011) satisfy the level game property (the payoff to all players of a component sum up to the payoff to the component in a game where components are the players) the values from our new classes don't satisfy this property in general. The absence of this characteristic reveals that in our new classes the players are more independent from a nesting component.

A level structure with only two levels coincides with a coalition structure. Thus all presented axioms and so the given axiomatizations coincide with related axioms and axiomatizations formulated for coalition structures. Therefore we deal in this paper with coalition structures only marginally and concentrate on level structures so that in this respect the reader has to transform adequate results by his own.

The remainder of this paper is structured as follows. Section 2 provides some preliminaries and section 3 presents the axioms. In the main part of this paper we introduce in section 4 the weighted Shapley alliance levels values, in section 5 the weighted Shapley collaboration levels values and, as a special case, in section 6 the Shapley collaboration levels value. An example in section 7 compares the proposed values, section 8 gives the conclusion and an appendix (section 9) provides all the proofs.

2 Preliminaries

We denote by $\mathbb{R}$ the real numbers and by $\mathbb{R}_{++}$ the set of all positive real numbers. Let $\mathfrak{U}$ be a countably infinite set, the universe of all players, and $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N is the power set of N. Subsets $S \subseteq N$ are called **coalitions**, $v(S)$ is the **worth** of coalition S and the set of all nonempty subsets of S is denoted by Ω^S. The set of all TU-games with player set N is denoted by $\mathbb{V}^N$. The **restriction** of (N, v) to the player set $S \in \Omega^N$ is denoted by (S, v).

Let $(N, v) \in \mathbb{V}^N$ and $S \subseteq N$. The **dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), \text{ if } S \in \Omega^N, \text{ and} \\ 0, \text{ if } S = \emptyset. \end{cases}$$

A game (N, u_T), $T \in \Omega^N$, with $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise for all $S \subseteq N$ is called an **unanimity game**. It is well-known that any coalition function v on N has a unique presentation

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \tag{1}$$

We call a coalition $S \subseteq N$ **active** in v if $\Delta_v(S) \neq 0$. Player $i \in N$ is called a **null player** in v if $v(S \cup \{i\}) = v(S)$, $S \subseteq N \backslash \{i\}$; players $i, j \in N$, $i \neq j$, are called (mutually)

dependent (Nowak and Radzik, 1995) in v if $v(S \cup \{i\}) = v(S) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$.

A **coalition structure** $\mathcal{B}$ on N is a partition of the player set N, i.e. a collection of nonempty, pairwise disjoint, and mutually exhaustive subsets of N. Each $B \in \mathcal{B}$ is called a **component** and $\mathcal{B}(i)$ denotes the component that contains a player $i \in N$. A **level structure** (Winter, 1989) on N is a finite sequence $\underline{\mathcal{B}} := \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ of coalition structures $\mathcal{B}^r$, $0 \leq r \leq h+1$, on N such that:

- $\mathcal{B}^0 = \big\{\{i\} \colon i \in N\big\}$.
- $\mathcal{B}^{h+1} = \{N\}$.
- For each r, $0 \leq r \leq h$, $\mathcal{B}^r$ is a refinement of $\mathcal{B}^{r+1}$, i.e. $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$ for all $i \in N$.

$\mathcal{B}^r$ is called the r-th **level** of $\underline{\mathcal{B}}$; $\overline{\mathcal{B}}$ is the set of all components $B \in \mathcal{B}^r$ of all levels $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$; $\mathcal{B}^r(B^k)$ is the component of the r-th level which contains the component $B^k \in \mathcal{B}^k$, $0 \leq k \leq r \leq h+1$.

The collection of all level structures with player set N is denoted by $\mathbb{L}^N$. A TU-game $(N,v) \in \mathbb{V}^N$ together with a level structure $\underline{\mathcal{B}} \in \mathbb{L}^N$ is an **LS-game** $(N,v,\underline{\mathcal{B}})$. The set of all LS-games on N is defined by $\mathbb{VL}^N$. Each TU-game (N,v) corresponds to an LS-game $(N,v,\underline{\mathcal{B}_0})$ with a **trivial level structure** $\underline{\mathcal{B}_0} := \{\mathcal{B}^0, \mathcal{B}^1\}$ and each LS-game $(N,v,\underline{\mathcal{B}_1})$, $\underline{\mathcal{B}_1} := \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$, corresponds to a game with coalition structure (Aumann and Drèze, 1974), discussed as "games with a priori unions" in Owen (1977).

Let $(N,v,\underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and $T \in \Omega^N$: From a level structure on N follows a level structure on T by eliminating the players in $N \backslash T$. With coalition structures $\mathcal{B}^r|_T := \{B \cap T \colon B \in \mathcal{B}^r, B \cap T \neq \emptyset\}$, $0 \leq r \leq h+1$, the new level structure on T is given by $\underline{\mathcal{B}}|_T := \{\mathcal{B}^0|_T, ..., \mathcal{B}^{h+1}|_T\} \in \mathbb{L}^T$ and $(T,v,\underline{\mathcal{B}}|_T) \in \mathbb{VL}^T$ is called the **restriction** of $(N,v,\underline{\mathcal{B}})$ to player set T.

