



Munich Personal RePEc Archive

# **Granger causality testing in mixed-frequency Vars with possibly (co)integrated processes**

Alain Hecq and Thomas Goetz

Maastricht University, Deutsche Bundesbank

27 June 2018

Online at <https://mpa.ub.uni-muenchen.de/87746/>  
MPRA Paper No. 87746, posted 12 July 2018 13:30 UTC

# GRANGER CAUSALITY TESTING IN MIXED-FREQUENCY VARs WITH POSSIBLY (CO)INTEGRATED PROCESSES

Thomas B. Götz<sup>\*1</sup> and Alain W. Hecq<sup>†2</sup>

<sup>1</sup>Deutsche Bundesbank

<sup>2</sup>Maastricht University

June 27, 2018

## Abstract

We analyze Granger causality testing in mixed-frequency VARs with possibly (co)integrated time series. It is well known that conducting inference on a set of parameters is dependent on knowing the correct (co)integration order of the processes involved. Corresponding tests are, however, known to often suffer from size distortions and/or a loss of power. Our approach, which boils down to the mixed-frequency analogue of the one by Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996), works for variables that are stationary, integrated of an arbitrary order, or cointegrated. As it only requires an estimation of a mixed-frequency VAR in levels with appropriately adjusted lag length, after which Granger causality tests can be conducted using simple standard Wald test, it is of great practical appeal. We show that the presence of non-stationary and trivially cointegrated high-frequency regressors (Götz et al., 2013) leads to standard distributions when testing for causality on a parameter subset, without any need to augment the VAR order. Monte Carlo simulations and two applications involving the oil price and consumer prices as well as GDP and industrial production in Germany illustrate our approach.

**JEL Codes:** C32

**JEL Keywords:** Mixed frequencies; Granger causality; Hypothesis testing, Vector autoregressions; Cointegration

---

<sup>\*</sup>Thomas B. Götz, Deutsche Bundesbank, Economics Department, Macroeconomic Analysis and Projection Division, P.O. Box 100602, 60006 Frankfurt am Main, Germany. Email: thomas.goetz@bundesbank.de; the views expressed in this paper are solely ours and should not be interpreted as reflecting the views of the Deutsche Bundesbank.

<sup>†</sup>Alain W. Hecq, Maastricht University, School of Business and Economics, Department of Quantitative Economics, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: a.hecq@maastrichtuniversity.nl

# 1 Introduction

“The world is mixed-frequent” a young researcher said when presenting his paper on forecasting with a mixed-frequency (MF) time series model.<sup>1</sup> It not only shows that MF models constitute a popular and widely studied topic in time series econometrics, it is simply an omnipresent fact applied and theoretical researchers need to deal with, and they do: by now, it has become standard to properly account for the mismatch in publication frequencies among (macroeconomic) time series, instead of aggregating high-frequency (HF) observations using predetermined aggregation schemes (Silvestrini and Veredas, 2008). The set of MF models ranges from single-regression models (e.g., the, by this time, routinely used MIDAS model; see Ghysels et al., 2004 or Ghysels et al., 2007 and, for the unrestricted version, Foroni et al., 2015b) over factor models (see Mariano and Murasawa, 2003, Marcellino and Schumacher, 2010 and Blasques et al., 2016) to vector autoregressive (VAR) models (see, most notably, Ghysels, 2016, Schorfheide and Song, 2015 and Chiu et al., 2011).<sup>2</sup>

Particularly the latter model class, MF-VAR models, has received a lot of attention recently, predominantly in two related fields of application: forecasting (Schorfheide and Song, 2015 and Götz and Hauzenberger, 2017, among others) and Granger causality testing (Ghysels et al., 2016, Götz et al., 2016 and Ghysels et al., 2017). Both topics are of immense interest to practitioners at, e.g., central banks, who routinely forecast key variables like the gross domestic product (GDP) using a variety of, usually higher-frequent, indicators, or investigate causal patterns between the time series they monitor. We will focus on Granger causality (GC) testing, generally introduced in Granger (1969), whereby both concepts are obviously related due to the way GC is usually defined. The three papers mentioned above cover different aspects of GC testing within a MF-VAR: Ghysels et al.

---

<sup>1</sup>Unfortunately, the name of said researcher as well as the conference he presented at slipped the authors’ memories; as if choosing the adjective “young” was not enough for proving that one grew older.

<sup>2</sup>There exists a multitude of sub-variants of each model class, e.g., hybrid versions of MIDAS LASSO (Siliverstovs, 2017), Markov-switching MF models (Foroni et al., 2015a) and MF-VARs with time-varying parameters and stochastic volatility (Götz and Hauzenberger, 2017 or Cimadomo and D’Agostino, 2016), among many others.

(2016) discuss the general theory of associated hypothesis tests in detail, which – while asymptotically valid – suffer from size distortions and a loss of power in case the number of HF observations is large relative the low-frequency (LF) period. Götz et al. (2016) and Ghysels et al. (2017) then introduce various ways to overcome these implications of the *curse of dimensionality*. Yet, all three papers have one assumption in common, i.e., they remain in a stationary time series environment after properly transforming the series; in this paper we allow the variables to be integrated or cointegrated.

If we knew the order of integration of the variables under consideration as well as whether the series are cointegrated, we could transform an initial MF-VAR in levels to a model in differences or to an error correction model (ECM) and test for GC using the methods of Ghysels et al. (2016) or Götz et al. (2016) and Ghysels et al. (2017), depending on the frequency mismatch. In practice, though, we usually do not know the precise (co)integration order, and appropriate tests are required beforehand; tests that – in the case of unit roots – tend to have rather low power or that – in the case of cointegration – may suffer from severe size distortions (Ghysels and Miller, 2015). Instead of testing for GC in a system, that is a-priori transformed based on the outcomes of more or less error-prone pre-tests, we aim at a methodology that allows estimation of the MF-VAR in levels and leads to valid and standard inference procedures.

In this paper, we thus extend the (somewhat classical) methodology of Toda and Yamamoto (1995), Sims et al. (1990), Dolado and Lütkepohl (1996), Toda and Phillips (1993, 1994) as well as Lütkepohl and Reimers (1992) among others to the MF case. Starting from a MF-VAR in levels, in which the series may be stationary, integrated of an arbitrary order, or cointegrated, we propose – like in the common-frequency counterparts – not take differences or re-write the model into an ECM format. In order to conduct asymptotically valid inference on a subset of coefficients (in accordance with GC), we could instead apply a suitable adjustment of the lag length (see Toda and Yamamoto, 1995 or Dolado and Lütkepohl, 1996). The price one has to pay for intentionally overfitting the model is inefficiency due to a loss of power. In the MF case, though, we

show that for the stacked, observation-driven MF-VAR system of Ghysels (2016),<sup>3</sup> the necessary adjustment is small at worst and in some cases even entirely superfluous. This causes the corresponding inefficiencies to be lower than in the common-frequency case or to be absent altogether.

To be more precise, and to highlight the most important consequence of our approach, consider testing for GC from a high- to a low-frequency variable, i.e., the arguably more interesting case in terms of nowcasting a LF series (e.g., quarterly GDP) using HF indicators with eventual leading properties (e.g., monthly surveys). Using our simple methodology one can apply a standard Wald test on a MF-VAR estimated in levels without any need to adjust the lag length, irrespective of the (co)integration order of the series involved. Key for this finding is the presence of “trivial” cointegrating relationships among the stacked HF series (Götz et al., 2013) and a suitable application of Theorem 1 of Toda and Phillips (1993). With respect to testing the reverse direction, only a small adjustment of the system suffices to rely on an asymptotic  $\chi^2$ -distribution.

The rest of the paper is organized as follows. In Section 2 we describe the model, introduce GC testing within the MF-VAR and outline the different augmentations ensuring valid inference in levels, irrespective of the order of (co)integration. We present the theoretical background for our approach and confirm our findings using Monte Carlo simulations in Section 3. An empirical analysis in Section 4 illustrates our approach for German data involving the consumer price index and the oil price as well as GDP and industrial production. Section 5 concludes.

---

<sup>3</sup>We leave an analysis of the same research question for parameter-driven MF-VAR models à la Schorfheide and Song (2015) for future research. A discussion on these alternative model specifications follows below.

## 2 GC Testing in MF-VARs

### 2.1 The Model

Let us assume a two-variable MF system, where  $y_t$  represents the LF variable running from  $t = 1, \dots, T$ . The HF variable  $x$  appears  $m$  times as often, implying  $m = 3$  for a month/quarter- or  $m = 4$  for a quarter/year-example. We write  $x_{t-i/m}^{(m)}$  for a specific HF observation, whereby  $i = m - 1$  ( $i = 0$ ) represents the beginning (end) of a LF period  $t$ . The LF and HF lag and difference operators are denoted  $L^i$  and  $\Delta^i$  as well as  $L^{j/m}$  and  $\Delta^{j/m}$ , respectively. Hence,  $\Delta^i y_t = y_t - L^i y_t = y_t - y_{t-i}$  and  $\Delta^{j/m} x_t = x_t - L^{j/m} x_t = x_t - x_{t-j/m}$ . Also,  $L^{1/m} x_{t-(m-1)/m} = x_{t-1}$ . For this rather standard notation in the MF literature, see also Clements and Galvão (2008) or Miller (2014). Furthermore, we denote an integrated process of integer order  $d$  by  $I(d)$  and a cointegrated process of order  $d, b$  by  $CI(d, b)$ . Finally,  $vec$  represents the operator stacking the columns of a matrix,  $\otimes$  the Kronecker product,  $I_k$  the identity matrix of dimension  $k$  and  $\mathbf{0}_{i \times j}$  an  $(i \times j)$ -matrix of zeros.

