Growth: Scale or Market-Size Effects?

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Abstract

Is the supply of researchers or the demand for technologies more important for innovation? The supply of research labor captures a scale effect, whereas the demand from production labor for technologies captures a market-size effect. We find that both the scale effect and the market-size effect are important for innovation and their relative importance depends on the relative intensity of lab-equipment R&D and knowledge-driven R&D in the innovation process.

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1 Introduction

In an influential study, Jones (1999) shows that the R&D-based growth model features a scale effect, which implies that a larger labor force causes a higher growth rate of technologies. Intuitively, with a larger labor force, there is more labor for R&D. Acemoglu (2002) shows that the R&D-based growth model also features a market-size effect under which the growth rate of technologies is increasing in the amount of labor that uses the technologies. Therefore, the scale effect and the market-size effect are closely related. Acemoglu (2002) writes, "[s]ince the scale effect is related to the market size effect [...], one might wonder whether, once we remove the scale effect, the market size effect will also disappear."

This study disentangles the scale effect and the market-size effect. The supply of research labor determines the scale effect, whereas the demand from production labor for technologies determines the market-size effect. In a Schumpeterian growth model that features both lab-equipment R&D and knowledge-driven R&D, we find that the growth rate of technologies is generally increasing in both research labor and production labor. Therefore, both the scale effect and the market-size effect matter to innovation. However, their relative importance depends on the relative intensity of lab-equipment R&D and knowledge-driven R&D. Under knowledge-driven R&D that uses research labor as input, only the scale effect matters to innovation. Under lab-equipment R&D that uses final good as input, only the market-size effect matters to innovation. In general, the importance of the scale effect relative to the market-size effect is increasing in the intensity of research labor relative to final good in the innovation process. Extending our analysis to a semi-endogenous growth model, we find that the scale effect and the market-size effect are still present but affect the long-run level of technologies, instead of the long-run growth rate of technologies. We also confirm our results in a hybrid growth model that features both endogenous and semi-endogenous growth.

This study relates to the literature on innovation and economic growth. Romer (1990) develops the seminal R&D-based growth model in which new products drive innovation. Segerstrom *et al.* (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992) develop the Schumpeterian model in which higher-quality products drive innovation. Jones (1999) shows that these seminal studies feature a scale effect and discusses two approaches of removing this scale effect.[1] Acemoglu (2002) develops an R&D-based growth model of directed technical change and argues that "the scale effect and the market size effect [...] are distinct". He shows that the market-size effect exists even without the scale effect on growth; however, his formulation maintains the scale effect on level. Our study complements Acemoglu (2002) by showing the different determinants of the scale and market-size effects and the importance of the relative intensity of two conventional R&D specifications.

2 A Schumpeterian growth model

We consider the Schumpeterian model. Previous studies often assume that the R&D sector uses either research labor (i.e., knowledge-driven R&D) or final good (i.e., lab-equipment R&D). We specify a generalized R&D process that uses both research labor and final good.

[1] See Laincz and Peretto (2006), Cozzi (2017a, 2017b) and Peretto (2018) for recent studies.

2.1 Household

The representative household has the following utility function:

$$U = \int_0^\infty e^{-\rho t} \ln c_t dt, \tag{1}$$

where the parameter $\rho > 0$ is the discount rate and c_t denotes consumption at time t. The household supplies m units of manufacturing labor and s units of research labor. Research labor s determines the supply of an input for innovation and captures the scale effect. Production labor m uses invented technologies and determines the market size of innovation.

The household maximizes utility subject to the following asset-accumulation equation:

$$\dot{a}_t = r_t a_t + w_{m,t} m + w_{s,t} s - c_t. \tag{2}$$

a_t is the real value of assets (i.e., the share of monopolistic firms). r_t is the real interest rate. $w_{m,t}$ and $w_{s,t}$ are respectively the real wage rates of manufacturing labor and research labor. Standard dynamic optimization yields

$$\frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{3}$$

2.2 Final good

Competitive firms produce final good y_t using the following Cobb-Douglas aggregator:

$$y_t = \exp\left(\int_0^1 \ln x_t(i) di\right), \tag{4}$$

where $x_t(i)$ is intermediate good $i \in [0, 1]$. The conditional demand function for $x_t(i)$ is

$$x_t(i) = \frac{y_t}{p_t(i)}, \tag{5}$$

where $p_t(i)$ is the price of $x_t(i)$.