A **TU-value** ϕ is an operator that assigns to any $(N,v) \in \mathbb{V}^N$ a payoff vector $\phi(N,v) \in \mathbb{R}^N$, an **LS-value** φ is an operator that assigns payoff vectors $\varphi(N,v,\underline{\mathcal{B}}) \in \mathbb{R}^N$ to all LS-games $(N,v,\underline{\mathcal{B}}) \in \mathbb{VL}^N$.

We define $\mathrm{W}^N := \{f \colon N \to \mathbb{R}_{++}\}$ with $\mathrm{w}_i := \mathrm{w}(i)$ for all $\mathrm{w} \in \mathrm{W}^N$ and $i \in N$ as the set of all positive weight systems on the player set N, we define $\mathcal{W}^{2^N} := \{f \colon 2^N \backslash \emptyset \to \mathbb{R}_{++}\}$ with $w_S := w(S)$ for all $w \in \mathcal{W}^{2^N}$, $S \in \Omega^N$, as the set of all positive weight systems on all non-empty coalitions $S \subseteq N$ and we define $\mathcal{W}^{\overline{\mathcal{B}}} := \{f \colon \overline{\mathcal{B}} \to \mathbb{R}_{++}\}$ with $w_B := w(B)$ for all $w \in \mathcal{W}^{\overline{\mathcal{B}}}$, $B \in \overline{\mathcal{B}}$, as the set of all positive weight systems on the components of all levels r, $0 \leq r \leq h$, of a level structure $\underline{\mathcal{B}}$.

Let $(N,v) \in \mathbb{V}^N$ and $\mathrm{w} \in \mathrm{W}$. The (simply) **weighted Shapley value**[2] Sh^{w} (Shapley, 1953a) is defined by

$$Sh_i^{\mathrm{w}}(N,v) := \sum_{S \subseteq N,\, S \ni i} \frac{\mathrm{w}_i}{\sum_{j \in S} \mathrm{w}_j} \Delta_v(S) \quad \text{for all } \ i \in N.$$

If all weights are equal we obtain, as a special case of a weighted Shapley value, the **Shapley value** Sh (Shapley, 1953b), given by

$$Sh_i(N,v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all } \ i \in N.$$

[2]We desist from possibly null weights as in Shapley (1953a) or Kalai and Samet (1987).

We introduce the Shapley levels value with a formula presented in Calvo, Lasaga and Winter (1996, eq. (1)). Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and for all $T \in \Omega^N$, $T \ni i$,

$$K_T(i) := \prod_{r=0}^{h} K_T^r(i), \quad \text{where}$$

$$K_T^r(i) := \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\, B \cap T \neq \emptyset\}|}.$$

The **Shapley Levels value** Sh^L (Winter, 1989) is given by

$$Sh_i^L(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} K_T(i) \Delta_v(T) \text{ for all } i \in N. \tag{2}$$

It is easy to see that Sh^L coincides with Sh if $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$.

All values above are or coincide with payoff vectors from the **Harsanyi set** (Hammer, 1977; Vasil'ev, 1978), also called **selectope** (Derks, Haller and Peters, 2000), where the payoffs are obtained by distributing the dividends. The payoffs ϕ_i^p in this set, titled **Harsanyi payoffs**, are defined by

$$\phi_i^p(N, v) := \sum_{S \subseteq N, S \ni i} p_i^S \Delta_v(S), i \in N,$$

where the p_i^S are non-negative weights in a sharing system $p = (p_i^S)_{S \in \Omega^N, i \in S}$ and sum up to 1 for each coalition S. The collection P^N on N of all such **dividend share systems** p is given by

$$P^N := \left\{ p = (p_i^S)_{S \in \Omega^N, i \in S} \Big| \sum_{i \in S} p_i^S = 1 \text{ and } p_i^S \geq 0 \text{ for each } S \in \Omega^N \text{ and all } i \in S \right\}.$$

3 Axioms

We refer to the following axioms for LS-values which are simple adaptions of standard-axioms for TU-values:

Efficiency, E. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, we have $\sum_{i \in N} \varphi_i(N, v, \underline{\mathcal{B}}) = v(N)$.

Null player, N. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ and $i \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$.

Null player out, NO[3]**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ and $j \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N \backslash \{j\}, v, \underline{\mathcal{B}}|_{N \backslash \{j\}})$ for all $i \in N \backslash \{j\}$.

Additivity, A. For all $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$, we have $\varphi(N, v, \underline{\mathcal{B}}) + \varphi(N, v', \underline{\mathcal{B}}) = \varphi(N, v + v', \underline{\mathcal{B}})$.

Winter (1989) introduced a symmetry between components axiom to characterize his value. There the sum of the payoffs to all players of a component equals the sum of the payoffs to all players of another component if both components are in the same level,

[3]This axiom is an extension from "null player out" in Derks and Haller (1999).

are subsets of the same component one level higher and both components are symmetric players in a game where the components are the players. Besner (2018) used a similar axiom to characterize the weighted Shapley support levels values. Unlike as before there the components must be dependent in the game where the components are the players. Then the sums of the payoffs to all players of both components are in the same proportion as the weights of the components. Here we present a new similar axiom. The only difference as before: now all players of the components must be dependent in the originally game instead of the components in the game with the components as players.

