Two classes of weighted values for coalition structures with extensions to level structures

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Abstract

In this paper we introduce two new classes of weighted values for coalition structures with related extensions to level structures. The values of both classes coincide on given player sets with Harsanyi payoffs and match therefore adapted standard axioms for TU-values which are satisfied by these values.

Characterizing elements of the values from the new classes are a new weighted proportionality within components property and a null player out property, but on different reduced games for each class. The values from the first class, we call them weighted Shapley alliance coalition structure values (weighted Shapley alliance levels values), satisfy the null player out property on usual reduced games. By contrast, the values from the second class, named as weighted Shapley collaboration coalition structure values (weighted Shapley collaboration levels values) have this property on new reduced games where a component decomposes in the components of the next lower level if one player of this component is removed from the game. The first class contains as a special case the Owen value (Shapley levels value) and the second class includes a new extension of the Shapley value to coalition structures (level structures) as a special case.

Keywords Cooperative game · Weighted Shapley coalition structure values · Weighted Shapley levels values · Dividends · Null player out

1 Introduction

Whereas e. g. governments, firms or political organizations are mostly structured strong hierarchical and in a statical manner supply chains or electricity and other networks have often a more dynamical and not so strong top down frame work. For hierarchical organized structures Winter (1989) developed a model, called level structure, that is an extension of a coalition structure (Aumann and Drèze, 1974). Therefore Winter (1989) introduced his value, we name it Shapley levels value, in accordance, e. g., with Álvarez-Mozos et al. (2017). This value is an extension of the Owen value (Owen, 1977), itselves an extension

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of the Shapley value (Shapley, 1953b). A level structure consists of ordered partitions (the levels) of the original player set. Each partition consists of disjoint coalitions of players (the components) such that each component of a level higher that contains a player from a component of the lower level must contain all players of this component. So in a level structure the lowest level contains all the singletons as components and the highest level contains the grand coalition as the only component.

Vidal-Puga (2012) introduced a weighted value for coalition structures, extended by Gómez-Rúa and Vidal-Puga (2011) to level structures. There the weights are given by the size of the coalitions. The size of coalitions may be a considerable magnitude that is to take into account by sharing the payoff in a cooperative game. But often also other factors, e. g. in the context of cost allocation, play a decisive role: the size of firms or departments, political influence, fixed costs of the units, spent time of working members and so on. Recently Besner (2018b) extended their value to a class of values with arbitrary exogenously given weights, called weighted Shapley hierarchy levels values which extend the weighted Shapley values (Shapley, 1953a). These values don't satisfy the null player property in general.

Also Levy and McLean (1989) and McLean (1991) extended the weighted Shapley values to coalition structures for arbitrary weights which not depend on the coalition function. The main class of these values, in Dragan (1992) called McLean weighted coalition structure values, is in Besner (2018a) extended to level structures, called weighted Shapley support levels values. There the hierarchy of the level structure is treated more statically. If a sub-coalition of a component deals with other coalitions outside of this component the sub-coalition is supported by the weight of the whole component. In general, without further statement, it is not clear how to go on if a player is removed from the player set meaning what are the weights of the new components in the new game. In this respect, the Shapley levels value has an exceptional character within this class of values. It can be easily be treated also dynamically since the used weights are in a game where some players are removed the same as before, they are always equal.

To handle also weighted values for level structures in the same manner we introduce the class of weighted Shapley alliance levels values (weighted Shapley alliance coalition structure values). There weights, similar as by the weighted Shapley hierarchy levels values, are assigned not only to components, each subset of a component owns a weight too. If a player is removed from a component the remaining players build a new component with the original weight of the coalition of these players. Thus here the coalitions within a component can act more independently in the corset of a level structure. They can act with players outside of the component without consultation of the whole component but all involved players of a component form always an alliance. The weighted Shapley alliance levels values coincide with payoff vectors, called Harsanyi payoffs, from the Harsanyi set (Hammer, 1977; Vasil'ev, 1978) and inherit so for fixed player sets all properties of these payoff vectors adapted to level structures. Interestingly, the Shapley levels value (Owen value) is also a special case of the Shapley alliance levels values and our axiomatization gives so deeper insight into this value too. A characterizing element of the weighted Shapley alliance levels values is a null player out property. If we delete a null player this doesn't influence the payoff to the other players.

In some situations it is possible that if a player leaves a component the component loses its cohesion. For such situations the values from our second new class, called weighted

Shapley collaboration levels values (weighted Shapley collaboration coalition structure values) are recommended. Especially if a null player is removed from a component and the coalition of the remaining players of this component is smashed in the next smaller components. Then the payoff to all players is the same as before in the game with the complete player set. If some players of a component B are involved in a bargaining situation with players from outside of B all players which form the next largest sub-component of B collaborate together. So also here the players of a component can act more independently. Here, similar as by the weighted Shapley support levels values, only the components need a weight. As a special case of these values we single out a new extension of the Shapley value to level structures and name it Shapley collaboration levels value (Shapley collaboration caolition structure value).

Whereas the Shapley levels value, the weighted Shapley support levels values and the weighted Shapley hierarchy levels values satisfy the level game property (the payoff to all players of a component sum up to the payoff to the component in a game where components are the players) the values from our new classes don't satisfy this property in general. The absence of this characteristic reveals that in our new classes the players are more independent from a nesting component.

A level structure with only two levels coincides with a coalition structure. Thus all presented axioms and so the given axiomatizations coincide with related axioms and axiomatizations formulated for coalition structures. Therefore we deal in this paper with coalition structures only marginally and concentrate on level structures so that in this respect the reader has to transform adequate results by his own.

After the introduction section 2 provides some preliminaries. In the main part of the paper we introduce in section 3 the weighted Shapley alliance levels values, in section 4 the weighted Shapley collaboration levels values and, as a special case, in section 5 the Shapley collaboration levels value. An example in section 6 compares the proposed values, section 7 gives the conclusion and an appendix (section 8) provides all the proofs.