**Remark 1** *In principle, we could allow for higher dimensional multivariate systems by, e.g., considering  $n_l$  LF and  $n_h$  HF series, where the HF series may have different sampling frequencies  $m_j, j = 1, \dots, n_h$ . Firstly, though, analyzing GC in a system with more than two variables opens the door for multi-horizon causality and thus to causal chains (see, e.g., Lütkepohl, 1993). Secondly, such an extension would complicate the notation and illustration of results.*

The observation-driven MF-VAR of Ghysels (2016) is constructed by first stacking the HF observations corresponding to one  $t$ -period together with the observation for  $y$  yielding  $Z_t = (y_t, X_t^{(m)'})'$ , where  $X_t^{(m)} = (x_t^{(m)}, x_{t-1/m}^{(m)}, \dots, x_{t-(m-1)/m}^{(m)})'$ . Then, a dynamic structural equations model for  $Z$  can be written as

$$A_c Z_t = A_1^* Z_{t-1} + \dots + A_p^* Z_{t-p} + u_t^*, \quad (1)$$

where coefficients in  $A_c$  govern the evolution within the HF process  $x$  as well as so-called *nowcasting causality* (Götz and Hecq, 2014).  $u_t^*$  is an independently and identically distributed (i.i.d.) error term with  $\mathbb{E}(u_t) = \mathbf{0}_{(m+1) \times 1}$ ,  $\mathbb{E}(u_t^* u_t^{*'}) = \Sigma_{u^*}$ , the latter being a diagonal matrix of dimension  $m + 1$  with  $(1, 1)$ -element  $\sigma_y^2$  and  $\sigma_x^2$  (the variances of the processes  $y$  and  $x$ ) on the remainder of the diagonal. After pre-multiplying (1) by  $A_c^{-1}$  we obtain the reduced-form MF-VAR( $p$ ):

$$Z_t = A_1 Z_{t-1} + \dots + A_p Z_{t-p} + u_t = AZ_{-p} + u_t \quad (2)$$

with  $A_j = A_c^{-1} A_j^*$ ,  $j = 1, \dots, p$ ,  $u_t = A_c^{-1} u_t^*$ ,  $A = (A_1, \dots, A_p)$  and  $Z_{-p} = (Z'_{t-1}, \dots, Z'_{t-p})'$ . Note that we exclude an intercept, mostly for ease of notation.<sup>4</sup> Let us make the following assumptions:

**Assumption 1**  $Z_t$  is generated by the MF-VAR( $p$ ) in (2), whereby (i)  $Z_t$  is  $I(d)$  and may or may not be  $CI(d, b)$ ;  $u_t$  is an i.i.d. sequence of  $(m + 1)$ -dimensional random vectors with  $\mathbb{E}(u_t) = \mathbf{0}_{(m+1) \times 1}$ ,  $\mathbb{E}(u_t u_t') = \Sigma_u$ , where  $\Sigma_u > 0$  such that  $\mathbb{E}|u_{jt}|^{2+\delta} < \infty$  for some  $\delta > 0$ .

**Remark 2** As in Toda and Yamamoto (1995) and Dolado and Lütkepohl (1996), we initially assume the lag order  $p$  to be a-priori known or estimated via some standard selection criterion. Indeed, one may expect biases resulting from such pre-tests to affect all approaches more or less equally.

**Remark 3** As far as the difference in frequencies between  $y$  and  $x$ , captured by the parameter  $m$ , is concerned, we primarily focus on rather small values, i.e.,  $m \leq 4$ . Firstly, the corresponding cases are usually of more interest in typical macroeconomic applications (see, e.g., Section 4). Secondly, the size of the MF-VAR in (2) grows rapidly with  $m$ . Consequently, inefficiencies resulting from the inherent curse of dimensionality will most likely dominate any effects from testing for GC using one or the other approach.

---

<sup>4</sup>One may think of the processes to be demeaned a-priori. As in Dolado and Lütkepohl (1996), the theory remains valid if deterministic terms of any sort are present.

Thirdly, small values of  $m$  allow us to rely on standard asymptotic theory for Wald test (Ghysels et al., 2016), even in the benchmark case.<sup>5</sup>

**Remark 4** *Assumption 1 implies that the data are truly generated at mixed frequencies. Alternatively, one could base the analysis on a common HF data generating process (DGP) and obtain model (2) by temporally aggregating the LF series (see, inter alia, Ghysels and Miller, 2015, Zdrozny, 2016 or Koelbl and Deistler, 2018). Our goal, however, is not to study which causality patterns can be preserved when moving from a latent HF system to a MF one,<sup>6</sup> but to find an asymptotically valid approach to conduct inference in a model that is directly estimable by practitioners. Additionally, we base our analysis on a model with observable data only, not involving latent processes of any kind. Such parameter-driven MF-VAR models (see, e.g., Schorfheide and Song, 2015) certainly have their merits, but do not lend themselves easily to a (co)integration-order-robust way of GC testing. The model in (2), however, does, which is why we consider that model whenever we refer to a MF-VAR.*

## 2.2 Standard Approach

GC testing within a MF-VAR boils down to testing a set of zero restrictions on the coefficient matrices  $A = (A_1, \dots, A_p)$ . In particular, testing for Granger non-causality in both directions implies testing the following null hypotheses in system (2):

$$H_0^{HF \rightarrow LF} : A_i^{(1,2)} = A_i^{(1,3)} = \dots = A_i^{(1,m+1)} = 0 \quad \forall i = 1, \dots, p, \quad (3)$$

$$H_0^{LF \rightarrow HF} : A_i^{(2,1)} = A_i^{(3,1)} = \dots = A_i^{(m+1,1)} = 0 \quad \forall i = 1, \dots, p, \quad (4)$$

---

<sup>5</sup>One could – as mentioned above – rely on the approaches of Götz et al. (2016) or Ghysels et al. (2017) in case  $m$  is comparably large. However, the performance of these tests in a situation, where the (co)integration order of the variables is unknown, is unclear and should be inspected first.

<sup>6</sup>In fact, as shown by Ghysels et al. (2016), depending on the aggregation scheme used, Granger non-causality does not get preserved when moving from a HF- to a MF-VAR, making it impossible to evaluate different GC test approaches. We comment in more detail on this issue when discussing our simulation results.



where  $A_i^{(r,c)}$  denotes the  $(r, c)$ -element of matrix  $A_i$ .<sup>7</sup> Of course, if we knew the process  $Z$  to be  $I(0)$ , we could just estimate model (2) in levels and apply a standard Wald test: for  $a_p = \text{vec}(A_1, \dots, A_p)$  and a suitably constructed matrix  $R$ , we can rewrite both null hypotheses as

$$H_0 : Ra_p = \mathbf{0}_{mp \times 1}$$

and compute the Wald statistic as

$$W = (R\hat{a}_p)'(R\hat{\Omega}R')^{-1}(R\hat{a}_p)$$

with  $\hat{\Omega} = (A'A)^{-1} \otimes \hat{\Sigma}_u$  and  $\hat{\Sigma}_u = (\hat{u}'\hat{u})/(T - p)$  being a consistent estimator of  $\Sigma_u$ . For a stationary MF-VAR, Ghysels et al. (2016) show the Wald statistic to be asymptotically  $\chi^2(mp)$ -distributed. We refer to testing for GC in this way as “standard test”.

In case we knew the process  $Z_t$  to be  $I(d)$ , we could achieve stationarity of the MF-VAR by differencing  $d$  times. Here, however, an additional ambiguity is added to the situation due to the presence of mixed frequencies: while LF differences are surely being applied to  $y$ , one could either use  $\Delta$  or  $\Delta^{1/m}$  for  $x$ . Indeed, both transformations applied  $d$  times yield a stationary process, yet have consequences on the dynamics of the system and thereby on conducting inference. Somewhat similarly, in the additional presence of cointegration between  $y$  and  $x$  we can follow the lines of either Götz et al. (2013) or Ghysels and Miller (2015), who derived alternative specifications of a MF-VECM.<sup>8</sup> Again, which specification is chosen in the end has implications for the construction of GC tests and may affect their performance, especially in finite samples. At the very least, it affects the way in which trivial cointegrating relationships among the HF series themselves enter the model (Götz et al., 2013).

Obviously, the (co)integration orders of the series are not known a-priori and a battery

---

<sup>7</sup>The alternative hypotheses, of course, imply that at least one of the respective coefficients is non-zero.

<sup>8</sup>For the single-regression counterpart Miller (2014) and Götz et al. (2014) developed MF-ECMs, whereas Miller (2016) focused on efficient estimation of a cointegrating vector in a MF scenario.

of pre-tests are usually performed. With respect to tests for the order of integration (usually the ones based on Dickey and Fuller, 1979, Phillips, 1987 or Phillips and Perron, 1988),<sup>9</sup> however, power tends to be rather low against a (trend) stationary series. As for tests on cointegration, Ghysels and Miller (2015) show that depending on the (often unknown) aggregation scheme underlying the series, one may have to expect severe size distortions.<sup>10</sup> Given the pitfalls of such pre-tests, an approach for GC testing in levels irrespective of the (co)integration orders of the variables would be highly valuable.

**Remark 5** *Like Dolado and Lütkepohl (1996), we focus on  $d = 1$  in this paper to simplify notation and discussion. On the one hand it is indeed the most important case in practice, on the other hand the approach and theory extend quite straightforwardly to  $d > 1$  (see Toda and Yamamoto, 1995 for the common-frequency case). We will mention any changes due to larger  $d$  in footnotes.*

## 2.3 TY/DL Approach

In the common-frequency framework, Toda and Yamamoto (1995) as well as Dolado and Lütkepohl (1996) show the following simple strategy to achieve the desired outcome: instead of transforming the VAR, estimate it in levels, but augment the regressor set by an additional lag, i.e.,  $Z_{t-(p+1)}$ . Subsequently, test for Granger non-causality on the original coefficients (corresponding to  $Z_{t-1}, \dots, Z_{t-p}$ ) in the modified model. The reason why this approach leads to valid inference, also for  $I(1)$  process that are not cointegrated, goes back to an early contribution by Sims et al. (1990). They showed that parameters that can be re-written as coefficients on zero-mean  $I(0)$  regressors, have a standard asymptotic distribution. Here, it is important to notice that one does not need to re-write the model

---

<sup>9</sup>Note that as such tests are done for each variable individually, the MF nature of the variables plays a minor role here.

<sup>10</sup>To be precise, if the variables are aggregated with identical schemes (e.g., both end-of-period sampled or averaged), size distortions are null at best and mild at most; if, however, the underlying aggregation schemes differ, size distortions can be very severe (Ghysels and Miller, 2015). One can mitigate these problems by constructing a trace test on a MF-VECM or bootstrap the critical values of a LF residual-based test. However, even then there are instances in which size and power issues – albeit small – may occur.

accordingly, it is enough if it is theoretically possible to do so.

