Weighted Shapley hierarchy levels values

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Abstract

In this paper we present a new class of values for cooperative games with level structure. We use a multi-step proceeding, suggested first in Owen (1977), applied to the weighted Shapley values. Our first axiomatization is an extension of the axiomatization given in Myerson (1980) and Hart and Mas-Colell (1989) respectively by efficiency and weighted balanced contributions. The second axiomatization extends the axiomatization of the weighted Shapley values by weighted standardness for two player games and consistency, also introduced by Hart and Mas-Colell (1989). As a corollary we obtain a new axiomatization of the Shapley levels value.

Keywords Game theory · Cooperative game · Consistency · Level structure · (Weighted) Shapley (levels) value · Weighted balanced contributions

1 Introduction

Presenting his famous value for coalition structures, Owen (1977) suggested an extension of his value that

> "... deals with union structure hierarchies, i. e., the possibility that, inside each union, there may be some groups who are closer together then the remaining members of the union. For example, some of the union members may belong to a certain clan, and will therefore make a common front against the other union members."

Winter (1989) was the first one who putted this idea into action by constructing a model for hierarchical structures, called level structure. Then he defined his value, we call it Shapley levels value, by a set of axioms, which are extensions of axioms of an axiomatization of the Owen value (Owen, 1977) and so finally of the Shapley value (Shapley, 1953b). Another axiomatization of the Shapley levels value is presented in Calvo, Lasaga

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and Winter (1996, Theorem 2). They extended the axiomatization of the Shapley value in Myerson (1980) and Hart and Mas-Colell (1989) respectively by efficiency and balanced contributions to level structures to axiomatize the Shapley levels value.

With this in mind, Vidal-Puga (2012) defined initially a weighted value for coalition structures and Gómez-Rúa and Vidal-Puga (2011) extended it to level structures. Their value is an extension of a special case of the weighted Shapley values (Shapley, 1953a). They axiomatized this value by efficiency and an adaption of a particular case of weighted balanced contributions, called balanced per capita contributions, in the end an adaption of a special case of an axiomatization of the weighted Shapley values, also presented in Myerson (1980) and Hart and Mas-Colell (1989). By this value, all coalitions have been assigned weights given by the cardinality of the coalitions. In difference to the Shapley levels value, if we regard games where some coalitions are the players such acting coalitions are only be treated as symmetric if they are symmetric in these games and contain the same amount of players in the original game.

But sometimes this approach is not adequate, e.g. if some coalitions have a lack of ability to perform or not the same political power and so on. To be no longer restricted to that limitation we introduce arbitrary exogenously given weights for the coalitions which become prominent if the coalitions act as players. So we can introduce the weighted Shapley hierarchy levels values. It turned out, by defining weights of coalitions appropriately, the axiomatization of the value from Gómez-Rúa and Vidal-Puga (2011) can be easily transferred to an axiomatization given by efficiency and weighted balanced group contributions, a general extension of weighted balanced contributions in Myerson (1980).

Hart and Mas-Colell (1989) presented another axiomatization of the weighted Shapley values. They used the weighted standardness property, that means that in a two player game the cooperation benefit has to be divided proportionally to the weights of the players, and a consistency property. Consistency requires the payout to a player in a reduced game be the same as the payoff to this player in the original game if the reduced game is defined suitably. As main part of the paper, we extend this axiomatization to level structures to characterize the weighted Shapley hierarchy levels values.

The plan of the paper is as follows. Some preliminaries are given in section 2, in section 3 are introduced the weighted Shapley hierarchy levels values and an axiomatization in the sense of Calvo, Lasaga and Winter (1996, Theorem 2), section 4 presents a new consistency property with an related axiomatization and section 5 gives a short conclusion.

2 Preliminaries

In some definitions and notations we will follow with Besner (2018a). Let $\mathbb{R}$ be the real numbers, $\mathbb{R}_{++}$ the set of all positive real numbers, $\mathfrak{U}$ a countably infinite set, the universe of all possible players, and $\mathcal{N}$ the set of all finite subsets $N \subseteq \mathfrak{U}$, $N \neq \emptyset$. A **TU-game** is a pair (N, v) with a player set $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, $2^N := \{S\colon S \subseteq N\}$. We call $v(S)$ the **worth** of **coalition** $S \subseteq N$, we denote by Ω^S the set of all nonempty subsets of S and the set of all TU-games with player set N is denoted by $\mathbb{V}^N$. (S, v) is the **restriction** of (N, v) to $S \in \Omega^N$ and a player $i \in N$ is called a **null player** in v if $v(S \cup \{i\}) = v(S)$, $S \subseteq N \backslash \{i\}$.