2 Preliminaries

In this paper some definitions and notations will follow with Besner (2018a). Let a countably infinite set $\mathfrak{U}$ of all potential players be given and let $\mathcal{N}$ the set of all finite subsets of $\mathfrak{U}$ which are non-empty. A **TU-game** can be defined as a pair (N, v) where $N \in \mathcal{N}$ is a player set and v a **coalition function** from the power set 2^N into the real numbers $\mathbb{R}$ such that $v(\emptyset) = 0$. We call subsets $S \subseteq N$ **coalitions** with a **worth** $v(S)$; Ω^S denotes the set of all nonempty subsets of $S \subseteq N$. If we regard only a restricted player set $S \in \Omega^N$ we denote by (S, v) the **restriction** of (N, v). We denote by $\mathbb{V}^N$ the set of all TU-games with player set N.

A player $i \in N$ is called a **null player** in v if $v(S \cup \{i\}) = v(S)$, $S \subseteq N \backslash \{i\}$; we call two different players $i, j \in N$ (mutually) **dependent** (Nowak and Radzik, 1995) in v if $v(S \cup \{k\}) = v(S)$ for all $S \subseteq N \backslash \{i, j\}$ and $k \in \{i, j\}$. For all $S \subseteq N$ and $T \in \Omega^N$ a game $(N, u_T) \in \mathbb{V}^N$ with $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called an **unanimity game**. We define the **dividends** $\Delta_v(S)$ (Harsanyi, 1959) by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), & \text{if } S \in \Omega^N, \text{ and} \\ 0, & \text{if } S = \emptyset. \end{cases} \tag{1}$$

It is well-known that any coalition function v on N can be uniquely presented by

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \tag{2}$$

Coalitions $S \subseteq N$ are called **active** in v if $\Delta_v(S) \neq 0$.

A **coalition structure** on N is a partition $\mathcal{B} := \{B_1, ..., B_m\}$ of the player set N where $B_k \neq \emptyset$, $1 \leq k \leq m$, $B_k \cap B_\ell = \emptyset$, $1 \leq k < \ell \leq m$, and $\bigcup_{k=1}^m B_k = N$. Each $B \in \mathcal{B}$ is called a **component** and $\mathcal{B}(i)$ denotes the component that contains a player $i \in N$.

By Winter (1989), each finite sequence $\underline{\mathcal{B}} := \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ of coalition structures $\mathcal{B}^r$, $0 \leq r \leq h+1$, on N is called a **level structure** on N if $\mathcal{B}^0 = \big\{\{i\} \colon i \in N\big\}$, $\mathcal{B}^{h+1} = \{N\}$ and each $\mathcal{B}^r$ is a refinement of $\mathcal{B}^{r+1}$ for all r, $0 \leq r \leq h$, that means that $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$ for all $i \in N$. We denote by $\mathbb{L}^N$ the set of all level structures with player set N; we call $\mathcal{B}^r$ the r-th **level** of $\underline{\mathcal{B}}$; for a component $B^k \in \mathcal{B}^k$, $0 \leq k \leq r \leq h+1$, $\mathcal{B}^r(B^k)$ is the component of the r-th level that contains B^k and by $\overline{\mathcal{B}}$ we denote the set of all components $B \in \mathcal{B}^r$ of all levels $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$.

An **LS-game** is a triple $(N, v, \underline{\mathcal{B}})$ where $(N, v) \in \mathbb{V}^N$ and $\underline{\mathcal{B}} \in \mathbb{L}^N$. We denote by $\mathbb{VL}^N$ the set of all LS-games on N. Please note that we have for each $(N, v, \underline{\mathcal{B}_0}) \in \mathbb{VL}^N$ with a **trivial level structure** $\underline{\mathcal{B}_0} := \{\mathcal{B}^0, \mathcal{B}^1\}$ a corresponding TU-game (N, v) and for each $(N, v, \underline{\mathcal{B}_1}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}_1} := \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$, a corresponding game with coalition structure (Aumann and Drèze, 1974; Owen, 1977).

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and $T \in \Omega^N$. From a level structure on N follows a level structure on T by eliminating the players in $N \backslash T$. With coalition structures $\mathcal{B}^r|_T := \{B \cap T : B \in \mathcal{B}^r, B \cap T \neq \emptyset\}$, $0 \leq r \leq h+1$, the new level structure on T is given by $\underline{\mathcal{B}}|_T := \{\mathcal{B}^0|_T, ..., \mathcal{B}^{h+1}|_T\} \in \mathbb{L}^T$ and $(T, v, \underline{\mathcal{B}}|_T) \in \mathbb{VL}^T$ is called the **restriction** of $(N, v, \underline{\mathcal{B}})$ to player set T.

A **TU-value** ϕ is a map that assigns to any $(N, v) \in \mathbb{V}^N$ a vector $\phi(N, v) \in \mathbb{R}^N$, an **LS-value** φ is a map that assigns to any $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ a vector $\varphi(N, v, \underline{\mathcal{B}}) \in \mathbb{R}^N$.

Let $\mathbb{R}_{++}$ the set of all positive real numbers. We define $\mathrm{W}^N := \{f : N \to \mathbb{R}_{++}\}$ with $\mathrm{w}_i := \mathrm{w}(i)$ for all $\mathrm{w} \in \mathrm{W}^N$ and $i \in N$ as the set of all positive weight systems on the player set N, we define $\mathcal{W}^{2^N} := \{f : 2^N \backslash \emptyset \to \mathbb{R}_{++}\}$ with $w_S := w(S)$ for all $w \in \mathcal{W}^{2^N}$, $S \in \Omega^N$, as the set of all positive weight systems on all non-empty coalitions $S \subseteq N$ and we define $\mathcal{W}^{\overline{\mathcal{B}}} := \{f : \overline{\mathcal{B}} \to \mathbb{R}_{++}\}$[1] with $w_B := w(B)$ for all $w \in \mathcal{W}^{\overline{\mathcal{B}}}$, $B \in \overline{\mathcal{B}}$, as the set of all positive weight systems on the components of all levels r, $0 \leq r \leq h$, of a level structure $\underline{\mathcal{B}}$.