2.3 Intermediate goods

There is a unit continuum of monopolistic industries producing differentiated intermediate goods. The production function of the industry leader in industry $i \in [0, 1]$ is

$$x_t(i) = z^{q_t(i)} m_t(i), \tag{6}$$

where the parameter $z > 1$ is the quality step size, $q_t(i)$ is the number of quality improvements that have occurred in industry i as of time t, and $m_t(i)$ is manufacturing labor employed in industry i. Given the productivity level $z^{q_t(i)}$, the marginal cost of the leader in industry i is $w_{m,t}/z^{q_t(i)}$. The monopolistic price is

$$p_t(i) = \mu \frac{w_{m,t}}{z^{q_t(i)}}, \tag{7}$$

where the markup $\mu \in (1, z)$ is a policy parameter determined by the government.[2] The wage payment is

$$w_{m,t} m_t(i) = \frac{1}{\mu} p_t(i) x_t(i) = \frac{1}{\mu} y_t, \tag{8}$$

and the monopolistic profit is

$$\pi_t(i) = p_t(i) x_t(i) - w_{m,t} m_t(i) = \frac{\mu - 1}{\mu} y_t. \tag{9}$$

2.4 R&D

Equation (9) shows that $\pi_t(i) = \pi_t$. Therefore, the value of inventions is the same across industries such that $v_t(i) = v_t$.[3] The no-arbitrage condition that determines v_t is

$$r_t = \frac{\pi_t + \dot{v}_t - \lambda_t v_t}{v_t}, \tag{10}$$

which states that the rate of return on v_t is equal to r_t. The return on v_t is the sum of monopolistic profit π_t, capital gain $\dot{v}_t$ and expected capital loss $\lambda_t v_t$, where λ_t is the arrival rate of innovation.[4]

Competitive entrepreneurs recruit research labor s_t and devote R_t units of final good to perform innovation. The arrival rate of innovation is

$$\lambda_t = \varphi (s_t)^{1-\alpha} \left(\frac{R_t}{Z_t} \right)^{\alpha}, \tag{11}$$

where $\varphi > 0$ is a productivity parameter and Z_t denotes aggregate technology. The parameter $\alpha \in [0, 1]$ is the intensity of final good relative to research labor in the innovation process. Knowledge-driven R&D is captured by $\alpha = 0$, whereas lab-equipment R&D is captured by $\alpha = 1$. The first-order conditions for $\{s_t, R_t\}$ are $(1 - \alpha)\lambda_t v_t = w_{s,t} s_t$ and

$$\alpha \lambda_t v_t = R_t \Leftrightarrow \alpha \varphi s^{1-\alpha} \left(\frac{R_t}{Z_t} \right)^{\alpha - 1} \frac{v_t}{Z_t} = 1, \tag{12}$$

which uses (11) and $s_t = s$.

[2] Grossman and Helpman (1991) and Aghion and Howitt (1992) assume that the markup is equal to the quality step size z, due to limit pricing between current and previous quality leaders. Here we follow Evans *et al.* (2003) to consider price regulation under which the regulated markup ratio is $\mu \in (1, z)$.

[3] We follow the standard approach in the literature to focus on the symmetric equilibrium. See Cozzi *et al.* (2007) for a theoretical justification for the symmetric equilibrium to be the unique rational-expectation equilibrium in the Schumpeterian model.

[4] When the next innovation occurs, the previous technology becomes obsolete. This is known as the Arrow replacement effect; see Cozzi (2007) for a discussion.

2.5 Economic growth

Aggregate technology Z_t is defined as

$$Z_t \equiv \exp\left(\int_0^1 q_t(i)di \ln z\right) = \exp\left(\int_0^t \lambda_\omega d\omega \ln z\right), \tag{13}$$

which uses the law of large numbers. Differentiating the log of Z_t with respect to time yields the growth rate of technology given by

$$g_t \equiv \frac{\dot{Z}_t}{Z_t} = \lambda_t \ln z. \tag{14}$$

Substituting (6) into (4) yields the aggregate production function given by

$$y_t = \exp\left(\int_0^1 q_t(i)di \ln z + \int_0^1 \ln m_t(i)di\right) = Z_t m. \tag{15}$$

Thus, the growth rate of output y_t is also g_t, which is determined by λ_t as shown in (14).