$\mathcal{B} := \{B_1, ..., B_m\}$ is said to be a **coalition structure** on N if $B_k \cap B_\ell = \emptyset$, $1 \leq k < \ell \leq$

m, $B_k \neq \emptyset$ for all k, $1 \leq k \leq m$, and $\bigcup_{k=1}^{m} B_k = N$. We call each $B \in \mathcal{B}$ a **component** and $\mathcal{B}(i)$ denotes the component that contains the player $i \in N$.

A finite sequence $\underline{\mathcal{B}} := \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ of coalition structures $\mathcal{B}^r$, $0 \leq r \leq h+1$, on N is called a **level structure** (Winter, 1989) on N if $\mathcal{B}^r$ is a refinement of $\mathcal{B}^{r+1}$ for each r, $0 \leq r \leq h$, i.e. $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$ for all $i \in N$, $\mathcal{B}^0 = \big\{\{i\}: i \in N\big\}$ and $\mathcal{B}^{h+1} = \{N\}$. $\mathcal{B}^r$ denotes the r-th **level** of $\underline{\mathcal{B}}$ and $\mathcal{B}^r(B^k)$ denotes the component of the r-th level that contains the component $B^k \in \mathcal{B}^k$, $0 \leq k \leq r \leq h+1$. The set of all level structures with player set N is denoted by $\mathbb{L}^N$. A triple $(N, v, \underline{\mathcal{B}})$ consisting of a TU-game $(N, v) \in \mathbb{V}^N$ and a level structure $\underline{\mathcal{B}} \in \mathbb{L}^N$ is called an **LS-game**. The collection of all LS-games on N is denoted by $\mathbb{VL}^N$. We note that each LS-game $(N, v, \underline{\mathcal{B}_0})$ with a **trivial level structure** $\underline{\mathcal{B}_0} := \{\mathcal{B}^0, \mathcal{B}^1\}$ corresponds to a TU-game (N, v) and each LS-game $(N, v, \underline{\mathcal{B}_1})$, $\underline{\mathcal{B}_1} := \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$, corresponds to a game with coalition structure as introduced in Aumann and Drèze (1974) and (Owen (1977). Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ and $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$:

- From a level structure on N follows a **restricted** level structure on $T \in \Omega^N$ by eliminating the players in $N \backslash T$. With coalition structures $\mathcal{B}^r|_T := \{B \cap T: B \in \mathcal{B}^r, B \cap T \neq \emptyset\}$, $0 \leq r \leq h+1$, the new level structure on T is given by $\underline{\mathcal{B}}|_T := \{\mathcal{B}^0|_T, ..., \mathcal{B}^{h+1}|_T\} \in \mathbb{L}^T$ and $(T, v, \underline{\mathcal{B}}|_T) \in \mathbb{VL}^T$ is called the **restriction** of $(N, v, \underline{\mathcal{B}})$ to player set T.
- We define $\underline{\mathcal{B}^r} := \big\{\mathcal{B}^{r^0}, ..., \mathcal{B}^{r^{h+1-r}}\big\} \in \mathcal{L}^{\mathcal{B}^r}$, $0 \leq r \leq h$, as the **induced *r*-th level structure** from $\underline{\mathcal{B}}$ by considering the components $B \in \mathcal{B}^r$ as players, where $\mathcal{B}^{r^k} := \big\{\{B \in \mathcal{B}^r : B \subseteq B'\}: B' \in \mathcal{B}^{r+k}\big\}$, $0 \leq k \leq h+1-r$ is a coalition structure such that each component is a set of all components of the r-th level which are subsets of the same component of the $(r+k)$-th level. The **induced *r*-th level game** $\big(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}\big) \in \mathbb{VL}^{\mathcal{B}^r}$ is given by
$$v^r(\mathcal{Q}) := v(\bigcup_{B \in \mathcal{Q}} B) \text{ for all } \mathcal{Q} \subseteq \mathcal{B}^r.[1] \tag{1}$$
- We define $\underline{\mathcal{B}_r} := \big\{\mathcal{B}^0, ..., \mathcal{B}^r, \{N\}\big\} \in \mathcal{L}^N, 0 \leq r \leq h$, as the **$r$-th cut level structure** from $\underline{\mathcal{B}}$ if we cut out all levels between the r-th and the $(h+1)$-th level. $(N, v, \underline{\mathcal{B}_r})$ is called the **r-th cut** of $(N, v, \underline{\mathcal{B}})$. Note that we have for each $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ also $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$.