For all $(N, v) \in \mathbb{V}^N$ and $\mathrm{w} \in \mathrm{W}$ the (simply) **weighted Shapley values**[2] Sh^{w} (Shapley, 1953a) are defined by

$$Sh_i^{\mathrm{w}}(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\mathrm{w}_i}{\sum_{j \in S} \mathrm{w}_j} \Delta_v(S) \quad \text{for all } \; i \in N.$$

If all weights are equal we obtain as a special case of a weighted Shapley value the **Shapley value** Sh (Shapley, 1953b), given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all } \; i \in N.$$

[1]Note that $\mathcal{W}^{\overline{\mathcal{B}}}$ is a subset of $\mathcal{W}^{2^N}$.

[2]We make no use from null weights as suggested in Shapley (1953a) or Kalai and Samet (1987).

We introduce the Shapley levels value with a formula (Calvo, Lasaga and Winter, 1996, eq. (1)). Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and for all $T \in \Omega^N$, $T \ni i$,

$$K_T(i) := \prod_{r=0}^{h} K_T^r(i), \quad \text{where } K_T^r(i) := \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\, B \cap T \neq \emptyset\}|}.$$

Then the **Shapley Levels value** Sh^L (Winter, 1989) is defined by

$$Sh_i^L(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N,\, T \ni i} K_T(i)\Delta_v(T) \text{ for all } i \in N. \tag{3}$$

Remark 2.1. *If $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$, Sh^L coincides with Sh.*

Hammer (1977) and Vasil'ev (1978) presented a family of TU-values called **Harsanyi set** also known as **selectope** (Derks, Haller and Peters, 2000). For the definition of the TU-values in this set, called **Harsanyi payoffs**, we need a **dividend share system** $p = (p_i^S)_{S \in \Omega^N, i \in S}$ where $\sum_{i \in S} p_i^S = 1$ and $p_i^S \geq 0$ for all $S \in \Omega^N$ and all $i \in S$. The set of all dividend share systems p is denoted by P^N. Then for each $p \in P^N$ we have a Harsanyi payoff $\phi^p \in \mathbb{V}^N$, given by

$$\phi_i^p(N, v) := \sum_{S \subseteq N,\, S \ni i} p_i^S \Delta_v(S), i \in N.$$

The following axioms are simple adaptions of standard-axioms for TU-values:

Efficiency, E. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, we have $\sum_{i \in N} \varphi_i(N, v, \underline{\mathcal{B}}) = v(N)$.

Null player, N. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ and $i \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$.

Null player out, NO[3]**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ and $j \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N \backslash \{j\}, v, \underline{\mathcal{B}}|_{N \backslash \{j\}})$ for all $i \in N \backslash \{j\}$.

Additivity, A. For all $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$, we have
$\varphi(N, v, \underline{\mathcal{B}}) + \varphi(N, v', \underline{\mathcal{B}}) = \varphi(N, v + v', \underline{\mathcal{B}})$.

Winter (1989) introduced a symmetry between components axiom to characterize his value. There the sum of the payoffs to all players of a component equals the sum of the payoffs to all players of another component if both components are in the same level, are subsets of the same component one level higher and both components are symmetric players in a game where the components are the players. Besner (2018a) used a related axiom, called weighted proportionality between components, to characterize the weighted Shapley support levels values. Unlike as before there the components must be dependent in the game where the components are the players. Then the sums of the payoffs to all players of both components are in the same proportion as the weights of the components. Here we present a new similar axiom. The only difference as before: now all players of the components must be dependent in the originally game instead of the components in the game with the components as players.

[3]This axiom is an extension from null player out in Derks and Haller (1999).

Weighted proportionality within components, WPWC[4]**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\overline{\mathcal{B}}}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and all $i \in B_k \cup B_\ell$ are dependent in v, we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}.$$

The following axiom is a special case of the previous one.

Dependency within components, DWC. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and all $i \in B_k \cup B_\ell$ are dependent in v, we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}).$$

3 Weighted Shapley alliance levels values

The Shapley levels value, the weighted Shapley support levels values and the weighted Shapley hierarchy levels values satisfy the level game property. That means that the total payoff to all players of a component coincides with the payoff to the component if we would play a game where the components are the players itselves. But this is not the case[5] for our two new classes of weighted values. The values in these classes allow the players, within the hierarchy of the level structure, to act more independent. So in the following class they can form subgroups with an own weight within the components containing them.

E. g., looking at a game where the whole world is the grand coalition. The world splits up in political unions like the European Union (EU) and countries which remain fully autonomous. Within the EU many countries are organized as a federal state or a comparable system and so on. But within the EU are also powerful subgroups possible like the euro area. Assume that we have found a weight system for the political influence and power of all countries, states and so on and all possible cooperations of these units. Using a weighted Shapley support levels value the euro area throws, outside of the EU, the same weight as the whole EU into the balance! Instead, the following class of values assigns the euro area exactly the weight it has itself. The structure of the level structure determines here that always the involved players within a component act together as a single unit outside of the component.

Definition 3.1. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{2^N}$[6] *and for all* $T \subseteq N$, $T \ni i$,

$$A_{w,T}(i) := \prod_{r=0}^{h} A^r_{w,T}(i), \text{ where } A^r_{w,T}(i) := \frac{w_{\mathcal{B}^r(i) \cap T}}{\sum_{\substack{B \in \mathcal{B}^r :\, B \subseteq \mathcal{B}^{r+1}(i), \\ B \cap T \neq \emptyset}} w_{B \cap T}}. \tag{4}$$

The ***weighted Shapley alliance levels value*** Sh^{wAL} *is given by*

$$Sh^{wAL}_i(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} A_{w,T}(i) \Delta_v(T) \text{ for all } i \in N. \tag{5}$$

[4]In Nowak and Radzik (1995) the basic version of this axiom for TU-values is called ω-mutual dependence.