From (3) and (10), the balanced-growth value of an invention is

$$v_t = \frac{\pi_t}{\rho+\lambda} = \frac{\mu-1}{\mu}\frac{Z_t m}{\rho+\lambda}, \tag{16}$$

which uses (9) and (15). Equation (16) shows that v_t is increasing in production labor m, capturing the market-size effect in Acemoglu (2002). Substituting (16) into (12) yields

$$\lambda = \alpha\varphi s^{1-\alpha}\left(\frac{R_t}{Z_t}\right)^{\alpha-1}\frac{\mu-1}{\mu}m - \rho. \tag{17}$$

Substituting $s_t = s$ into (11) yields

$$\lambda = \varphi s^{1-\alpha}\left(\frac{R_t}{Z_t}\right)^{\alpha}. \tag{18}$$

Combining (17) and (18) yields

$$(\rho+\lambda)^\alpha \lambda^{1-\alpha} = \left(\alpha\frac{\mu-1}{\mu}\right)^\alpha \varphi s^{1-\alpha} m^\alpha, \tag{19}$$

which determines the unique steady-state equilibrium λ.

Equation (19) shows that the arrival rate λ of innovation is increasing in production labor m (i.e., the market-size effect) and research labor s (i.e., the scale effect). Therefore, the equilibrium growth rate g in (14) is also increasing in the market-size effect m and the scale effect s. Proposition 1 summarizes this result.

Proposition 1 *Economic growth is increasing in the market-size effect and the scale effect.*

Considering a zero discount rate $\rho \to 0$, we can simplify (19) to

$$\lim_{\rho \to 0} \lambda = \left(\alpha \frac{\mu - 1}{\mu} \right)^{\alpha} \varphi s^{1-\alpha} m^{\alpha}. \tag{20}$$

Substituting (20) into (14) yields

$$\lim_{\rho \to 0} g = \left(\alpha \frac{\mu - 1}{\mu} \right)^{\alpha} \varphi s^{1-\alpha} m^{\alpha} \ln z, \tag{21}$$

which shows that the importance of the market-size effect m relative to the scale effect s on growth is increasing in the intensity α of final good relative to research labor in the innovation process. Equation (19) shows that this result is robust to $\rho > 0$.[5] Intuitively, as α increases, R&D spending R_t becomes more important for innovation relative to research labor s_t; consequently, the market-size effect, which determines the value of inventions, becomes more important relative to the scale effect in determining innovation. Proposition 2 summarizes this result.

Proposition 2 *The importance of the market-size effect relative to the scale effect on economic growth is increasing in the intensity of final good relative to research labor in the innovation process.*

Finally, we consider knowledge-driven R&D given by $\alpha = 0$ and lab-equipment R&D given by $\alpha = 1$. Under knowledge-driven R&D, the arrival rate of innovation is $\lambda^{KD} = \varphi s$ and the growth rate of technology is $g^{KD} = \varphi s \ln z$. Therefore, only the scale effect s matters under knowledge-driven R&D because innovation is solely determined by the supply of research labor in this case.[6] Under lab-equipment R&D, the arrival rate of innovation is $\lambda^{LE} = \varphi m(\mu - 1)/\mu - \rho$, and the growth rate of technologies is $g^{LE} = \lambda^{LE} \ln z$. Therefore, only the market-size effect m matters under lab-equipment R&D because innovation is determined by the demand for technologies in this case.[7] Proposition 3 summarizes these results.

Proposition 3 *Under knowledge-driven R&D, only the scale effect matters to innovation. Under lab-equipment R&D, only the market-size effect matters to innovation.*

[5] One can apply the approximation $\ln(X) \approx X - 1$ to (19) to show that $\partial \lambda / \partial m \approx \alpha$ and $\partial \lambda / \partial s \approx 1 - \alpha$.

[6] This result is robust to allowing s to be allocated between research s_r and production s_x. For example, one can modify (6) as $x_t(i) = z^{q_t(i)} [m_t(i)]^{\beta} [s_{x,t}(i)]^{1-\beta}$ to confirm that g^{KD} is still independent of m.