Providing more practical appeal to this result, let us consider the situation of common-frequency VAR, where  $Z_t^{CF} = (y_t, x_t)'$  is  $I(1)$  and  $CI(1, b)$  for some  $b$ . A well-known way to re-write this model in accordance with the finding of Sims et al. (1990) is the ECM format: due to cointegration, all coefficients (which are transformations of the parameters in the original VAR in levels) are assigned to stationary regressors. In the absence of cointegration, however, one of the regressors (depending on how we transform the original model, i.e., which lag  $i \in 1, \dots, p$  we capture the long-run term with) will remain  $I(1)$ . Now, imagine we add  $Z_{t-(p+1)}$  to the VAR in levels and re-write the model as follows:

$$\begin{aligned} Z_t^{CF} &= \sum_{i=1}^p A_i Z_{t-i}^{CF} + A_{p+1} Z_{t-(p+1)}^{CF} + \varepsilon_t \\ \Leftrightarrow \Delta_{p+1} Z_t^{CF} &= \sum_{i=1}^p A_i \Delta_{(p+1)-i} Z_{t-i}^{CF} - \Pi Z_{t-(p+1)}^{CF} + \varepsilon_t, \end{aligned}$$

where  $\Pi = I - \sum_{i=1}^p A_i$  and  $\Delta_j Z_t^{CF} = Z_t^{CF} - Z_{t-j}^{CF}$ . Hence, no matter whether the series are cointegrated, the standard Wald test applies to the coefficients corresponding to the first  $p$  (stationary) regressors.<sup>11</sup> Said differently – using the terminology in Sims et al. (1990) or Toda and Phillips (1994) – in case of an  $I(1)$  system, one needs “enough cointegration” or additional lags to account for the nondegenerate stochastic trends.

The MF analogue of this approach is thus to replace  $Z_t^{CF}$  by  $Z_t$  and estimating the MF-VAR in levels, thereby obtaining  $\hat{a}_{p+1} = \text{vec}(A_1, \dots, A_p, A_{p+1}) \equiv \text{vec}(A^*)$ . The modified Wald test is then still applied on the  $mp$  elements corresponding to the GC-relevant elements in  $A$ ; in terms of null hypothesis and Wald test:

$$\begin{aligned} H_0^* : \quad & R^* a_{p+1} = \mathbf{0}_{mp \times 1}, \\ W^* &= (R^* \hat{a}_{p+1})' (R^* \hat{\Omega}^* R^{*'})^{-1} (R^* \hat{a}_{p+1}), \end{aligned}$$

---

<sup>11</sup>We refer to Dolado and Lütkepohl, 1996, p. 372 for this illustrating example and Theorem 1 of the same paper for the formal result.

where  $\hat{\Omega}^* = (A^*A^*)^{-1} \otimes \hat{\Sigma}_u^*$  and  $\hat{\Sigma}_u^*$  being a consistent estimator of the covariance matrix in the modified MF-VAR. We refer to  $W^*$  based on Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996) as “TY/DL-test”.

Of course, the robustness of the TY/DL-test to the (co)integration order of the system does not come freely. Intentionally over-fitting the model in this way quickly leads to inefficiencies in case the adjustment is not necessary. This cost is obviously higher in a MF-VAR as an extra lag adds  $(n_l + mn_h)^2$  (in the two-variable system  $(m + 1)^2$ ) coefficients to be estimated. But there are ways to decrease these costs or to get rid of them altogether...

## 2.4 Mixed-Frequency Approach

Under the null hypothesis, it is clear that one cannot do better than designing an asymptotically valid inference method. While the TY/DL-test does so irrespectively of the (co)integration order of the series involved, it may – in small samples – suffer from size distortions and inefficiencies by intentionally over-fitting the model. We aim for an approach that may keep such issues at bay, while still providing an asymptotically valid test. We propose two approaches: the “MF-dep-test”, indicating that this procedure depends on the GC testing direction, and an alternative “MF-indep-test”.

### 2.4.1 MF-dep-test

**Testing  $H_0^{HF \leftrightarrow LF}$ :** We start by considering the test direction from the high- to the low-frequency series, i.e., the arguably more interesting case in practice given that one is often interested in evaluating the effects of a HF indicator (e.g., a survey variable) on a LF aggregate (e.g., GDP). We propose the following procedure:

- Estimate the MF-VAR in (2) in levels and obtain  $\hat{a}_p$  as in Section 2.2, i.e., without augmenting the system with an additional lag.
- Consider the  $mp$  GC-relevant coefficients, i.e., the ones appearing in (3). Construct

two Wald statistics corresponding to (i) the  $(1, 2)$ -element of each autoregressive matrix being equal to zero and (ii) the  $(1, 3)$  up to  $(1, m + 1)$ -elements of each  $A$ -matrix being jointly equal to zero.<sup>12</sup> For accordingly constructed selection matrices  $R^{HF1}$  and  $R^{HF2}$ , this boils down to

$$\begin{aligned} H_0^{HF1} : & \quad R^{HF1} a_p = \mathbf{0}_{p \times 1}, \\ H_0^{HF2} : & \quad R^{HF2} a_p = \mathbf{0}_{p(m-1) \times 1}, \\ W^{HFj} = & \quad (R^{HFj} \hat{a}_p)' (R^{HFj} \hat{\Omega}^{HFj} R^{HFj'})^{-1} (R^{HFj} \hat{a}_p) \end{aligned}$$

for  $j = 1, 2$  and properly constructed matrices  $\hat{\Omega}^{HFj}$ .

- Compare the corresponding  $p$ -values of the  $\chi^2(p)$ - and  $\chi^2(p(m - 1))$ -distributed test statistics to  $\alpha/2$ , where  $\alpha$  represents the significance level, i.e., apply a Bonferroni correction to account for the fact that we want test both null hypotheses jointly (Dunn, 1961).
- Reject  $H_0^{HF \rightarrow LF}$  if you reject  $H_0^{HF1}$ ,  $H_0^{HF2}$  or both; otherwise do not reject  $H_0^{HF \rightarrow LF}$ .

In this case, we can hold on to the original model in (2), i.e., no intentional over-fitting is necessary! This means we can stick to the usual and simple MF-VAR( $p$ ) model in levels, a remarkably convenient outcome, especially for applied work. Hopefully avoiding eventual inefficiencies comes at a rather cheap price: all we have to do is compute two Wald statistics instead of one.

The intuition for this finding rests on the fact that the stacked VAR structure provides us with “enough [or] sufficient cointegration”, in the sense of Toda and Phillips (1994). To be more precise, the HF variables – provided they are  $I(1)$  – are trivially cointegrated with each other, i.e.,  $m - 1$  cointegrating relationships are a-priori known (Götz et al., 2013). Hence, the absence of additional cointegration just forces us to test at most  $m - 1$

---

<sup>12</sup>For  $d > 1$ , we propose to apply tests on (i) the elements  $(1, 2)$  up to  $(1, 1 + d)$  and (ii) the elements  $(1, d + 2)$  up to  $(1, m + 1)$  of each autoregressive matrix. In case  $d \geq m$  one does not get around adding  $X_{t-(p+1)}^{(m)}$  to the MF-VAR, i.e., what would be done for the TY/DL-test. But such a case should hardly occur in practice.

coefficients at a time.<sup>13</sup> Loosely speaking, one could say that we apply the argument of Toda and Yamamoto (1995) and Dolado and Lütkepohl (1996) “backwards”: instead of looking at  $m + 1$  coefficients, of which  $m$  are tested on, we look at  $m$  and test on  $m - 1$  (two times).

Importantly, note in the presence of additional cointegration between  $x$  and  $y$ , the extra cointegrating relationship compensates for the one missing linear combination among the trivial ones; the standard approach, i.e., testing zero restrictions on all  $m$  coefficients per autoregressive matrix jointly, would thus have sufficed (Lütkepohl and Reimers, 1992).

**Testing  $H_0^{LF \rightarrow HF}$ :** Let us now consider the reverse test direction, i.e., the one from the low- to the high-frequency series. Albeit being less common, this situation is still of interest. Apart from being complete on the issue, there are (macro)economic examples such as quarterly capacity utilization rates (e.g., in Germany), which may affect indicators like monthly industrial production. Here is what we propose for this test direction:

- Augment the MF-VAR in (2) by adding the regressor  $y_{t-(p+1)}$  to each equation.<sup>14</sup>

$$Z_t = \sum_{i=1}^p A_i Z_{t-i} + A_{p+1}^{(:,1)} y_{t-(p+1)} + v_t$$

with  $A_{p+1}^{(:,1)}$  being an  $(m + 1)$ -vector (the notation resembling the similarity to the first column of matrix  $A_{p+1}$  used for the TY/DL-test). Estimate the model in levels, thereby obtaining  $\hat{a}_{p+\frac{1}{m+1}} = \text{vec}(\hat{A}_1, \dots, \hat{A}_p, \hat{A}_{p+1}^{(:,1)}) = \text{vec}(A^{LF})$ .

- Consider the usual  $mp$  GC-relevant coefficients, i.e., the ones appearing in (4). For an accordingly constructed selection matrix  $R^{LF}$ , the null hypothesis and Wald

---

<sup>13</sup>Alternatively to splitting the  $m$  coefficients as proposed here, one may test  $m - 1$  coefficients two times, once element (1, 2) up to (1,  $m$ ) of each  $A$ -matrix and once elements (1, 3) up to (1,  $m+1$ ). However, the presence of overlapping coefficients makes the subsequent Bonferroni correction overly conservative. Likewise, testing each of the  $m$  sets of coefficients individually is asymptotically valid, yet complicates a joint consideration of the  $m$  respective null hypotheses.

<sup>14</sup>Generally, i.e., also for  $d > 1$ , one has to add regressors  $y_{t-(p+1)}, \dots, y_{t-(p+d)}$  to each equation.

statistic read

$$H_0^{LF} : \quad R^{LF} a_{p+\frac{1}{m+1}} = \mathbf{0}_{mp \times 1},$$

$$W^{LF} = (R^{LF} \hat{a}_{p+\frac{1}{m+1}})' (R^{LF} \hat{\Omega}^{LF} R^{LF'})^{-1} (R^{LF} \hat{a}_{p+\frac{1}{m+1}})$$

with  $\hat{\Omega}^{LF} = (A^{(LF)'} A^{(LF)})^{-1} \otimes \hat{\Sigma}_v^{LF}$ , the latter being a consistent estimator of the covariance matrix in the augmented model.

- Inspect the  $p$ -value of the  $\chi^2(mp)$ -distributed test statistic to decide upon  $H_0^{LF \leftrightarrow HF}$ .

