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Topology and Invertible Maps*

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I study connected manifolds and prove that a proper map $f: M \to M$ is globally invertible when it has a nonvanishing Jacobian and the fundamental group $\pi_1(M)$ is finite. This includes finite and infinite dimensional manifolds. Reciprocally, if $\pi_1(M)$ is infinite, there exist locally invertible maps that are not globally invertible. The results provide simple conditions for unique solutions to systems of simultaneous equations and for unique market equilibrium. Under standard desirability conditions, it is shown that a competitive market has a unique equilibrium if its reduced excess demand has a nonvanishing Jacobian. The applications are sharpest in markets with limited arbitrage and strictly convex preferences: a nonvanishing Jacobian ensures the existence of a unique equilibrium in finite or infinite dimensions, even when the excess demand is not defined for some prices, and with or without short sales.

1. INTRODUCTION

This paper gives a global invertibility theorem, obtained by using algebraic topology. The inverse function theorem gives a simple condition for the local invertibility of a smooth map $f: M \to M$. If f has a nonvanishing Jacobian at a point, then it is locally invertible. The moral is that a linear approximation to the map predicts its local behavior. What about global behavior?

A finite fundamental group is the crucial link between local and global invertibility. The results are as follows. Let M be a connected compact manifold with a finite fundamental group $\pi_1(M)$. Then a smooth map $f: M \to M$ with a nonvanishing Jacobian is globally invertible. Reciprocally, when $\pi_1(M)$ is not finite, there are locally invertible maps $f: N \to M$ that are not globally invertible. These results also hold when M is not

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compact, provided that f is proper,[1] and includes infinite dimensional Banach manifolds. Manifolds with finite fundamental groups are not difficult to find: examples are Euclidean space, open convex sets, the sphere S^n, $n \geq 2$, all Grassmanian manifolds $G^{n,k}$ with $n > 2$, and all Banach spaces.

A simple example illustrates the role of topology in obtaining global invertibility. Any rotation from the circle S^1 to itself is a locally invertible map with a nonvanishing Jacobian. Yet most rotations are not globally invertible. What fails is the topology: the fundamental group of the circle $\pi_1(S^1)$ is the integers Z, which is not a finite group.

Finite fundamental groups are the crucial link between local and global invertibility. The example of the circle provided above is typical. I show below that the condition on the fundamental group is necessary as well as sufficient to go from local to global invertibility in the following sense: if the group $\pi_1(M)$ of a manifold M is *not* finite,[2] then there exist locally invertible maps $f: N \to M$ that are not globally invertible.

The results of this paper are simple but have useful applications. These include conditions for unique solutions to systems of simultaneous nonlinear equations. The results imply a simple condition for unique market equilibrium. Under standard desirability conditions, a competitive market has a unique equilibrium if its reduced excess demand has a nonvanishing Jacobian. When markets have limited arbitrage the results are sharper. *Limited arbitrage* is a condition on endowments and preferences that was introduced and shown to be necessary and sufficient for the existence of equilibrium, the core and social choice in finite or infinite economies.[3] With strictly convex preferences, a nonvanishing Jacobian defined on part of the price space—the intersection of "market cones"—ensures the existence and uniqueness of an equilibrium. This covers economies that were neglected in the literature on unique equilibrium, holding equally with finite or infinitely many markets and with or without short sales.

2. DEFINITIONS AND BACKGROUND

Unless otherwise specified, all manifolds are smooth[4] connected and without boundary and maps are smooth. The manifolds considered here may be compact or not, and they may be finite or infinite dimensional. In the latter case they are Banach manifolds [2]. A map $f: M \to N$ is *locally*

[1] These results were discussed in Chichilnisky [3]. A special case of this result, when M is simply connected and finite dimensional, was known to Hadamard (1906).

[2] I consider fundamental groups that are finitely generated but not finite.

[3] See Chichilnisky [4–9].

