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August 2018

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August 21, 2018

Abstract

This paper presents a comprehensive analysis of patent licensing in a Cournot oligopoly with general demand and looks at both outside and incumbent innovators. The licensing policies considered are upfront fees, unit royalties and combinations of fees and royalties (FR policies). It is shown that (i) royalties unambiguously ensure full diffusion of the innovation while diffusion is limited under upfront fees, (ii) the Cournot price is higher under royalties compared to upfront fees and the price could even exceed the post-innovation monopoly price, (iii) for generic values of magnitudes of the innovation, when the industry size is relatively large, royalties are superior to upfront fees for the innovator and (iv) for any m, there is always a non empty subset of m-drastic innovations such that for relatively large industry sizes, upfront fee policy results in higher consumer surplus as well as welfare compared to both royalty and FR policies.

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1 Introduction

A patent confers monopoly rights upon an innovator for the use of an innovation for a specified period of time. An innovator can disseminate a patented innovation by licensing it to other users. Licensing policies in practice take diverse forms, but they can be broadly classified into three categories: upfront fees, royalties and policies that combine both fees and royalties. Studies have documented prevalence of royalties in licensing practices. For example, in his study of corporate licensing Rostoker (1984) finds that royalties are used in 39% of the transactions, while the frequency for fees is 13% and for combinations of fees and royalties, it is 46%. In their survey of university licensing, Thursby et al. (2001) find royalties are most frequently used: 81% of respondents "almost always" use royalties, while 16% report "often" using royalties.

The theoretical literature of patent licensing can be traced back to Arrow (1962) who argued that a perfectly competitive industry provides a higher incentive to innovate than a monopoly. Licensing under oligopoly was first studied by Kamien and Tauman (1984, 1986) and Katz and Shapiro (1985, 1986). The early literature concluded that licensing by means of upfront fees dominates royalty licensing for an innovator who is an outsider to the industry (Kamien and Tauman, 1984, 1986; Kamien et al., 1992). It has been subsequently argued that royalties can be explained by factors such as informational asymmetry (Gallini and Wright, 1990; Beggs, 1992), product differentiation (Muto, 1993; Wang and Yang, 1999), or integer constraint of number of licenses (Sen, 2005). The literature has also considered innovators who are one of the incumbent firms in the industry. An incumbent innovator has an additional incentive to use royalties, as they provide the innovator with a competitive edge by raising the effective cost of its rival (see Shapiro, 1985; Wang, 1998; Kamien and Tauman, 2002).

This paper presents a comprehensive analysis of patent licensing in a Cournot oligopoly in which all competing firms initially have the same unit cost of production. An innovator has a patent on a technological innovation that reduces the unit cost. The innovator can license its innovation to some or all firms in the oligopoly. We characterize optimal licensing policies for the innovator where

(a) the demand structure of the Cournot oligopoly is general;

(b) the innovator can either be an outsider, or one of the incumbent firms in the industry;

(c) the licensing policies are pure upfront fees, pure royalties or combinations of upfront fees and royalties.

Our approach to pure royalty policies is more general than the existing literature. It is usually assumed (see, e.g., Kamien and Tauman, 1986) that the innovator sets a rate of acceptable royalty and all firms choose to have licenses. By contrast, we allow the innovator to set the rate of royalty as well as the number of licenses. We show that in this expanded set of policies, it is always optimal for the innovator to sell licenses to all firms, that is, royalties ensure full diffusion of the innovation. We also show that in spite of full diffusion, the market price under royalty is high and sometimes it may be even higher than the post-innovation monopoly price (Proposition 1).

Under pure upfront fee policy, the number of firms who receive a license depends on the significance of the innovation. An innovation is *m-drastic* if (a) when m or more firms have the innovation, all firms who do not have it drop out of the market and (b) when $m-1$ or fewer firms have the innovation, all firms (regardless of whether it has the innovation or not) are active in the market. We show that when the innovator uses only upfront fees, it is not optimal to offer more than m licenses for an m-drastic innovation, a result which has been obtained in Kamien et al. (1992) in the case of an outside innovator. We further qualify the nature of optimal upfront fee policies by showing that the set of m-drastic innovations can be partitioned into two intervals separated by a threshold such that for relatively large industry sizes, innovations of magnitude below the threshold are licensed to m firms and all non-licensees drop out of the market, whereas innovations of higher magnitude are licensed to $m-1$ firms and all firms are active in the market (Proposition 2).

Comparing pure royalties with pure upfront fees we show that upfront fees result in lower market price, so consumer surplus is always higher under upfront fees. On the other hand, for generic magnitudes of the innovation, royalties are superior to upfront fees for the innovator when the industry size is relatively large. This result is an extension of Sen (2005) where it was shown for an outside innovator with linear demand.

To compare welfare between royalty and upfront fee policies, we observe that a key difference in the implications of these two policies is in their nature of diffusion: there is full diffusion of the innovation under royalties while diffusion is generally limited under upfront fees. If this limited diffusion leads to a smaller oligopoly consisting only of the firms who have the innovation (that is, the price falls below the pre-innovation unit cost and all non-licensees drop out of the market), then upfront fees give higher welfare. However, if non-licensees are active in the market under an upfront fee policy, their higher cost of production adversely affects welfare. In such a case, royalty policy (where all firms have the innovation) can give higher welfare. We show that for any m, the same partition of the set of m-drastic innovations that identifies optimal upfront fee policies also determines the welfare ordering between royalties and fees for relatively large sizes of industry. When innovations are of lower magnitude, the unique optimal upfront fee policy results in a smaller oligopoly of only the efficient firms and gives higher welfare than royalty policy. On the other hand, when innovations are of higher magnitude, all firms are active and royalties give higher welfare (Proposition 3).

For licensing policies that are combinations of upfront fees and royalties (FR policies), we show that there is always an optimal policy where all firms, except perhaps one, obtain licenses. Furthermore, for relatively large sizes of industry, any optimal FR policy must have positive royalties and the market price does not fall below the pre-innovation unit cost. These results extend Sen and Tauman (2007) to general demand framework. We also show that for any m, there is always a non empty subset of m-drastic innovations for which both the consumer surplus and the welfare is higher under upfront fee compared to FR policy when industry size is relatively large (Proposition 4). This subset is precisely the interval where the optimal upfront fee policy has m licensees, which results in market price lower than the pre-innovation unit cost so that all non-licensees drop out of the market. Thus a general conclusion that emerges from our analysis is that while diffusion is limited under pure upfront fee policy, there is always a non empty subset of m-drastic innovations for which the consumer surplus as well as the welfare are higher under upfront fee compared to both royalty and FR policies.

Of the existing literature, the papers most closely related to our work are Kamien et al. (1992) [KOT] and Sen and Tauman (2007) [ST]. KOT study patent licensing in a Cournot oligopoly under a general demand framework. However, their analysis is restricted to outside innovators and the policies studied are pure upfront fees and pure royalties. More general policies in combinations of fees and royalties are studied in ST for both outside and incumbent innovators, but the demand is restricted to be linear. Seeking to fill these gaps in the literature, this paper presents a more complete analysis of the patent licensing problem in a Cournot oligopoly. The main contributions of this paper are (a) extension of previous literature from linear to general demand for both outside and incumbent innovators, (b) characterization of optimal pure royalties and identifying their adverse effect on consumer surplus, (c) characterization of optimal pure upfront fees for large industry sizes and (d) welfare comparison of royalties and upfront fees. In addition to these new results, we extend several results of both KOT and ST to more general settings. Our main conclusions are:

(i) The diffusion of the innovation is unambiguously maximum under pure royalties but limited under pure upfront fees; under FR policies, there are optimal combinations that result in almost full diffusion but there could be other optimal policies with limited diffusion.

(ii) The Cournot price is higher under pure royalties compared to pure upfront fees and it can sometimes even exceed the post-innovation monopoly price.

(iii) For generic values of magnitudes of the innovation, when the industry size is relatively large, pure royalty policy is superior to pure upfront fee policy for the innovator.

(iv) The set of m-drastic innovations can be partitioned into two intervals separated by a threshold such that for relatively large industry sizes under upfront fee policy, innovations of magnitude below the threshold are licensed to m firms, all non-licensees drop out of the market and the welfare is higher compared to royalty policy, whereas innovations of magnitude above the threshold are licensed to $m-1$ firms, all firms are active and the welfare is lower compared to royalty policy.

(v) Under combinations of fees and royalties, for relatively large sizes of industry any optimal policy must include a positive royalty and the Cournot price is at least as large as the pre-innovation unit cost.

(vi) There is always a non empty subset of m-drastic innovations such that for relatively large industry sizes, under upfront fee policy m licenses are sold that results in the Cournot price falling below the pre-innovation unit cost and higher consumer surplus as well as welfare compared to both royalty and FR policies.

The paper is organized as follows. We present the model in Section 2 and derive the results in Section 3. We conclude in Section 4. Most proofs are relegated to the Appendix.

2 The model

Consider a homogeneous good Cournot oligopoly where $\overline{N}$ is the set of competing firms. Initially any firm $i \in \overline{N}$ produces under constant marginal cost $c > 0$. An innovator I has a patent for a new process innovation that reduces the per unit cost from c to $c-\varepsilon$ ($0<\varepsilon<c$), so ε is the magnitude of the innovation.[1] For $i \in \overline{N}$, let q_i be the quantity produced by firm i and $Q = \sum_{i\in\overline{N}} q_i$.

Assumptions

A1 The price function or the inverse demand function $p(Q)$: $\mathrm{R}_{++} \to \mathrm{R}_{+}$ is non-increasing and $\exists\, \overline{Q} > 0$ such that $p(Q)$ is decreasing and twice continuously differentiable for $Q \in (0, \overline{Q})$.

A2 $\overline{p} \equiv \lim_{Q\uparrow 0} p(Q) > c$ and $\exists\ 0 < Q^c < Q^{c-\varepsilon} < \overline{Q}$ such that $p(Q^c) = c > p(Q^{c-\varepsilon}) = c - \varepsilon > p(\overline{Q})$.

A3 $p(Q)$ is log-concave for $Q \in (0, \overline{Q})$.

Assumptions A1-A3 imply A4.

A4 For $p \in (0, \overline{p})$, the price elasticity $\eta(p) := -pQ'(p)/Q(p)$ is non-decreasing.

We also assume A5, which ensures a certain comparative-statics result.

A5 The revenue function $\gamma(Q) := p(Q)Q$ is strictly concave for $Q \in (0, \overline{Q})$.

The existence and uniqueness of Cournot equilibrium is ensured by Assumptions A1-A3 (Badia et al., 2014), or alternatively by assumptions A1, A2, A4, A5 (KOT). We assume either [A1-A3, A5] or [A1-A2, A4-A5] holds.[2]

Examples In addition to linear demand, an example of demand functions covered by our analysis include the constant elasticity inverse demand function $p(Q) = s/Q^t$ (where $s > 0$ and $0 < t < 1$) which satisfy [A1-A2, A4-A5]. Another example is $p(Q) = \max\{(a-Q)^t, 0\}$ (where $a, t > 0$ and $c < a^t$), which satisfies [A1-A3, A5].

2.1 Two cases: outside and incumbent innovators

Denote by $N \equiv \{1, \dots, n\}$ the set of all firms other than the innovator I, where $n \geq 1$. We consider two cases:

(i) Outside innovator: The innovator I is an outsider to the industry, i.e., it is not one of the firms in $\overline{N}$. For this case, $\overline{N} = N$.