A **TU-value** ϕ is an operator that assigns a payoff vector $\phi(N, v) \in \mathbb{R}^N$ to any $(N, v) \in \mathbb{V}^N$, an **LS-value** φ is an operator that assigns a payoff vector $\varphi(N, v, \underline{\mathcal{B}}) \in \mathbb{R}^N$ to any $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$.

We define $\mathrm{W} := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$ with $\mathrm{w}_i := \mathrm{w}(i)$ for all $\mathrm{w} \in \mathrm{W}$ and $i \in \mathfrak{U}$ as the set of all positive weight systems on the universe of all players and we define $\mathcal{W}^{2^N} := \{f : 2^N \backslash \emptyset \to \mathbb{R}_{++}\}$ with $w_S := w(S)$ for all $w \in \mathcal{W}^{2^N}$, $S \in \Omega^N$, as the set of all positive weight systems on all coalitions $S \in \Omega^N$.

The (simply) **weighted Shapley values**[2] Sh^{w} (Shapley, 1953a) are defined by
$$Sh_i^{\mathrm{w}}(N, v) := \sum_{S \subseteq N, S \ni i} \frac{\mathrm{w}_i}{\sum_{j \in S} \mathrm{w}_j} \Delta_v(S) \;\; \text{for all } i \in N \text{ and } \mathrm{w} \in \mathrm{W}. \tag{2}$$

The **Shapley value** Sh (Shapley, 1953b) distributes the dividends $\Delta_v(S)$ equally among all players of a coalition S and is a special case of a weighted Shapley value, given by

[1] In Owen (1977) we have the special case $r = 1$. There such a game is called as quotient game.
[2] We desist from possibly null weights as in Shapley (1953a) or Kalai and Samet (1987).

$$Sh_i(N,v) := \sum_{S\subseteq N,\, S\ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all} \quad i \in N.$$

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N, \mathcal{B} = \mathcal{B}_h$, $T \in \Omega^N$, $T \ni i$, and

$$K_T(i) := \prod_{r=0}^{h} K_T^r(i), \quad \text{where } K_T^r(i) := \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\, B \cap T \neq \emptyset\}|}.$$

The **Shapley Levels value**[3] Sh^L (Winter, 1989) is defined by

$$Sh_i^L(N,v,\underline{\mathcal{B}}) := \sum_{T\subseteq N,\, T\ni i} K_T(i)\Delta_v(T) \text{ for all } i \in N.$$

Obvious coincides Sh^L with Sh if $h = 0$ and it is well-known that Sh^L coincides with the Owen value (Owen, 1977) if $h = 1$.

We refer to the following axioms for LS-values which are adaptions of standard-axioms for TU-values (with the exception of the last axiom):[4]

Efficiency, E. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, we have $\sum_{i\in N} \varphi_i(N,v,\underline{\mathcal{B}}) = v(N)$.

Null player, N. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ and $i \in N$ a null player in v, we have $\varphi_i(N,v,\underline{\mathcal{B}}) = 0$.

Additivity, A. For all $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$, we have
$\varphi(N,v,\underline{\mathcal{B}}) + \varphi(N,v',\underline{\mathcal{B}}) = \varphi(N,v+v',\underline{\mathcal{B}})$.

Balanced group contributions, BGC (Calvo, Lasaga and Winter, 1996)**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \le r \le h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, we have

$$\sum_{i\in B_k} \varphi_i(N,v,\underline{\mathcal{B}}) - \sum_{i\in B_k} \varphi_i(N\backslash B_\ell, v, \underline{\mathcal{B}}|_{N\backslash B_\ell}) = \sum_{i\in B_\ell} \varphi_i(N,v,\underline{\mathcal{B}}) - \sum_{i\in B_\ell} \varphi_i(N\backslash B_k, v, \underline{\mathcal{B}}|_{N\backslash B_k}).$$

This axiom states that for two components B_k, B_ℓ of the same level which belong to the same component one level higher the contribution of B_ℓ to the total payoff of all players from B_k equals the contribution of B_k to the total payoff of all players from B_ℓ.