Here, the situation is a bit different, because the LF variable  $y$  is not part of any trivial cointegrating relationship. Hence, we will not get around performing some sort of adjustment along the lines of the TY/DL-test. Yet, in contrast to how the straightforward extension described in the previous subsection works, we want to limit the amount of over-fitting as much as possible. As we only require  $y_{t-(p+1)}$  for being able to re-write the model in an ECM-fashion (see the example in Section 2.3), we propose to merely add this regressor to each equation of the system. This implies an addition of merely  $m + 1$  coefficients to be estimated, in contrast to the  $(m + 1)^2$  for the TY/DL-test.

#### 2.4.2 MF-indep-test

In contrast to the approach of Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996), the MF-dep-test depends on the direction of GC we are interested in. If one aims for GC testing in both directions using the same estimated model, one does not get around an adjustment similar to the TY/DL-test. To be precise, one would need to add at least  $y_{t-(p+1)}$  and *one* HF observation from period  $t - (p + 1)$ , e.g.,  $x_{t-(p+1)}^{(m)}$ . Of course, one would sacrifice efficiency as far as testing for GC from  $x$  to  $y$  is concerned. Still, the amount of over-fitting is smaller than for the TY/DL-test, as one needs to estimate  $2(m + 1)$  additional coefficients instead of  $(m + 1)^2$ .

## 2.5 Overview

Now that we have introduced each of the potential ways to test for GC in both directions, Table 1 provides a small overview. To this end, revisit the most general reduced-form MF-VAR( $p$ ) in (2), i.e.,  $Z_t = AZ_{-p} + u_t$  with  $A = (A_1, \dots, A_p)$  and  $Z_{-p} = (Z'_{t-1}, \dots, Z'_{t-p})'$ , for the vector  $Z_t = (y_t, X_t^{(m)})'$  with  $X_t^{(m)} = (x_t^{(m)}, x_{t-1/m}^{(m)}, \dots, x_{t-(m-1)/m}^{(m)})'$ . Let the set of coefficients corresponding to testing for GC from the HF to the LF series in the standard approach, i.e., the ones in, be denoted as

$$A^{HF \rightarrow LF} = A_1^{(1,2)}, \dots, A_p^{(1,m+1)}, \dots, A_p^{(1,2)}, \dots, A_p^{(1,m+1)},$$

and likewise for the reverse direction, i.e., the ones in (4),

$$A^{LF \rightarrow HF} = A_1^{(2,1)}, \dots, A_1^{(m+1,1)}, \dots, A_p^{(2,1)}, \dots, A_p^{(m+1,1)}.$$

Finally, let “ $C \setminus D$ ” denote “ $C$  without  $D$ ”.

## 3 Theoretical Background and Simulations

### 3.1 Partially cointegrated systems

The validity of the proposed approach in this paper – at least as far as testing the direction from the HF to the LF series is concerned – rests on the theoretical framework in Toda and Phillips (1994), which we revisit here. While our approach for testing the reverse direction may also be validated using similar grounds, the methodology underlying both the MF-dep- and the MF-indep-test is, in fact, broadly identical to the TY/DL-procedure. Hence, we refer to the respective papers in this case.

Recall that we consider a two-variable MF system, where we want to test whether the  $m$  series corresponding to the HF variable Granger cause  $y$ . Now, let  $\Pi_\theta$  be the  $(m+1) \times r$ -matrix of  $r$  cointegrating vectors and let  $\Pi_{\theta, g_1: g_2}$  denote its rows  $g_1$  up to  $g_2$ .



Given a set of fairly mild assumptions, some of which are already encompassed by the ones we stated at the beginning of Section 2 and the others easily transferable (see Toda and Phillips, 1994, p. 261), the following holds:

**Theorem 1** *Suppose we are under the following null hypothesis:*

$$H_0 : A_i^{(1,g_1)} = \dots = A_i^{(1,g_2)} = 0 \quad \forall i$$

for  $2 \leq g_1 \leq g_2$ . Then, if  $\text{rank}(\Pi_{\theta,g_1:g_2}) = g (\leq n_g = g_2 - g_1 + 1)$ , we obtain the following for the corresponding test statistic:

$$W^{HF} \rightarrow^d \chi^2(n_g(p-1) + g) + \mathbb{1}_{(g < n_g)} TNP,$$

where *TNP* denotes a term (nonstandard distribution) that depends on nuisance parameters, that in turn depend on the long-run covariance matrix. Note that *TNP* cancels for  $g = n_g$ , though (labeled using the indicator function).

This Theorem has a series of implications, directly related to our situation. Corrolary 1 shows that the standard approach works even if the series are  $I(1)$ , provided they are cointegrated. In case cointegration between  $x$  and  $y$  is absent, though, the test converges to a mixture of a chi-square and a nonstandard distribution, the latter depending on nuisance parameters (Corrolary 2).

**Corrolary 1** *Suppose there is cointegration between the  $I(1)$ -series  $x$  and  $y$  and we test the entire set of coefficients corresponding to the HF series, i.e.,  $g_1 = 2$  and  $g_2 = m + 1$ , implying  $n_g = m$ . Then, as  $\text{rank}(\Pi_{\theta,2:m+1}) = m$  (see Götzt et al., 2013) we have that  $W^{HF} \rightarrow^d \chi^2(mp)$ .*

**Corrolary 2** *Suppose there is no cointegration between the  $I(1)$ -series  $x$  and  $y$  and we test the entire set of coefficients corresponding to the HF series, i.e.,  $g_1 = 2$  and  $g_2 = m + 1$ , implying  $n_g = m$ . Then, as  $\text{rank}(\Pi_{\theta,2:m+1}) = m - 1$  (see Götzt et al., 2013) we have that  $W^{HF} \rightarrow^d \chi^2(mp - 1) + TNP$ .*

Now, the MF-dep-test rests on computing two Wald statistics, that are subsequently combined using the Bonferroni approach. For each of the individual tests, we have the following for the cointegration- and the no-cointegration-case (Corrolary 3):

**Corrolary 3** *Suppose the series  $x$  and  $y$  are  $I(1)$  and we test two sets of coefficients corresponding to the HF series: (i) for  $g_1 = 2$  and  $g_2 = 2$ , implying  $n_g = 1$  and (ii) for  $g_1 = 3$  and  $g_2 = m + 1$ , implying  $n_g = m - 1$ . For (i)  $\text{rank}(\Pi_{\theta,2:2}) = 1$  s.t.  $W^{HF} \rightarrow^d \chi^2(p)$  and for (ii)  $\text{rank}(\Pi_{\theta,3:m+1}) = m - 1$  s.t.  $W^{HF} \rightarrow^d \chi^2((m - 1)p)$ .*

## 3.2 Monte Carlo study

### 3.2.1 Setup

In order to investigate size and power properties of the various test versions in finite samples, we conduct a series of simulation experiments. In particular, we assess the sensitivity of the results with respect to the sample size ( $T = 50, 150, 250$ ) and the frequency discrepancy ( $m = 2, 3, 4$ ). All simulations are based on a 10,000 replications of the respective DGP and plot the rejection frequencies of the test statistics in Table 1.

We start by describing our MF-DGP, i.e., the data are truly generated at mixed frequencies (see Remark 3), flexibly incorporating the different possible features of the data. Due to the potential presence of cointegration, through which GC in one direction is present by construction, we need to differentiate the DGP for both test directions. Hence, let  $y_t$  and  $x_t^{(m)}$  be generated by one of the following systems, depending on whether we inspect...

...GC from  $x$  to  $y$ , i.e.,

$$y_t = \rho y_{t-1} + \sum_{j=0}^{m-1} \lambda_j \Delta x_{t-1-j/m}^{(m)} + \epsilon_{y,t}, \quad (5)$$

$$x_{t-j/m}^{(m)} = \theta y_t + v_{x,t-j/m}^{(m)}, \text{ where } v_{x,t-j/m}^{(m)} = (\alpha + 1)v_{x,t-(j+1)/m}^{(m)} + \epsilon_{x,t-j/m}^{(m)}, \quad (6)$$

..., or GC from  $y$  to  $x$ , i.e.,

$$y_t = \theta x_t^{(m)} + v_{y,t}, \text{ where } v_{y,t} = (\alpha + 1)v_{y,t-1} + \epsilon_{y,t}, \quad (7)$$

$$x_{t-j/m}^{(m)} = \rho x_{t-(j+1)/m}^{(m)} + \delta_j \Delta y_{t-1} + \epsilon_{x,t-j/m}^{(m)}, \quad (8)$$

where  $\epsilon_{y,t}, \epsilon_{x,t-j/m}^{(m)} \sim N(0, 1)$ ,<sup>15</sup>  $j = 0, \dots, m-1$  and  $-2 \leq \alpha \leq 0$ .

Note that (5) contains a U-MIDAS-type (Forni et al., 2015b) impact of the HF series on  $y$ , and that (8) features a similar effect of past LF-differences on  $x$ .<sup>16</sup> This setup allows us to look at the consequences of  $I(1)$ -ness as well as cointegration simultaneously or in isolation. In case  $-1 < \rho < 1$  (i.e.,  $x$  or  $y$  is  $I(0)$  depending on the DGP), the value of  $\alpha$  determines whether the other series is  $I(0)$  as well ( $-2 < \alpha < 0$ ) or whether it is  $I(1)$  ( $|\alpha + 1| = 1$ ). In case  $|\rho| = 1$  (i.e.,  $x$  or  $y$  is  $I(1)$  depending on the DGP),  $\alpha$  controls the presence or absence of cointegration, respectively. In the cointegrated case,  $\theta$  then governs the cointegrating relationship.

After some manipulations, both DGPs can be re-written into a reduced-form MF-VAR(2) in levels, i.e.,

$$Z_t = A_1 Z_{t-1} + A_2 Z_{t-2} + u_t,$$

where  $u_t = A^* u_t^*$  with  $u_t^* = (\epsilon_{y,t}, \epsilon_{x,t}^{(m)}, \epsilon_{x,t-1/m}^{(m)}, \dots, \epsilon_{x,t-(m-1)/m}^{(m)})' \sim N(\mathbf{0}_{m+1 \times 1}, I_{m+1})$  such that  $\Sigma_u = A^* A^*$ ; precise formulae for  $A_1, A_2$  and  $A^*$  under both DGPs are being delegated to Appendix B.

