Necessary and sufficient conditions for global uniqueness of equilibria

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5 May 2008

NECESSARY AND SUFFICIENT CONDITIONS FOR GLOBAL UNIQUENESS OF EQUILIBRIA

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Abstract. We study the problem of finding necessary and sufficient conditions that guarantee global uniqueness of equilibria in a pure exchange economy. We show that for every economy to have a unique equilibrium it is necessary and sufficient that (i) there are no critical economies and (ii) a compact set of economies has a compact set of equilibria.

1. Introduction

One of the first results that need to be established in modeling the structure of markets is the existence of an equilibrium price system for an economy. The classical works of Arrow and Debreu have shown that all economies have at least one equilibrium although maybe not a unique one; an Edgeworth box can illustrate economies with a continuum of them. Debreu [5] pointed out in his Nobel Prize Lecture that "the explanation of equilibrium given by a model of the economy would be complete if the equilibrium were unique, and the search for satisfactory conditions guaranteeing uniqueness has been actively pursued

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Date: May 5, 2008.

The author wishes to thank Yves Balasko, Michael Singer and Leo Butler for many useful conversations.

Research support from the Mexican Council of Science and Technology is gratefully acknowledged.

JEL classification: D50, D51.

[...] However, the strength of the conditions that were proposed made it clear by the late sixties that global uniqueness was too demanding a requirement and that one would have to be satisfied with local uniqueness."

In this paper we study the problem of global uniqueness and give conditions that the equations describing the equality of supply and demand must satisfy in order to admit exactly one solution.

Arrow and Hahn [1] have written a survey containing sufficient conditions for a unique equilibrium. The most general sufficient condition known on the excess demand function, has been provided by Chichilnisky [3] where she establishes that, under certain desirability conditions, if the reduced excess demand $\hat{Z}$ has a nonvanishing Jacobian, then $\hat{Z}$ is globally invertible. So in this case the market has a unique equilibrium.

A generic necessary condition is given by the index theorem. Dierker [6] shows that, under certain desirability assumptions, if the Jacobian of the reduced excess supply function is positive at all Walras equilibria, then there is exactly one equilibrium. Varian [17] remarked that, conversely, if there is only one Walras equilibrium and the economy is regular, then the index theorem also implies that the Jacobian evaluated at the equilibrium is positive. Thus, generically the condition is necessary and sufficient.

To specify necessary and sufficient conditions we use Balasko's [2] setting. The Walras correspondence assigns to every economy its set of equilibrium prices. Having exactly one equilibrium for each economy is the same as asking when the Walras correspondence is a differentiable bijection with a differentiable inverse. However, working with functions is easier than with correspondences. We can instead ask for **necessary and sufficient** conditions that guarantee that the projection from the equilibrium manifold to the set of economies is a differentiable bijection and its inverse is also differentiable.

Our result is the following: for every economy to have a unique equilibrium price system it is necessary and sufficient that (i) there are no critical economies and (ii) a compact set of economies has a compact set of equilibrium prices.

2. The Market

Consider an economy with l goods so that the consumption space is $\mathbb{R}^l_{++}$. Prices are in $S = \{p = (p_1, \dots p_l) \in \mathbb{R}^l_{++} : p_l = 1\}$. There are $i = 1, \dots, m$ agents each of which is characterized by a smooth individual demand function $f_i : S \times (0, \infty) \to \mathbb{R}^l_{++}$ that satisfies Walras law, and by an initial endowment $\omega_i \in \mathbb{R}^l_{++}$. We suppose that an economy is parametrized by $\omega = (\omega_1, \dots, \omega_m) \in \Omega = (\mathbb{R}^l_{++})^m$.

The excess demand function $Z_\omega : S \to \mathbb{R}^l$ of an economy ω is given by

$$Z_\omega(P) = \sum_{i=1}^{m} [f_i(P, P \cdot \omega_i) - \omega_i]$$

We will also write the evaluation $Z(\omega, P) = Z_\omega(P)$. The reduced excess demand function $\hat{Z} : \Omega \times S \to \mathbb{R}^{l-1}$ is obtained from Z by deleting the last component.

Definition 1. *The* ***equilibrium manifold*** $\Gamma \subset \Omega \times S$ *is defined by*

$$\Gamma = \{(\omega, P) \in \Omega \times S : \hat{Z}(\omega, P) = 0\}$$

Balasko [2] shows that Γ is indeed a manifold and has dimension lm. Furthermore, it is contractible so it is connected and simply connected.

Definition 2. *The natural* ***projection map*** π *is the projection* $\Omega \times S \to \Omega$ *restricted to the equilibrium manifold* Γ. *We simply write* $\pi : \Gamma \to \Omega$.

Definition 3. *An economy* $\omega \in \Omega$ *is* ***regular*** *(resp.* ***critical****) if and only if* ω *is a regular (resp. critical) value of the projection* $\pi : \Gamma \to \Omega$.