Standardness, $\mathbf{ST^0}$. (Hart and Mas-Colell, 1989) For all $(N, v) \in \mathbb{V}^N$, $N = \{i, j\}$, $i \neq j$, we have

$$\varphi_i(N,v) = v(\{i\}) + \frac{1}{2}\big[v(\{i,j\}) - v(\{i\}) - v(\{j\})\big].$$

Standardness leads to sharing the surplus by cooperating in two player games that the surplus is divided equally. The following axiom stands for dividing the surplus proportionally to the weights of the players in two player games.

Weighted standardness, $\mathbf{WS^0}$. (Hart and Mas-Colell, 1989) For all $(N, v) \in \mathbb{V}^N$, $N = \{i, j\}$, $i \neq j$, and $\mathrm{w} \in \mathrm{W}$, we have

$$\varphi_i(N,v) = v(\{i\}) + \frac{\mathrm{w}_i}{\mathrm{w}_i + \mathrm{w}_j}\big[v(\{i,j\}) - v(\{i\}) - v(\{j\})\big].$$

The next axiom has the meaning that the sum of the payoffs to all players in a component coincides with the payoff to this component if the component is regarded as a player in an induced level game.

[3] This formula is presented in Calvo, Lasaga and Winter (1996, eq. (1))

[4] In the case of using a subdomain, we require an axiom to hold when all games belong to this subdomain. If there are used weights for some coalitions, these weights are still valid in the subdomain.

Level game property, LG (Winter, 1989). For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $B \in \mathcal{B}^r$, $0 \leq r \leq h+1$, we have

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_B(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}).$$

3 Weighted Shapley hierarchy levels values

Owen (1977) suggested for his value a two-step approach where the first step consists of using the Shapley value on the quotient game. In the second stage the Shapley value is applied to the result of the first step again. Vidal-Puga (2012) adapted the same procedure for the value ζ, an extension of a weighted Shapley value to coalition structures where the weights are given by the size of the coalitions. Gómez-Rúa and Vidal-Puga (2011) presented an extension to level structures and replaced the two-step approach by a multi-step mechanism as suggested also in Owen (1977). It is still not known and there is no hint in Gómez-Rúa and Vidal-Puga (2011) that their value can easily generalized to an extension of the weighted Shapley values in general. The only problem is to know how to deal with arbitrary exogenously given weights for coalitions. The substantial idea behind our definition, similar suggested in Owen (1977) and implemented as an algorithm in Gómez-Rúa and Vidal-Puga (2011) is as follows:

In a first step we distribute the worth of the grand coalition $v(N)$ among the components B^h of the $h-$th level as players using a weighted Shapley value. In the second step each payoff to a component B^h from the first step is divided by a weighted Shapley value among all components B^{h-1} (the new players) which are subsets of the component B^h and so on for all levels. In the last step we distribute the payoff to components B^1 from the first level among the original players $i \in N$. Therefore we need a weight system for the coalitions. For notational parsimony we give the following notation.

Notation 3.1. Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $i \in N$ and $T \in \Omega^{\mathcal{B}^k(i)}$, $0 \leq k \leq h$. We denote by $\mathcal{T}_i^k := \{B \in \mathcal{B}^k \colon B \subseteq \mathcal{B}^{k+1}(i),\ B \neq \mathcal{B}^k(i)\} \cup \{T\}$ the set that contains all components of the k-th level which are subsets of the component $\mathcal{B}^{k+1}(i)$ where component $\mathcal{B}^k(i)$ is replaced by coalition T.

Definition 3.2. *Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $w \in \mathcal{W}^{2^N}$, $i \in N$ and let for all k, $0 \leq k \leq h$, $T \in \Omega^{\mathcal{B}^k(i)}$, $\mathcal{T}_i^k$ the set from notation 3.1, $\mathrm{w}^{k,T} \in \mathrm{W}$ a weight system with coalitions S as players such that $\mathrm{w}_S^{k,T} := w_S$, $w_S \in w$, for all $S \in \mathcal{T}_i^k$, $\bar{v}_i^{h+1} := v$ and $\bar{v}_i^k$ given by*

$$\bar{v}_i^k(T) := Sh_T^{\mathrm{w}^{k,T}}(\mathcal{T}_i^k, \tilde{v}_i^k) \text{ for all } T \in \Omega^{\mathcal{B}^k(i)} \tag{3}$$

where $\tilde{v}_i^k$ is specified by

$$\tilde{v}_i^k(\mathcal{Q}) := \bar{v}_i^{k+1}\Big(\bigcup_{S \in \mathcal{Q}} S\Big) \text{ for all } \mathcal{Q} \subseteq \mathcal{T}_i^k.$$