Weighted proportionality within components, WPWC[4]**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\overline{\mathcal{B}}}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and all $i \in B_k \cup B_\ell$ are dependent[5], we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}.$$

The following axiom is a special case of the previous one.

Dependency within components, DWC. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and all $i \in B_k \cup B_\ell$ are dependent in v, we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}).$$

4 Weighted Shapley alliance levels values

The Shapley levels value, the value in Gómez-Rúa and Vidal-Puga (2011) and the Shapley support levels values (Besner, 2018) satisfy the level game property. That means that the total payoff to all players of a component coincides with the payoff to the component if we would play a game where the components are the players itselves. But this is not the case[6] for our two new classes of weighted values[7]. The values in these classes allow the players, within the hierarchy of the level structure, to act more independently. So in the following class they can form subgroups with an own weight within the components containing them.

E. g., looking to a game where the whole world is the grand coalition. The world splits up in political unions like the European Union (EU) and countries which remain fully autonomous. Within the EU many countries are organized as a federal state or a comparable system and so on. But within the EU are also powerful subgroups possible like the euro area. Assume that we have found a weight system for the political influence

[4]In Nowak and Radzik (1995) the basic version of this axiom for TU-values is called "ω-mutual dependence".

[5]A coalition S such that all players $i \in S$ are dependent in v is called a "coalition of partners" in v (Kalai and Samet, 1987), known also as "partnership".

[6]We show that this property is not satisfied in our example in section 7.

[7]Both classes are a special case of values for level structures proposed in Besner (2016) as f-weighted-ALS-Shapley-values and f-weighted-SLS-Shapley-values respectively. Also exists a working paper (Besner, 2017) where these classes are discussed.

and power of all countries, states and so on and all possible cooperations of these units. Using a weighted Shapley support levels value the euro area throws, outside of the EU, the same weight as the whole EU into the balance! Instead, the following class of values assigns the euro area exactly the weight it has itself. The structure of the level structure determines here that always the involved players within a component act together as a single unit outside of the component.

Definition 4.1. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{2^N}$[8] *and for all* $T \subseteq N$, $T \ni i$,

$$A_{w,T}(i) := \prod_{r=0}^{h} A^r_{w,T}(i), \quad \text{where} \tag{3}$$

$$A^r_{w,T}(i) := \frac{w_{\mathcal{B}^r(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^r:\, B\subseteq\mathcal{B}^{r+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}.$$

The ***weighted Shapley alliance levels value*** Sh^{wAL} *is given by*

$$Sh^{wAL}_i(N, v, \underline{\mathcal{B}}) := \sum_{T\subseteq N, T\ni i} A_{w,T}(i)\Delta_v(T) \text{ for all } i \in N. \tag{4}$$

Of particular interest in the next theorem are the null player out property and the weighted proportionality within components property. In many cases it seems convincing that if a null player who obtain a payoff of zero should not affect the payoff to the other players if he leaves the game. The property that all players of two components are dependent means that no one outside of these components is interested to join a coalition of these players if not all players of these components are contained in the coalition and all players of both components must act together to obtain anything at all. Then the property that the sum of the payoffs to players of the first component in proportion to the weight of the first component equals the sum of the payoffs to players of the second component in proportion to the weight of the second component cannot be too bad for a weighted value for level structures.

Theorem 4.2. *Let* $w \in \mathcal{W}^{2^N}$. *The weighted Shapley alliance levels value* Sh^{wAL} *satisfies* ***E****,* ***N****,* ***NO****,* ***A*** *and* ***WPWC****.*

For the proof, see appendix 9.1. We obtain an axiomatization of the weighted Shapley alliance levels values which corresponds in case of a trivial level structure to an axiomatization of the weighted Shapley values too.

Theorem 4.3. *Let* $w \in \mathcal{W}^{2^N}$. Sh^{wAL} *is the unique LS-value that satisfies* ***E****,* ***NO****,* ***WPWC*** *and* ***A****.*

For the proof, see appendix 9.2. We have an interesting special case if the weights are the size of the components. It seems quite naturally that forming components within a hierarchical structure has no effect to the payoff to players if the component possesses as a whole the same weight as the parts in the sum.

Proposition 4.4. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ *and* $\overline{w} \in \mathcal{W}^{2^N}$ *such that* $\overline{w}_S = |S|$ *for all* $S \in \Omega^{B^h}$ *for all* $B^h \in \mathcal{B}^h$, $\overline{w}_S \in \overline{w}$. *Then we have*

$$Sh^{\overline{w}AL}_i(N, v, \underline{\mathcal{B}}) = Sh_i(N, v) \text{ for all } i \in N.$$

For the proof, see appendix 9.3.

[8] In fact, we need only weights for coalitions $S \in \Omega^{B^h}$ for all $B^h \in \mathcal{B}^h$.