(ii) Incumbent innovator: The innovator I is one of the incumbent firms in $\overline{N}$. For this case, $\overline{N} = \{I\} \cup N$.

It will be useful to define the indicator variable

$$\lambda = \begin{cases} 0 \text{ if } I \text{ is an outside innovator,} \\ 1 \text{ if } I \text{ is an incumbent innovator} \end{cases} \tag{1}$$

By (1), $|\overline{N}| = n + \lambda$.

[1]Like most papers in the existing literature, firms are assumed to operate under constant returns to scale. There is a small literature that has looked at the patent licensing problem under returns to scale (see, e.g., Sen and Stamatopoulos, 2009a; 2016).

[2]The assumption A5 is needed to show that the equilibrium profit of a licensee is decreasing in the rate of royalty; this result is used to characterize optimal pure royalty policies. When [A1-A3] holds, A5 can be dropped for the analysis of all policies other than pure royalty policies. For other sufficient conditions on the existence of Cournot equilibrium, see, e.g., Novshek (1985), Gaudet and Salant (1991).

2.2 Licensing policies

The innovator I chooses a licensing policy and licenses its innovation to k firms in N, where $k \in \{0, 1, \ldots, n\}$. A firm that becomes a licensee uses the innovation and pays I according to the chosen policy. We consider the following policies.

- *Pure royalty (R) policy*: Under a pure royalty policy $(k, r)_R$, I commits to license the innovation to at most k firms at a unit royalty $r \geq 0$. Any licensee pays r to I for every unit it produces. If more than k firms are willing to pay r and purchase a license, k of them are chosen at random to be licensees.

- *Pure upfront fee (F) policy*: Under a pure upfront fee policy $(k, f)_F$, I commits to license the innovation to at most k firms at an upfront fee $f \geq 0$. Any licensee pays the fee f upfront to I. If more than k firms are willing to pay f and purchase a license, k of them are chosen at random to be licensees.

- *Upfront fee plus royalty (FR) policy*: Under an FR policy $(k, r, f)_{FR}$, I commits to sell at most k licenses at an upfront fee $f \geq 0$ and rate of royalty $r \geq 0$. Any licensee pays the fee f upfront and pays r for every unit it produces. If more than k firms are willing to purchase a license, k of them are chosen at random to be licensees. Note that for $k < n$, this policy will yield the innovator the same revenue as the policy of announcing royalty r and then auctioning off k licenses.

When an innovation of magnitude ε is licensed with rate of royalty $r \geq 0$, the effective marginal cost of a licensee is $c - \varepsilon + r$. A firm has marginal cost c without a license. Assuming that no firm will accept a policy that raises its marginal cost, we restrict $r \leq \varepsilon$.

Define $\delta := \varepsilon - r$. The variable $\delta \in [0, \varepsilon]$ is the *effective magnitude* of the innovation when the rate of royalty is r. Henceforth the policies will be expressed in terms of δ. We shall denote FR policies by $(k, \delta, f)_{FR}$, pure royalty policies by $(k, \delta)_R$ and pure upfront fee policies by $(k, f)_F$. Note that a pure royalty is a special FR policy with $f = 0$ and a pure upfront fee is a special FR policy with $\delta = \varepsilon$.

2.3 The licensing game

For $\lambda \in \{0, 1\}$, the strategic interaction between I and the firms in N is modeled as the *licensing game* Γ_λ that has the following stages.

Stage 1: I offers to license the innovation using an FR policy $(k, \delta, f)_{FR}$.

Stage 2: Firms decide whether to purchase a license or not and the set of licensees is determined.

Stage 3: Firms in $\overline{N}$ compete in quantities. If a firm becomes a licensee under an FR policy $(k, \delta, f)_{FR}$ and produces q, it pays $f + rq = f + (\varepsilon - \delta)q$ to I.

Payoffs of firms Under a licensing policy that has royalty r, the effective marginal cost of a licensee is $c - \varepsilon + r = c - \delta$. The marginal cost of a non-licensee is c. If I is an incumbent innovator (i.e., $\lambda = 1$), it produces with the new technology, so its marginal cost is $c - \varepsilon$.

Let $S \subseteq N$ be the set of licensees, so that $N \setminus S$ is the set of non-licensees. Let f be the fee that a licensee pays upfront. Let q_j be the quantity produced by firm j and

$Q = \sum_{j \in \overline{N}} q_j = \sum_{j \in N} q_j + \lambda q_I$. Then the payoffs of firms in N are

$$\pi_j = \begin{cases} [p(Q) - (c - \delta)]q_j - f & \text{if } j \in S \\ [p(Q) - c]q_j & \text{if } j \in N \setminus S \end{cases} \tag{2}$$

Firm I's payoff has three components: (i) its profit in the Cournot oligopoly (if $\lambda = 1$), (ii) royalty revenue $\sum_{j \in S}(\varepsilon - \delta)q_j$ and (iii) revenue $|S|f$ received upfront from fees. So firm I's payoff is

$$\pi_I = \lambda[p(Q) - (c - \varepsilon)]q_I + \sum_{j \in S}(\varepsilon - \delta)q_j + |S|f \tag{3}$$

We confine to Subgame Perfect Nash Equilibrium (SPNE) outcomes of Γ_λ.

2.4 The Cournot oligopoly subgame $\mathcal{C}^n_\lambda(k, \delta)$

To determine SPNE of Γ_λ, consider first Stage 3 of this game. For $j \in S$, f is paid upfront and it is not affected by the choice of q_j. Similarly, note for firm I that only the first term of (3) is affected by its choice of q_I. Therefore if there are k licensees, in Stage 3 of Γ_λ a Cournot oligopoly game is played with $n + \lambda$ firms where

(i) k firms (licensees) have unit cost $c - \delta$,
(ii) $n - k$ firms (non-licensees) have unit cost c and
(iii) $\lambda \in \{0, 1\}$ firms have unit cost $c - \varepsilon$.

Denote this subgame by $\mathcal{C}^n_\lambda(k, \delta)$. To determine SPNE of Γ_λ, we need to characterize Cournot-Nash Equilibrium[3] of $\mathcal{C}^n_\lambda(k, \delta)$ for all[4] k and δ.

2.5 Drastic and non drastic innovations

The notion of *drastic* innovations (Arrow, 1962) is useful for the analysis of patent licensing. A cost-reducing innovation is drastic if the monopoly price under the new technology does not exceed the old marginal cost c; otherwise it is non drastic.

To classify drastic and non drastic innovations, it will be useful to define

$$\theta \equiv c/\eta(c) = -Q(c)/Q'(c) \tag{4}$$

Furthermore, for $1 \leq k \leq n$, define the function $H^k : (0, \overline{p}) \to R$ as

$$H^k(p) := p[1 - 1/k\eta(p)] \tag{5}$$

The following property of $H^k(p)$ follows immediately from assumption A4 that $\eta(p)$ is non-decreasing.

Observation 1 *Let $p, \widetilde{p} \in (0, \overline{p})$ and suppose $H^k(\widetilde{p}) > 0$. Then $H^k(p) > H^k(\widetilde{p})$ for $p > \widetilde{p}$ and $H^k(p) < H^k(\widetilde{p})$ for $p < \widetilde{p}$.*

To see the interpretation of the function $H^k(p)$, consider a k-firm oligopoly. Corresponding to price p, the quantity is $Q(p)$. If all firms produce equal quantities, then to have industry

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[3] A Cournot-Nash Equilibrium will be denoted simply by *equilibrium* from now onwards; equilibrium price, profit etc. will be denoted by Cournot price, Cournot profit etc.

[4] When there is no licensing, the game $\mathcal{C}^n_\lambda(n, 0)$ is played in Stage 3 of Γ_λ.

output $Q(p)$, each firm must produce $Q(p)/k$. Thus, $H^k(p)$ is the marginal revenue of a firm in a k-firm oligopoly when each firm produces $Q(p)/k$.

To characterize drastic innovations consider a monopolist who has unit cost $c-\varepsilon$. If either [A1-A3], or [A1-A2, A4-A5] holds, then there exists a unique profit maximizing price $p_M(\varepsilon)$ and it satisfies $H^1(p_M(\varepsilon)) = c - \varepsilon$, where H^k is given in (5).

An innovation of magnitude ε is drastic if and only if $p_M(\varepsilon) \leq c$. Since $c - \varepsilon > 0$, by Observation 1, $p_M(\varepsilon) \leq c$ if and only if $H^1(p_M(\varepsilon)) = c-\varepsilon \leq H^1(c) \Leftrightarrow \varepsilon \geq c/\eta(c) \equiv \theta$. Thus, an innovation of magnitude ε is drastic if $\varepsilon \geq \theta$ and it is non drastic if $\varepsilon < \theta$.

m-firm natural oligopoly: For $1 \leq m \leq n + \lambda - 1$ we say that an equilibrium of $\mathcal{C}^n_\lambda(k, \delta)$ results in an *m-firm natural oligopoly* if at that equilibrium, only m firms produce positive output and the remaining $n+\lambda-m$ firms drop out of the market. An 1-firm natural oligopoly is called a *natural monopoly.*

Remark Consider an incumbent innovator of a drastic innovation. By not licensing the innovation at all, it can drive all other firms out of the market to become a natural monopolist. The same outcome can be achieved by licensing policies that have sufficiently high royalties. Under such policies it is not worthwhile for the licensees (as well as non-licensees) to produce and the innovator becomes a natural monopolist. Such licensing policies are redundant since they do not give any licensing revenue to the innovator. For the rest of the paper we shall only consider licensing policies that give positive licensing revenue to the innovator.

2.6 Equilibrium of $\mathcal{C}^n_\lambda(k, \delta)$

Recall that $\theta = c/\eta(c)$. For $\lambda \in \{0, 1\}$, define

$$\delta_\lambda(k) := \begin{cases} (\theta - \lambda\varepsilon)/k & \text{if } k \geq 1 \\ \theta & \text{if } k = 0 \end{cases} \tag{6}$$

As we show in Lemma 1, $\delta_\lambda(k)$ is the threshold which determines whether or not the non-licensees are active in the market. Specifically, in the game $\mathcal{C}^n_\lambda(k, \delta)$ all non-licensees are active in the market if $\delta < \delta_\lambda(k)$ and they drop out if $\delta \geq \delta_\lambda(k)$.

Lemma 1 *Consider the set of licensing policies that give a positive licensing revenue to the innovator. For any $k \in \{1, \ldots, n\}$ and $\delta \in [0, \varepsilon]$, the subgame $\mathcal{C}^n_\lambda(k, \delta)$ has a unique equilibrium. The licensees always obtain positive Cournot profit in equilibrium. Specifically:*

(i) *The Cournot price $p^n_\lambda(k, \delta)$ is continuous and decreasing in δ.*

(ii) *If $\delta < \delta_\lambda(k)$, then $c < p^n_\lambda(k, \delta) < \overline{p}$ and $p^n_\lambda(k, \delta)$ is the unique solution of $H^{n+\lambda}(p) = c - (k\delta + \lambda\varepsilon)/(n + \lambda)$ over $p \in (0, \overline{p})$. All firms obtain positive Cournot profit. The Cournot output and profit of firm I and any non-licensee, as well as the Cournot price, depend only on the product $k\delta$.*

(iii) *If $\delta \geq \delta_\lambda(k)$, then $c - \delta < p^n_\lambda(k, \delta) \leq c$ [equality iff $\delta = \delta_\lambda(k)$] and $p^n_\lambda(k, \delta)$ is the unique solution of $H^{k+\lambda}(p) = c - (k\delta + \lambda\varepsilon)/(k + \lambda)$ over $p \in (0, \overline{p})$. A $(k + \lambda)$-firm natural oligopoly is created, k licensees and firm I (if $\lambda = 1$) obtain positive Cournot profit and the $n - k$ non-licensees drop out of the market.*

Proof Follows from the first order conditions. ■

2.7 Classification of innovations

Based on Lemma 1, it will be helpful for our analysis to classify innovations by extending the notion of drastic innovation as follows.

k-drastic innovation: Let $\lambda \in \{0, 1\}$. For $k \geq 1$, an innovation is *k-drastic for the game* Γ_λ if k is the minimum integer such that if k firms use the innovation, all other firms drop out of the market and a k-firm natural oligopoly is created.