**Remark 6** *To show that these DGPs nest the case of a cointegrated system with trivial cointegrating relationships, consider the example of  $m = 3$  and  $\rho = 1$  such that both  $y$  and  $x$  are  $I(1)$ . Using the formulae for  $A_1, A_2$  and  $A^*$  in the Appendix, the reduced-form MF-VARs can be re-written into the following VECMs (Götz et al., 2013 or Ghysels and Miller, 2015).*

<sup>15</sup>Hence,  $\sigma_y^2 = \sigma_x^2 = 1$ .

<sup>16</sup>Note that one could leave the impact of the lagged series constant over one  $t$ -period, significantly simplifying notation. We, however, thought this specification to be of more empirical value, especially in light of the commonly used U-MIDAS model. Moreover, one could also consider lags of LF differences in (5) or of HF differences in (8), somewhat unnecessarily complicating the notation, though.

For GC from  $x$  to  $y$ , i.e., the DGP in (5) and (6):

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t^{(3)} \\ \Delta x_{t-1/3}^{(3)} \\ \Delta x_{t-2/3}^{(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ -\theta\alpha(\alpha^2 + 3\alpha + 3) & \alpha(\alpha^2 + 3\alpha + 3) & 0 & 0 \\ -\theta\alpha(\alpha + 2) & \alpha(\alpha + 2) + 1 & -1 & 0 \\ -\theta\alpha & \alpha + 1 & 0 & -1 \end{bmatrix}}_{\Pi} \begin{bmatrix} y_{t-1} \\ x_{t-1}^{(3)} \\ x_{t-4/3}^{(3)} \\ x_{t-5/3}^{(3)} \end{bmatrix} + \begin{bmatrix} 0 & \lambda_0 & \lambda_1 & \lambda_2 \\ 0 & \theta\lambda_0 & \theta\lambda_1 & \theta\lambda_2 \\ 0 & \theta\lambda_0 & \theta\lambda_1 & \theta\lambda_2 \\ 0 & \theta\lambda_0 & \theta\lambda_1 & \theta\lambda_2 \end{bmatrix} \begin{bmatrix} \Delta y_{t-2} \\ \Delta x_{t-2}^{(3)} \\ \Delta x_{t-7/3}^{(3)} \\ \Delta x_{t-8/3}^{(3)} \end{bmatrix} + u_t.$$

For GC from  $y$  to  $x$ , i.e., the DGP in (7) and (8):

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t^{(3)} \\ \Delta x_{t-1/3}^{(3)} \\ \Delta x_{t-2/3}^{(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & -\theta\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{\Pi} \begin{bmatrix} y_{t-1} \\ x_{t-1}^{(3)} \\ x_{t-4/3}^{(3)} \\ x_{t-5/3}^{(3)} \end{bmatrix} + \begin{bmatrix} \theta(\delta_0 + \rho\delta_1 + \rho^1\delta_2) & 0 & 0 & 0 \\ \delta_0 + \rho\delta_1 + \rho^1\delta_2 & 0 & 0 & 0 \\ \delta_0 + \rho\delta_1 & 0 & 0 & 0 \\ \delta_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta y_{t-2} \\ \Delta x_{t-2}^{(3)} \\ \Delta x_{t-7/3}^{(3)} \\ \Delta x_{t-8/3}^{(3)} \end{bmatrix} + u_t.$$

Indeed then, for  $\alpha = 0$  the matrix  $\Pi$  (in both DGPs) merely contains the trivial cointegrating relationships among the HF variable itself. For  $-2 < \alpha < 0$ , though, there is an additional cointegrating relationship between  $y$  and  $x$  of the form  $(-\theta, 1)$  or  $(1, -\theta)$ , respectively. To see this, note that one can write  $\Pi = \Pi_\alpha \Pi_\theta$ , where  $\Pi_\alpha$  and  $\Pi_\theta$  are both  $(4 \times 3)$ -matrices and where the columns of  $\Pi_\theta$  contain the aforementioned cointegrating

*relationships.*

Now, the null hypotheses for both test directions (under their respective DGP) are easily seen to be:

$$H_0^{HF \rightarrow LF} : \lambda_j = 0 \quad \forall j = 0, \dots, m-1,$$

$$H_0^{LF \rightarrow HF} : \delta_j = 0 \quad \forall j = 0, \dots, m-1.$$

We will consider three cases: (a)  $y$  and  $x$  are  $I(0)$  by setting  $\rho = 0.8$  and  $\alpha = -0.5$ , (b)  $y$  and  $x$  are  $I(1)$  and cointegrated by setting  $\rho = 1$  and  $\alpha = -0.5$ , (c)  $y$  and  $x$  are  $I(1)$  but not cointegrated by setting  $\rho = 1$  and  $\alpha = 0$ . Throughout the simulations,  $\theta = 0.5$ . We set the GC-determining coefficients to  $\lambda = T^{-0.5}\lambda^*$  for  $\lambda^* = \{0, 1, 2\}$ ; likewise for  $\delta$ . Note that values of zero imply no GC (size), whereas non-zero values imply GC (power).<sup>17</sup> The results are summarized in (a) Tables 2 and 3, (b) Tables 4 and 5 and (c) Tables 6 and 7.

### 3.2.2 Results

The stationary scenario is obviously favourable for the standard test, which is asymptotically  $\chi^2(mp)$ -distributed. While incurring some size distortions for small  $T$ , it leads to the correct size for larger sample sizes. Furthermore, this test is the most powerful one for each  $(T, m, \lambda^*/\delta^*)$ -combination as it does not feature any alteration to the estimated model. The TY/DL- and the MF-indep-test perform almost identically, the latter having slightly higher rejection rates under the alternative, because  $m^2 - 1$  fewer parameters need to be estimated. Compared to the standard test, though, power is clearly inferior. Interestingly, the MF-dep-test seems to cope somewhat better with size distortions arising from small  $T$ , particularly when testing the empirically more interesting GC-direction from  $x$  to  $y$ . Granted, it still falls short of the standard test's power, yet it beats the TY/DL- and MF-indep-tests, clearly so again for testing  $HF \rightarrow LF$ . Overall, the MF-dep-test presents a more than compelling approach for  $I(0)$ -MF-series.

---

<sup>17</sup>This specification of the GC-determining coefficients is derived from the Monte Carlo study in Dolado and Lütkepohl (1996) for the common-frequency case.

Let us turn to the more interesting cases, in which  $x$  and  $y$  are non-stationary, and start with the cointegrated scenario (b). From Corollary 1 it follows that the standard test is in fact still favoured in this situation. Together with the trivial cointegrating relationships among the HF series, the presence of additional cointegration between  $x$  and  $y$  implies “enough cointegration” (Toda and Phillips, 1994) to yield a  $\chi^2(mp)$ - distribution of the standard Wald test. As a consequence, the outcomes in Tables 4 and 5 are qualitatively identical to the ones for the stationary case (a) in Tables 2 and 3.

Finally, inspecting the non-stationary and non-cointegrated (w.r.t.  $x$  and  $y$ ) case (c) in Tables 6 and 7 reveals that the standard test does not have a  $\chi^2(mp)$  under  $H_0$  (Corrolary 2). Indeed, even for large  $T$ , actual size clearly exceeds the nominal level of 5%. The TY/DL- and MF-indep-tests, on the other hand, constitute asymptotically valid tests, although they are – as before – somewhat oversized for a small sample size. In terms of power, though, they perform rather well, more so given that size-adjusted power of the standard test would surely turn out to be smaller than the size-unadjusted figures presented here.

As far as the MF-dep-test is concerned, let us discuss the outcomes for the two GC test directions separately. For testing GC from  $y$  to  $x$  the approach is almost identical to the MF-indep-test (and the TY/DL-test for that matter); only that fewer parameters need to be estimated (revisit Section 2.4). In terms of size, the outcomes are thus by and large comparable, whereas power is either as high as or even slightly higher when using the MF-dep-test. For testing GC from  $x$  to  $y$ , asymptotic validity of the MF-dep-test is also confirmed (see Corrolary 3 and – for the Bonferroni correction – Dunn, 1961), but more noteworthy the test appears far less oversized than its competitors: even for  $T = 150$  and  $m = 4$  empirical size coincides with the nominal one. The cost of this presumably controlling effect the Bonferroni correction has is a loss of power, which is most pronounced for cases closer to the common-frequency setup, i.e., the smaller  $m$  is. Indeed, for larger  $m$  the benefits in terms of efficiency counterweigh the conservative nature of the Bonferroni adjustment.

To sum up, the MF approaches present competitive and easy-to-implement alternatives to the existing TY/DL-approach. Here, the MF-indep-test yields marginally better results throughout the entire analysis, which is not too surprising given the only slight adjustment in test design. The MF-dep-test is mostly the preferred choice, although in the absence of cointegration between  $x$  and  $y$  the merits of a correctly sized test – also for small samples – stand against somewhat lower power. From an empirical and conservative standpoint, however, the MF-dep-test is the dominant strategy.

**Remark 7** *We also experimented with an alternative, equally accepted way to perceive the DGP underlying the observable MF data (Ghysels and Miller, 2015, Ghysels et al., 2016 or Miller, 2014): that the data are all generated at the common HF, yet the ones observable at the LF contain latent observations. To be more precise, we considered a DGP that is directly derived from the setup in Dolado and Lütkepohl (1996) for the common-frequency case; see Appendix C for explicit formulae. In line with most macroeconomic applications, we temporally aggregated the  $mT$  HF-observations of  $y$  using the simple average of the  $m$  values corresponding to each  $t$ -period.*

*While the outcomes were broadly in line with the ones for the MF-DGP as far as GC from  $y$  to  $x$  are concerned (with somewhat lower power, though), the results for testing GC from  $x$  to  $y$  revealed the shortcomings of such a DGP: as shown by Ghysels et al. (2016), depending on the aggregation scheme, Granger non-causality will not be preserved when moving from a HF- to a MF-VAR. In particular, “a crucial condition for non-causality preservation is that the information for the [variable that is caused by the other under  $H_A$ , i.e.,  $y$ ] is not lost by temporal aggregation” (p. 216). This condition is only satisfied in some specific, simple cases and surely not in the scenarios of most interest for us here (averaging  $y$ ,  $I(1)$ -ness, potential cointegration).<sup>18</sup>*

---

<sup>18</sup>The results of the Monte Carlo experiments based on a HF-DGP are not displayed here to save on space, but are available upon request.