[5]We show that this property is not satisfied in our example in section 6.

[6]In fact, we need only weights for coalitions $S \in \Omega^{B^h}$ for all $B^h \in \mathcal{B}^h$.

Of particular interest in the next theorem are the null player out and the weighted proportionality within components property. In many cases it seems naturally that if a null player, obtaining a payoff of zero, should not affect the payoff to the other players if he leaves the game. If all players of two components are dependent means that no one outside of these components is interested to join a coalition of these players if not all players of these components are contained in the coalition and inside of both components all players must act together to obtain anything at all. Then the property that the sum of the payoffs to players of the first component in proportion to the weight of the first component equals the sum of the payoffs to players of the second component in proportion to the weight of the second component cannot be too bad for a weighted value for level structures.

Theorem 3.2. *Let $w \in \mathcal{W}^{2^N}$. The weighted Shapley alliance levels value Sh^{wAL} satisfies* ***E****,* ***N****,* ***NO****,* ***A*** *and* ***WPWC****.*

For the proof, see appendix 8.1. We obtain an axiomatization of the weighted Shapley alliance levels values which corresponds in case of a trivial level structure to an axiomatization of the weighted Shapley values.

Theorem 3.3. *Let $w \in \mathcal{W}^{2^N}$. Sh^{wAL} is the unique LS-value that satisfies* ***E****,* ***NO****,* ***WPWC*** *and* ***A****.*

For the proof, see appendix 8.2. We have an interesting special case if the weights are the size of the components.

Proposition 3.4. *Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ and $\overline{w} \in \mathcal{W}^{2^N}$ such that $\overline{w}_S = |S|$ for all $S \in \Omega^{B^h}$, $B^h \in \mathcal{B}^h$, $\overline{w}_S \in \overline{w}$. Then we have*

$$Sh_i^{\overline{w}AL}(N, v, \underline{\mathcal{B}}) = Sh_i(N, v) \text{ for all } i \in N.$$

For the proof, see appendix 8.3.

3.1 A new characterization of the Shapley levels value and the Owen value

If all weights are equal, the coefficients $A_{w,T}(i)$ from def. 3.1 equal the $K_T(i)$ in definition (3) of the Shapley levels value. Thus the Shapley levels value (Owen value) is a special case of a weighted Shapley alliance levels value (weighted Shapley alliance coalition structure value). We obtain, if we replace in the proof of theorem 3.3 **WPWC** by **DWC** the following corollary.

Corollary 3.5. *Sh^L is the unique LS-value that satisfies* ***E****,* ***NO****,* ***DWC*** *and* ***A****.*

Remark 3.6. *Since a level structure $(N, v, \underline{\mathcal{B}_1}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}_1} := \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$, coincides with a coalition structure on N we obtain a new axiomatization of the Owen value (Owen, 1977) if we adapt* ***E****,* ***NO****,* ***DWC*** *and* ***A*** *to games with a coalition structure.*

4 Weighted Shapley collaboration levels values

Mostly, if players form hierarchical structured coalitions (here called components) and a player of such a component is removed from the game, the remaining players of this component form a new component and so the structure of the level structure remains largely intact. In the preliminaries we called the new level structure a restriction of the old one. But sometimes it is thinkable that the component loses its cohesion. We will not go so far that the whole level structure breaks apart completely. The cohesion from components outside of the broken one and all complete components which are subsets of the remaining player set of the broken component remains unchanged. So we introduce an internally, by the remaining components, induced restriction of the old level structure.

Definition 4.1. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ *and* $T \in \Omega^N$*: We denote by* $\mathcal{B}^r_{\mathcal{I}}|_T$, $0 \leq r \leq h+1$*, the coalition structure on* T*, named* ***internally induced*** r***-th level*** *of* $(N, v, \underline{\mathcal{B}})$ *to player set* T*, given by*

$$\mathcal{B}^r_{\mathcal{I}}|_T := \begin{cases} \{T\}, & \text{if } r = h+1, \\ \big\{B \in \overline{\mathcal{B}} : B \subseteq (B^r \cap T),\, B^r \in \mathcal{B}^r,\, B \nsubseteq B' \in \overline{\mathcal{B}}, B' \subseteq (B^r \cap T)\big\}, & \text{else.} \end{cases}$$

With the level structure $\underline{\mathcal{B}_{\mathcal{I}}}|_T = \{\mathcal{B}^0_{\mathcal{I}}|_T, ..., \mathcal{B}^{h+1}_{\mathcal{I}}|_T\} \in \mathbb{L}^T$ *the LS-game* $(T, v, \underline{\mathcal{B}_{\mathcal{I}}}|_T) \in \mathbb{VL}^T$ *is called the* ***internally induced restriction*** *of* $(N, v, \underline{\mathcal{B}})$ *to player set* T*.*

The internally induced r-th level $\mathcal{B}^r_{\mathcal{I}}|_T$ of a level structure to a player set T consists always of all largest components of the original level structure which are subsets of T and subsets of a component of the r-th level of the original level structure.

Now we can formulate a new null player out axiom that uses internally restrictions.

Internal (induced restriction) **null player out, INO.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $j \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N\backslash\{j\}, v, \underline{\mathcal{B}_{\mathcal{I}}}|_{N\backslash\{j\}})$ for all $i \in N\backslash\{j\}$.

In the following class of weighted Shapley levels values subgroups of a component which are a component of a lower level are the actors in the new game if some players leave the original component, they don't form a new alliance as by the weighted Shapley alliance levels values. For instance, let's have a game where the whole world is the grand coalition in 1990-91. In a highly simplified scheme and not always historically correct we look at the Warsaw Pact, formally also known as the "Treaty of Friendship, Cooperation and Mutual Assistance" as a component of the second highest level. In 1990 the former German Democratic Republic withdraw the Warsaw Pact and left the game by an accession to West Germany. The truncated Warsaw Pact was splitted up completely in different countries as independent components shortly thereafter. Within the Soviet Union (USSR) itself a similar scenario occurred: The withdraw of Lithuania and then also of Estonia and Latvia from the USSR ended with the dissolution of the USSR.