[4] I.e., C^k, with $k \geq 2$.

invertible when for each $x \in M$ there exists neighborhoods U_x and U_y of x and of $y = f(x)$, respectively, such that f maps U_x one to one and onto U_y. A map $f: M \to N$ is *globally invertible* when it is one to one and $f(M) = N$. Given two topological spaces, X and Y, a continuous map $f: X \to Y$ is *proper* when the inverse image of every compact set C, $f^{-1}(C)$, is compact.

EXAMPLE 1. If X and Y are open convex open sets, then $f: X \to Y$ is proper if $x^n \to x \in \partial X \Rightarrow f(x^n) \to y \in \partial Y$.[5] If X is an open convex set and $Y = R^N$, then the map $f: X \to R^N$ is proper when $x^n \to x \in \partial X$ implies $\|f(x)\| \to \infty$.

DEFINITION 1. Let $l_2(Z, \mu)$ be the space of all square integrable sequences of real numbers, i.e., of all square integrable functions from the integers to the reals $f: Z \to R$ such that $\sum_{t=1}^{\infty} f(t)^2 \mu(t) < \infty$.

Remark 1. l_2 is a Banach space, indeed it is a Hilbert space with the inner product defined by $\langle f, g \rangle = \Sigma f(t) \cdot g(t) \mu(t)$ [2]. When μ is a finite measure on Z, i.e., $\Sigma_t \mu(t) < \infty$, then $l_2(Z, \mu)$ contains unbounded sequences.

The following presents concepts and results of algebraic topology that can be found in any standard textbook, e.g. Spanier [12] or Greenberg [11]. All topological spaces are assumed to be connected and locally path connected.

DEFINITION 2. Given two topological spaces, X and Y, X is a covering space of Y if there exists a continuous onto map $\theta: X \to Y$ such that each $y \in Y$ has a neighborhood U_y whose inverse image $\theta^{-1}(U_y)$ is the disjoint union of sets in X, each of which is homeomorphic to U_y. The map θ is called a *covering map*. When the inverse image $\theta^{-1}(y)$ of each point $y \in Y$ contains exactly $k \geq 1$ points, then the covering is called a *k-fold covering*.

EXAMPLE 2. The map $\theta(r) = e^{i\pi r}$ from the line R to the circle S^1 is a covering map that makes the line a covering space of the circle. The map $\theta(e^{i\pi r}) = e^{i2\pi r}$ makes the circle a two-fold covering of itself.

DEFINITION 3. The first homotopy group of X, also called its *fundamental group*, is denoted $\pi_1(X)$.

EXAMPLE 3. This group is zero, and therefore finite, whenever X is convex or contractible, for example, $X = R^k$ or R^k_+, any k. The fundamental group of the circle S^1 is Z, the group of integers, and is not a finite group. All other spheres S^n, $n > 1$, have zero (and therefore finite) fundamental groups, i.e. $\pi_1(S^n) = 0$. All Grassmanian manifolds (other than the circle S^1) have fundamental groups equal to Z_2, the group of

[5] ∂M denotes the boundary of the set M.

integers modulo 2. This group has two elements $Z_2 = \{0, 1\}$, and therefore the fundamental groups of all of the Grassmanian manifolds other than S^1 are finite.

Intuitively, $\pi_1(X)$ is the space of all loops in X, where a loop is a continuous map $\phi : S^1 \to X$, under the following natural equivalence relation: two loops ϕ_1 and ϕ_2 are equivalent if and only if one is a continuous deformation of the other; i.e., there exists a continuous map $F : S^1 \times [0,1] \to X$ such that $\forall x \in X$, $F(x,0) = \phi_1(x)$, and $F(x,1) = \phi_2(x)$. The group operation in $\pi_1(X)$ is defined by running one loop after the other sequentially, thus obtaining another loop; for formal definitions see [11].