Then the ***weighted Shapley hierarchy levels value*** *Sh^{wHL} is defined by*

$$Sh_i^{wHL}(N, v, \mathcal{B}) := \bar{v}_i^0(\{i\}) \text{ for all } i \in N.$$

Remarks 3.3. *Sh^{wHL} coincides with Sh^{w} if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$ and $w_{\{i\}} = \mathrm{w}_i$ for all $i \in N$. Since Sh^{w} is efficient and additive, it is obvious, that Sh^{wHL} meets* ***E****,* ***A*** *and* ***LG****.*

Similar to Calvo, Lasaga and Winter (1996), who characterized the Shapley levels value by efficiency and balanced group contributions, Gómez-Rúa and Vidal-Puga (2011) characterized their value by efficiency and balanced per capita contributions. This axiom is a special case of the following one by limiting the weights to the size of the relevant components.

Weighted balanced group contributions, WBGC[5]**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $w \in \mathcal{W}^{2^N}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, we have

$$\frac{\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) - \sum_{i \in B_k} \varphi_i(N \backslash B_\ell, v, \underline{\mathcal{B}}|_{N \backslash B_\ell})}{w_{B_k}} = \frac{\sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}) - \sum_{i \in B_\ell} \varphi_i(N \backslash B_k, v, \underline{\mathcal{B}}|_{N \backslash B_k})}{w_{B_\ell}}.$$

In difference to **BGC**, here the contributions of one component to the players of the other component and vice versa are proportional to the weights of the components.

Theorem 3.4. *Let $w \in \mathcal{W}^{2^N}$. Sh^{wHL} is the unique LS-value that satisfies **E** and **WBGC**.*

Proof. It is clear that def. 3.2 is equivalent to the algorithm given in Gómez-Rúa and Vidal-Puga (2011, section 3) if we use as weights the cardinality of the coalitions. Thus the rest of the proof of theorem 3.4 is omitted because it's only an exercise to adapt the proofs of proposition 7 and theorem 8 in Gómez-Rúa and Vidal-Puga (2011), using weighted balanced contributions (Myerson, 1980; Hart and Mas-Colell, 1989) for the weighted Shapley values in proposition 7 and replacing the given weights (size of the components) by arbitrary weights in theorem 8. □

Remark 3.5. *If all weights are equal, we use in fact in line (3) the Shapley value and **WBGC** equals **BGC**. So, by Calvo, Lasaga and Winter (1996, Theorem 2), def. 3.2 determines in this case the Shapley levels value Sh^L.*

Remark 3.6. *As shown by example 1 in Gómez-Rúa and Vidal-Puga (2011), Sh^{wHL} doesn't satisfy **N** in general.*

4 Level-consistency

Hart and Mas-Colell (1989) introduced a reduced game of a TU-game.

Definition 4.1. *Let $(N, v) \in \mathbb{V}^N$, $R \in \Omega^N$ and ϕ a TU-value. The **reduced game** $(R, v_R^\phi) \in \mathbb{V}^R$ is given for all $S \in \Omega^R$ by*

$$v_R^\phi(S) := v\big(S \cup R^c\big) - \sum_{j \in R^c} \phi_j\big(S \cup R^c, v\big) \tag{4}$$

where $R^c := N \backslash R$.

Remark 4.2. *If ϕ is efficient eq. (4) can be simplified to*

$$v_R^\phi(S) := \sum_{j \in S} \phi_j\big(S \cup R^c, v\big).$$

[5] This axiom extends weighted balanced contributions (Myerson, 1980).

The interpretation of a reduced game must be seen in context with a TU-value that must have a special property. So Hart and Mas-Colell (1989) formulated the following axiom to characterize the weighted Shapley values.

Consistency, C. For all $(N, v) \in \mathbb{V}^N$, $R \in \Omega^N$, we have $\phi_i(N, v) = \phi_i(R, v_R^{\phi})$ for all $i \in R$.