So we have, greatly simplified, nearly a situation where the following class of values can be recommended. If a component that is part of a component C one level higher leaves the game, the cohesion of the remaining components B within C is deleted and each such component is bargaining with its own weight. The structure and the power of components outside of C and within a component B remain unconcerned.

Definition 4.2. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\underline{B}}$ *and for all* $T \subseteq N$, $T \ni i$,

$$C_{w,T}(i) := \prod_{r=0}^{h} C^r_{w,T}(i), \;\; \textit{with } C^r_{w,T}(i) := \frac{w_{\mathcal{B}^r_T(i)}}{\sum_{B \in \widehat{\mathcal{B}^{r+1}_T(i)}} w_B},$$

where $\mathcal{B}_T^r(i)$ *is the largest component of all components* $\mathcal{B}^\ell(i)$, $0 \leq \ell \leq r$, *with* $\mathcal{B}^\ell(i) \subseteq T$, $\mathcal{B}_T^{h+1}(i) := T$ *and*

$$\widehat{\mathcal{B}_T^{r+1}(i)} := \begin{cases} \{\mathcal{B}_T^r(i)\}, \ \text{if } \mathcal{B}_T^r(i) = \mathcal{B}_T^{r+1}(i), \\ \{B \in \overline{\mathcal{B}} : \ B \subsetneq \mathcal{B}_T^{r+1}(i), \ B \not\subseteq B' \in \overline{\mathcal{B}}, \ B' \subsetneq \mathcal{B}_T^{r+1}(i)\}, \ \text{else}, \end{cases}$$

is the set of all largest components which are subsets of $\mathcal{B}_T^{r+1}(i)$. *The* ***weighted Shapley collaboration levels value*** Sh^{wCL} *is given by*

$$Sh_i^{wCL}(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} C_{w,T}(i) \Delta_v(T) \text{ for all } i \in N.$$

A difference between the weighted Shapley alliance levels values and the weighted Shapley collaboration levels values lies in satisfying a different null player out property.

Theorem 4.3. *Let* $w \in \mathcal{W}^{\overline{\mathcal{B}}}$. *The weighted Shapley collaboration levels value* Sh^{wCL} *satisfies* ***E****,* ***N****,* ***INO****,* ***A*** *and* ***WPWC****.*

The proof is omitted because it is completely analogous to the proof of theorem 3.2. Also the following axiomatization extends an axiomatization of the weighted Shapley values and resembles theorem 3.3 of the weighted Shapley alliance levels values.

Theorem 4.4. *Let* $w \in \mathcal{W}^{\overline{\mathcal{B}}}$. Sh^{wCL} *is the unique LS-value that satisfies* ***E****,* ***INO****,* ***WPWC*** *and* ***A****.*

The proof is omitted because it is completely analogous to the proof of theorem 3.3. Also here we obtain an interesting special case if the weights are the size of the components.

Proposition 4.5. *Let* $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ *and* $\overline{w} \in \mathcal{W}^{\overline{\mathcal{B}}}$ *such that* $\overline{w}_B = |B|$ *for all* $B \in \overline{\mathcal{B}}$, $\overline{w}_B \in \overline{w}$. *Then we have*

$$Sh_i^{\overline{w}CL}(N, v, \underline{\mathcal{B}}) = Sh_i(N, v) \text{ for all } i \in N.$$

Again the proof is omitted because it is completely analogous to the proof of proposition 3.4.

5 The Shapley collaboration levels value

As a special case of the weighted Shapley collaboration levels values we present an extension of the Shapley value to level structures.

Definition 5.1. *Let* $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathbb{L}^N$, $v \in \mathbb{VL}^N$, *and for all* $T \subseteq N$, $T \ni i$,

$$C_T(i) := \prod_{r=0}^{h} C_T^r(i), \ \text{with } C_T^r(i) := \frac{1}{\left|\widehat{\mathcal{B}_T^{r+1}(i)}\right|},$$

where $\mathcal{B}_T^r(i)$ *is the largest component* $\mathcal{B}^\ell(i)$, $0 \leq \ell \leq r$, $\mathcal{B}^\ell(i) \subseteq T$, $\mathcal{B}_T^{h+1}(i) := T$ *and* $\widehat{\mathcal{B}_T^{r+1}(i)} := \{B \in \overline{\mathcal{B}} : \ B \subseteq \mathcal{B}_T^{r+1}(i), \ B \not\subseteq B' \in \overline{\mathcal{B}}, \ B' \subsetneq \mathcal{B}_T^{r+1}(i)\}$ *is the set of all largest components which are subsets of* $\mathcal{B}_T^{r+1}(i)$. *The* ***Shapley collaboration levels value*** Sh^{CL} *is given by*

$$Sh_i^{CL}(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} C_T(i) \Delta_v(T) \text{ for all } i \in N.$$

Def. 5.1 coincides with def. 4.2 if all weights are equal. If we replace **WPWC** by **DWC** in theorem 4.4, we obtain, similar to corollary 3.5 the following corollary.

Corollary 5.2. *Sh^{CL} is the unique LS-value that satisfies **E**, **INO**, **DWC** and **A**.*

6 Example

In this section we give a numerical example to compare the sharings for different values. We recall the level game property Winter (1989) that means that the sum of the payoffs to all players of a component equals the payoff to this component in a game where the components are the players. For this we need the definition of an induced level game where components are the players.