DEFINITION 4. A map between topological spaces $f : X \to Y$ defines a homeomorphism of the corresponding fundamental groups $f_* : \pi_1(X) \to \pi_1(Y)$.[6] The map f_* is called a *monomorphism* when it is one to one.

THEOREM 1. *Let $p : X \to Y$ be a covering map. Then $p_* : \pi_1(X) \to \pi_1(Y)$ is a monomorphism. See* [11], *p.* 19.

DEFINITION 5. Two covering spaces $p : X \to Y$ and $p' : X' \to Y$ are equivalent when there is a unique homeomorphism $\phi : X \to X'$ such that $p \circ \phi = p'$.

THEOREM 2. *Any manifold M has a covering space $p : X \to M$ with $\pi_1(X) = 0$, called its universal covering space. See* [11], *p.* 23, (6.7).

THEOREM 3. *Let $p : X \to Y$ be a covering space. For any subgroup H of $\pi_1(Y)$ there exists a covering space $p : X \to Y$ unique up to an equivalence, such that $H = p_* \pi_1(X)$. See* [11], *p.* 24, (6.9).

THEOREM 4. *Inverse Function Theorem* [2]. *Let M and N be two manifolds of the same dimension, $f : M \to N$ a smooth map, and $y = f(x)$. If the Jacobian of f is nonvanishing at x, there exist neighborhoods U_x and U_y of x and y, respectively, such that $f/U_x : U_x \to U_y$ is a diffeomorphism.*[7]

3. RESULTS

This section establishes the global invertibility of maps with nonvanishing Jacobian on compact manifolds, and then extends this to proper maps on noncompact manifolds.

THEOREM 5. *Let M be a compact, connected manifold with a finite fundamental group $\pi_1(M)$, and let $f : M \to M$ be a smooth map. If the Jacobian of f is nonvanishing, then f is globally invertible.*

[6] I.e., $\forall \alpha, \beta \in \pi_1(X)$, $f_*(\alpha * \beta) = f(\alpha) * f(\beta)$, where $*$ denotes the group operation in $\pi_1(X)$.

[7] A diffeomorphism is a one-to-one onto map that is smooth and has a smooth inverse.

Proof. By the implicit function theorem, the image $f(M)$ is an open set in M. I will show that under the assumptions $f(M)$ is closed as well. Consider two sequences $\{x^n\} \subset M$ and $\{y^n\} \subset M$ such that $y^n = f(x^n)$, $x^n \to x$ and $y^n \to y$. Since M is compact and the Jacobian of f does not vanish, this Jacobian is bounded away from zero. Therefore there exists δ, $\varepsilon > 0$ such that M can be covered by a family of δ-neighborhoods[8] $\{U_x\}$ on each of which f is a diffeomorphism, and the image under f of each U_x covers an ε-neighborhood in M. Since $y^n \to y$, for n large enough y is contained in such an ε neighborhood of y^n, so that by construction $y \in f(M)$. Therefore the image $f(M)$ is closed. Since $f(M)$ is both open and closed and M is connected, $f(M) = M$, i.e., the map f is onto.

The next step is to show that f is a covering map. By the inverse function theorem, if $f(x) = y$, there exist neighborhoods U_x and U_y of x and y, respectively, such that the restriction of the map f on U_x, $f/U_x : U_x \to U_y$, is one to one and onto. By the continuity of the map f, for any $y \in M$, the set $f^{-1}(y)$ is closed; since M is compact, the set $f^{-1}(y)$ is also compact, and by the inverse function theorem it is 0-dimensional. Therefore for any $y \in M$, the set $f^{-1}(y)$ consists of finitely many points $\{x_i\}$ $i = 1, \ldots, k$. We may then choose a neighborhood U_y of y such that $f^{-1}(U_y)$ consists of a union of disjoint neighborhoods of each diffeomorphic to U_y; compactness of M implies that $f^{-1}(U_y) = \bigcup_{x_i \in f^{-1}(y)} \{U_x\}$. This implies that f is a covering map from M onto M.[9]