4.1 A new characterization of the Shapley levels value and the Owen value

If all weights are equal, the coefficients $A_{w,T}(i)$ from def. 4.1 equal the $K_T(i)$ in eq. (2) from the definition of the Shapley levels value. Thus the Shapley levels value (Owen value) is a special case of a weighted Shapley alliance levels value (weighted Shapley alliance coalition structure value). We obtain, if we replace in the proof of theorem 4.3 **WPWC** by **DWC** the following corollary.

Corollary 4.5. *Sh^L is the unique LS-value that satisfies **E**, **NO**, **DWC** and **A**.*

Remark 4.6. *Since a level structure $(N, v, \underline{\mathcal{B}_1}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}_1} := \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$, coincides with a coalition structure on N we obtain a new axiomatization of the Owen value (Owen, 1977) if we adapt **E**, **NO**, **DWC** and **A** to games with a coalition structure.*

5 Weighted Shapley collaboration levels values

Mostly, if players form hierarchical structured coalitions (here called components) and a player of such a component is removed from the game, the remaining players of this component form a new component that replaces the old component and so the structure of the level structure remains largely intact. In the preliminaries we called the new level structure a restriction of the old one. But sometimes it is thinkable that the component loses its cohesion. We will not go so far that the whole level structure breaks apart completely. The cohesion from components outside of the broken one and all complete components which are subsets of the remaining player set of the broken component remains unchanged. So we introduce an internally, by the remaining components, induced restriction of the old level structure.

Definition 5.1. *Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ and $T \in \Omega^N$: We denote by $\mathcal{B}^r_{\mathcal{I}}|_T$, $0 \leq r \leq h+1$, the coalition structure on T, named **internally induced r-th level** of $(N, v, \underline{\mathcal{B}})$ to player set T, given by*

$$\mathcal{B}^r_{\mathcal{I}}|_T := \begin{cases} \{T\}, \ \text{if } r = h+1, \\ \big\{B \in \overline{\mathcal{B}} :\ B \subseteq (B^r \cap T),\ B^r \in \mathcal{B}^r,\ B \nsubseteq B' \in \overline{\mathcal{B}}, B' \subseteq (B^r \cap T)\big\}, \ \text{else.} \end{cases}$$

*With the level structure $\underline{\mathcal{B}_{\mathcal{I}}}|_T = \{\mathcal{B}^0_{\mathcal{I}}|_T, ..., \mathcal{B}^{h+1}_{\mathcal{I}}|_T\} \in \mathbb{L}^T$ the LS-game $(T, v, \underline{\mathcal{B}_{\mathcal{I}}}|_T) \in \mathbb{VL}^T$ is called the **internally induced restriction** of $(N, v, \underline{\mathcal{B}})$ to player set T.*

The internally induced r-th level $\mathcal{B}^r_{\mathcal{I}}|_T$ of a level structure to a player set T consists always of all largest components of the original level structure which are subsets of T and subsets of a component of the r-th level of the original level structure.

Now we can formulate a new null player out axiom that uses internally restrictions.

Internal (induced restriction) **null player out, INO.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $j \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N \backslash \{j\}, v, \underline{\mathcal{B}_{\mathcal{I}}}|_{N \backslash \{j\}})$ for all $i \in N \backslash \{j\}$.

In the following class of weighted Shapley levels values the subgroups of a component which are a component of a lower level can act more independently. For instance, we look at the regions of Europe, the regions merge to independent countries and most of the countries are members of the EU or autonomous nations like Norway. The Nordic Council is a geo-political forum where Denmark, Finland, Iceland, Norway and Sweden are full members. Assume also here, e. g., that we have found a weight system for the political influence and power of all countries, states and so on which build a component in the sense of a level structure. Denmark, Finland and Sweden are also members of the EU, but neither within the Nordic Council nor in the EU these countries act together as a component. So the coalition of these countries have not a weight support, given by the EU, as by the weighted Shapley support levels values, nor they form an alliance with an own weight as by the weighted Shapley alliance levels values. Within the Nordic Council each of these countries act autonomous as a single component and thus each country owns only its own weight. The following class of values supports such situations.

Definition 5.2. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\underline{\mathcal{B}}}$ *and for all* $T \subseteq N$, $T \ni i$,

$$C_{w,T}(i) := \prod_{r=0}^{h} C^r_{w,T}(i), \text{ with} \tag{5}$$

$$C^r_{w,T}(i) := \frac{w_{\mathcal{B}^r_T(i)}}{\sum_{B \in \widehat{\mathcal{B}^{r+1}_T(i)}} w_B},$$

where $\mathcal{B}^r_T(i)$ *is the largest component of all components* $\mathcal{B}^\ell(i)$, $0 \leq \ell \leq r$, *with* $\mathcal{B}^\ell(i) \subseteq T$, $\mathcal{B}^{h+1}_T(i) := T$ *and*

$$\widehat{\mathcal{B}^{r+1}_T(i)} := \begin{cases} \{\mathcal{B}^r_T(i)\}, & \text{if } \mathcal{B}^r_T(i) = \mathcal{B}^{r+1}_T(i), \\ \{B \in \overline{\mathcal{B}} : B \subsetneq \mathcal{B}^{r+1}_T(i),\, B \not\subseteq B' \in \overline{\mathcal{B}},\, B' \subsetneq \mathcal{B}^{r+1}_T(i)\}, & \text{else,} \end{cases}$$

is the set of all largest components which are subsets of $\mathcal{B}^{r+1}_T(i)$. *The* ***weighted Shapley collaboration levels value*** Sh^{wCL} *is given by*

$$Sh^{wCL}_i(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} C_{w,T}(i)\Delta_v(T) \text{ for all } i \in N. \tag{6}$$

A difference between the weighted Shapley alliance levels values and the weighted Shapley collaboration levels values lies in satisfying a different null player out property.