A drastic innovation is 1-drastic and any non drastic innovation is k-drastic for some integer $k \geq 2$. Note that for a k-drastic innovation: (a) if k or more firms have the innovation, then all other firms drop out of the market and (b) if $k-1$ or less firms have the innovation, then all firms including all non-licensees are active in the market.

Recall that as $\delta := \varepsilon - r$, under a policy with royalty r, the effective magnitude of the innovation is δ. From Lemma 1 it follows that for any $k \geq 2-\lambda$, an innovation of effective magnitude δ is $(k+\lambda)$-*drastic for* Γ_λ if $\delta_\lambda(k) \leq \delta < \delta_\lambda(k-1)$. In this case if the number of licensees is k, a $(k+\lambda)$-firm natural oligopoly emerges[5] and $p^n_\lambda(k, \delta) \leq c$. Moreover $p^n_\lambda(k, \delta) = c$ iff $\delta = \delta_\lambda(k)$.

Exact and non-exact innovation: Let $\lambda \in \{0, 1\}$. For $k \geq 2-\lambda$, an innovation of magnitude ε is *exact* $(k+\lambda)$-*drastic* if $\varepsilon = \delta_\lambda(k)$ (so that $p^n_\lambda(k, \varepsilon) = c$) and it is *non-exact* $(k+\lambda)$-*drastic* if $\delta_\lambda(k) < \varepsilon < \delta_\lambda(k-1)$ (so that $p^n_\lambda(k, \varepsilon) < c$). Note that an innovation is non-exact for all but countably many magnitudes. Henceforth we shall derive the results for generic values of ε and restrict to non-exact innovations.

Remark Observe from (6) that for $k \geq 1$,

$$\varepsilon \gtreqless \delta_\lambda(k) \Leftrightarrow \varepsilon \gtreqless \theta/(k+\lambda) \text{ for } \lambda \in \{0, 1\} \tag{7}$$

It follows from (7) that for any $\lambda \in \{0, 1\}$ and $k \geq 2-\lambda$, an innovation of magnitude ε is $(k+\lambda)$-drastic for Γ_λ if $\theta/(k+\lambda) \leq \varepsilon < \theta/(k+\lambda-1)$. It is exact if $\varepsilon = \theta/(k+\lambda)$ and non-exact if $\theta/(k+\lambda) < \varepsilon < \theta/(k+\lambda-1)$.

2.8 The payoff structure

Let $\overline{q}^n_\lambda(k, \delta)$, $\underline{q}^n_\lambda(k, \delta)$ and $\widehat{q}^n_\lambda(k, \delta)$ be the respective Cournot outputs of a licensee, a non-licensee and firm I for the game $\mathcal{C}^n_\lambda(k, \delta)$. Let $\overline{\phi}^n_\lambda(k, \delta)$, $\underline{\phi}^n_\lambda(k, \delta)$ and $\widehat{\phi}^n_\lambda(k, \delta)$ be the corresponding Cournot profits. Note that

$$\overline{\phi}^n_\lambda(k, \delta) = [p^n_\lambda(k, \delta) - c + \delta]\overline{q}^n_\lambda(k, \delta), \;\; \underline{\phi}^n_\lambda(k, \delta) = [p^n_\lambda(k, \delta) - c]\underline{q}^n_\lambda(k, \delta),$$
$$\widehat{\phi}^n_\lambda(k, \delta) = [p^n_\lambda(k, \delta) - c + \varepsilon]\widehat{q}^n_\lambda(k, \delta) \tag{8}$$

When I offers a policy $(k, \delta, f)_{FR}$ for $k < n$, in equilibrium at least $k+1$ firms decide to have a license and k of them are chosen at random to be licensees. The equilibrium upfront fee f for $k < n$ is

$$f^n_\lambda(k, \delta) = \overline{\phi}^n_\lambda(k, \delta) - \underline{\phi}^n_\lambda(k, \delta) \tag{9}$$

For $k = n$, the equilibrium upfront fee is

$$f^n_\lambda(n, \delta) = \overline{\phi}^n_\lambda(n, \delta) - \underline{\phi}^n_\lambda(n-1, \delta) \tag{10}$$

[5] This $(k+\lambda)$-firm natural oligopoly consists of k licensees and the innovator (if $\lambda = 1$).

This is because under the policy $(n, \delta, f)_{FR}$, each firm is guaranteed to have a license, so a firm can reduce the number of licensees from n to $n-1$ by choosing to not have a license.

Henceforth we denote FR policies by $(k, \delta)_{FR}$ where it will be implicit that upfront fees are given by (9) and (10). Pure upfront fee policies are denoted by $(k)_F$ where upfront fees are obtained by taking $\delta = \varepsilon$ in (9) and (10).

If I is an outsider (i.e., $\lambda = 0$), I's payoff is simply its licensing revenue. If I is an incumbent firm (i.e., $\lambda = 1$), I's payoff is the sum of its licensing revenue and its Cournot profit in the resulting oligopoly game. Under an FR policy $(k, \delta)_{FR}$, in equilibrium, the licensing revenue of I consists of: (i) upfront fees $kf^n_\lambda(k, \delta)$ and (ii) royalty payment $kr\overline{q}^n_\lambda(k, \delta) = k(\varepsilon - \delta)\overline{q}^n_\lambda(k, \delta)$. If $\lambda = 1$, I also obtains its Cournot profit $\widehat{\phi}^n_\lambda(k, \delta)$. Hence the payoff of I under policy $(k, \delta)_{FR}$ is

$$\Pi^n_\lambda(k, \delta)_{FR} = kf^n_\lambda(k, \delta) + k(\varepsilon - \delta)\overline{q}^n_\lambda(k, \delta) + \lambda\widehat{\phi}^n_\lambda(k, \delta) \tag{11}$$

Under a pure royalty policy $(k, \delta)_R$, I's payoff consists of the royalty payment and its own Cournot profit, so it is

$$\Pi^n_\lambda(k, \delta)_R = k(\varepsilon - \delta)\overline{q}^n_\lambda(k, \delta) + \lambda\widehat{\phi}^n_\lambda(k, \delta) \tag{12}$$

As a pure upfront fee policy $(k)_F$ is a special FR policy $(k, \varepsilon)_{FR}$ (that is, $r = 0$), under this policy I obtains

$$\Pi^n_\lambda(k)_F = \Pi^n_\lambda(k, \varepsilon)_{FR} = kf^n_\lambda(k, \varepsilon) + \lambda\widehat{\phi}^n_\lambda(k, \varepsilon) \tag{13}$$

Under any policy (FR, R or F), the problem of I is to choose $k \in \{1, \ldots, n\}$ and/or $\delta \in [0, \varepsilon]$ to maximize (11), (12) or (13), respectively. Since these functions are continuous and bounded, these maximization problems have a solution, i.e., there exists an optimal FR, an optimal pure royalty and an optimal pure upfront fee policy.

3 The main results

Consider a monopolist who produces with unit cost $c - \varepsilon$. The profit of this monopolist at price p is

$$G(p) := (p - c + \varepsilon)Q(p) \tag{14}$$

Note that the unique maximum of $G(p)$ is attained at the monopoly price $p = p_M(\varepsilon)$ and $G(p_M(\varepsilon))$ equals $\pi_M(\varepsilon)$ (the monopoly profit under cost $c - \varepsilon$).

Lemma 2 *Consider an innovation of magnitude ε and let $n \geq 2$.*

(i) *The payoff of I is bounded above by $\pi_M(\varepsilon)$ under any licensing policy* (*pure royalty, pure upfront fee or FR policy*).

(ii) *For a drastic innovation, an incumbent innovator obtains $\pi_M(\varepsilon)$ by using its innovation exclusively and an outside innovator obtains $\pi_M(\varepsilon)$ by selling only one license using a pure upfront fee.*

(iii) *For a non drastic innovation,* (a) *licensing is superior to not licensing for the innovator and* (b) *the innovator obtains lower than $\pi_M(\varepsilon)$ under any licensing policy.*

Proof See the Appendix. ■

In view of Lemma 2(ii), for the rest of the paper we focus on non drastic innovations. Before proceeding, a remark on drastic innovations.

Remark Lemma 2 shows that an outside innovator of a drastic innovation obtains the monopoly profit by selling only one license using a pure upfront fee. However, this is not the unique optimal policy. Sen and Stamatopoulos (2009b) have shown that an outside innovator of a drastic innovation can obtain the monopoly profit through $n-1$ or n FR policies. For any drastic innovation of magnitude ε, let $\delta^{\varepsilon}(k) := \varepsilon/k + (1-1/k)(c - p_M(\varepsilon))$ and let $S_t := \{(k, \delta^{\varepsilon}(k)) | k = 1, \ldots, t\}$. The set of optimal FR policies is either S_{n-1} or S_n. Whenever $k \geq 2$ for any such policy, the rate of royalty is positive.

3.1 Pure royalty policies

The following proposition determines optimal pure royalty policies, together with the diffusion of the innovation (that is, the number of firms who obtain a license) and the post-innovation price.

Proposition 1 *Consider a non drastic innovation of magnitude* ε, *i.e.,* $\varepsilon \in (0, \theta)$.

(I) *Any optimal pure royalty policy has* $k = n$ *for both outside and incumbent innovators. All firms produce positive Cournot output.*

(II) *For an incumbent innovator, the unique optimal pure royalty policy has* $r = \varepsilon$. *The Cournot price exceeds* c *but is lower than the post-innovation monopoly price.*

(III) $\exists\, \underline{\varepsilon} \in (0, \theta)$ *such that for an outside innovator:*

- (i) *If* $\varepsilon \in (0, \underline{\varepsilon})$, *then the unique optimal pure royalty policy has* $r = \varepsilon$ *and the result is the same as* (II).
- (ii) *If* $\varepsilon \in (\underline{\varepsilon}, \theta)$, $\exists\, N(\varepsilon)$ (*which is a weakly increasing step function*) *such that*
  - (a) *If* $n > N(\varepsilon)$, *then the unique optimal pure royalty policy has* $r = \varepsilon$ *and the result is the same as* (II).
  - (b) *If* $n < N(\varepsilon)$, *then any optimal pure royalty policy has positive royalty and the Cournot price is at least as large as the post-innovation monopoly price.*

Proof See the Appendix. ■

Remark It is usually assumed in the existing literature that under pure royalty policy, the innovator sells licenses to all firms. Proposition 2 shows that even when the innovator is allowed to sell arbitrary number of licenses, under pure royalty it is optimal to sell licenses to all firms. Thus, the diffusion of innovations is maximum under pure royalty. However, the Cournot price can be high and it may even exceed the post-innovation monopoly price. An illustration of this result is given in the following example.