Definition 6.1. *Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ and $T \in \Omega^N$. We define for each level r, $0 \leq r \leq h$, the level structure $\underline{\mathcal{B}^r} := \{\mathcal{B}^{r^0}, ..., \mathcal{B}^{r^{h+1-r}}\} \in \mathbb{L}^{\mathcal{B}^r}$ as the induced **r-th level structure** from $\underline{\mathcal{B}}$ by considering the components $B \in \mathcal{B}^r$ as players. There all levels from the original level structure lower then r are dropped and we have $\mathcal{B}^{r^k} := \{\{B \in \mathcal{B}^r : B \subseteq B^{r+k}\}$ for all $B^{r+k} \in \mathcal{B}^{r+k}\}$, $0 \leq k \leq h+1-r$. The induced **r-th level game** $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathbb{VL}^{\mathcal{B}^r}$, where $\mathcal{B}^r$ is the player set with $B \in \mathcal{B}^r$ as players, is given by*

$$v^r(\mathcal{T}) := v\Big(\bigcup_{B \in \mathcal{T}} B\Big) \text{ for all } \mathcal{T} \subseteq \mathcal{B}^r.$$

Remark 6.2. *For a level structure $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathbb{L}^N$ and an induced r-th level structure $\underline{\mathcal{B}^r}$ related components have the same weights. So we have for all r, k, $0 \leq r \leq k \leq h$, $B^{r^{k-r}} \in \mathcal{B}^{r^{k-r}}$, $\mathcal{B}^{r^{k-r}} \in \underline{\mathcal{B}^r}$, $B^k \in \mathcal{B}^k$, $\mathcal{B}^k \in \underline{\mathcal{B}}$,*

$$w_{B^{r^{k-r}}} = w_{B^k} \text{ with } B^{r^{k-r}} := \{B \in \mathcal{B}^r : B \subseteq B^k\} \text{ and } w_{B^{r^{k-r}}} \in \mathcal{W}^{\underline{\mathcal{B}^r}}, \ w_{B^k} \in \mathcal{W}^{\underline{\mathcal{B}}}.$$

It follows the desired property.

Level game property, LG (Winter, 1989). For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $B \in \mathcal{B}^r$, $0 \leq r \leq h+1$, we have

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_B(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}).$$

Let now $(N, u_S, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $w \in \mathcal{W}^{2^N}$ and $w' \in \mathcal{W}^{\underline{\mathcal{B}}}$, where $N := \{1, 2, 3, 4, 5\}$, and $\underline{\mathcal{B}} = \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3\}$, with $\mathcal{B}^1 := \{\{1, 2\}, \{3, 4\}, \{5\}\}$, $\mathcal{B}^2 := \{\{1, 2, 3, 4\}, \{5\}\}$ and u_S the unanimity game with carrier $S := \{1, 2, 3, 5\}$ (see Figure 1). Assume that the weights are exogenously given as shown in Table 1 and reflect, e. g., the political or market power or something else of the coalitions. For the weight system w' we use the same weights for the components as given in the weight system w so that we have $w'_B = w_B$ for all $B \in \underline{\mathcal{B}}$; for calculating the weighted Shapley value Sh^{w} we use the weight system $\mathrm{w} \in \mathrm{W}$, given by $\mathrm{w}_i := w_{\{i\}}$ for all $i \in N$.

Table 1: Weights of the coalitions

S	{1}	{2}	{3}	{4}	{5}	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
w_S	1	1	1	1	5	3	3	3	3	3	3	5	5	5	5	7

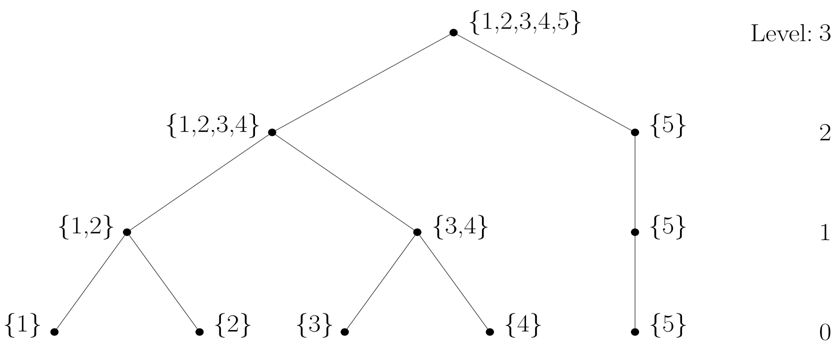

Figure 1: Structure of the components in different levels

We obtain Table 2 where Sh^{wHL} is the weighted Shapley hierarchy levels value, presented in Besner (2018b) and Sh^{wSL} is the weighted Shapley support levels value, presented in Besner (2018a).

Table 2: Comparison of different values

Value	Payoff to player $1,2,3,4,5$	$\varphi_{\{1,2,3,4\}}(\mathcal{B}^2, v^2, \underline{\mathcal{B}}^2)$	$\sum_{i\in\{1,2,3,4\}} \varphi_i(N, v, \underline{\mathcal{B}})$
$Sh(N,v)$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}$	-	-
$Sh^{w}(N,v)$	$\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, 0, \frac{5}{8}$	-	-
$Sh^{L}(N,v,\underline{\mathcal{B}})$	$\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, 0, \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$Sh^{wHL}(N,v,\underline{\mathcal{B}})$	$\frac{7}{48}, \frac{7}{48}, \frac{5}{24}, \frac{1}{12}, \frac{5}{12}$	$\frac{7}{12}$	$\frac{7}{12}$
$Sh^{wSL}(N,v,\underline{\mathcal{B}})$	$\frac{7}{48}, \frac{7}{48}, \frac{7}{24}, 0, \frac{5}{12}$	$\frac{7}{12}$	$\frac{7}{12}$
$Sh^{wAL}(N,v,\underline{\mathcal{B}})$	$\frac{3}{16}, \frac{3}{16}, \frac{1}{8}, 0, \frac{1}{2}$	$\frac{7}{12}$	$\frac{1}{2}$
$Sh^{wCL}(N,v,\underline{\mathcal{B}})$	$\frac{1}{6}, \frac{1}{6}, \frac{1}{9}, 0, \frac{5}{9}$	$\frac{7}{12}$	$\frac{4}{9}$
$Sh^{CL}(N,v,\underline{\mathcal{B}})$	$\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, 0, \frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$