Theorem 5.3. *Let* $w \in \mathcal{W}^{\overline{\mathcal{B}}}$. *The weighted Shapley collaboration levels value* Sh^{wCL} *satisfies* ***E***, ***N***, ***INO***, ***A*** *and* ***WPWC***.

The proof is omitted because it is completely analogous to the proof of theorem 4.2. Also the following axiomatization extends an axiomatization of the weighted Shapley values and resembles theorem 4.3 of the weighted Shapley alliance levels values.

Theorem 5.4. *Let* $w \in \mathcal{W}^{\overline{\mathcal{B}}}$. Sh^{wCL} *is the unique LS-value that satisfies* ***E***, ***INO***, ***WPWC*** *and* ***A***.

The proof is omitted because it is completely analogous to the proof of theorem 4.3. Also here we obtain an interesting special case if the weights are the size of the components.

Proposition 5.5. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ *and* $\overline{w} \in \mathcal{W}^{\overline{\mathcal{B}}}$ *such that* $\overline{w}_B = |B|$ *for all* $B \in \overline{\mathcal{B}}$, $\overline{w}_B \in \overline{w}$. *Then we have*

$$Sh_i^{\overline{w}CL}(N, v, \underline{\mathcal{B}}) = Sh_i(N, v) \text{ for all } i \in N.$$

Again the proof is omitted because it is completely analogous to the proof of proposition 4.4.

6 The Shapley collaboration levels value

As a special case of the weighted Shapley collaboration levels values we can present an extension of the Shapley value to level structures.

Definition 6.1. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ *and for all* $T \subseteq N$, $T \ni i$,

$$C_T(i) := \prod_{r=0}^{h} C_T^r(i), \text{ with}$$
$$C_T^r(i) := \frac{1}{\left|\widehat{\mathcal{B}_T^{r+1}(i)}\right|},$$

where $\mathcal{B}_T^r(i)$ *is the largest component* $\mathcal{B}^\ell(i)$ *of all level* ℓ, $0 \leq \ell \leq r$, $\mathcal{B}^\ell(i) \subseteq T$, $\mathcal{B}_T^{h+1}(i) := T$ *and* $\widehat{\mathcal{B}_T^{r+1}(i)} := \left\{B \in \overline{\mathcal{B}} \colon B \subseteq \mathcal{B}_T^{r+1}(i),\ B \nsubseteq B' \in \overline{\mathcal{B}},\ B' \subsetneq \mathcal{B}_T^{r+1}(i)\right\}$ *is the set of all largest components which are subsets of* $\mathcal{B}_T^{r+1}(i)$. *The* ***Shapley collaboration levels value*** Sh^{CL} *is given by*

$$Sh_i^{CL}(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} C_T(i) \Delta_v(T) \text{ for all } i \in N.$$

Def. 6.1 coincides with def. 5.2 if all weights are equal. If we replace **WPWC** by **DWC**, we obtain, similar to corollary 4.5 the following corollary.

Corollary 6.2. Sh^{CL} *is the unique LS-value that satisfies* ***E****,* ***INO****,* ***DWC*** *and* ***A****.*

7 Example

In this section we give a numerical example to compare the sharing for different values presented in this paper where we refer to example 1 in Gómez-Rúa and Vidal-Puga (2011). To show that the new classes of values don't satisfy the level game property, this means that the total payoff of all players of a component equals the payoff to this component in a game where the components are the players we set this property, introduced in Winter (1989): Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ and $T \in \Omega^N$. We define for each level r, $0 \leq r \leq h$, the level structure $\underline{\mathcal{B}^r} := \left\{\mathcal{B}^{r^0}, ..., \mathcal{B}^{r^{h+1-r}}\right\} \in \mathbb{L}^{\mathcal{B}^r}$ as the induced ***r*-th level structure** from $\underline{\mathcal{B}}$ by considering the components $B \in \mathcal{B}^r$ as players. There all levels from the original level structure lower then r are dropped and we have $\mathcal{B}^{r^k} := \{\{B \in \mathcal{B}^r : B \subseteq B^{r+k}\}$ for all $B^{r+k} \in \mathcal{B}^{r+k}\}$, $0 \leq k \leq h + 1 - r$.