Example 1 Consider a Cournot oligopoly of $n \geq 2$ firms with an outside innovator (i.e., $\lambda = 0$) and inverse demand $p(Q) = s/Q^t$ where $s > 0$ and $0 < t < 1$. This demand function has constant elasticity $\eta(p) = 1/t$. The initial unit cost is $c > 0$ and the magnitude of the innovation is ε.

For this example, $\theta = c/\eta(c) = tc$. Hence any innovation of magnitude $\varepsilon \in (0, tc)$ is a non drastic innovation. From Proposition 1(I) we know that any optimal pure royalty policy must have $k = n$. Consider a pure royalty policy $(n, \delta)_R$ where $\delta \in [0, \varepsilon]$. Under this policy, the Cournot output of each firm is $(1/n)[s(n-t)/n(c-\delta)]^{1/t}$. Consequently, the payoff of the innovator is $\Pi_0^n(n, \delta)_R = (\varepsilon - \delta)[s(n-t)/n(c-\delta)]^{1/t}$. The derivative of this function with respect to δ is $-[s(n-t)/n(c-\delta)]^{1/t}[tc - \varepsilon + (1-t)\delta]/t(c-\delta)$. Since $\varepsilon < tc$, $0 < t < 1$ and $\delta \leq \varepsilon < c$, this derivative is negative, implying that $\Pi_0^n(n, \delta)_R$ is decreasing for $\delta \in [0, \varepsilon]$. So this payoff is maximized at $\delta = 0$ and the unique optimal pure royalty policy is $r = \varepsilon$.

Under the optimal pure royalty policy, the Cournot price is $nc/(n-t)$. The monopoly price under cost $c - \varepsilon$ is $p_M(\varepsilon) = (c-\varepsilon)/(1-t)$. The Cournot price exceeds the monopoly price iff $n < t(c-\varepsilon)/tc - \varepsilon$. For instance, if $t = 0.5$, $c = 2$ and $\varepsilon = 0.84$, then $\varepsilon < tc$ and we have $t(c-\varepsilon)/(tc-\varepsilon) = 3.625$. In this case for $n = 2, 3$, the Cournot price under the optimal pure royalty policy exceeds the post-innovation monopoly price.

3.2 Pure upfront fee policies

The next result characterizes optimal pure upfront fee policies. It is shown that diffusion of the innovation is generally limited under these policies.

Proposition 2 *Consider a non drastic innovation of magnitude ε which is $(m+\lambda)$-drastic for some $m \geq 2 - \lambda$ and non-exact, i.e., $\theta/(m+\lambda) < \varepsilon < \theta/(m+\lambda-1)$. The following hold under pure upfront fee policy.*

(I) *Under any optimal pure upfront fee policy, the number of licenses does not exceed m and the Cournot price is lower than the post-innovation monopoly price.*

(II) *For sufficiently large industry sizes, it is optimal for the innovator to offer either $m-1$ or m licenses. Specifically, $\exists\ \varepsilon_\lambda^*(m) \in (\theta/(m+\lambda), \theta/(m+\lambda-1))$ and a function $M_\lambda(\varepsilon)$ such that*:

  (a) *If $\varepsilon \in (\theta/(m+\lambda), \varepsilon_\lambda^*(m))$, then for all $n > M_\lambda(\varepsilon)$, the unique optimal policy is to offer $k = m$ licenses. The Cournot price falls below c; an $(m+\lambda)$-firm natural oligopoly is created and all non-licensee firms drop out of the market.*

  (b) *If $\varepsilon \in (\varepsilon_\lambda^*(m), \theta/(m+\lambda-1))$, then for all $n > M_\lambda(\varepsilon)$, the unique optimal policy is to offer $k = m-1$ licenses. The Cournot price exceeds c and all firms produce positive Cournot output.*

Proof See the Appendix. ■

Proposition 3 shows that for any m, the set of m-drastic innovations can be partitioned by a threshold $\varepsilon_\lambda^*(m)$ such that under pure upfront fee, innovations of relatively smaller magnitude are licensed to m firms, which creates a smaller oligopoly consisting only of the firms who have the innovation, whereas innovations of relatively larger magnitude are licensed to $m-1$ firms in which case all firms are active in the market. As we show in the next proposition, the welfare ordering between upfront fee and royalty policies also depends on this threshold.

3.3 Pure royalty versus pure upfront fee

Comparing pure upfront fee with pure royalty, the next proposition shows that upfront fee always results in lower price, so consumers would prefer upfront fee; on the other hand, for sufficiently large industry sizes the innovator prefers royalty over upfront fee. Welfare comparison between the two policies shows that whether upfront fee or royalty gives higher welfare depends on the magnitude of the innovation. Before we formally state the proposition, some observations on welfare will be useful.

At any price y, the consumer surplus is given by

$$\Omega(y) = \int_0^{Q(y)} p(q)\mathrm{d}q - yQ(y) \tag{15}$$

A monopolist who has unit cost $c - \varepsilon$ obtains profit $G(y) = (y - c + \varepsilon)Q(y)$ at price y. So in a monopoly with unit cost $c - \varepsilon$, the welfare (the sum of consumer surplus and the profit of the monopolist) at price y is

$$\Psi(y) = \Omega(y) + G(y) = \int_0^{Q(y)} p(q)\mathrm{d}q - (c - \varepsilon)Q(y) \tag{16}$$

Note that $\Psi(y)$ is decreasing for $y \in [0, c - \varepsilon]$.

Consider a licensing policy with number of licensees k and rate of royalty $r = \varepsilon - \delta$. This policy results in Cournot price $p^n_\lambda(k, \delta)$. Since upfront fees are lump-sum transfers from the licensees to the innovator, they do not affect the sum of payoffs. So the sum of payoffs of the innovator and all firms consists of (i) the Cournot profit of the innovator plus royalty payments, (ii) the Cournot profits of licensees and (iii) the Cournot profits of non-licensees. By (8) and (14), the sum of payoffs is

$$\lambda\widehat{\phi}^n_\lambda(k,\delta) + (\varepsilon-\delta)k\overline{q}^n_\lambda(k,\delta) + k\overline{\phi}^n_\lambda(k,\delta) + (n-k)\underline{\phi}^n_\lambda(k,\delta) = G(p^n_\lambda(k,\delta)) - (n-k)\varepsilon\underline{q}^n_\lambda(k,\delta) \tag{17}$$

Using (16), the welfare (the sum of consumer surplus and payoffs of all firms including the innovator) under this policy is given by

$$W^n_\lambda(k,\delta) = \Omega(p^n_\lambda(k,\delta)) + G(p^n_\lambda(k,\delta)) - (n-k)\varepsilon\underline{q}^n_\lambda(k,\delta) = \Psi(p^n_\lambda(k,\delta)) - (n-k)\varepsilon\underline{q}^n_\lambda(k,\delta) \tag{18}$$

Observe from the last term of (18) that having active non-licensees in the market adversely affects the welfare.

Proposition 3 *Consider a non drastic innovation of magnitude ε which is $(m + \lambda)$-drastic for some $m \geq 2 - \lambda$ and non-exact, i.e., $\theta/(m + \lambda) < \varepsilon < \theta/(m + \lambda - 1)$.*

(I) *The Cournot price under any optimal pure upfront fee policy is always lower than the Cournot price under any optimal pure royalty policy.*

(II) *For sufficiently large industry sizes, the unique optimal pure royalty policy (selling licenses to all firms using royalty $r = \varepsilon$) is superior for the innovator than any pure upfront fee policy.*

(III) $\exists\ \varepsilon^*_\lambda(m) \in (\theta/(m + \lambda), \theta/(m + \lambda - 1))$ *and a function $N_\lambda(\varepsilon)$ such that*

(a) *If* $\varepsilon \in (\theta/(m+\lambda), \varepsilon^*_\lambda(m))$, *then for all* $n > N_\lambda(\varepsilon)$, *the welfare under the unique optimal pure upfront fee policy* ($k = m$) *is higher than the welfare under the unique optimal pure royalty policy.*

(b) *Let* $\varepsilon \in (\varepsilon^*_\lambda(m), \theta/(m+\lambda-1))$, *then for all* $n > N_\lambda(\varepsilon)$, *the welfare under the unique optimal pure upfront fee policy* ($k = m-1$) *is higher than the welfare under the unique optimal pure royalty policy.*

Proof See the Appendix. ■

To see why upfront fee always results in lower price compared to royalty, note from Proposition 2 that the Cournot price under upfront fee is always lower than the post-innovation monopoly price $p_M(\varepsilon)$. So for cases where the price under royalty is at least $p_M(\varepsilon)$, clearly upfront fee results in lower price. For cases where the price under royalty is lower than $p_M(\varepsilon)$, the unique optimal royalty policy has $r = \varepsilon$ (see Proposition 1) so that all firms other than the innovator effectively operate with the pre-innovation unit cost c. Since under upfront fee there are at least some firms (the licensees) who have the reduced cost $c - \varepsilon$, the resulting price is lower.

Part (II) is proved by first showing that $G(c) = \varepsilon Q(c)$ forms an upper bound for the payoffs of the innovator from both royalty and upfront fee policies. While the payoff from royalty can be made arbitrarily close to $G(c)$ by increasing the industry size, the payoff from upfront fee stays bounded away from $G(c)$ as industry size increases for any non-exact innovation. This result is an extension of Sen (2005) who proved it in the special case of an outside innovator with linear demand. Note that if the magnitude ε of the innovation is a random variable with a continuous distribution on $(0, c]$, then with probability 1 an innovation is non-exact. In this case for industries with relatively large number of competing firms, selling licenses to all firms with per unit royalty $r = \varepsilon$ is superior to any pure upfront fee policy with probability 1.

To see the welfare implications observe that a key distinction between royalty and upfront fee policies is in the nature of diffusion. While diffusion is maximum under royalty, it is limited under upfront fee. If this limited diffusion creates a smaller oligopoly consisting only of the efficient firms (that is, the Cournot price falls below c and non-licensees drop out of the market), then the last term of (18) is zero and by the monotonicity of the function Ψ, the welfare under upfront fee exceeds $\Psi(c)$. Since the Cournot price under royalty always exceeds c, by (18), the welfare under royalty is lower than $\Psi(c)$, so upfront fee gives higher welfare (part (III)(a)).

If non-licensees are active in the market under an upfront fee policy, every unit they produce requires an additional cost of magnitude ε that adversely affects the welfare, as represented by the last term of (18). Since the diffusion of the innovation is maximum (that is, $k = n$) under royalty, this adverse effect is absent there. This difference between the two policies becomes dominant for relatively large sizes of industry which explains why the welfare under royalty is higher (part (III)(b)). Note that as royalties raise unit costs, licensees are relatively inefficient under royalty; however, the additional cost they incur for each unit is paid to the innovator as royalty. So unlike the case of active non-licensees, the inefficiency of licensees does not have any additional effect on welfare. Its only effect on the welfare is through the market price.

3.4 FR policies: some general results

The next proposition derives some general properties of optimal FR policies. It extends the results of Sen and Tauman (2007) to general demand. In addition, it is shown that for any $m \geq 2 - \lambda$ there is always a non empty subset of $(m + \lambda)$-drastic innovations for which the welfare is higher under pure upfront fee compared to FR policies when the industry size is relatively large.