Discussion: The third column gives the payoff to coalition $\{1,2,3,4\}$ in the second level game where coalition $\{1,2,3,4\}$ is a player. We see, in difference to the other presented values for level structures, Sh^{wAL}, Sh^{wCL} and Sh^{CL} don't match the level game property. Sh^{wHL} is the only value that doesn't satisfy the null player property. Player 3 is supported by player 4 in the weight $w_{\{3,4\}}$. Whereas by Sh^{wHL} player 3 passes a share on to player 4, by Sh^{wSL} player 3 takes it all. All weights of coalitions which contain player 4 do not play a role for computing Sh^{wAL} and Sh^{wCL}. So it is no surprise that the payoff to all other players wouldn't change by these values if player 4 would leave the game. In addition, we see that all presented values are different, especially Sh^{wHL}, Sh^{wSL} and Sh^{wAL} don't coincide although they all contain the Shapley levels value! Also of interest is the fact that players 1 and 2 are treated symmetrically by all values. Both players are dependent in the coalition function and so symmetric as well, are part of the same component in the first level and all coalitions which are joined with only one of these players have as joined coalition for both players the same weights in the related cases.

7 Conclusion

In an increasingly networked world where sharing of information, data, knowledge, expensive ressources and so on is a matter of fact it is increasingly important for all stakeholders to share in the benefits. To distribute generated surpluses the presented two new classes of LS-values, together with the weighted Shapley hierarchy levels values and the weighted Shapley support levels values, contain alternatives to the Shapley levels value if there exist exogenously given weights for some coalitions.

The weighted Shapley hierarchy levels values and the weighted Shapley support levels values meet the level game property but subcoalitions of a component always depend on the weight of the whole component. An detailed examination of the weighted Shapley hierarchy levels values shows that by these values, and so also for the Shapley levels value, the worth of many coalitions, e. g. coalitions which are the union of proper subsets of pairwise disjoint components, is not relevant for the payoff to all players!

In this regard by the values of our two new classes (the Shapley levels value is here an exception) the players are more independent from the nesting components. The new classes offer also an alternative if it is desired that if a null player is removed from the game there is no change in the payoff to the other players.

The weighted Shapley alliance levels values contain the Shapley levels value as a special case. So we have three different classes, the weighted Shapley hierarchy levels values, the weighted Shapley support levels values and the weighted Shapley alliance levels values which contain all the Shapley levels values. Thus the different axiomatizations by these classes open different perspectives on the Shapley levels value too.

But a level structure is more then just a sequence of coalition structures, the coalition structures are ordered in a certain way. Thus we could present a new internally induced restriction, which should be used for example in the case that a component splits in the components next in size if one player quits the component. The weighted Shapley collaboration levels values satisfy the null player out property for internally induced restrictions. So we have found a situation where the Shapley levels value fails and a new extension of the Shapley value, called Shapley collaboration levels value, is recommended if all weights are equal or there aren't any weights at all respectively.

8 Appendix

The following lemma is used in the proofs of theorem 3.2 and theorem 3.3.

Lemma 8.1. *Besner (2018a, lemma 7.3) Players $i, j \in N$, $i \neq j$, are dependent in $v \in \mathbb{V}^N$, iff $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, for all $S \subseteq N \backslash \{i, j\}$.*

8.1 Proof of theorem 3.2

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{2^N}$ and $A^r_{w,T}$ the expressions according to def. 3.1.

- **E, N, A**: Let $T \in \Omega^N$, $j \in T$. It is clear, by induction on r, that

$$\sum_{i \in \mathcal{B}^{r+1}(j),\, i \in T} \prod_{\ell=0}^{r} A^r_{w,T}(i) = 1.$$

By $\sum_{i\in T} A_{w,T}(i) = 1$ and $A_{w,T}(i) > 0$, $i \in T$, the $A_{w,T}(i)$ form a dividend share system $p \in P^N$. Thus Sh^{wAL} coincides with a Harsanyi payoff. It is well-known that a Harsanyi payoff satisfies efficiency, additivity and the null player property for TU-values. Thus it is obvious that Sh^{wAL} matches **E, A** and **N**.

• **NO**: It is well-known that each coalition $S \in \Omega^N$, containing a null player $j \in N$ in v, is not active in v. In eq. (5) we have only to consider active coalitions. But for these coalitions there is no change in the weights. Thus we have $Sh_i^{wAL}(N, v, \underline{\mathcal{B}}) = Sh_i^{wAL}(N\backslash\{j\}, v, \underline{\mathcal{B}}|_{N\backslash\{j\}})$ for all $i \in N\backslash\{j\}$.

• **WPWC**: Let $k, \ell \in N$, $0 \leq r \leq h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and all players $i \in \mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)$ be dependent in v. We obtain

$$\begin{aligned}
\sum_{i\in\mathcal{B}^r(k)} \frac{Sh_i^{wAL}(N,v,\underline{\mathcal{B}})}{w_{\mathcal{B}^r(k)}} &\underset{\substack{Def.\\ 3.1}}{=} \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{i\in\mathcal{B}^r(k)} \sum_{\substack{T\subseteq N,\\ T\ni i}} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\right]\Delta_v(T)\\
&\underset{\substack{Lem.\\ 8.1}}{=} \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{i\in\mathcal{B}^r(k)} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\right]\Delta_v(T)\\
&= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \sum_{i\in\mathcal{B}^r(k)} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\right]\Delta_v(T)\\
&= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T)\left[\prod_{j=r}^{h} \frac{w_{\mathcal{B}^j(k)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} w_{B\cap T}} \cdot \sum_{i\in\mathcal{B}^r(k)} \prod_{j=0}^{r-1} \frac{w_{\mathcal{B}^j(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}^{7}\right]\\
&= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r}^{h} \frac{w_{\mathcal{B}^j(k)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\\
&= \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r+1}^{h} \frac{w_{\mathcal{B}^j(k)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} w_{B\cap T}}\\
&= \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r+1}^{h} \frac{w_{\mathcal{B}^j(\ell)\cap T}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(\ell),\\ B\cap T\neq\emptyset}} w_{B\cap T}} = \sum_{i\in\mathcal{B}^r(\ell)} \frac{Sh_i^{wAL}(N,v,\underline{\mathcal{B}})}{w_{\mathcal{B}^r(\ell)}}.
\end{aligned}$$

□

Convention 8.2. To avoid cumbersome case distinctions in the proof of theorem 3.3 using **WPWC**, if there is only one single player assessed in isolation, she is defined as dependent by herself. Then **WPWC** is trivially satisfied.