If $T = \bigcup_{B \subseteq T} B$, $B \in \mathcal{B}^r$, and we want to stress this property, T is denoted by T^r. Each such T^r is related to a coalition of all players $B \in \mathcal{B}^r$, $B \subseteq T^r$, in the induced r-th level structure, denoted by $\mathcal{T}^r := \{B \in \mathcal{B}^r : B \subseteq T^r\}$ and vice versa. Then the induced **r-th level game** $\left(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}\right) \in \mathbb{VL}^{\mathcal{B}^r}$, where $\mathcal{B}^r$ is the player set with $B \in \mathcal{B}^r$ as players, is given by

$$v^r(\mathcal{T}^r) := v(T^r) \text{ for all } \mathcal{T}^r \in \Omega^{\mathcal{B}^r}.$$

For a level structure $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathbb{L}^N$ and an induced r-th level structure $\underline{\mathcal{B}^r}$ related components have the same weights. So we have for all r, k, $0 \leq r \leq k \leq h$, $B^{r^{k-r}} \in \mathcal{B}^{r^{k-r}}$, $\mathcal{B}^{r^{k-r}} \in \underline{\mathcal{B}^r}$, $B^k \in \mathcal{B}^k$, $\mathcal{B}^k \in \underline{\mathcal{B}}$,

$$w_{B^{r^{k-r}}} = w_{B^k} \text{ with } B^{r^{k-r}} := \{B \in \mathcal{B}^r \colon B \subseteq B^k\} \text{ and } w_{B^{r^{k-r}}} \in \mathcal{W}^{\overline{\mathcal{B}^r}},\ w_{B^k} \in \mathcal{W}^{\overline{\mathcal{B}}}.$$

We present the desired property.

Level game property, LG (Winter, 1989). For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $B \in \mathcal{B}^r$, $0 \leq r \leq h+1$, we have

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_B(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}).$$

We come finally to the example: Let $(N, u_S, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $w \in \mathcal{W}^{2^N}$ and $w' \in \mathcal{W}^{\underline{\mathcal{B}}}$, where $N := \{1,2,3,4\}$ and $\underline{\mathcal{B}} = \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3\}$, with $\mathcal{B}^1 := \{\{1\},\{2,3\},\{4\}\}$, $\mathcal{B}^2 := \{\{1,2,3\},\{4\}\}$ and u_S the unanimity game with carrier $S := \{1,2,4\}$. Assume that the weights are exogenous given and reflect, e. g., the political power or something else of the coalitions. For the weight system w' we use the same weights for the components as given in the weight system w so that we have $w'_B = w_B$ for all $B \in \underline{\mathcal{B}}$; for calculating the weighted Shapley value Sh^{w} we use the weight system w, given by $\mathrm{w}_i := w_{\{i\}}$ for all $i \in N$ and $w_{\{i\}} \in w$. Since $S = \{1,2,4\}$ is the only active coalition in the game for computing we need only the following given weights:

$$w_{\{1\}} = w_{\{2\}} = w_{\{3\}} = 1, \qquad w_{\{4\}} = 4, \qquad w_{\{1,2\}} = w_{\{2,3\}} = 3, \qquad w_{\{1,2,3\}} = 5.$$

We obtain Table 1 where ϕ^L is the value for level structures presented in Gómez-Rúa and Vidal-Puga (2011).

Table 1: Payoffs of different values

Value	Payoff to player $1,2,3,4$	$\varphi_{\{1,2,3\}}(\mathcal{B}^2, v2, \underline{\mathcal{B}^2})$	$\sum_{i \in \{1,2,3\}} \varphi_i(N, v, \underline{\mathcal{B}})$
$Sh(N,v)$	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}$	-	-
$Sh^{\mathrm{w}}(N,v)$	$\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}$	-	-
$Sh^L(N,v,\underline{\mathcal{B}})$	$\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\phi^L(N,v,\underline{\mathcal{B}})$	$\frac{1}{4}, \frac{5}{12}, \frac{1}{12}, \frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$Sh^{wSL}(N,v,\underline{\mathcal{B}})$	$\frac{5}{36}, \frac{5}{12}, 0, \frac{4}{9}$	$\frac{5}{9}$	$\frac{5}{9}$
$Sh^{wAL}(N,v,\underline{\mathcal{B}})$	$\frac{3}{14}, \frac{3}{14}, 0, \frac{4}{7}$	$\frac{5}{9}$	$\frac{3}{7}$
$Sh^{wCL}(N,v,\underline{\mathcal{B}})$	$\frac{1}{6}, \frac{1}{6}, 0, \frac{2}{3}$	$\frac{5}{9}$	$\frac{1}{3}$
$Sh^{CL}(N,v,\underline{\mathcal{B}})$	$\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$

We see, in difference to the other presented values for level structures, the Shapley alliance levels values and the Shapley collaboration levels values don't match in general the level game property.

8 Conclusion

The rapidly increasing volume of collected data and global networking make it possible and necessary to share benefits between cooperating participants, often structured hierarchical. To distribute generated surpluses the presented two new classes of LS-values, together with the weighted Shapley support levels values, contain alternatives to the Shapley levels value if there exist exogenous given weights for some coalitions.

The weighted Shapley support levels values meet the level game property but subcoalitions of a component always depend on the weight of the whole component. Our two new classes of values offer an alternative, particularly if some players are removed from the game, especially null players. Missing the level game property is the price that is to pay. But this is a sign that the level game does here not reflect the complete bargaining on subcoalitions, they are more independent from nesting components.