[7] If we assume that s can be allocated to production s_x and specify $x_t(i) = z^{q_t(i)} [m_t(i)]^{\beta} [s_{x,t}(i)]^{1-\beta}$, then $g^{LE} = [\varphi m^{\beta} s^{1-\beta} (\mu - 1)/\mu - \rho] \ln z$. Although innovation is also determined by s in this case, its effect works through the market size (i.e., the demand from production labor $s_x = s$ for technologies).

3 A scale-invariant Schumpeterian growth model

In this section, we allow for population growth and convert the model into a semi-endogenous growth model. In this case, we assume that research labor is $s_t \equiv sL_t$ and production labor is $m_t \equiv mL_t$, where $s + m \leq 1$ and population L_t increases at an exogenous growth rate $n > 0$. Then, we modify the innovation process in (11) as follows:

$$\lambda_t = \frac{\varphi(s_t)^{1-\alpha}}{Z_t^{\phi}} \left(\frac{R_t}{Z_t}\right)^{\alpha}, \tag{22}$$

where the parameter $\phi > 0$ and the new term Z_t^{ϕ} capture an increasing-difficulty effect of R&D similar to Segerstrom (1998). The rest of the model is the same as in Section 2. We will show that R_t/Z_t is proportional to m_t and increases at the rate n in the long run. Therefore, $(s_t)^{1-\alpha}(R_t/Z_t)^{\alpha}$ also increases at the rate n. Then, a steady-state arrival rate λ of innovation requires that Z_t^{ϕ} also grows at the rate n in the long run. Therefore, the long-run growth rate of aggregate technology Z_t is $g = n/\phi$, and the steady-state arrival rate of innovation is $\lambda = g/\ln z = n/(\phi \ln z)$.[8]

Substituting (16) into $\alpha\lambda_t v_t = R_t$ yields

$$\frac{R_t}{Z_t} = \frac{\mu - 1}{\mu}\frac{\alpha\lambda}{\rho + \lambda} m_t, \tag{23}$$

which shows that R_t/Z_t is proportional to m_t in the long run. Substituting (23) into (22) yields the long-run level of technology (per capita) as follows:

$$\frac{Z_t^{\phi}}{L_t} = \frac{\varphi(s_t/L_t)^{1-\alpha}(m_t/L_t)^{\alpha}}{\lambda}\left(\frac{\mu - 1}{\mu}\frac{\alpha\lambda}{\rho + \lambda}\right)^{\alpha} = \frac{\varphi s^{1-\alpha} m^{\alpha}}{\lambda}\left(\frac{\mu - 1}{\mu}\frac{\alpha\lambda}{\rho + \lambda}\right)^{\alpha}, \tag{24}$$

where $\lambda = n/(\phi \ln z)$ is determined by exogenous parameters. Equation (24) shows that the long-run level of technology is increasing in the market-size effect m and the scale effect s. Furthermore, the relative importance of the market-size effect m and the scale effect s on innovation is determined by the relative intensity α of final good and research labor in innovation. Under knowledge-driven R&D (i.e., $\alpha = 0$), only the scale effect s matters to innovation. Under lab-equipment R&D (i.e., $\alpha = 1$), only the market-size effect m matters to innovation. All these results are the same as before, except the effect on innovation is reflected in the long-run level of technology instead of the long-run growth rate of technology.

3.1 Labor allocation

In this section, we extend the semi-endogenous growth model by allowing the factor input s to be allocated between research s_r and production s_x. Specifically, we modify (6) as follows:

$$x_t(i) = z^{q_t(i)}[m_t(i)]^{\beta}[s_{x,t}(i)]^{1-\beta}, \tag{25}$$

[8] Alternatively, one can achieve long-run endogenous growth despite population growth by replacing Z_t^{ϕ} in (22) with L_t, where L_t captures a dilution effect in the spirit of Laincz and Peretto (2006). In this case, (19) is the same as before except for $s^{1-\alpha}m^{\alpha}$ being replaced by $(s_t/L_t)^{1-\alpha}(m_t/L_t)^{\alpha}$.