The meaning of this axiom is as follows: A coalition of players R^c leaves the game. In the reduced game each coalition S that is a subset of the coalition R of the remaining players obtains the worth of the old coalition $S \cup R^c$ less the payoff that the removed players obtain in the restricted game played on $S \cup R^c$. Then a TU-value is consistent if each player in R obtains in the reduced and in the original game the same payoff. It turned out that the weighted Shapley values have got an intense connection to this axiom.

Theorem 4.3. *(Hart and Mas-Colell, 1989) Let* $\mathrm{w} \in \mathrm{W}$. Sh^{w} *is the unique TU-value that satisfies* ***C*** *and* ***WS***^{0}.

If we exchange $\mathbf{WS}^0$ by $\mathbf{ST}^0$ this theorem characterizes the Shapley value. By adding **LG**, Winter (1992) extended this axiomatization for the Shapley value to the Owen value (see proof of theorem 3 in Winter (1992)). Similar Huettner (2015) transferred the related axiomatization of Feldman (1999) and Ortmann (1999) for the proportional value to his proportional value for coalition structures. Our new reduced game for level structures extends the reduced game for TU-games in Hart and Mas-Colell (1989) too but it is no extension of the reduced game used in Winter (1992) and Huettner (2015) respectively.

Definition 4.4. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $R \in \Omega^N$ *such that* $R = \bigcup_{B \in \mathcal{B}^h} B$ *and* φ *an LS-value. The* ***reduced level game*** $(R, v_R^{\varphi}, \underline{\mathcal{B}}|_R) \in \mathbb{VL}^R$ *is given for all* $S \in \Omega^R$ *by*

$$v_R^{\varphi}(S) := v(S \cup R^c) - \sum_{B \in \mathcal{B}^h,\, B \not\subseteq R} \varphi_B\big(\mathcal{B}^h|_{S \cup R^c}, v^h, \underline{\mathcal{B}^h}|_{S \cup R^c}\big) \tag{5}$$

where $R^c := N \backslash R$ *and* $\big(\mathcal{B}^h|_{S \cup R^c}, v^h, \underline{\mathcal{B}^h}|_{S \cup R^c}\big)$ *is the induced* h*-th level game restricted to* $S \cup R^c$.

Remark 4.5. *If* $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, *def. 4.4 coincides with def. 4.1.*

Remark 4.6. *If* φ *is efficient eq. (5) can be simplified to*

$$v_R^{\varphi}(S) := \sum_{B \in \mathcal{B}^h,\, B \subseteq R} \varphi_{S \cap B}\big(\mathcal{B}^h|_{S \cup R^c}, v^h, \underline{\mathcal{B}^h}|_{S \cup R^c}\big).$$

The following axiom extends **C**.

Level-consistency, LC. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $R = \bigcup_{B \in \mathcal{B}^h} B$, we have

$$\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(R, v_R^{\varphi}, \underline{\mathcal{B}}|_R) \text{ for all } i \in R. \tag{6}$$

R is a coalition of all players of some components of the h-th level. All players of the other components of the h-th level leave the game. In the reduced level game each coalition $S \in \Omega^R$ obtains the worth of the old coalition $S \cup R^c$ less the payoff that the removed components obtain in the induced h-th level game restricted to $S \cup R^c$. Then an LS-value is consistent if each player in R obtain in the reduced level game the same payoff as in the original game.

Remark 4.7. *If* $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, ***LC*** *coincides with* ***C***.

We adapt $\mathbf{WS}^0$ to level structures.

Weighted standardness, WS. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ with $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$, $N = \{i, j\}$, $i \neq j$, and $w \in \mathcal{W}^{2^N}$, we have

$$\varphi_i(N, v, \underline{\mathcal{B}}) = v(\{i\}) + \frac{w_{\{i\}}}{w_{\{i\}} + w_{\{j\}}} \big[v(\{i, j\}) - v(\{i\}) - v(\{j\}) \big].$$

To characterize the weighted Shapley hierarchy levels values we need a very weak property that states that if the h-th level consists only of one component we can remove this level from the level structure without consequences.

Inessential (last) **level property, IL.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $h \geq 1$, and $\mathcal{B}^h = \mathcal{B}^{h+1}$ we have $\varphi(N, v, \underline{\mathcal{B}}) = \varphi(N, v, \underline{\mathcal{B}_{h-1}})$.