Proposition 4 *Consider a non drastic innovation of magnitude ε that is non-exact and let $n \geq 3$.*

(I) *There always exists an optimal FR policy where $k = n - 1$ or n. Moreover if $\varepsilon < \theta/(n + \lambda - 2)$, then any optimal FR policy must have $k = n - 1$ or n.*

(II) *If $\varepsilon > \theta/(n + \lambda - 1)$, then under any optimal FR policy:*

   (a) *the innovator obtains at least $G(c) = \varepsilon Q(c)$;*

   (b) *the Cournot price is at least c;*

   (c) *there is positive royalty.*[6]

(III) *Suppose $\theta/(m + \lambda) < \varepsilon < \theta/(m + \lambda - 1)$ for some integer $m \geq 2 - \lambda$. Then $\exists\ \varepsilon^*_\lambda(m) \in (\theta/(m + \lambda), \theta/(m + \lambda - 1))$ and a function $\widetilde{N}_\lambda(\varepsilon)$ such that if $\varepsilon \in (\theta/(m + \lambda), \varepsilon^*_\lambda(m))$, then for all $n > \widetilde{N}_\lambda(\varepsilon)$, the welfare under the unique optimal pure upfront fee policy $(m)_F$ is higher than the welfare under any optimal FR policy.*

Proof See the Appendix for the proofs of parts (I)-(II). For part (III), take $\widetilde{N}_\lambda(\varepsilon)$ sufficiently large so that $\varepsilon > \theta/(\widetilde{N}_\lambda(\varepsilon) + \lambda - 1)$ and $\widetilde{N}_\lambda(\varepsilon) > M_\lambda(\varepsilon)$ where $M_\lambda(\varepsilon)$ is given in Proposition 2(II)(a). Since for $n > \widetilde{N}_\lambda(\varepsilon)$, any optimal FR policy results in Cournot price at least c (by part (II)), using (18) and the monotonicity of the function Ψ, the welfare under any optimal FR policy is at most $\Psi(c)$. By Proposition 2(II)(a), for $n > \widetilde{N}_\lambda(\varepsilon)$, the unique optimal pure upfront fee policy has $k = m$ that results in Cournot price lower than c and all non-licensees drop out of the market. Again by (18) and the monotonicity of Ψ it follows that the welfare under the unique optimal pure upfront fee policy exceeds $\Psi(c)$, which proves (III). ■

From Proposition 4(III) and Proposition 3(III)(a) it follows that for any $m \geq 2 - \lambda$, there is a non empty interval $(\theta/(m+\lambda), \varepsilon^*_\lambda(m))$ such that if the magnitude ε of an $(m+\lambda)$-drastic innovation is in this interval, then for all relatively large sizes of industry the unique optimal pure upfront fee results in Cournot price lower than c and gives higher welfare than any optimal pure royalty as well as optimal FR policies.

Remark 1 If $\varepsilon < \theta/(n + \lambda)$, then $\varepsilon < \delta_\lambda(k)$ for $k = n - 1, n$ (by (7)) and any optimal FR policy has $k = n - 1$ or n (by Proposition 4(I)). By Lemma 1, for any $\delta \in [0, \varepsilon]$, a policy $(n - 1, \delta)_{FR}$ or $(n, \delta)_{FR}$ results in a Cournot price more than c. Together with Proposition 4(II)(b), this shows that any optimal FR policy can result in Cournot price lower than c only if $\theta/(n + \lambda) < \varepsilon < \theta/(n + \lambda - 1)$.

[6] However, note that for an outside innovator, setting only royalty and no upfront fee cannot be an optimal FR policy in this case. This is because under such a policy an outside innovator obtains less than $G(c)$.

Remark 2 Proposition 4 asserts that with $k = n-1$ or n, one can always achieve an optimal FR policy and for $\varepsilon < \theta/(n+\lambda-2)$, any optimal FR policy must have $k = n-1$ or n. For $\varepsilon > \theta/(n+\lambda-2)$, there might be optimal FR policies with smaller number of licenses $k \leq n-2$. However, if there is such an optimal FR policy, it must be $(k, \delta_\lambda(k))_{FR}$ which gives[7] Cournot price exactly c. This is illustrated in the next example.

Example 2 Consider a Cournot oligopoly with 5 firms and an outside innovator (i.e., $n = 5, \lambda = 0$). The initial unit cost is $c > 0$ and the magnitude of the innovation is ε. The inverse demand is linear, given by $p(Q) = \max\{a-Q, 0\}$ where $0 < c < a$. For this example, $\theta \equiv c/\eta(c) = a - c$, and $G(c) = \varepsilon(a-c)$. Let $\theta/2 < \varepsilon < \theta$ (i.e., the innovation is 2-drastic). Consider all policies $(k, \delta)_{FR}$ where $k \in \{1, \ldots, 5\}$ and $\delta \in [0, \varepsilon]$.

First let $k = 1$. Over all policies $(1, \delta)_{FR}$, the best policy for I is $(1, \varepsilon)_{FR}$ (a pure upfront fee policy). Under this policy I obtains $\Pi_0^5(1, \varepsilon)_{FR} = \varepsilon[2(a-c) + 4\varepsilon]/6 < \varepsilon(a-c)$.

Next consider $k \in \{2, 3, 4\}$. For any such k, over all policies $(k, \delta)_{FR}$, the best policy is $(k, \theta/k)_{FR}$ that gives payoff $\varepsilon(a-c)$ to I, i.e., $\Pi_0^5(2, \theta/2)_{FR} = \Pi_0^5(3, \theta/3)_{FR} = \Pi_0^5(4, \theta/4)_{FR} = \varepsilon(a-c)$.

Finally let $k = 5$. Over all policies $(5, \delta)_{FR}$, the best policy is $(5, \delta^*)_{FR}$ where $\delta^* \equiv [3\varepsilon + 2(a-c)]/21$ with $\Pi_0^5(5, \delta^*)_{FR} = 5[4(a-c)^2 + 138\varepsilon(a-c) + 9\varepsilon^2]/756 < \varepsilon(a-c)$.

Therefore there are 3 optimal FR policies: $(2, \theta/2)_{FR}, (3, \theta/3)_{FR}$ and $(4, \theta/4)_{FR}$. Under each of these policies, I obtains $G(c) = \varepsilon(a-c)$.

In general, under linear demand $p(Q) = \max\{a-Q, 0\}$, in an n-firm Cournot oligopoly with outside innovator ($n \geq 3$) there is a function $1 < q(n) \leq 2$ such that for $(a-c)/q(n) < \varepsilon < (a-c)$, there are $n-2$ optimal FR policies and under each of these policies I obtains $\varepsilon(a-c)$. This multiplicity was exogenously resolved in Sen and Tauman (2007) by assuming that whenever the innovator is indifferent between different policies, it chooses the one where the number of licensees is maximum.

In conclusion, this paper has presented a comprehensive analysis of patent licensing in a Cournot oligopoly. As the Cournot oligopoly model is a useful benchmark to study various aspects of industrial economics such as differentiated products and entry and exit in markets, the results of this paper can be useful to understand how licensing of patents interacts with other determinants of a market structure. However, our analysis has some realistic limitations. One limitation is with respect to the nature of licensing policies. We assume that royalty payments are based on a unit royalty where a licensee pays a royalty for each unit that it produces. By designing more complicated royalty schemes the innovator may get a higher licensing revenue. For instance, Erutku and Richelle (2007) consider combinations of upfront fees and royalties where royalties are based on the individual output of a licensee as well as the industry output. They show that such policies enable an outside innovator to obtain the monopoly profit. Other alternative forms of royalty licensing include ad valorem royalty where a licensee pays a fraction of its revenue to the innovator. In a recent paper Colombo and Filippini (2015) study combinations of fees and ad valorem royalties in a differentiated Bertrand duopoly and find that ad valorem royalties can lead to higher welfare compared to unit royalties.

The static framework of our model also has some limitations. For licensing policies that

[7] This is because any policy $(k, \delta)_{FR}$ with $\delta > \delta_\lambda(k)$ cannot be optimal as it gives payoff lower than $G(c)$ to I, whereas for any policy $(k, \delta)_{FR}$ with $\delta < \delta_\lambda(k)$, the policy $(n-1, \widetilde{\delta})_{FR}$ where $\widetilde{\delta} = k\delta/(n-1)$ does strictly better for I.

include a fee, we implicitly assume that a potential licensee pays the fee upfront. However, in actual licensing practices a financially constrained firm may not be able to pay the entire fee upfront and might prefer a more flexible contract that allows it to pay part of the fee from its future revenue. A richer framework with multiple periods will be needed to adequately model this kind of situation. There might be issues with the royalty payments as well. For a royalty policy to be effective, outputs that licensees sell should be perfectly observable by the innovator, which may not always be the case in practice.

Acknowledgements

We express our sincere gratitude to an anonymous reviewer and an anonymous associate editor for their helpful comments and suggestions. For their comments on earlier versions of the paper, we are also grateful to the seminar participants at 3rd GAEL Conference on "Product differentiation and innovation on related markets" at Grenoble and "Workshop on innovations, patents and licensing" at 25th Summer Festival on Game Theory, Stony Brook.

Appendix

Some intermediary lemmas

Lemma A1 *Let* $\beta(Q,q) := 2p'(Q)+qp''(Q)$. *Then* $\beta(Q,q) < 0$ *for* $Q \in (0,\overline{Q})$ *and* $q \in [0,Q]$.

Proof Since $p' < 0$, $\beta(Q,q) < 0$ if $p'' \leq 0$. So let $p'' > 0$. Then for any $q \in [0,Q]$, $\beta(Q,q) \leq 2p'(Q) + Qp''(Q) = \gamma''(Q) < 0$ (since by assumption A5, $\gamma(Q) := Qp(Q)$ is strictly concave). ■

The proof of the next lemma follows along the same lines of the comparative static analysis of Dixit (1986).

Lemma A2 $\phi^n_\lambda(k,\delta)$ (*the Cournot profit of a licensee*) *is increasing in* δ.