If we use the restriction of an LS-game (the extended restriction as normally used for coalition structures), the weighted Shapley alliance levels values satisfy the null player out property and contain the Shapley levels value as a special case. So this paper also opens different perspectives on the Shapley levels value. But a level structure is more then just a sequence of coalition structures, the coalition structures are ordered in a certain way. Thus we could present a new internally induced restriction, which should be used for example in the case that a component splits in the components next in size if one player quits the component. The weighted Shapley collaboration levels values satisfy the null player out property for internally induced restrictions. So we have found a situation where the Shapley levels value fails and a new extension of the Shapley value, called Shapley collaboration levels value, fits best.

Since an LS-game $(N, v, \underline{\mathcal{B}_1})$ corresponds to a game with coalition structure the transfer of definitions, axioms and axiomatizations to coalition structures is left to the reader.

9 Appendix

The following lemma is used in the proofs of theorem 4.2 and theorem 4.3.

Lemma 9.1. *Besner (2018, lemma 7.3) Players $i, j \in N$, $i \neq j$, are dependent in $v \in \mathcal{G}^N$, iff $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, for all $S \subseteq N \backslash \{i, j\}$.*

9.1 Proof of theorem 4.2

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{2^N}$ and $A^r_{w,T}$ the expressions according to def. 4.1.

- **E, N, A**: Let $T \in \Omega^N$, $j \in T$. It is easy to show, by induction on r, that

$$\sum_{i \in \mathcal{B}^{r+1}(j),\, i \in T} \prod_{\ell=0}^{r} A^r_{w,T}(i) = 1.$$

So $\sum_{i \in T} A_{w,T}(i) = 1$ and, since $A_{w,T}(i) > 0$, $i \in T$, the $A_{w,T}(i)$ form a dividend share system $p \in P^N$ and Sh^{wAL} coincides with a Harsanyi payoff on a fixed player set. Therefore

Sh^{wAL} satisfies all for level structures simply adapted axioms on a fixed player set which are, as related TU-axioms, satisfied by a Harsanyi payoff, in particular **E, N** and **A** are well-known matched axioms.

• **NO**: It is well-known that each coalition $S \in \Omega^N$, containing a null player $j \in N$ in v, is not active in v. In eq. (4) we have only to consider active coalitions. But for these coalitions there is no change in the weights. Thus we have $Sh_i^{wAL}(N, v, \underline{\mathcal{B}}) = Sh_i^{wAL}(N\backslash\{j\}, v, \underline{\mathcal{B}}|_{N\backslash\{j\}})$ for all $i \in N\backslash\{j\}$.

• **WPWC**: Let $k, \ell \in N$, $0 \leq r \leq h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and all players $i \in \mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)$ be dependent in v. We obtain

$$\begin{aligned}
\sum_{i\in\mathcal{B}^r(k)} \frac{Sh_i^{wAL}(N,v,\underline{\mathcal{B}})}{w_{\mathcal{B}^r(k)}} &\underset{\substack{Def.\\4.1}}{=} \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{i\in\mathcal{B}^r(k)} \sum_{\substack{T\subseteq N,\\ T\ni i}} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\right]\Delta_v(T)\\
&\underset{\substack{Lem.\\9.1}}{=} \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{i\in\mathcal{B}^r(k)} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\right]\Delta_v(T)\\
&= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \sum_{i\in\mathcal{B}^r(k)} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\right]\Delta_v(T)\\
&= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T)\left[\prod_{j=r}^{h} \frac{w_{\mathcal{B}^j(k)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} w_{B\cap T}} \cdot \sum_{i\in\mathcal{B}^r(k)} \prod_{j=0}^{r-1} \frac{w_{\mathcal{B}^j(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}{}^{9}\right]\\
&= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r}^{h} \frac{w_{\mathcal{B}^j(k)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\\
&= \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r+1}^{h} \frac{w_{\mathcal{B}^j(k)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\\
&= \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r+1}^{h} \frac{w_{\mathcal{B}^j(\ell)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(\ell),\\ B\cap T\neq\emptyset}} w_{B\cap T}} = \sum_{i\in\mathcal{B}^r(\ell)} \frac{Sh_i^{wAL}(N,v,\underline{\mathcal{B}})}{w_{\mathcal{B}^r(\ell)}}.
\end{aligned}$$

□

Convention 9.2. To avoid cumbersome case distinctions in the proof of theorem 4.3 using **WPWC**, if there is only one single player assessed in isolation, she is defined as dependent by herself. Then **WPWC** is trivially satisfied.

9.2 Proof of theorem 4.3

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{2^N}$, $S \in \Omega^N$ arbitrary and φ an LS-value that satisfies all axioms of theorem 4.3 and **N**, because **E** and **NO** imply obvious **N**. Due

[9] The last sum always equals 1, if $r = 0$, we have an empty product, which is equal, by convention, to the multiplicative identity 1.

to theorem 4.2, property (1) and **A**, it is sufficient to show that φ is uniquely defined on the game $v_S := \Delta_v(S) \cdot u_S$.