The last step is to show that f is globally invertible. We know by Theorem 3 that for each subgroup H of $\pi_1(M)$ there exists a covering $\theta : X \to M$, which is unique up to equivalence, such that $\theta_*(\pi_1(X)) = H$. Now let $H = \pi_1(M)$. The identity map $i : M \to M$ defines a covering such that $i_*(\pi_1(M)) = \pi_1(M)$. We already saw that $f : M \to M$ is a covering map, so that $f_* : \pi_1(M) \to \pi_1(M)$ is a monomorphism by Theorem 1. Since the first homotopy group $\pi_1(M)$ is finite, and $f_* : \pi_1(M) \to \pi_1(M)$ is one to one, then f_* must be an onto, so that $f_*(\pi_1(M)) = \pi_1(M)$. Therefore both maps f and i satisfy $f_*(\pi_1(M)) = i_*(\pi_1(M)) = \pi_1(M)$; it follows from Theorem 3 that f and i define equivalent coverings. Since i is the identity map, f must be a onefold covering of M, i.e., f is a globally invertible map, as we wished to prove. ∎

The following provides an extension of Theorem 5 to paracompact manifolds, such as, for example, R^N.

[8] We can choose the coordinate patches that define the manifold M so that a δ-neighborhood in M is the image of a ball of radius δ in the linear space that is the model for M.

[9] Ehresman (1947) proved that for compact manifolds a differential submersion is a locally trivial fibration; in connected finite dimensional manifolds this implies it is a covering map.

COROLLARY 1. *Let M be a paracompact connected manifold*[10] *with $\pi_1(M)$ finite, and $f: M \to M$ a proper map. When the Jacobian of f is nonvanishing, f is globally invertible.*

Proof. First we check that since f is proper and its Jacobian is nonvanishing, then $f(M) = M$. We know that $f(M)$ is an open set by the inverse function theorem. Next we show that $f(M)$ is closed. Let $y^n \in f(M)$, $y^n \to y$. I will show that $y \in f(M)$. Since $y^n \to y$, the set $\{y^n\} \cup y$ is compact, and since f is proper, the set $f^{-1}(\{y^n\} \cup y)$ is compact as well. Let $\{x^n\}$ satisfy $f(x^n) = y^n$; then the set $\{x^n\}$ is contained in the compact set $f^{-1}(\{y^n\} \cup y)$. Therefore we may apply the proof of Theorem 5, implying that $y \in f(M)$, so that $f(M)$ is closed. Since $f(M)$ is open and closed and M is connected, $f(M) = M$.

Since f is a proper map, $f^{-1}(y)$ is a compact set, and therefore, by the inverse function theorem, $f^{-1}(y)$ consists of finitely many points. The same argument as in Theorem 5 establishes, therefore, that $f: M \to M$ is a covering map, and that f is globally invertible. ▮

COROLLARY 2. *Let C be a convex open region of R and assume that $f: C \to R^N$ satisfies $x^n \to x \in \partial C \Rightarrow \|f(x^n)\| \to \infty$. Then if the Jacobian of f is nonvanishing, the map f is globally invertible.*

The results extend also to infinite dimensional manifolds, provided they are Banach manifolds, so the inverse function theorem holds [2]:

COROLLARY 3. *Let M be a connected Banach manifold with $\pi_1(M)$ finite. Let $f: M \to M$ be a proper map. If its Frechet derivative is invertible, then f is globally invertible.*

Proof. The argument is similar to those in Theorem 5: since f is proper, the condition on its Frechet derivative ensures that the image $f(M)$ is open and closed in M, so that $f(M) = M$. The rest of the proof follows that of Theorem 5 without modification. ▮

A partial converse of the above results can be given. A finite fundamental group $\pi_1(M)$ is necessary for ensuring that a locally invertible map is globally invertible:

COROLLARY 4. *Let $\pi_1(M)$ be a finitely generated Abelian group. If $\pi_1(M)$ is infinite, then there exists a locally invertible map $f: N \to M$ that is not globally invertible.*

Proof. If $\pi_1(M)$ is infinite, it contains a strict subgroup H that is isomorphic to $\pi_1(M)$. By Theorem 3 we know that there exists a covering space $p: N \to M$ such that $p_*(\pi_1(N)) = H$, and the covering map p is not globally invertible. Yet the covering map p is a locally invertible map, by definition of a covering. ▮

[10] A manifold is paracompact if it can be covered by a countable set of precompact charts.

4. APPLICATIONS

4.1. *Grassmanians and Simultaneous Equations*

This section describes two simple applications of the global invertibility results: one to compact manifolds, and the other to open regions in R^N.

COROLLARY 5. *If $G^{k,n}$ denotes the classical Grassmanian manifold of k planes in R^n and $n > 2$, $n > k$. Then any smooth map $f: G^{k,n} \to G^{k,n}$ with nonvanishing Jacobian is globally invertible.*

Proof. When $n > 2$, the fundamental group $\pi_1(G^{k,n})$ is finite. Indeed, $\pi_1(G^{k,n}) = Z_2$ for all n, k except $G^{1,2} = S^1$. ∎

COROLLARY 6. *Consider a system of equations that define a smooth nonlinear map $f: X \to R^N$, where $X \subset R^N$ is open and convex. If the map f is proper and has a nonvanishing Jacobian, there is a unique solution to the problem $f(x) = 0$.* ∎

4.2. *Market Equilibrium*

Consider the positive vectors in the unit sphere: $\Delta = \{p \in R^N : p \gg 0$ and $\Sigma_{i=1}^N p_i^2 = 1\}$. This represents the set of relative prices of a market that contains N commodities. An excess demand function for a market economy is a smooth map $Z: \Delta \to R^N$ satisfying $\forall p \in \Delta, \langle p, Z(p)\rangle = 0$. The condition is derived from the requirement that all traders have balanced budgets, and is described by saying that the value of demand equals the value of supply. A standard desirability condition of the excess demand function Z is:

DEFINITION 6. *Desirability condition*: if $p^j \to p \in \partial\Delta$ then $\|Z(p^j)\| \to \infty$.

Similar conditions are in, e.g., [10].

Remark 2. The desirability condition implies that the map $Z: \Delta \to R^N$ is proper. However, Z does not generally have a nonvanishing Jacobian.

A market is said to be at an equilibrium when supply equals demand in all markets. Formally:

DEFINITION 7. A *market equilibrium* p^* is a zero of the excess demand function $Z: \Delta \to R^N$, i.e., $p^* \in Z^{-1}(0)$.

Observe that when the vector $Z(p)$ has all but one coordinate equal to zero, then $Z(p)$ is the zero vector, because $\forall p \in \Delta,\ \langle p, Z(p)\rangle = 0$. Therefore, to identify a market equilibrium it suffices to find a zero of another map, the composition map $\tilde{Z}_i = \pi_i \circ Z: \Delta \to R^{N-1}$, where $i \in$

$\{1,\ldots,N\}$ and $\pi_i : R^N \to R^{N-1}$ is the projection map on the $N-1$ coordinates other than i. Observe that the composition map $\tilde{Z}_i = \pi_i \circ Z : \Delta \to R^{N-1}$ maps an open convex set of R^{N-1} into R^{N-1}, and that under the desirability condition, $\tilde{Z}_i$ is a proper map. $\tilde{Z}_i$ is called a *reduced excess demand.*