Proof First, let $\delta < \delta_\lambda(k)$. By Lemma 1, all firms produce positive output and the industry output (dropping n, k, λ for brevity) is $Q(\delta) = k\overline{q}(\delta) + (n-k)\underline{q}(\delta) + \lambda\widehat{q}(\delta)$. The quantities are determined by the first order conditions of profit maximization. From (2) and (3), the respective first order conditions of a licensee, a non-licensee and firm I (if $\lambda = 1$) are

$$p'(Q(\delta))\overline{q}(\delta) + p(Q(\delta)) - (c-\delta) = 0, \;\; p'(Q(\delta))\underline{q}(\delta) + p(Q(\delta)) - c = 0,$$
$$p'(Q(\delta))\widehat{q}(\delta) + p(Q(\delta)) - (c-\varepsilon) = 0 \tag{19}$$

The respective Cournot profits of a licensee, non-licensee and firm I (if $\lambda = 1$) are

$$\overline{\phi}(\delta) = [p(Q(\delta))-(c-\delta)]\overline{q}(\delta), \;\; \underline{\phi}(\delta) = [p(Q(\delta))-c]\underline{q}(\delta), \;\; \widehat{\phi}(\delta) = [p(Q(\delta))-(c-\varepsilon)]\widehat{q}(\delta) \tag{20}$$

Denote $\overline{z} \equiv \mathrm{d}\overline{q}(\delta)/\mathrm{d}\delta$, $\underline{z} \equiv \mathrm{d}\underline{q}(\delta)/\mathrm{d}\delta$, $\widehat{z} \equiv \mathrm{d}\widehat{q}(\delta)/\mathrm{d}\delta$ and $\overline{\Delta} \equiv \mathrm{d}\overline{\phi}(\delta)/\mathrm{d}\delta$. Totally differentiating $\overline{\phi}$ with respect to δ and using (19), from (20) we have

$$\overline{\Delta} = \frac{\partial\overline{\phi}}{\partial\overline{q}}\overline{z} + \frac{\partial\overline{\phi}}{\partial\underline{q}}\underline{z} + \lambda\frac{\partial\overline{\phi}}{\partial\widehat{q}}\widehat{z} + \frac{\partial\overline{\phi}}{\partial\delta} = p'(Q(\delta))[(k-1)\overline{z} + (n-k)\underline{z} + \lambda\widehat{z}]\overline{q}(\delta) + \overline{q}(\delta) \tag{21}$$

Denote

$$\psi_\lambda(Q) := (n-1+\lambda)p'(Q) + \beta(Q,Q), \tau(Q,q) := p'(Q) + qp''(Q),$$
$$\text{and } \nu_\lambda(Q,q) := (n-k+1+\lambda)p'(Q) + (Q-kq)p''(Q)$$

By Lemma A1, $\psi_\lambda(Q) < 0$. Totally differentiating both sides of each expression of (19) with respect to δ, we have the following, where $Q, \overline{q}, \underline{q}, \widehat{q}$ are all functions of δ, i.e., $Q = Q(\delta), \overline{q} = \overline{q}(\delta)$ etc.

$$\tau(Q,\overline{q})[k\overline{z} + (n-k)\underline{z} + \lambda\widehat{z}] + p'(Q)\overline{z} + 1 = 0, \;\; \tau(Q,\underline{q})[k\overline{z} + (n-k)\underline{z} + \lambda\widehat{z}] + p'(Q)\underline{z} = 0,$$

$$\tau(Q,\widehat{q})[k\overline{z} + (n-k)\underline{z} + \lambda\widehat{z}] + p'(Q)\widehat{z} = 0$$

The system of equations has the unique solution

$$\overline{z} = -\nu_\lambda(Q,\overline{q})/p'(Q)\psi_\lambda(Q), \; \underline{z} = k\tau(Q,\underline{q})/p'(Q)\psi_\lambda(Q), \; \widehat{z} = k\tau(Q,\widehat{q})/p'(Q)\psi_\lambda(Q)$$

Applying this solution in (21), we have

$$\overline{\Delta} = 2\overline{q}[(n-k+1+\lambda)p'(Q) + (Q-k\overline{q}/2)p''(Q)]/\psi_\lambda(Q)$$

$$= 2\overline{q}[\beta(Q, Q-k\overline{q}/2) + (n-k+\lambda-1)p'(Q)]/\psi_\lambda(Q)$$

There are three possibilities: (a) $k \le n-1$, (b) $(k=n, \lambda=1)$, or (c) $(k=n, \lambda=0)$. Since $\psi_\lambda(Q) < 0$, note from Lemma A1 that $\overline{\Delta} > 0$ under (a) or (b). For (c), we have $n\overline{q} = Q$ and $\overline{\Delta} = \overline{q}\beta(Q,Q)/\psi_\lambda(Q) > 0$.

Finally, let $\delta \ge \delta_\lambda(k)$. Then all non-licensee firms produce zero output and the industry output is $Q(\delta) = k\overline{q}(\delta) + \lambda\widehat{q}(\delta)$. Taking $k = n$ in the previous case, we obtain $\overline{\Delta} > 0$. This completes the proof. ■

Lemma A3 *The following limiting properties hold for $\lambda \in \{0,1\}$, where $Q^n_\lambda(k,\delta) := Q(p^n_\lambda(k,\delta))$ and G is given in* (14).

- (i) (a) $lim_{n\to\infty} p^n_\lambda(n,0) = c$ *and* (b) $lim_{n\to\infty} Q^n_\lambda(n,0) = Q(c)$.
- (ii) (a) $lim_{n\to\infty} \lambda\widehat{q}^n_\lambda(n,0) = \lambda G(c)/\theta$, (b) $lim_{n\to\infty} n\overline{q}^n_\lambda(n,0) = Q(c) - \lambda G(c)/\theta$ *and* (c) $lim_{n\to\infty} \lambda\widehat{\phi}^n_\lambda(n,0) = \lambda\varepsilon G(c)/\theta$.
- (iii) *If* $\varepsilon < \delta_\lambda(m-1)$, *then for any* $k \le m-1$: (a) $lim_{n\to\infty} p^n_\lambda(k,\varepsilon) = c$, (b) $lim_{n\to\infty} \underline{q}^n_\lambda(k,\varepsilon) = 0$, (c) $lim_{n\to\infty} n\underline{q}^n_\lambda(k,\varepsilon) = Q(c) - (k+\lambda)G(c)/\theta$ *and* (d) $lim_{n\to\infty} n\underline{\phi}^n_\lambda(k,\varepsilon) = 0$.

Proof (i) Taking $(k,\delta) = (n,0)$ in Lemma 1, $p^n_\lambda(n,0)$ is the unique solution of $H^{n+\lambda}(p) = c - \lambda\varepsilon/(n+\lambda)$ (see (5)). By (5), this equation reduces to

$$p = [c - \lambda\varepsilon/(n+\lambda)]/[1 - 1/(n+\lambda)\eta(p)] \tag{22}$$

Since $\eta(p)$ is bounded for $p \in (0,\overline{p})$, the right side of (22) converges to c as $n \to \infty$. This proves (a). Part (b) is immediate.

(ii) For $\lambda = 0$ (outside innovator), we have $\widehat{q}^n_\lambda(n,0) = 0$ (I produces zero) and the result trivially holds, so let $\lambda = 1$. By (19),

$$\lim_{n\to\infty} [p'(Q^n_\lambda(n,0))\widehat{q}^n_\lambda(n,0)) + p(Q^n_\lambda(n,0)) - (c-\varepsilon)] = 0 \tag{23}$$

Since $\lim_{n\to\infty} Q_\lambda^n(n,0) = Q(c)$ (by part (i)), the left side of (23) equals

$$p'(Q(c))\lim_{n\to\infty}\widehat{q}_\lambda^n(n,0) + p(Q(c)) - (c-\varepsilon) = p'(Q(c))\lim_{n\to\infty}\widehat{q}_\lambda^n(n,0) + \varepsilon$$

Then (23) implies $\lim_{n\to\infty}\widehat{q}_\lambda^n(n,0) = -\varepsilon/p'(Q(c)) = -\varepsilon Q'(c) = \varepsilon\eta(c)Q(c)/c = \varepsilon Q(c)/\theta = G(c)/\theta$ which proves (a). Parts (b),(c) follow from (a) and part (i) by noting that $Q_\lambda^n(n,0) = n\overline{q}_\lambda^n(n,0) + \lambda\overline{q}_\lambda^n(n,0)$ and $\widehat{\phi}_\lambda^n(n,0) = (p_\lambda^n(n,0) - c + \varepsilon)\widehat{q}_\lambda^n(n,0)$.

(iii) Let $\varepsilon < \delta_\lambda(m-1)$. Taking $\delta = \varepsilon$ in Lemma 1, it follows that for $k \le m-1$, the Cournot price $p_\lambda^n(k,\varepsilon)$ is more than c and it is the unique solution of $H^{k+\lambda}(p) = c - \varepsilon$. Using the expression of H from (5), this equation can be written as

$$p = [c - (k+\lambda)\varepsilon/(n+\lambda)]/[1 - 1/(n+\lambda)\eta(p)] \tag{24}$$

Since $\eta(p)$ is bounded for $p \in (0,\overline{p})$, the right side of (22) converges to c as $n \to \infty$. This proves (a).

To prove (b), note by (19) that $\lim_{n\to\infty}[p'(Q_\lambda^n(k,\varepsilon))\underline{q}_\lambda^n(k,\varepsilon)) + p(Q_\lambda^n(k,\varepsilon)) - c] = 0$. Since $\lim_{n\to\infty} Q_\lambda^n(k,\varepsilon) = Q(c)$, we have $p'(Q(c))\lim_{n\to\infty}\underline{q}_\lambda^n(k,\varepsilon) = 0$, which proves (b).

To prove (c), taking limits from the first order conditions in (19), and using result (a):

$$\lim_{n\to\infty}[\overline{q}_\lambda^n(k,\varepsilon) - \underline{q}_\lambda^n(k,\varepsilon)] = -\lim_{n\to\infty}\varepsilon/p'(Q_\lambda^n(k,\varepsilon)) = -\varepsilon/p'(Q(c)) = -\varepsilon Q'(c) = G(c)/\theta \text{ and}$$

$$\lim_{n\to\infty}\lambda\widehat{q}_\lambda^n(k,\varepsilon) = -\lambda\lim_{n\to\infty}[p_\lambda^n(k,\varepsilon) - c + \varepsilon]/p'(Q_\lambda^n(k,\varepsilon)) = -\lambda\varepsilon/p'(Q(c)) = \lambda G(c)/\theta \tag{25}$$

Using the definition of $Q_\lambda^n(k,\varepsilon)$ we have

$$\lim_{n\to\infty} n\underline{q}_\lambda^n(k,\varepsilon) = \lim_{n\to\infty}[Q_\lambda^n(k,\varepsilon) - k(\overline{q}_\lambda^n(k,\varepsilon) - \underline{q}_\lambda^n(k,\varepsilon)) - \lambda\widehat{q}_\lambda^n(k,\varepsilon)]$$

Using (25) and the result of part (a) in the limits above, we have $\lim_{n\to\infty} n\underline{q}_\lambda^n(k,\varepsilon) = Q(c) - (k+\lambda)G(c)/\theta$. This proves (c).

Part (d) follows from (a) and (c) by noting that $n\underline{\phi}_\lambda^n(k,\varepsilon) = [p_\lambda^n(k,\varepsilon) - c]n\underline{q}_\lambda^n(k,\varepsilon)$. ■

Lemma A4 *Consider an innovation of magnitude ε. Let $n \ge 2$, $k \in \{1,\ldots,n\}$ and $\delta \in [0,\varepsilon]$.*

- (i) *If $k \le n-1$, then $\Pi_\lambda^n(k,\delta)_{FR} < G(p_\lambda^n(k,\delta))$ for $\delta < \delta_\lambda(k)$ and $\Pi_\lambda^n(k,\delta)_{FR} = G(p_\lambda^n(k,\delta))$ for $\delta \ge \delta_\lambda(k)$.*
- (ii) *$\Pi_\lambda^n(n,\delta)_{FR} < G(p_\lambda^n(n,\delta))$ for $\delta < \delta_\lambda(n-1)$ and $\Pi_\lambda^n(n,\delta)_{FR} = G(p_\lambda^n(n,\delta))$ if $\delta \ge \delta_\lambda(n-1)$.*
- (iii) *Let $k \le n-2$, $\delta < \delta_\lambda(k)$ and $\widetilde{\delta} := k\delta/(n-1)$. Then $\Pi_\lambda^n(n-1,\widetilde{\delta})_{FR} > \Pi_\lambda^n(k,\delta)_{FR}$.*
- (iv) *Let*

$$\alpha_\lambda^n(k,\delta) := G(p_\lambda^n(k,\delta)) - k\overline{\phi}_\lambda^n(k,\delta) \tag{26}$$

If $p_\lambda^n(k,\delta) \le p_M(\varepsilon)$, then $\alpha_\lambda^n(k,\delta)$ is decreasing in δ.