All players $j \in N\backslash S$ are null players in v_S and so φ is unique on v_S for all $j \in N\backslash S$ by **N**. All players $i \in S$, possibly using conv. 9.2, are dependent in v_S and, by **NO**, we obtain

$$\varphi_i(N, v_S, \underline{\mathcal{B}}) = \varphi_i(S, v_S, \underline{\mathcal{B}}|_S) \text{ for all } i \in S.$$

So we can use an induction on the restriction to the player set S on the size m, $0 \leq m \leq h$, for all levels r, $0 \leq r \leq h$, with $m := h - r$.

Initialisation: Let $m = 0$ and so $r = h$. We get for an arbitrary $i \in S$

$$\begin{aligned}
\sum_{\substack{B \in \mathcal{B}^h|_S,\\ B \cap S \neq \emptyset}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) \underset{\textbf{(WPBC)}}{=} \sum_{\substack{B \in \mathcal{B}^h|_S,\\ B \cap S \neq \emptyset}} \frac{w_B}{w_{\mathcal{B}^h|_S(i)}} \sum_{j \in \mathcal{B}^h|_S(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) \underset{\textbf{(E)}}{=} \Delta_v(S) \\
\Leftrightarrow \quad \sum_{j \in \mathcal{B}^r|_S(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) = \left[\prod_{k=h-m}^{h} \frac{w_{\mathcal{B}^k|_S(i)}}{\sum_{\substack{B \in \mathcal{B}^k|_S:\, B \subseteq \mathcal{B}^{k+1}|_S(i),\\ B \cap S \neq \emptyset}} w_B}\right] \Delta_v(S). \qquad (7)
\end{aligned}$$

Induction step: Assume that eq. (7) holds to φ with an arbitrary $m-1$, $0 \leq m-1 \leq h-1$ (IH). It follows for an arbitrary $i \in S$

$$\begin{aligned}
\sum_{\substack{B \in \mathcal{B}^r|_S,\, B \cap S \neq \emptyset,\\ B \subseteq \mathcal{B}^{r+1}|_S(i)}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) &\underset{\textbf{(WPBC)}}{=} \sum_{\substack{B \in \mathcal{B}^r|_S,\, B \cap S \neq \emptyset,\\ B \subseteq \mathcal{B}^{r+1}|_S(i)}} \frac{w_B}{w_{\mathcal{B}^r|_S(i)}} \sum_{j \in \mathcal{B}^r|_S(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) \\
&\underset{(IH)}{=} \left[\prod_{k=h-m+1}^{h} \frac{w_{\mathcal{B}^k|_S(i)}}{\sum_{\substack{B \in \mathcal{B}^k|_S:\, B \subseteq \mathcal{B}^{k+1}|_S(i),\\ B \cap S \neq \emptyset}} w_B}\right] \Delta_v(S) \\
\Leftrightarrow \quad \sum_{j \in \mathcal{B}^r|_S(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) &= \left[\prod_{k=h-m}^{h} \frac{w_{\mathcal{B}^k|_S(i)}}{\sum_{\substack{B \in \mathcal{B}^k|_S:\, B \subseteq \mathcal{B}^{k+1}|_S(i),\\ B \cap S \neq \emptyset}} w_B}\right] \Delta_v(S).
\end{aligned}$$

So φ is uniquely defined on v_S (take $m = h$ and so $r = 0$). □

9.3 Proof of proposition 4.4

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and $\overline{w} \in \mathcal{W}^{2^N}$ such that $\overline{w}_S = |S|$ for all $S \in \Omega^{B^h}$ and all $B^h \in \mathcal{B}^h$, $\overline{w}_S \in \overline{w}$. We have only to show that

$$A_{\overline{w},T}(i) = \frac{1}{|T|} \text{ for all } T \subseteq N,\, T \ni i.$$

For all $T \subseteq N$, $T \ni i$, and $0 \leq r \leq h$ the set $\widetilde{\mathcal{B}^{r+1}_T(i)} := \{B \cap T : B \in \mathcal{B}^r, B \subseteq \mathcal{B}^{r+1}(i), B \cap T \neq \emptyset\}$ is a partition of $\mathcal{B}^{r+1}(i) \cap T$. So we have

$$\sum_{B \in \widetilde{\mathcal{B}^{r+1}_T(i)}} \overline{w}_{B \cap T} = \sum_{\substack{B \in \mathcal{B}^r:\, B \subseteq \mathcal{B}^{r+1}(i),\\ B \cap T \neq \emptyset}} \overline{w}_{B \cap T} = |\mathcal{B}^{r+1}(i) \cap T|.$$

By eq. (3) we get $A_{\overline{w},T}(i) = \dfrac{1}{|T|}$ as desired. □

9.4 Logical independence

All axiomatizations must also hold if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$. In this case all axioms, used for axiomatization in this paper, coincide with usual axioms for TU-values. So the given axiomatizations coincide in this case with axiomatizations of the weighted Shapley values and the Shapley value, respectively. It is well-known or easy to proof that in this case the used axioms are logical independent. Therefore all axioms for LS-values must be also logical independent in the given axiomatizations.

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