The last three properties fit best with the weighted Shapley hierarchy levels values:

Theorem 4.8. *Let $w \in \mathcal{W}^{2^N}$. Sh^{wHL} satisfies **IL**, **WS** and **LC**.*

Proof. Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$ and $w \in \mathcal{W}^{2^N}$:

- **IL**: If $\mathcal{B}^h = \mathcal{B}^{h+1}$, we have in notation 3.1 for all $i \in N$ and $T \in \Omega^{\mathcal{B}^h(i)}$, $\mathcal{T}_i^h = \{T\}$. It follows in line (3), $\bar{v}_i^h(T) = Sh_T^{\mathrm{w}^{h,T}}(\{T\}, \tilde{v}_i^h) = v(T)$ for all $T \in \Omega^{\mathcal{B}^h(i)}$. Thus, if $h \geq 1$, there has been no change in line (3) for all $k, 0 \leq k \leq h-1$, if we drop the h-th level and **IL** is shown.
- **WS**: By def. (3.2) and line (2), the claim is obvious.
- **LC**: Let $i \in N$, $R = \bigcup_{B \in \mathcal{B}^h} B$, $R \supseteq \mathcal{B}^h(i)$. By **IL**, it is sufficient to show **LC** on level structures $\underline{\mathcal{B}_h}$ with $\mathcal{B}^h \neq \mathcal{B}^{h+1}$. If $R = N$ or $|N| = 1$, eq. (6) is trivially satisfied.
  Let now $|N| \geq 2$ and $R \neq N$. By Hart and Mas-Colell (1989), Sh^{w} satisfies **C** for all $\mathrm{w} \in \mathrm{W}$. For each $T \in \Omega^{\mathcal{B}^h(i)}$ we have a set $\mathcal{T}_i^h$ from notation 3.1 and a related $\mathcal{T}_i^h|_R$ on the restricted level structure $\underline{\mathcal{B}}|_R$. Thus follows by eq. (3) for $k = h$

$$\bar{v}_i^h(T) = Sh_T^{\mathrm{w}^{h,T}}(\mathcal{T}_i^h, \tilde{v}_i^h) \underset{\substack{\mathbf{C},\\ rem.\,4.5}}{=} Sh_T^{\mathrm{w}^{h,T}}\big(\mathcal{T}_i^h|_R, (\widetilde{v_R^{\varphi}})_i^h\big) \tag{7}$$

  where $(\widetilde{v_R^{\varphi}})_i^h$ is given by $(\widetilde{v_R^{\varphi}})_i^h(\mathcal{Q}) := v_R^{\varphi}(\bigcup_{S \in \mathcal{Q}} S)$ for all $\mathcal{Q} \subseteq \mathcal{T}_i^h|_R$. Thus there is no difference in eq. (3) between both games for all $k \neq h$ too and **LC** is shown. □

It transpires that the weighted Shapley hierarchy levels values are characterized by the following theorem.

Theorem 4.9. *Let $w \in \mathcal{W}^{2^N}$. Sh^{wHL} is the unique TU-value that satisfies **IL**, **WS** and **LC**.*

Proof. By theorem 4.8, it is sufficient to show uniqueness. Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$, $w \in \mathcal{W}^{2^N}$, $i \in N$ arbitrary and φ an LS-values that satisfies **IL**, **WS** and **LC**. By **LC**, we have

$$\varphi_j(N, v, \underline{\mathcal{B}}) = \varphi_j(\mathcal{B}^h(i), v_{\mathcal{B}^h(i)}^{\varphi}, \underline{\mathcal{B}}|_{\mathcal{B}^h(i)}) \text{ for all } j \in \mathcal{B}^h(i)$$

where $v_{\mathcal{B}^h(i)}^{\varphi}$ is unique by **WS**, **LC**, rem. 4.7, rem. 4.5 and theorem 4.3.