Proof (i)-(ii): Using (8), (9) and the function G defined in (14), from (11)

$$\Pi_\lambda^n(k,\delta)_{FR} = G(p_\lambda^n(k,\delta)) - n\underline{\phi}_\lambda^n(k,\delta) - \varepsilon(n-k)\underline{q}_\lambda^n(k,\delta) \text{ for } k \in \{1,\ldots,n-1\} \tag{27}$$

Using (8), (10) and (14) in (11), we have

$$\Pi^n_\lambda(n,\delta)_{FR} = G(p^n_\lambda(n,\delta)) - n\underline{\phi}^n_\lambda(n-1,\delta) \tag{28}$$

(i) follows from (27) and (ii) from (28) by noting that $\underline{q}^n_\lambda(k,\delta)$, $\underline{\phi}^n_\lambda(k,\delta)$ are positive if $\delta < \delta_\lambda(k)$ and zero otherwise (Lemma 1).

(iii) Since $\delta < \delta_\lambda(k)$, by Lemma 1, the Cournot output of a non-licensee is positive. Moreover the Cournot output and profit of a non-licensee, as well as the Cournot price, depend only on $k\delta$. Since $(n-1)\widetilde{\delta} = k\delta$, by (27) we have $\Pi^n_\lambda(n-1,\widetilde{\delta})_{FR} - \Pi^n_\lambda(k,\delta)_{FR} = \varepsilon(n-k-1)\underline{q}^n_\lambda(k,\delta) > 0$.

(iv) Since $G(p)$ is increasing for $p < p_M(\varepsilon)$ and $p^n_\lambda(k,\delta)$ is decreasing in δ, it follows that if $p^n_\lambda(k,\delta) \leq p_M(\varepsilon)$, then $G(p^n_\lambda(k,\delta))$ is decreasing in δ. As $\overline{\phi}^n_\lambda(k,\delta)$ is always increasing in δ (Lemma A2), the proof is complete. ■

Proofs of results

Proof of Lemma 2 By Lemma A4(i)-(ii), the payoff of I cannot exceed $G(p) \leq \pi_M(\varepsilon)$, which proves (i). Part (ii) is immediate from the definition of drastic innovations. Part (iii)(a) is immediate for an outside innovator. If an incumbent innovator of a non drastic innovation sells the license to all other firms using a pure royalty policy with $r = \varepsilon$ (i.e., $\delta = 0$), the Cournot profits stay the same as the case of no licensing. As all firms produce positive Cournot output for a non drastic innovation, the innovator obtains positive royalty payments in addition to its Cournot profit, showing the superiority of licensing.

Finally, for part (iii)(b), note that for a policy $(k,\delta)_{FR}$ if either (a) $k \in \{1,\ldots,n-1\}$ and $\delta < \delta_\lambda(k)$, or (b) $k = n$ and $\delta < \delta_\lambda(n-1)$, then the result is immediate from Lemma A4(i)-(ii). So let (a) $k \in \{1,\ldots,n-1\}$ and $\delta \geq \delta_\lambda(k)$, or (b) $k = n$ and $\delta \geq \delta_\lambda(n-1)$ (implying $\delta > \delta_\lambda(n)$, since $\delta_\lambda(n-1) > \delta_\lambda(n)$). Then by Lemma 1, $p^n_\lambda(k,\delta) \leq c$. As the innovation is non drastic, $p_M(\varepsilon) > c$, so that $p^n_\lambda(k,\delta) < p_M(\varepsilon)$. Then Lemma A4(i)-(ii) imply $\Pi^n_\lambda(k,\delta)_{FR} = G(p^n_\lambda(k,\delta)) < G(p_M(\varepsilon)) = \pi_M(\varepsilon)$. ■

Proof of Proposition 1

Proof of (I) If $\delta \geq \delta_\lambda(k)$, then $p^n_\lambda(k,\delta) \leq c$ and all non-licensees produce zero (Lemma 1). Using (8), by (12) for this case we have $\Pi^n_\lambda(k,\delta)_R = \alpha^n_\lambda(k,\delta)$ (given by (26)). As $p^n_\lambda(k,\delta) \leq c < p_M(\varepsilon)$, by Lemma A4(iv), $\Pi^n_\lambda(k,\delta)_R$ is decreasing for $\delta \geq \delta_\lambda(k)$.

So it is sufficient to consider policies $(k,\delta)_R$ where $\delta \leq \delta_\lambda(k)$. Then from (12) we have

$$\Pi^n_\lambda(k,\delta)_R \leq (\varepsilon-\delta)[Q(p^n_\lambda(k,\delta)) - \lambda\widehat{q}^n_\lambda(k,\delta)] + \lambda\widehat{\phi}^n_\lambda(k,\delta) \tag{29}$$

Consider the policy $(n,\widetilde{\delta})_R$ such that $n\widetilde{\delta} = k\delta$. Since the Cournot price as well as Cournot outputs and profits of firm I and a non-licensee depend only on the product $k\delta$ in this case (Lemma 1), from (12) we have

$$\begin{aligned}\Pi^n_\lambda(n,\widetilde{\delta})_R &= (\varepsilon-\widetilde{\delta})[Q(p^n_\lambda(n,\widetilde{\delta})) - \lambda\widehat{q}^n_\lambda(n,\widetilde{\delta})] + \lambda\widehat{\phi}^n_\lambda(n,\widetilde{\delta}) \\ &= (\varepsilon-\widetilde{\delta})[Q(p^n_\lambda(k,\delta)) - \lambda\widehat{q}^n_\lambda(k,\delta)] + \lambda\widehat{\phi}^n_\lambda(k,\delta) \end{aligned} \tag{30}$$

Since $\widetilde{\delta} = (k/n)\delta \leq \delta$ (equality iff $k = n$) from (29) and (30) it follows that $\Pi^n_\lambda(n,\widetilde{\delta})_R \geq \Pi^n_\lambda(k,\delta)_R$ (equality iff $k = n$), which proves (I).

Proof of (II)-(III) By (I), consider a royalty policy $(n,\delta)_R$ for $\delta \in [0,\varepsilon]$. By Lemma 1, the resulting Cournot price $p_\lambda^n(n,\delta)$ satisfies $H^{n+\lambda}(p_\lambda^n(n,\delta)) = c - (n\delta + \lambda\varepsilon)/(n+\lambda) \geq c - \varepsilon > 0$. Using the property of H given in Observation 1, we conclude that $p_\lambda^n(n,\delta) \leq p_M(\varepsilon)$ iff $H^{n+\lambda}(p_M(\varepsilon)) \geq c - (n\delta + \lambda\varepsilon)/(n+\lambda)$. By (5), $H^k(p) = [(k-1)p + H^1(p)]/k$. Since $H^1(p_M(\varepsilon)) = c - \varepsilon$, we have $H^{n+\lambda}(p_M(\varepsilon)) = [(n+\lambda-1)p_M(\varepsilon) + (c-\varepsilon)]/(n+\lambda)$ and $p^n(n,\delta) \leq p_M(\varepsilon)$ iff $(n+\lambda-1)(p_M(\varepsilon)-c) \geq (1-\lambda)\varepsilon - n\delta$. This inequality always strictly holds if $\delta = \varepsilon$, so that $p_\lambda^n(n,\varepsilon) < p_M(\varepsilon)$. Next observe that $p_\lambda^n(n,0) \leq p_M(\varepsilon)$ iff $\omega_\lambda^n(\varepsilon) := (1-\lambda)\varepsilon - (n+\lambda-1)(p_M(\varepsilon)-c) \leq 0$.

Observation A1 *$p_\lambda^n(n,0) > c$.*
Proof Note from Lemma 1 that $H^{n+\lambda}(p_\lambda^n(n,0)) = c - \lambda\varepsilon/(n+\lambda) > c - \varepsilon > 0$. Since $\theta \equiv c/\eta(c)$ and $\varepsilon \in (0,\theta)$, by (5), $H^{n+\lambda}(c) = c - \theta/(n+\lambda) < c - \lambda\varepsilon/(n+\lambda)$. Then the result follows by using the property of H from Observation 1. ■

Proof of (II) Let $\lambda = 1$ (incumbent innovator). Note that $\omega_1^n(\varepsilon) < 0$. Hence $p_1^n(n,0) < p_M(\varepsilon)$ and therefore $p_1^n(n,\delta) < p_M(\varepsilon)$ for all $\delta \in [0,\varepsilon]$. By (12), $\Pi_1^n(n,\delta)_R = \alpha_1^n(n,\delta)$ (given in (26)). Then by Lemma A4(iv), $\Pi_1^n(n,\delta)_R$ is decreasing in δ and it is maximized at $\delta = 0$ (i.e. maximum royalty $r = \varepsilon$). By Observation A1, $p_\lambda^n(n,0) > c$. This proves (II).

Proof of (III) Let $\lambda = 0$ (outside innovator). Then $\omega_0^n(\varepsilon) = \varepsilon - (n-1)(p_M(\varepsilon)-c)$. Since $p_M(\varepsilon)$ is decreasing, $\omega_0^n(\varepsilon)$ is increasing in ε. Note that $\varepsilon \in (0,\theta)$. As $p_M(\theta) = c < p_M(0)$, we have $\omega_0^n(0) < 0 < \omega_0^n(\theta) = \theta$. Hence for every $n \geq 2$, $\exists\ \overline{\varepsilon}(n) \in (0,\theta)$ such that $p_0^n(n,0) \lesseqqgtr p_M(\varepsilon)$ iff $\varepsilon \lesseqqgtr \overline{\varepsilon}(n)$. As $\omega_0^n(\varepsilon)$ is decreasing in n for any $\varepsilon \in [0,\theta)$, it follows that $\overline{\varepsilon}(n)$ is increasing in n and the following hold:

Observation A2
(i) *If $\varepsilon \in (0,\overline{\varepsilon}(2))$, then for all $n \geq 2 : p_0^n(n,0) < p_M(\varepsilon)$ and consequently $p_0^n(n,\delta) < p_M(\varepsilon)$ for all $\delta \in [0,\varepsilon]$.*
(ii) *If $\varepsilon \in (\overline{\varepsilon}(2),\theta)$, then $\exists\ N(\varepsilon)$ (which is a weakly increasing step function) such that $p_0^n(n,0) \leq p_M(\varepsilon)$ iff $n \geq N(\varepsilon)$. If $n > N(\varepsilon)$, then $p_0^n(n,\delta) < p_M(\varepsilon)$ for all $\delta \in [0,\varepsilon]$. If $n < N(\varepsilon)$, then $\exists\ \delta^*(n) \in (0,\varepsilon)$ such $p_0^n(n,\delta) \lesseqqgtr p_M(\varepsilon) \Leftrightarrow \delta \gtreqqless \delta^*(n)$.*

By (12), $\Pi_0^n(n,\delta)_R = \alpha_0^n(n,\delta)$ (given in (26)). Taking $\underline{\varepsilon} \equiv \overline{\varepsilon}(2)$, parts (III)(i) and (III)(ii)(a) follow from Observations A2, A1 and Lemma A4(iv).