## 4 Application

To illustrate our approach with actual data, we consider two empirical applications. The first one involves the weekly WTI oil price traded in New York and denoted in US-Dollar per barrel (OIL) as well as the monthly consumer price index in Germany (CPI), i.e.,  $m = 4$ .<sup>19</sup> The series were downloaded from the internal database of the Deutsche Bundesbank and refer to the period from January 1991 to May 2018.<sup>20</sup> According to economic theory one would usually expect OIL to Granger cause CPI. First, OIL appears in the consumption basket of Germany consumers, be it directly (through, e.g., car fuel or energy production) or indirectly (through affecting other goods). Second, being a global indicator OIL may affect several countries and eventual uncertainties may also have implications for the German economy.

With respect to the integration order and especially the presence of a cointegrating relationship between the two series, the situation is, however, not at all clear a-priori. To illustrate this ambiguity we conducted a series of tests one would often pursue in practice. Standard Dickey-Fuller tests for a unit root in the log-levels of the series revealed that both series are in fact  $I(1)$ , with only marginal (and thus negligible) indications of a potential  $I(2)$ -ness of CPI. We use the Schwartz information criterion to determine the lag order of  $p = 1$ . As far as cointegration tests are concerned, the conclusions are somewhat discordant: on the one hand, the trace and maximum-eigenvalue-tests of Johansen (1991) both indicate – using the critical values of MacKinnon et al. (1999) and applied to the MF-VAR(1) – the presence of cointegration between  $x$  and  $y$  on top of the  $m - 1 = 3$  trivial long-run terms. On the other hand, the Engle and Granger (1987) two-step procedure – applied to MF data – points toward the no-cointegration scenario.<sup>21</sup> Clearly, a GC testing procedure that is robust to the cointegration order would thus be desirable.

---

<sup>19</sup>The underlying series for OIL is actually based on working days, which have been aggregated to the weekly frequency. Some months get assigned four weeks, whereas some even have five weeks. To simplify notation we delete the first weekly observation in the latter case.

<sup>20</sup>The data was downloaded on 21 June, 2018 implying that OIL would have been available for June even. Due to the publication delay of CPI, though, we consider a balanced dataset ending in May.

<sup>21</sup>Hereby, it did not matter which of the weekly observations of OIL we place into a potential cointegrating term.



To this end we compare the four different GC test approaches for both test directions.<sup>22</sup> We start by estimating the five-dimensional MF-VAR(1) in log-levels and apply the usual Wald test for the four respective coefficient estimates. It turns out that the  $p$ -values corresponding to the test statistics  $W_{OIL \rightarrow CPI}$  and  $W_{CPI \rightarrow OIL}$  are 0.001 and 0.013, respectively. The standard tests thus indicate bi-directional GC between CPI and OIL, a somewhat surprising result. For the TY/DL-test we simply estimate a MF-VAR(2) in log-levels and apply a Wald test on the original lag-one-coefficients. The  $p$ -values associated with  $W_{OIL \rightarrow CPI}^*$  and  $W_{CPI \rightarrow OIL}^*$  turn out to be 0.0001 and 0.199, respectively. Maybe cointegration between the two series is, in fact, absent, causing the standard test not to deliver asymptotically valid inference for the GC test from LF to HF; power may be fine as far as the detected causal link from OIL to CPI is concerned, however.

Let us see whether the MF-tests, which often tended to be less oversized or even more powerful, back up the finding of the TY/DL-test. For the MF-indep-test we simply add  $(CPI_{t-2}, OIL_{t-2})'$  to each equation of the MF-VAR(1), estimate the system and apply a Wald test on the same coefficients as before. The  $p$ -values become 0.0001 and 0.122, respectively. Finally, for the MF-dep-test and the test direction from CPI to OIL, we merely add  $y_{t-2}$  to each row of the MF-VAR(1) and apply the same procedure as before. The resulting  $p$ -value equals not less than 0.275; recalling that this was an instance, in which the MF-dep-test was clearly outperforming its competitors, it puts all the more weight on Granger non-causality from CPI to OIL. Testing the reverse direction boils down to two Wald tests on the original MF-VAR(1) in log-levels: once on the  $OIL_{t-1}$ -coefficient and once on the coefficients corresponding to  $OIL_{t-5/4}$ ,  $OIL_{t-6/4}$  and  $OIL_{t-7/4}$  jointly. The  $p$ -values of the two individual tests, which are based on a 0.025 level due to the Bonferroni correction, turn out to be 0.001 each. Overall, we thus find overwhelming evidence for uni-directional GC from OIL to CPI, in line with economic theory.

For the second application we consider quarterly GDP and the monthly industrial production index (IP) in Germany, i.e.,  $m = 3$ . Again, the series originate from the

---

<sup>22</sup>All calculations are easily doable in a software such as EViews, making the methods very appealing to practitioners, without the need for advanced programming skills.

database of the Deutsche Bundesbank and – being downloaded mid-June and achieving balancedness in view of publication lags – cover the period from January 1991 to March 2018.<sup>23</sup> Consequently,  $T = 109$  implying a much shorter sample size than before; recall that some approaches suffered from size distortions for small  $T$ . Economic theory may actually support bi- or uni-directional (from IP to GDP) causality. On the one hand, the HF series is an important determinant of the LF series and, particularly in Germany, has a large share in the country's National Accounts. On the other hand, GDP may cause IP as high economic activity last period may imply a continuously thriving economy with full order books that will keep industrial output expanding, too.

Like before, standard procedures are agreeing fully on the integration order,  $I(1)$ , as well as the lag length,  $p = 1$ , but disagree with respect to cointegration: the Johansen (1991) procedures again find  $m$  cointegrating relationships, i.e., two trivial ones and one additional long-run term for GDP and IP. The MF Engle and Granger (1987) approach, however, yields no cointegration. Going through the different GC test options – rather quickly – draws the following picture: the standard test finds bi-directional GC, whereby a  $p$ -value of 0.042 for  $W_{GDP \rightarrow IP}$  puts more doubt on this causal link than in the reverse direction ( $p$ -value of 0.0001). In fact, all tests clearly detect GC from IP to GDP. For the reverse direction, however, the MF-dep-test overwhelmingly rejects the null as well, the MF-indep-test is somewhat torn in the middle ( $p$ -value of 0.092) and the TY/DL-test concludes no GC ( $p$ -value of 0.487). It is to be expected that the small sample size affects the outcomes, as eventual inefficiencies are the main drivers behind differences between the three MF-tests, particularly for LF-to-HF GC tests. Given that the MF-dep-test is the sparsest one in this respect and that it showed quite robust results, we conclude bi-directional GC between GDP and IP.

---

<sup>23</sup>GDP for the second quarter of 2018 gets released in the middle of August.

## 5 Conclusion

In this paper we extended the method of Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996) on testing for causality in levels-VARs to the mixed-frequency scenario. Based on the fact that transformations of a VAR, that result from non-stationarities and/or cointegration among the series, rest on corresponding tests that are prone to size distortions and/or losses of power, we aimed for an approach that works independently from the variables' (co)integration properties. Apart from the straightforward application of the TY/DL-test – based on the aforementioned papers – to a mixed-frequency VAR, we propose two further test approaches that exploit the stacked nature of the VAR vector in the observation-driven model we employ (Ghysels, 2016). These methods come at smaller or even zero costs in terms of intentionally over-fitting the model; when testing for Granger causality in the empirically more common direction from the high- to the low-frequency series one mixed-frequency test approach is even based on the standard MF-VAR( $p$ ) in (log-)levels without any adjustment. The price one has to pay is to compute two standard Wald tests (on subsets of the relevant coefficients) instead of one. For the other instances, minor extensions of the model are sufficient to ensure asymptotically valid inference.

A Monte Carlo study revealed that the MF approaches are indeed competitive and easy-to-implement alternatives to the TY/DL-procedure. While one of the mixed-frequency tests (MF-indep) yielded only marginally but consistently better results throughout the entire analysis, the other one (MF-dep) proved to be the overall preferred choice. In the absence of cointegration between  $x$  and  $y$ , however, the merits of a correctly sized test – also for a small sample size – stand against somewhat lower power. Two applications involving (i) the consumer price index and the oil price, for which uni-directional causality from the latter to the former was concluded, as well as (ii) GDP and industrial production, where bi-directional causality was detected, illustrated the different test options and their practical appeal. Here, the ambiguous outcomes of standard cointegration tests motivated the use of robust Granger causality tests, whose outcomes were then shown to potentially

deviate from the ones of the standard Wald test.

The analysis presented here can and should obviously be extended along a couple of lines. Firstly, of course, the effects of estimating the lag length should be analyzed, i.e., Remark 2 should be relaxed. Secondly, the observation-driven MF-VAR model we consider here is not the only way to model a system of time series sampled at varying frequencies. One could investigate the extension of the TY/DL- test within a parameter-driven MF-VAR model à la Schorfheide and Song (2015). However, one should keep in mind that the absence of trivial cointegrating relationships, due to the non-stacked design of the system, may diminish the scope for efficiency improvements. Finally, the extension toward systems of larger size – either owing to a larger set of variables or a larger frequency mismatch – may be fruitful. Hopefully, the present paper leads to a well-deserved revival of this literature, that was somewhat pushed in the background in recent years. Sometimes apparently, a small adjustment of a simple model suffices to conduct valid inference; why complicate matters?

## References

- Blasques, F., Koopman, S., Mallee, M., and Zhang, Z. (2016). Weighted maximum likelihood for dynamic factor analysis and forecasting with mixed frequency data. *Journal of Econometrics*, 193(2):405–417.
- Chiu, C. W. J., Eraker, B., Foerster, A. T., Kim, T. B., and Seoane, H. D. (2011). Estimating VAR’s sampled at mixed or irregular spaced frequencies : a Bayesian approach. Technical report.
- Cimadomo, J. and D’Agostino, A. (2016). Combining time variation and mixed frequencies: an analysis of government spending multipliers in italy. *Journal of Applied Econometrics*, 31(7):1276–1290. jae.2489.
- Clements, M. and Galvão, A. B. (2008). Macroeconomic forecasting with mixed-frequency data. *Journal of Business & Economic Statistics*, 26:546–554.
- Dickey, D. A. and Fuller, W. A. (1979). Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American Statistical Association*, 74(366):427–431.
- Dolado, J. J. and Lütkepohl, H. (1996). Making wald tests work for cointegrated var systems. *Econometric Reviews*, 15(4):369–386.
- Dunn, O. J. (1961). Multiple comparisons among means. *Journal of the American Statistical Association*, 56(293):52–64.
- Engle, R. F. and Granger, C. W. J. (1987). Co-integration and Error Correction: Representation, Estimation, and Testing. *Econometrica*, 55(2):251–276.
- Foroni, C., Gurin, P., and Marcellino, M. (2015a). Markov-switching mixed-frequency var models. *International Journal of Forecasting*, 31(3):692 – 711.