where $\beta \in (0,1)$. In the appendix, we derive the long-run level of technology (per capita) as

$$\frac{Z_t^{\phi}}{L_t} = \frac{\varphi s^{1-\alpha\beta} m^{\alpha\beta}}{\lambda}\Omega, \tag{26}$$

where $\lambda = n/(\phi \ln z)$ and the composite parameter Ω is defined as

$$\Omega \equiv \frac{\left(\frac{\alpha}{\mu}\right)^{\alpha} \left(\frac{1-\alpha}{1-\beta}\right)^{1-\alpha} \frac{\lambda(\mu-1)}{\rho+\lambda}}{\left[1 + \frac{1-\alpha}{1-\beta}\frac{\lambda(\mu-1)}{\rho+\lambda}\right]^{1-\alpha\beta}}.$$

Equation (26) shows that technology Z^{ϕ}/L_t is increasing in the market-size effect m and the scale effect s. The importance of m relative to s is increasing in α. The exponent on s is $1-\alpha\beta = 1-\alpha+\alpha(1-\beta)$, where $1-\alpha$ captures the scale effect from s_r and $\alpha(1-\beta)$ captures the market-size effect from s_x. Under knowledge-driven R&D (i.e., $\alpha = 0$), only the scale effect s matters to technology. Under lab-equipment R&D (i.e., $\alpha = 1$), only the market-size effect $m^{\beta} s^{1-\beta}$ matters, where $s^{1-\beta}$ captures the demand from production labor $(s_x)^{1-\beta}$ for technologies. All these results are the same as before.

3.2 Hybrid innovation

In this section, we extend the Schumpeterian growth model by modifying (22) as follows:

$$\lambda_t = \left(\frac{\theta}{Z_t^{\phi}} + \frac{1-\theta}{L_t}\right) \varphi (s_t)^{1-\alpha} \left(\frac{R_t}{Z_t}\right)^{\alpha}, \tag{27}$$

where the parameter $\theta \in [0,1]$ determines the importance of semi-endogenous growth relative to endogenous growth. This hybrid innovation originates from Cozzi (2017a). For simplicity, we focus on $\beta = 1$. Substituting (23) into (27) yields the following condition:

$$(\rho+\lambda)^{\alpha}\lambda^{1-\alpha} = \left(\theta\frac{L_t}{Z_t^{\phi}} + 1 - \theta\right)\left(\alpha\frac{\mu-1}{\mu}\right)^{\alpha} \varphi s^{1-\alpha} m^{\alpha}. \tag{28}$$

Whether the balanced growth path exhibits semi-endogenous growth or endogenous growth depends on the population growth rate n.

If n is below a threshold n^*, then L_t/Z_t^{ϕ} converges to zero. In this case, the steady-state arrival rate of innovation is *endogenous* and determined by

$$(\rho+\lambda)^{\alpha}\lambda^{1-\alpha} = (1-\theta)\left(\alpha\frac{\mu-1}{\mu}\right)^{\alpha} \varphi s^{1-\alpha} m^{\alpha}. \tag{29}$$

The threshold is defined as $n^* \equiv \phi\lambda^* \ln z$, where λ^* is the endogenous λ determined in (29).

If n is above n^*, then L_t/Z_t^{ϕ} converges to a positive steady state. The steady-state innovation arrival rate $\lambda = n/(\phi \ln z)$ is *semi-endogenous*. The long-run level of technology from (28) is

$$\frac{Z_t^{\phi}}{L_t} = \left\{\frac{(\rho+\lambda)^{\alpha}\lambda^{1-\alpha}}{\theta\varphi s^{1-\alpha} m^{\alpha}}\left[\frac{\mu}{\alpha(\mu-1)}\right]^{\alpha} - \frac{1-\theta}{\theta}\right\}^{-1}. \tag{30}$$

We see that λ^* in (29) and Z^ϕ/L_t in (30) are both increasing in the market-size effect m and the scale effect s. The importance of m relative to s is increasing in α. Under knowledge-driven R&D (i.e., $\alpha = 0$), only the scale effect s matters to innovation. Under lab-equipment R&D (i.e., $\alpha = 1$), only the market-size effect m matters to innovation. Thus, our results are robust to hybrid innovation with a new insight that whether the economy features endogenous growth or semi-endogenous growth depends on the population-growth threshold n^*, which is increasing in $s^{1-\alpha}m^\alpha$; i.e., a larger scale or market-size effect makes endogenous growth more likely by raising λ^* because semi-endogenous growth requires $\lambda = n/(\phi \ln z) > \lambda^*$.