The definitions provided here extend to markets in which the trading space rather than R^N is the infinite dimensional Hilbert space l_2 (see Chichilnisky and Heal [9]). l_2 has an inner product and a countable basis of coordinates, properties that all Hilbert spaces share, and is the closest to Euclidean space R^N to infinite dimensions. These properties allow one to treat the trading space the same as Euclidean space, and all of the definitions given above apply without modification. In particular, since the Hilbert space is self-dual, it is possible to find invertible operators from the price space to the commodity space: both are the same space H. This makes Hilbert spaces the preferred space for infinite dimensional markets:

THEOREM 6. *Under the desirability condition, if the reduced excess demand $\tilde{Z}_i : \Delta \to R^{N-1}$ has a nonvanishing Jacobian, then $\tilde{Z}_i$ is globally invertible. In this case the market has a unique equilibrium. This is also true in infinite dimensional markets when the Frechet derivative of $\tilde{Z}_i$ is an invertible operator.*

Proof. Observe that the image of Δ under the map $Z, Z(\Delta)$, is contractible[11] because Δ is contractible. The image $\tilde{Z}_i(\Delta)$ is also contractible, because $\tilde{Z}_i$ is Z composed with a projection. It follows that $\pi_1(\tilde{Z}_i(\Delta)) = 0$. Next observe that the image $\tilde{Z}_i(\Delta)$ is open under the hypothesis, by the inverse function theorem. In particular, $\tilde{Z}_i(\Delta)$ is a manifold. The desirability condition of Definition 5 implies that the map $\tilde{Z}_i$ is proper. The result now follows from Corollary 1 for the finite dimensional case, and from Corollary 3 for infinite dimensions. ∎

The following result applies to economies in which the excess demand function is not well defined at all prices, but only on a subset of prices:

COROLLARY 7. *If the demand function $Z : C \to R^N$ is defined on a convex subset of prices $C \subset \Delta$, and the desirability condition is satisfied in C, i.e., $p^j \to p \in \partial C \Rightarrow \|Z(p^j)\| \to \infty$, then there exists a unique market equilibrium price in C when the Jacobian of the reduced excess demand $\tilde{Z}_j$ is nonvanishing on C. When the market is infinite dimensional, $X = l_2$, the result obtains when the Frechet derivative of $\tilde{Z}_j$ is invertible.*

[11] A topological space X is contractible when there exists a continuous map $F : X \times [0,1] \to X$ and $x^0 \in X$ such that $\forall x \in X$, $F(x,0) = x$, and $F(x,1) = x^0$. A contractible space has a zero fundamental group, $\pi_1(X) = 0$.

4.3. *Limited Arbitrage and Uniqueness with Short Sales*

This subsection explicitly includes economies with short sales and with finite or infinitely many markets. These cases have been neglected in the literature on uniqueness of equilibrium.

The following uses a condition of *limited arbitrage*, which is defined on the preferences and endowments of a market. It was introduced and shown to be necessary and sufficient for the existence of a competitive equilibrium, the core and social choice in markets with or without short sales, in Chichilnisky [4–8], and in infinite dimensions in Chichilnisky and Heal [9].

Consider a market with trading space $X = R^N$ or $X = l_2$, where traders have property rights represented by vectors $\Omega_i \in X$ and preferences represented by strictly concave increasing utility functions $u_i : X \to R$.[12] For trader i define the *global cone* G_i as the set of directions along which utility never ceases to increase:

$$G_i = \{x \in X: \sim \exists \text{argmax}_{\lambda \geq 0} u_i(\Omega_i + \lambda x)\}$$

and the *market cone* as

$$D_i = \{y \in X: \forall x \in G_i, \langle y, x \rangle > 0\}.$$

DEFINITION 8. The market has *limited arbitrage* when $D = \bigcap_{i=1}^{H} D_i \neq \phi$.

Remark 3. The intersection $D = \bigcap_{i=1}^{H} D_i$ denotes the set of prices at which the excess demand of the economy is well defined. It has been shown that limited arbitrage, i.e., $D \neq \varnothing$, is equivalent to the existence of an equilibrium, the core and social choice [4–8].