- Foroni, C., Marcellino, M., and Schumacher, C. (2015b). Unrestricted mixed data sampling (MIDAS): MIDAS regressions with unrestricted lag polynomials. *Journal of the Royal Statistical Society Series A*, 178(1):57–82.
- Ghysels, E. (2016). Macroeconomics and the reality of mixed frequency data. *Journal of Econometrics*, 193(2):294 – 314.
- Ghysels, E., Hill, J. B., and Motegi, K. (2016). Testing for granger causality with mixed frequency data. *Journal of Econometrics*, 192(1):207 – 230.
- Ghysels, E., Hill, J. B., and Motegi, K. (2017). Testing a large set of zero restrictions in regression models, with an application to mixed frequency granger causality. Working paper.
- Ghysels, E. and Miller, J. I. (2015). Testing for cointegration with temporally aggregated and mixed-frequency time series. *Journal of Time Series Analysis*, 36(6):797–816. 10.1111/jtsa.12129.
- Ghysels, E., Santa-Clara, P., and Valkanov, R. (2004). The midas touch: Mixed data sampling regression models. CIRANO Working Papers 2004s-20, CIRANO.
- Ghysels, E., Sinko, A., and Valkanov, R. (2007). Midas regressions: Further results and new directions. *Econometric Reviews*, 26(1):53–90.
- Götz, T. and Hauzenberger, K. (2017). Large mixed-frequency vars with a parsimonious time-varying parameter structure. Working paper.
- Götz, T. B. and Hecq, A. (2014). Nowcasting causality in mixed frequency vector autoregressive models. *Economics Letters*, 122(1):74–78.
- Götz, T. B., Hecq, A., and Smeekes, S. (2016). Testing for granger causality in large mixed-frequency vars. *Journal of Econometrics*, 193(2):418 – 432.
- Götz, T. B., Hecq, A., and Urbain, J.-P. (2013). *Testing for Common Cycles in Non-Stationary VARs with Varied Frequency Data*, chapter 10, pages 361–393.

- Götz, T. B., Hecq, A., and Urbain, J.-P. (2014). Forecasting mixed frequency time series with ECM-MIDAS models. *Journal of Forecasting*, 33:198–213.
- Granger, C. W. J. (1969). Investigating causal relations by econometric models and cross-spectral methods. *Econometrica*, 37(3):424–438.
- Johansen, S. (1991). Estimation and hypothesis testing of cointegration vectors in gaussian vector autoregressive models. *Econometrica*, 59(6):1551–1580.
- Koelbl, L. and Deistler, M. (2018). A new approach for estimating var systems in the mixed-frequency case. *Statistical Papers*.
- Lütkepohl, H. (1993). *Testing for causation between two variables in higher dimensional VAR models*, pages 75–91. Studies in Applied Econometrics. Springer-Verlag, Heidelberg.
- Lütkepohl, H. and Reimers, H.-E. (1992). Granger-causality in cointegrated VAR processes The case of the term structure. *Economics Letters*, 40(3):263–268.
- MacKinnon, J., Haug, A., and Michelis, L. (1999). Numerical distribution functions of likelihood ratio tests for cointegration. *Journal of Applied Econometrics*, 14(5):563–77.
- Marcellino, M. and Schumacher, C. (2010). Factor MIDAS for Nowcasting and Forecasting with Ragged-Edge Data: A Model Comparison for German GDP. *Oxford Bulletin of Economics and Statistics*, 72(4):518–550.
- Mariano, R. S. and Murasawa, Y. (2003). A new coincident index of business cycles based on monthly and quarterly series. *Journal of Applied Econometrics*, 18(4):427–443.
- Miller, J. I. (2014). Mixed-frequency Cointegrating Regressions with Parsimonious Distributed Lag Structures. *Journal of Financial Econometrics*, 12(3):584–614.
- Miller, J. I. (2016). Conditionally Efficient Estimation of Long-Run Relationships Using Mixed-Frequency Time Series. *Econometric Reviews*, 35(6):1142–1171.

- Phillips, P. C. B. (1987). Time Series Regression with a Unit Root. *Econometrica*, 55(2):277–301.
- Phillips, P. C. B. and Perron, P. (1988). Testing for a unit root in time series regression. *Biometrika*, 75(2):335–346.
- Schorfheide, F. and Song, D. (2015). Real-time forecasting with a mixed-frequency VAR. *Journal of Business & Economic Statistics*, 33(3):366–380.
- Siliverstovs, B. (2017). Short-term forecasting with mixed-frequency data: a midasso approach. *Applied Economics*, 49(13):1326–1343.
- Silvestrini, A. and Veredas, D. (2008). Temporal aggregation of univariate and multivariate time series models: A survey. *Journal of Economic Surveys*, 22(3):458–497.
- Sims, C. A., Stock, J. H., and Watson, M. W. (1990). Inference in Linear Time Series Models with Some Unit Roots. *Econometrica*, 58(1):113–144.
- Toda, H. Y. and Phillips, P. C. B. (1993). Vector Autoregressions and Causality. *Econometrica*, 61(6):1367–1393.
- Toda, H. Y. and Phillips, P. C. B. (1994). Vector autoregression and causality: a theoretical overview and simulation study. *Econometric Reviews*, 13(2):259–285.
- Toda, H. Y. and Yamamoto, T. (1995). Statistical inference in vector autoregressions with possibly integrated processes. *Journal of Econometrics*, 66(1-2):225–250.
- Zadrozny, P. A. (2016). Extended YuleWalker identification of VARMA models with single- or mixed-frequency data. *Journal of Econometrics*, 193(2):438–446.



# A Tables & Figures

Table 1: Overview of GC tests

Test	Direction	Regress $Z_t$ on...	Tested Coefficients
Standard	HF $\rightarrow$ LF	$\dots Z_{-p}$	$A^{HF \rightarrow LF}$
	LF $\rightarrow$ HF		$A^{LF \rightarrow HF}$
TY/DL	HF $\rightarrow$ LF	$\dots Z_{-(p+1)}$	$A^{HF \rightarrow LF}$
	LF $\rightarrow$ HF		$A^{LF \rightarrow HF}$
MF-dep	HF $\rightarrow$ LF	$\dots Z_{-p}$	(i) $A_1^{(1,2)}, A_2^{(1,2)}, \dots, A_p^{(1,2)}$
	LF $\rightarrow$ HF	$\dots Z_{-p}, y_{t-(p+1)}$	(ii) $A^{HF \rightarrow LF} \setminus \{A_1^{(1,2)}, A_2^{(1,2)}, \dots, A_p^{(1,2)}\}$
MF-indep	HF $\rightarrow$ LF	$\dots Z_{-p}, \begin{pmatrix} y_{t-(p+1)} \\ x_{t-(p+1)}^{(m)} \end{pmatrix}$	$A^{HF \rightarrow LF}$
	LF $\rightarrow$ HF		$A^{LF \rightarrow HF}$

Note: Definitions of  $A^{HF \rightarrow LF}$ ,  $A^{LF \rightarrow HF}$  and “ $\setminus$ ” are provided in Section 2.5.

Table 2: Granger Causality Tests; HF  $\rightarrow$  LF; Both series  $I(0)$

Standard		TY/DL		MF-dep		MF-indep						
$m = 2$		$m = 2$		$m = 2$		$m = 2$						
$T$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$			
50	6.9	13.9	41.4	7.2	11.6	30.1	6.3	11.9	33.7	7.2	11.8	30.9
150	6.0	13.7	41.8	5.9	11.6	34.1	5.6	12.0	34.5	5.9	11.6	34.4
250	5.1	13.7	41.8	5.1	11.7	34.7	5.2	11.8	34.9	5.1	11.7	34.9
$m = 3$		$m = 3$		$m = 3$		$m = 3$						
$T$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$
50	7.6	16.8	48.9	7.9	13.0	30.4	6.7	14.0	40.4	7.7	13.2	33.7
150	5.7	15.0	50.2	5.9	11.7	35.5	5.5	12.9	42.7	6.0	11.9	37.0
250	5.5	15.3	50.3	5.6	12.3	37.1	5.2	13.5	42.7	5.6	12.5	37.9
$m = 4$		$m = 4$		$m = 4$		$m = 4$						
$T$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$
50	7.9	19.8	61.5	9.0	14.3	29.4	7.2	16.0	49.9	8.7	14.7	35.0
150	5.7	16.4	58.6	5.7	12.0	33.5	5.3	13.5	49.6	5.8	12.1	35.8
250	5.4	16.3	59.3	5.6	11.3	35.8	5.3	13.7	50.7	5.6	11.7	37.7

Note: The figures represent rejection rates of the various Granger Non-Causality tests described in Sections 2.2 - 2.4 (and summarized in Section 2.5). The underlying DGP is a reduced-form MF-VAR(2) in levels corresponding to Equations (5) and (6), implying that there is no Granger Causality from  $x$  to  $y$ , i.e., HF  $\rightarrow$  LF. Here,  $\rho = 0.8$ ,  $\alpha = -0.5$  and  $\theta = 0.5$  such that both series are stationary, i.e.,  $I(0)$ . Moreover,  $\lambda = T^{-0.5}\lambda^*$ , whereby  $\lambda^* = 0$  indicates the size and  $\lambda^* = \{1, 2\}$  the power of the test. The nominal level is set to 5%.