By **IL**, we have for each restricted k-th cut level structure $\underline{\mathcal{B}_k}|_{\mathcal{B}^k(i)}$, $k > 0$, from $\underline{\mathcal{B}}$ on $\mathcal{B}^k(i)$,

$$\varphi(\mathcal{B}^k(i), \hat{v}^{\varphi}_{\mathcal{B}^k(i)}, \underline{\mathcal{B}_k}|_{\mathcal{B}^k(i)}) = \varphi(\mathcal{B}^k(i), \hat{v}^{\varphi}_{\mathcal{B}^k(i)}, \underline{\mathcal{B}_{k-1}}|_{\mathcal{B}^k(i)})$$

where $\hat{v}^{\varphi}_{\mathcal{B}^k(i)}$ is recursive defined by

$$\hat{v}^{\varphi}_{\mathcal{B}^k(i)} := \begin{cases} v^{\varphi}_{\mathcal{B}^h(i)} \text{ if } k = h, \\ \left(\hat{v}^{\varphi}_{\mathcal{B}^{k+1}(i)}\right)^{\varphi}_{\mathcal{B}^k(i)} \text{ else.} \end{cases}$$

Hence, by an induction on the levels k, **IL** and **LC**, it follows that

$$\varphi_j(N, v, \underline{\mathcal{B}}) = \varphi_j(\mathcal{B}^1(i), \hat{v}^{\varphi}_{\mathcal{B}^1(i)}, \underline{\mathcal{B}_0}|_{\mathcal{B}^1(i)}) \text{ for all } j \in \mathcal{B}^1(i)$$

where $\hat{v}^{\varphi}_{\mathcal{B}^1(i)}$ is unique by **WS**, **LC**, rem. 4.7, rem. 4.5 and theorem 4.3. Therefore, by **WS**, **LC**, rem. 4.7, rem. 4.5 and theorem 4.3, φ is well defined and the proof is completed. □

To introduce a new axiomatization of the Shapley levels value we adapt $\mathbf{ST}^0$.

Standardness, ST. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ with $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$ and $N = \{i, j\}, i \neq j$, we have

$$\varphi_i(N, v, \underline{\mathcal{B}}) = v(\{i\}) + \frac{1}{2}\left[v(\{i, j\}) - v(\{i\}) - v(\{j\})\right].$$

Since **ST** is a special case of **WS**, we obtain the following corollary by theorem 4.9.

Corollary 4.10. *Sh^L is the unique TU-value that satisfies **IL**, **ST** and **LC**.*

Remark 4.11. *The axioms in theorem 4.9 are logically independent.*

Proof. Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \underline{\mathcal{B}_h}$ and $w \in \mathcal{W}^{2^N}$:

- **IL**: The LS-value φ defined by
$$\varphi := \begin{cases} \frac{1}{2} Sh^{wHL}, \text{ if } h \geq 1, \\ Sh^{wHL}, \text{ else,} \end{cases}$$
satisfies **WS** and **LC** but not **IL**.

- **WS**: Let $v \in \mathbb{V}^N, v(S) > 0$ for all $S \in \Omega^N$, and $\underline{\mathcal{B}} = \underline{\mathcal{B}_1}$. If we regard the proportional value for cooperative games with a coalition structure in Huettner (2015) in the notation for a level structure with $\underline{\mathcal{B}} = \underline{\mathcal{B}_1}$, this value satisfies **IL** and **LC** but not **WS**.

- **LC**: The LS-value φ defined by $\varphi(N, v, \underline{\mathcal{B}}) := Sh^{\mathrm{w}}(N, v)$ with $\mathrm{w}_i := w_{\{i\}}$ for all $i \in N$ satisfies **IL** and **WS** but not **LC**. □

5 Conclusion

It would seem reasonable in our view to use weights if operating players are in some sense not symmetric, e. g., if some players have not the same political, financial or military power, contribute more to the result and so on. Often players act not as lone fighters. They join forces and small groups form together larger groups and so on. So the weighted Shapley hierarchy levels values meet with the multi-stage mechanism a widespread idea how a hierarchical value has to operate.

That these values don't satisfy the null player property is not necessary a disadvantage. A null player receives a compensation since she supports in an acting coalition the other members with her contribution to the weight of the coalition. So a null player is only a null in the coalition function and not necessary in the weight system for coalitions. For further information on this area see also Gómez-Rúa and Vidal-Puga (2010) and Gómez-Rúa and Vidal-Puga (2011).

An emphasis of this paper is on consistency. Consistency plays an important role in many allocation concepts in cooperative game theory. The differences in the numerous varieties of the reduced games are decisive for the diversity of the different consistency characteristics. So the reduced level game is no extension of the reduced game in Winter (1992). There the original players of the game determine the worths in the coalition function of the reduced game. In the reduced level game the acting coalitions are decisive for the worths of the coalitions. If we have a trivial level structure the reduced level game coincides with the reduced game, suggested in Hart and Mas-Colell (1989), too.

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