The result below proves the uniqueness of an equilibrium in an economy with limited arbitrage, with finite or infinitely many commodities and without short sales:

COROLLARY 8. *Consider an economy with limited arbitrage and strictly convex preferences. If the reduced excess demand function $\tilde{Z}_i$ has a nonvanishing Jacobian on $D = \bigcap_{i=1}^{H} D_i \subset \Delta$,[13] then there exists a unique competitive equilibrium. This is also true in infinite dimensional markets ($X = l_2$) when the Frechet derivative of $\tilde{Z}_i$ is invertible.*

Proof. By Corollary 7 it suffices to show that under limited arbitrage the reduced excess demand is a proper map on a convex subset of Δ on which its Jacobian is nonvanishing.

By the definition of limited arbitrage, the set $D = \bigcap_{i=1}^{H} D_i \subset \Delta$ is not empty. D is a convex set since, under the assumptions on preferences,

[12] Or, more generally, by utilities whose indifferences have no half lines.

[13] In the infinite dimensional case, when the Frechet derivative is an invertible operator.

each set D_i is convex. If $y \in R^N$ is in the boundary of $D = \bigcap_{i=1}^{H} D_i$, then it is in the boundary of D_i, ∂D_i, for some trader i. Since preferences are strictly convex, this implies that for trader i the norm of the excess demand increases without ever reaching a maximum along the direction defined by y from Ω_i; see [7]. By Example 1, this implies that for all j, the reduced excess demand function $\tilde{Z}_j$ is proper on D. Therefore under the assumptions, by Corollary 7 there exists a unique competitive equilibrium. ∎

4.4. *Related Literature on Unique Equilibrium*

It seems worth comparing these results with other approaches to the uniqueness of equilibrium. A main difference is that the results presented here apply equally to finite or infinite dimensional markets and applies with or without short sales, while the existing literature concentrates instead on finite dimensional markets without short sales. For ease of comparison, the following discussion concentrates on economies with finite dimensions and without short sales.

The closest to Theorem 6 above in finite economies without short sales is Theorem 15 on p. 236 of Arrow and Hahn [1], whose proof is connected to the convergence to equilibrium of the global Newton method. However, the proof of Theorem 15, on p. 304 of [1], uses a "numeraire assumption" A.11.2 given on page 268.[14] No such condition is required in this paper, so that Theorem 6 above is strictly stronger than the results in [1]. Furthermore, Corollaries 7 and 8 above include markets in which the excess demand function is not defined on the whole price space, as assumed in Theorem 15 of Arrow and Hahn, and Corollary 8 above covers markets with short sales and which are finite or infinite dimensional.

Working also on finite economies without short sales, Dierker [10] assumes a desirability condition that is similar to that required here, and uses an index argument to show the uniqueness of equilibrium. His conditions and results are different: I assume that the Jacobian never vanishes in the interior of Δ, or on a convex subset $C \subset \Delta$, while [10] assumes that there is a price adjustment system that is stable at each equilibrium, or more generally that the Jacobian of the system has the same sign at each equilibrium. The result obtained here is stronger than those in [10]: I prove the global invertibility of the map $\tilde{Z}_i$ and hence uniqueness of equilibrium, while [10] proves only that the equilibrium is unique.

The results presented here are also different from other global invertibility results for finite dimensional economies, such as the Gale–Nikaido theorem, which apply to maps defined on closed cubes, and require a

[14]Assumption A.11.2 is used to show that as relative prices $|p(t)| \to \infty$, the excess demand for a specific good, the "numeraire," goes to plus infinity; see p. 304 of [1].

nonvanishing Jacobian on the interior of the cube, as well as similar conditions on the boundary of the cube. The difference is that I only require conditions on the interior of the price space Δ or on a convex subset $C \subset \Delta$, and I allow infinite dimensions; the other results do not. The desirability condition eliminates boundary equilibrium, so there is no need to study the boundary of the price space.

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