Table 4: Granger Causality Tests; HF  $\rightarrow$  LF; Both series  $I(1)$ ; Cointegration

	Standard		TY/DL		MF-dep		MF-indep					
	$m = 2$		$m = 2$		$m = 2$		$m = 2$					
$T$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$			
50	6.9	14.2	41.4	6.8	12.4	30.6	6.3	11.7	33.2	6.8	12.4	31.3
150	6.0	14.0	43.3	5.9	11.8	34.4	5.7	11.9	35.2	6.0	11.8	34.5
250	5.2	13.7	42.9	5.2	11.8	34.8	5.0	12.0	35.3	5.2	11.8	35.2
	$m = 3$		$m = 3$		$m = 3$		$m = 3$					
$T$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$
50	7.4	17.0	50.0	7.9	13.4	30.8	6.8	14.0	40.8	7.8	13.6	33.6
150	5.7	15.4	54.0	6.0	11.8	35.7	5.5	13.0	45.0	5.9	12.0	36.9
250	5.5	16.4	54.4	5.4	12.2	37.1	5.0	13.8	45.4	5.5	12.5	38.0
	$m = 4$		$m = 4$		$m = 4$		$m = 4$					
$T$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$
50	8.3	20.4	64.9	9.4	14.3	30.4	7.3	16.1	51.8	9.0	14.6	35.8
150	5.7	18.1	66.7	5.9	11.8	33.7	5.2	14.4	54.5	5.8	12.1	36.1
250	5.6	18.6	67.9	5.7	11.6	36.2	5.4	14.6	56.8	5.7	11.9	38.0

Note:  $\rho = 1$ ,  $\alpha = -0.5$  and  $\theta = 0.5$  such that both series are non-stationary, i.e.,  $I(1)$ . Moreover, there is an additional cointegrating relationship – on top of trivial ones among  $x$  itself – between  $y$  and  $x$  of the form  $(-\theta, 1)$ . For the rest see Table 2.

Table 5: Granger Causality Tests; LF  $\rightarrow$  HF; Both series  $I(1)$ ; Cointegration

Standard		TY/DL		MF-dep		MF-indep						
$m = 2$												
$T$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$			
50	8.5	23.4	68.5	7.6	20.4	60.7	7.6	21.7	63.6	7.6	20.8	62.0
150	5.9	22.4	72.4	5.6	20.4	67.6	5.7	20.7	68.7	5.6	20.5	67.8
250	5.7	22.5	72.5	5.3	20.5	68.2	5.4	20.8	69.2	5.3	20.5	68.4
$m = 3$												
$T$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	10.3	30.6	80.0	10.6	26.5	71.3	9.8	27.6	74.6	9.7	26.9	73.4
150	6.6	28.1	84.2	6.1	24.8	80.0	6.3	25.5	81.4	6.1	25.0	80.8
250	5.5	28.4	85.3	5.3	25.5	81.6	5.4	26.3	82.8	5.3	26.0	82.0
$m = 4$												
$T$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	12.5	36.3	86.3	13.4	31.9	77.5	12.1	33.4	81.7	12.2	32.5	80.5
150	6.7	33.4	90.7	6.7	29.9	86.1	6.8	30.8	87.8	6.7	30.5	86.9
250	5.9	32.0	91.7	6.1	28.9	88.5	6.1	30.0	89.5	6.2	29.4	89.0

Note:  $\rho = 1$ ,  $\alpha = -0.5$  and  $\theta = 0.5$  such that both series are non-stationary, i.e.,  $I(1)$ . Moreover, there is an additional cointegrating relationship – on top of trivial ones among  $x$  itself – between  $y$  and  $x$  of the form  $(1, -\theta)$ . For the rest see Table 3.



Table 7: Granger Causality Tests; LF  $\rightarrow$  HF; Both series  $I(1)$ ; No Cointegration

Standard		TY/DL		MF-dep		MF-indep						
$m = 2$		$m = 2$		$m = 2$		$m = 2$						
$T$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$			
50	14.0	26.0	61.8	8.5	18.8	52.1	8.5	19.3	53.9	8.4	18.8	52.9
150	11.3	24.8	64.6	5.7	17.0	57.2	5.8	17.6	57.9	5.7	16.8	57.7
250	11.3	24.8	64.6	5.7	17.1	57.7	5.7	17.5	58.3	5.7	17.2	57.9
$m = 3$		$m = 3$		$m = 3$		$m = 3$		$m = 3$				
$T$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	15.9	31.7	73.1	11.0	24.4	62.8	10.9	25.2	65.2	10.2	24.9	64.3
150	11.0	28.3	75.9	6.2	21.2	69.2	6.2	21.5	70.3	6.3	21.5	69.8
250	10.5	28.4	75.8	5.4	21.3	70.6	5.9	21.7	71.4	5.3	21.4	71.4
$m = 4$		$m = 4$		$m = 4$		$m = 4$		$m = 4$				
$T$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	17.9	36.9	80.4	13.5	29.8	70.8	12.9	30.2	74.3	12.7	29.7	73.3
150	11.2	31.4	82.8	6.6	24.3	76.7	7.1	25.0	78.3	6.6	24.7	77.9
250	10.7	30.2	83.7	6.2	24.1	79.4	6.3	24.4	80.2	6.4	24.4	80.0

Note:  $\rho = 1$ ,  $\alpha = 0$  and  $\theta = 0.5$  such that both series are non-stationary, i.e.,  $I(1)$ , but *not* additionally cointegrated. For the rest see Table 3 and 5.

## B MF-VAR Parameters

In the reduced-form MF-VAR(2) levels formulation, the formulae for  $A_1, A_2$  and  $A^*$  for both DGPs in Section 3.2 are as follows. For the DGP used to test GC from  $x$  to  $y$ , i.e., (5) and (6), we have

$$\begin{aligned}
 A_1 &= \begin{bmatrix} \rho & \lambda_0 & \lambda_1 & \dots & \lambda_{m-1} \\ \theta(\rho - (\alpha + 1)^m) & (\alpha + 1)^m + \theta\lambda_0 & \theta\lambda_1 & \dots & \theta\lambda_{m-1} \\ \theta(\rho - (\alpha + 1)^{m-1}) & (\alpha + 1)^{m-1} + \theta\lambda_0 & \theta\lambda_1 & \dots & \theta\lambda_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta(\rho - (\alpha + 1)^2) & (\alpha + 1)^2 + \theta\lambda_0 & \theta\lambda_1 & \dots & \theta\lambda_{m-1} \\ \theta(\rho - \alpha - 1) & \alpha + 1 + \theta\lambda_0 & \theta\lambda_1 & \dots & \theta\lambda_{m-1} \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} 0 & -\lambda_0 & -\lambda_1 & \dots & -\lambda_{m-1} \\ 0 & -\theta\lambda_0 & -\theta\lambda_1 & \dots & -\theta\lambda_{m-1} \\ 0 & -\theta\lambda_0 & -\theta\lambda_1 & \dots & -\theta\lambda_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\theta\lambda_0 & -\theta\lambda_1 & \dots & -\theta\lambda_{m-1} \\ 0 & -\theta\lambda_0 & -\theta\lambda_1 & \dots & -\theta\lambda_{m-1} \end{bmatrix}, \\
 A^* &= \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ \theta & 1 & \alpha + 1 & \dots & \dots & (\alpha + 1)^{m-1} \\ \theta & 0 & 1 & \dots & \dots & (\alpha + 1)^{m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \theta & 0 & 0 & \dots & \ddots & \alpha + 1 \\ \theta & 0 & 0 & \dots & \dots & 1 \end{bmatrix}.
 \end{aligned}$$



For the DGP used to test GC from  $y$  to  $x$ , i.e., (7) and (8), we have

$$\begin{aligned}
A_1 &= \begin{bmatrix} \alpha + 1 + \theta\tilde{\delta}^{m-1} & \theta(\rho^m - \alpha - 1) & & & & \\ \tilde{\delta}^{m-1} & \rho^m & & & & \\ \tilde{\delta}^{m-2} & \rho^{m-1} & & & & \\ \vdots & \vdots & & & & \\ \tilde{\delta}^1 & \rho^2 & & & & \\ \tilde{\delta}^0 & \rho & & & & \\ & & & & & & 0_{(m+1)\times(m-1)} \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -\theta\tilde{\delta}^{m-1} \\ -\tilde{\delta}^{m-1} \\ -\tilde{\delta}^{m-2} \\ \vdots \\ -\tilde{\delta}^1 \\ -\tilde{\delta}^0 \end{bmatrix} 0_{(m+1)\times m}, \\
A^* &= \begin{bmatrix} 1 & \theta & \theta\rho & \dots & \dots & \theta\rho^{m-1} \\ 0 & 1 & \rho & \dots & \dots & \rho^{m-1} \\ 0 & 0 & 1 & \dots & \dots & \rho^{m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \ddots & \rho \\ 0 & 0 & 0 & \dots & \dots & 1 \end{bmatrix},
\end{aligned}$$

where  $\tilde{\delta}^\xi = \sum_{i=0}^{\xi} \rho^i \delta_i$ .

## C High-Frequency DGP

To derive a common HF-DGP, we assume  $Z_{t-j/m}^{(m)} = (y_{t-j/m}^{(m)}, x_{t-j/m}^{(m)})'$  with  $j = 0, \dots, m-1$  and  $t = 1, \dots, T$ , is generated by one of the following VECMs, depending on whether we inspect...

...GC from  $x$  to  $y$ , i.e.,

$$\Delta^{1/m} Z_{t-j/m}^{(m)} = \begin{bmatrix} 0 & 0 \\ \theta & -\theta \end{bmatrix} Z_{t-(j+1)/m}^{(m)} + \begin{bmatrix} 0.5 & \lambda \\ 0.3 & 0.5 \end{bmatrix} \Delta^{1/m} Z_{t-(j+1)/m}^{(m)} + v_{t-j/m}^{(m)},$$

..., or GC from  $y$  to  $x$ , i.e.,

$$\Delta^{1/m} Z_{t-j/m}^{(m)} = \begin{bmatrix} -\theta & \theta \\ 0 & 0 \end{bmatrix} Z_{t-(j+1)/m}^{(m)} + \begin{bmatrix} 0.5 & 0.3 \\ \delta & 0.5 \end{bmatrix} \Delta^{1/m} Z_{t-(j+1)/m}^{(m)} + v_{t-j/m}^{(m)},$$

where  $v_{t-j/m}^{(m)} \sim N(\mathbf{0}_{2 \times 1}, I_2)$ . Similar to the MF-case,  $\lambda = (mT)^{-0.5} \lambda^*$  and  $\delta = (mT)^{-0.5} \delta^*$  for  $\lambda^*, \delta^* = \{0, 1, 2\}$ , where the sample size is adjusted to the HF, in which the system is specified.  $\theta$  governs the cointegrating relationship, yet – unlike in the MF-case – it controls the presence (e.g.,  $\theta = 1$ ) or absence ( $\theta = 0$ ) of cointegration altogether.