4 Conclusion

In this study, we find that both the supply of research labor that determines the scale effect and the demand from production labor for technologies that determines the market-size effect matter to innovation. Interestingly, the relative importance of these supply and demand factors depends on the relative intensity of lab-equipment R&D and knowledge-driven R&D in the innovation process. Therefore, this structural parameter has important empirical implications. For example, it determines whether an education policy that increases research labor at the expense of production labor stimulates or stifles economic growth. If the intensity of lab-equipment R&D is high relative to knowledge-driven R&D, then a policy that promotes apprenticeships, such as the European Alliance for Apprenticeships, may be more effective in stimulating economic growth.

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Appendix (not for publication)

In this appendix, we generalize the production function in (6) as follows:

$$x_t(i) = z^{q_t(i)}[m_t(i)]^{\beta}[s_{x,t}(i)]^{1-\beta}. \tag{A1}$$

From cost minimization, the marginal cost of production for the leader in industry i is

$$MC_t(i) = \frac{1}{z^{q_t(i)}}\left(\frac{w_{m,t}}{\beta}\right)^{\beta}\left(\frac{w_{s,t}}{1-\beta}\right)^{1-\beta}. \tag{A2}$$

Given $p_t(i) = \mu MC_t(i)$, the monopolistic profit and wage payments are respectively

$$\pi_t(i) = \frac{\mu-1}{\mu}p_t(i)x_t(i) = \frac{\mu-1}{\mu}y_t, \tag{A3}$$

$$w_{m,t}m_t(i) = \frac{\beta}{\mu}p_t(i)x_t(i) = \frac{\beta}{\mu}y_t, \tag{A4}$$

$$w_{s,t}s_{x,t}(i) = \frac{1-\beta}{\mu}p_t(i)x_t(i) = \frac{1-\beta}{\mu}y_t. \tag{A5}$$

The arrival rate λ_t of innovation is given by (22) with s_t replaced by $s_{r,t}$. The first-order conditions for $\{s_{r,t}, R_t\}$ are

$$(1-\alpha)\lambda_t v_t = w_{s,t}s_{r,t}, \tag{A6}$$

$$\alpha\lambda_t v_t = R_t. \tag{A7}$$

Substituting (A1) into (4) yields

$$y_t = Z_t(m_t)^{\beta}(s_{x,t})^{1-\beta}. \tag{A8}$$

From (3) and (10), the balanced-growth value of an invention is

$$v_t = \frac{\pi_t}{\rho+\lambda} = \frac{\mu-1}{\mu}\frac{Z_t(m_t)^{\beta}(s_{x,t})^{1-\beta}}{\rho+\lambda}, \tag{A9}$$

where the second equality uses (A3) and (A8). Substituting (A9) into (A7) yields

$$\frac{R_t}{Z_t} = \frac{\alpha\lambda}{\rho+\lambda}\frac{\mu-1}{\mu}(m_t)^{\beta}(s_{x,t})^{1-\beta}. \tag{A10}$$

Substituting (A5) and (A9) into (A6) yields

$$\frac{s_{r,t}}{s_{x,t}} = \frac{1-\alpha}{1-\beta}\frac{\lambda(\mu-1)}{\rho+\lambda}. \tag{A11}$$

Substituting (A10) and (A11) into (22) yields

$$\lambda = \frac{\varphi(s_{x,t})^{1-\alpha\beta}(m_t)^{\alpha\beta}}{Z_t^{\phi}}\left(\frac{\alpha}{\mu}\right)^{\alpha}\left(\frac{1-\alpha}{1-\beta}\right)^{1-\alpha}\frac{\lambda(\mu-1)}{\rho+\lambda}, \tag{A12}$$

which shows that a steady-state equilibrium λ requires Z_t^{ϕ} to grow at the rate n. Substituting (A11) into $s_{x,t} + s_{r,t} = s_t$ yields

$$s_t = \left[1 + \frac{1-\alpha}{1-\beta}\frac{\lambda(\mu-1)}{\rho+\lambda}\right]s_{x,t}. \tag{A13}$$

Substituting (A13) into (A12) yields (26).