8.2 Proof of theorem 3.3

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{2^N}$, $S \in \Omega^N$ arbitrary and φ an LS-value that satisfies all axioms of theorem 3.3 and **N**, because **E** and **NO** imply obvious **N**. Due

[7] The last sum always equals 1, if $r = 0$, we have an empty product, which is equal, by convention, to the multiplicative identity 1.

to theorem 3.2, property (2) and **A**, it is sufficient to show that φ is uniquely defined on the game $v_S := \Delta_v(S) \cdot u_S$.

All players $j \in N \backslash S$ are null players in v_S and so φ is unique on v_S for all $j \in N \backslash S$ by **N**. By lemma 8.1, all players $i \in S$, possibly using conv. 8.2, are dependent in v_S and, by **NO**, we obtain

$$\varphi_i(N, v_S, \underline{\mathcal{B}}) = \varphi_i(S, v_S, \underline{\mathcal{B}}|_S) \text{ for all } i \in S.$$

So we can use an induction on the restriction to the player set S on the size m, $0 \leq m \leq h$, for all levels r, $0 \leq r \leq h$, with $m := h - r$.

Initialisation: Let $m = 0$ and so $r = h$. We get for an arbitrary $i \in S$

$$\sum_{\substack{B \in \mathcal{B}^h|_S,\\ B \cap S \neq \emptyset}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) \underset{\text{(WPBC)}}{=} \sum_{\substack{B \in \mathcal{B}^h|_S,\\ B \cap S \neq \emptyset}} \frac{w_B}{w_{\mathcal{B}^h|_S(i)}} \sum_{j \in \mathcal{B}^h|_S(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) \underset{\text{(E)}}{=} \Delta_v(S)$$

$$\Leftrightarrow \sum_{j \in \mathcal{B}^r|_S(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) = \left[\prod_{k=h-m}^{h} \frac{w_{\mathcal{B}^k|_S(i)}}{\sum_{\substack{B \in \mathcal{B}^k|_S:\, B \subseteq \mathcal{B}^{k+1}|_S(i),\\ B \cap S \neq \emptyset}} w_B} \right] \Delta_v(S). \tag{6}$$

Induction step: Assume that eq. (6) holds to φ with an arbitrary $m-1$, $0 \leq m-1 \leq h-1$ (IH). It follows for an arbitrary $i \in S$

$$\begin{aligned}
\sum_{\substack{B \in \mathcal{B}^r|_S,\, B \cap S \neq \emptyset,\\ B \subseteq \mathcal{B}^{r+1}|_S(i)}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) &\underset{\text{(WPBC)}}{=} \sum_{\substack{B \in \mathcal{B}^r|_S,\, B \cap S \neq \emptyset,\\ B \subseteq \mathcal{B}^{r+1}|_S(i)}} \frac{w_B}{w_{\mathcal{B}^r|_S(i)}} \sum_{j \in \mathcal{B}^r|_S(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) \\
&\underset{(IH)}{=} \left[\prod_{k=h-m+1}^{h} \frac{w_{\mathcal{B}^k|_S(i)}}{\sum_{\substack{B \in \mathcal{B}^k|_S:\, B \subseteq \mathcal{B}^{k+1}|_S(i),\\ B \cap S \neq \emptyset}} w_B} \right] \Delta_v(S) \\
\Leftrightarrow \sum_{j \in \mathcal{B}^r|_S(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}|_S) &= \left[\prod_{k=h-m}^{h} \frac{w_{\mathcal{B}^k|_S(i)}}{\sum_{\substack{B \in \mathcal{B}^k|_S:\, B \subseteq \mathcal{B}^{k+1}|_S(i),\\ B \cap S \neq \emptyset}} w_B} \right] \Delta_v(S).
\end{aligned}$$

So φ is uniquely defined on v_S (take $m = h$ and so $r = 0$). □

8.3 Proof of proposition 3.4

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and $\overline{w} \in \mathcal{W}^{2^N}$ such that $\overline{w}_S = |S|$ for all $S \in \Omega^{B^h}$, $B^h \in \mathcal{B}^h$, $\overline{w}_S \in \overline{w}$. We have only to show that

$$A_{\overline{w},T}(i) = \frac{1}{|T|} \text{ for all } T \subseteq N,\, T \ni i.$$

For all $T \subseteq N$, $T \ni i$, and $0 \leq r \leq h$ the set $\widetilde{\mathcal{B}_T^{r+1}(i)} := \{B \cap T : B \in \mathcal{B}^r, B \subseteq \mathcal{B}^{r+1}(i), B \cap T \neq \emptyset\}$ is a partition of $\mathcal{B}^{r+1}(i) \cap T$. So we have

$$\sum_{B \in \widetilde{\mathcal{B}_T^{r+1}(i)}} \overline{w}_{B \cap T} = \sum_{\substack{B \in \mathcal{B}^r:\, B \subseteq \mathcal{B}^{r+1}(i),\\ B \cap T \neq \emptyset}} \overline{w}_{B \cap T} = |\mathcal{B}^{r+1}(i) \cap T|.$$

By line (4) we get $A_{\overline{w},T}(i) = \dfrac{1}{|T|}$ as desired. □

8.4 Logical independence

Let $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$. All our axioms coincide in this case with axioms for TU-values which characterize the weighted Shapley values or the Shapley values. There it is well-known or easy to prove that the axioms are logical independent. Thus it is clear that the axioms, used in our axiomatizations, must be logical independent too.

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