Recall that if $f : M \to N$ is a differentiable mapping between manifolds then $m \in M$ is a **critical point** of f if the deriviative $Df(m)$ of f at m is not surjective. A **critical value** of f is the image $f(m)$ of a critical point m of f. A **regular value** of f is a point in N which is not a critical value.

3. When a map is a diffeomorphism

We want to understand when the projection map from the equilibrium manifold to the set of economies is a differentiable bijection and its inverse is also differentiable. This is the notion of a diffeomorphism.

Definition 4. *A differentiable map* f *is said to be a* ***diffeomorphism*** *if the map is injective and surjective and its inverse is also differentiable.*

The Implicit Function Theorem will tell us that if the Jacobian of f is nonvanishing then f is a local homeomorphism. Of course this in itself, does not guarantee that f will be a bijection. The surprising fact is that if f is proper then it will be.

Definition 5. *A continuous map f is said to be **proper** if $f^{-1}(K)$ is compact whenever K is compact.*

We can ask for necessary and sufficient conditions that guarantee that f is a diffeomorphism between manifolds M and N. The answer to this question was known to Hadamard; in a modern language it can be stated as:

Theorem 1. [10],[11],[12] *Let M and N be connected, oriented n-dimensional manifolds of class C^1, without boundary, and suppose that N is simply connected. Then a C^1 map f from M to N is a diffeomorphism if and only if f is proper and the Jacobian of f never vanishes.*

Hadamard's result was rediscovered by Palais [15] (p.128-129) and Gordon [8],[9]. There is a generalised version of Theorem 1 developed by Ho that does not require manifolds to be finite-dimensional, orientable or even boundaryless.

Theorem 2. [14] *Let M and N be connected manifolds of class C^1 and suppose that N is simply connected. Then a C^1 map f from M to N is a diffeomorphism if and only if f is proper and the Jacobian of f never vanishes.*

4. Main results

We are now ready to prove our main result.

Theorem 3. *For every economy to have a unique equilibrium it is necessary and sufficient that (i) there are no critical economies and (ii) a compact set of economies has a compact set of equilibrium prices.*

Theorem 3 could be rephrased in the following way.

Theorem 4. *The projection map $\pi : \Gamma \to \Omega$ is a diffeomorphism if and only if π is proper and the Jacobian of π never vanishes.*

The results are an application of the ideas of Hadamard, Palais, Gordon and Ho, and the proof presents no new mathematical insight. However, we reproduce it with an economic interpretation for completeness.

Proof of Theorem 3. By a result of Balasko [2], the equilibrium manifold Γ is connected. Also notice that Ω is simply connected.

First suppose that $\pi : \Gamma \to \Omega$ is a diffeomorphism. Then the Jacobian of π will never vanish; so every economy is regular. But also, by assumption, the Walras correspondence π^{-1} is a continuous map so it must map a set of compact economies to a set of compact equilibrium prices. Hence π is proper.

Conversely, now suppose that (i) there are no critical economies and (ii) a compact set of economies has a compact set of equilibrium prices. We want to show that π is a diffeomorphism. Since there are no critical economies, the Implicit Function Theorem guarantees that the inverse is differentiable. All we need to show then is that is a bijection.

Since π is proper, we can use a result of Palais [16] that a proper map sends closed sets into closed sets, i.e. $\pi(\Gamma)$ is closed. But also, since there are no critical economies, π is a local homeomorphism so it also sends open sets to open sets, i.e. $\pi(\Gamma)$ is open. Hence $\pi(\Gamma)$ is an open, closed and nonempty subset of Ω. So $\pi(\Gamma) = \Omega$. This shows that π is surjective.

We now show that π is injective. Consider two points γ_1, γ_2 in the equilibrium manifold Γ such that $\pi(\gamma_1) = \pi(\gamma_2) = \omega$. Since Γ is connected, we can consider a path $\alpha(t)$ in Γ connecting γ_1 to γ_2. Then $\pi \circ \alpha(t)$ is a loop in Ω based in ω. We also know that Ω is simply connected, so we may use a homotopy $F(s,t)$ such that $F(0,t) = \pi \circ \alpha(t)$ and $F(1,t) = \omega$. Since we have seen that π is is surjective, proper and a local homeomorphism from Γ to Ω, then by a result of Ho [14] (p.239), π must be a covering projection. And every covering projection has the homotopy lifting property property (Hatcher, 2002, p.60). So there has to be a unique lifting $\tilde{F}(s,t)$ of $F(s,t)$ with $\tilde{F}(0,t) = \alpha(t)$. The lift of $\tilde{F}(1,t)$ must be a connected set containing both γ_1 and γ_2. But $\pi^{-1}(\omega)$ is discrete, so $\gamma_1 = \gamma_2$.

$\square$

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