Finally consider $\varepsilon > \underline{\varepsilon}$ and $n < N(\varepsilon)$. By Observation A2 and Lemma A4(iv), $\Pi_0^n(n,\delta)_R$ is decreasing for $\delta \in [\delta^*(n),\varepsilon]$. Hence any optimal pure royalty policy has $\delta \in [0,\delta^*(n)]$ (or $r \in [\varepsilon - \delta^*(n),\varepsilon]$ and in particular $r > 0$) and $p_0^n(n,\delta) \geq p_M(\varepsilon)$. This proves part (III)(ii)(b). ■

Proof of Proposition 2

Proof of (I) As $\delta_\lambda(m) = \theta/(m+\lambda) < \varepsilon < \theta/(m+\lambda-1) = \delta_\lambda(m-1)$, by the monotonicity of $\delta_\lambda(k)$, it follows that $\varepsilon > \delta_\lambda(k)$ for all $k \geq m$. If $m \geq n$, it is not even feasible to offer more than m licenses, so let $m < n$. The pure upfront fee policy $(k)_F$ is the FR policy $(k,\varepsilon)_{FR}$, so by Lemma A4(i), for $k \geq m$, the payoff of I under the policy $(k)_F$ is $\Pi_\lambda^n(k,\varepsilon)_{FR} = G(p_\lambda^n(k,\varepsilon))$.

Taking $\delta = \varepsilon$ in Lemma 1, for $k \geq m$, the Cournot price $p_\lambda^n(k,\varepsilon)$ is lower than c and it is the unique solution of $H^{k+\lambda}(p) = c - \varepsilon$. Since $H^{k+\lambda}(p)$ is increasing in k, $p_\lambda^n(k,\varepsilon)$ is decreasing for $k \geq m$. As $c < p_M(\varepsilon)$ and $G(p)$ is increasing for $p < p_M(\varepsilon)$, it follows that $G(p_\lambda^n(k,\varepsilon))$ is decreasing for $k \geq m$. This proves that it is not optimal for I to offer $k > m$ licenses.

To prove the second part of (I), if non-licensees drop out of the market under a policy, then the Cournot price is at most c, which is lower than $p_M(\varepsilon)$. So consider a pure upfront fee policy $(k)_F$ such that all firms are active. By Lemma 1, the resulting Cournot price $p^n_\lambda(k,\varepsilon)$ satisfies $H^{n+\lambda}(p^n_\lambda(k,\varepsilon)) = c-\varepsilon > 0$. Since $H^{n+\lambda}(p) = [(n+\lambda-1)p + H^1(p)]/(n+\lambda)$ (by (5)) and $H^1(p_M(\varepsilon)) = c-\varepsilon$, we have $H^{n+\lambda}(p_M(\varepsilon)) = [(n+\lambda-1)p_M(\varepsilon)+c-\varepsilon]/(n+\lambda) > c-\varepsilon$. Using the property of H from Observation 1 it follows that $p^n_\lambda(k,\delta) < p_M(\varepsilon)$.

Proof of (II) Consider sufficiently large sizes of industry so that $n > m$. Using (I), consider $k \leq m$. First let $k = m$. Note that $\Pi^n_\lambda(m,\varepsilon)_{FR} = G(p^n_\lambda(m,\varepsilon))$. As $H^{m+\lambda}(p^n_\lambda(m,\varepsilon)) = c-\varepsilon > 0$, by Observation 1 it follows that $p^n_\lambda(m,\varepsilon)$ is decreasing in ε. Since $G(p)$ is increasing for $p < p_M(\varepsilon)$ and $p^n_\lambda(m,\varepsilon) < c < p_M(\varepsilon)$, we have:

Observation A3
$G(p^n_\lambda(m,\varepsilon))$ is decreasing in $\varepsilon \in (\theta/(m+\lambda), \theta/(m+\lambda-1))$ and $G(p^n_\lambda(m,\theta/(m+\lambda))) = G(c)$.

Next consider $k \leq m-1 < n-1$. Taking $\delta = \varepsilon$ in (27), we have $\Pi^n_\lambda(k,\varepsilon)_{FR} = G(p^n_\lambda(k,\varepsilon)) - n\underline{\phi}^n_\lambda(k,\varepsilon) - \varepsilon(1-k/n)n\underline{q}^n_\lambda(k,\varepsilon)$. Taking limits using Lemma A3(iii), noting that $G(c) = \varepsilon Q(c)$, we have

$$L_\lambda(k,\varepsilon) := \lim_{n\to\infty} \Pi^n_\lambda(k,\varepsilon)_{FR} = G(c) - \varepsilon[Q(c) - (k+\lambda)\varepsilon G(c)/\theta] = (k+\lambda)\varepsilon G(c)/\theta \qquad (31)$$

Note that

Observation A4
(i) *$L_\lambda(m-1,\varepsilon)$ is increasing in $\varepsilon \in (\theta/(m+\lambda), \theta/(m+\lambda-1))$ and $L_\lambda(m-1,\theta/(m+\lambda-1)) = G(c)$.*
(ii) *For any $\varepsilon \in (\theta/(m+\lambda), \theta/(m+\lambda-1))$, $L_\lambda(k,\varepsilon)$ is increasing in $k \leq m-1$.*

From Observations A3 and A4(i), $\exists\ \varepsilon^*_\lambda(m) \in (\theta/(m+\lambda), \theta/(m+\lambda-1))$ such that $L_\lambda(m-1,\varepsilon) \gtreqless G(p^n_\lambda(m,\varepsilon)) \Leftrightarrow \varepsilon \gtreqless \varepsilon^*_\lambda(m)$. This, together with Observation A4(ii), proves part (II)(a)-(b). ∎

Proof of Proposition 3

Proof of (I) If non-licensees drop out of the market under a pure upfront fee policy, then the Cournot price does not exceed c. Since under any optimal pure royalty policy the Cournot price exceeds c (Proposition 2), in this case the price is lower under upfront fee.

Since the Cournot price under any optimal pure upfront fee policy is lower than $p_M(\varepsilon)$, in the case where the Cournot price under pure royalty is at least as large as $p_M(\varepsilon)$, the price is again lower under upfront fee.

Finally we look at the case where (a) all firms are active under upfront fee policy and (b) the Cournot price is lower than $p_M(\varepsilon)$ under pure royalty policy. We know from Lemma 1 that when (a) holds, if the number of licensees is k, the Cournot price $p^n_\lambda(k,\varepsilon)$ satisfies $H^{n+\lambda}(p^n_\lambda(k,\varepsilon)) = c-(k+\lambda)\varepsilon/(n+\lambda) > 0$. We know from Proposition 1 that when (b) holds, the optimal pure royalty has $r = \varepsilon$, so $\delta = 0$. The resulting Cournot price $p^n_\lambda(n,0)$ satisfies $H^{n+\lambda}(p^n_\lambda(n,0)) = c - \lambda\varepsilon/(n+\lambda) > c-(k+\lambda)\varepsilon/(n+\lambda)$. By Observation 1 it follows that $p^n_\lambda(k,\varepsilon) < p^n_\lambda(n,0)$, completing the proof of (I).

Proof of (II) By Proposition 1, for sufficiently large industry sizes the unique optimal pure royalty policy is $(n,0)_R$ (the policy of selling licenses to all n firms using royalty $r = \varepsilon$). By (12), $\Pi^n_\lambda(n,0)_R = \varepsilon n\overline{q}^n_\lambda(n,0) + \lambda\widehat{\phi}^n_\lambda(n,0) = \varepsilon n\overline{q}^n_\lambda(n,0) + \lambda(p^n_\lambda(n,0)-c+\varepsilon)\widehat{q}^n_\lambda(n,0)$. Taking limit and using Lemma A3(i)-(ii), we have $\lim_{n\to\infty}\Pi^n_\lambda(n,0)_R = \varepsilon Q(c) = G(c)$.

By Proposition 2(I), the number of licenses under any optimal pure upfront fee policy does not exceed m, so consider a pure upfront fee policy $(k)_F = (k, \varepsilon)_{FR}$ where $k \leq m$. Let the number of firms be sufficiently large: $n > m + 1 + \lambda$.

First let $k = m$. From the proof of Proposition 2 we know $\Pi^n_\lambda(m, \varepsilon)_{FR} = G(p^n_\lambda(m, \varepsilon)) < G(c)$. Observe that for any $n > m+1+\lambda \geq m+1$, the payoff $\Pi^n_\lambda(m, \varepsilon)_{FR}$ is independent of n. Denote $\tau_1(\varepsilon) \equiv G(c) - \Pi^n_\lambda(m, \varepsilon)_{FR} > 0$. Then

$$\Pi^n_\lambda(k, \varepsilon)_{FR} \leq G(c) - \tau_1(\varepsilon) \text{ for all } k \geq m \text{ and } n > m + 1 + \lambda \tag{32}$$

For $k \leq m - 1$, note from (31) that

$$\lim_{n\to\infty} \Pi^n_\lambda(k, \varepsilon)_{FR} = (k + \lambda)\varepsilon G(c)/\theta \leq (m + \lambda - 1)\varepsilon G(c)/\theta. \tag{33}$$

Let $\tau_2(\varepsilon) \equiv G(c) - (m + \lambda - 1)\varepsilon G(c)/\theta > 0$. Taking $\tau(\varepsilon) = \min\{\tau_1(\varepsilon), \tau_2(\varepsilon)\}$, from (32) and (33), we have $\lim_{n\to\infty} \Pi^n_\lambda(k, \varepsilon)_{FR} \leq G(c) - \tau(\varepsilon) < G(c) = \lim_{n\to\infty} \Pi^n_\lambda(n, 0)_R$ for any $k \leq m$. This completes the proof of part (II).

Proof of (III) Let $\lambda \in \{0, 1\}$ and $\varepsilon \in (\theta/(m+\lambda), \theta/(m+\lambda-1))$. Consider the functions $N(\varepsilon)$ (in Proposition 1) and $M_\lambda(\varepsilon)$ (in Proposition 2). Then for $n > \max\{N(\varepsilon), M_\lambda(\varepsilon)\}$, the unique optimal pure royalty policy is $(n, 0)_R$ that results in Cournot price $p^n_\lambda(n, 0)$. By (18), the welfare under this policy is

$$W^n_\lambda(n, 0) = \Psi(p^n_\lambda(n, 0)) \tag{34}$$

If $\varepsilon \in (\theta/(m + \lambda), \varepsilon^*_\lambda(m))$, then by Proposition 2(II)(a), the unique optimal pure upfront fee policy is $(m, \varepsilon)_{FR}$ which results in Cournot price $p^n_\lambda(m, \varepsilon) < c$ and all non-licensees drop out of the market so that $\underline{q}^n_\lambda(m, \varepsilon) = 0$. Using this in (18), the welfare under this policy is

$$W^n_\lambda(m, \varepsilon) = \Psi(p^n_\lambda(m, \varepsilon)) \tag{35}$$

Since $c - \varepsilon < p^n_\lambda(m, \varepsilon) < c < p^n_\lambda(n, 0)$ and $\Psi(y)$ is decreasing for $y \in [0, c - \varepsilon]$, part (III)(a) follows by (34) and (35).

If $\varepsilon \in (\varepsilon^*_\lambda(m), \theta/(m + \lambda - 1))$, then by Proposition 2(II)(b), the unique optimal pure upfront fee policy is $(m - 1, \varepsilon)_{FR}$ which results in Cournot price $p^n_\lambda(m - 1, \varepsilon)$. By (18), the welfare under this policy is

$$W^n_\lambda(m, \varepsilon) = \Psi(p^n_\lambda(m - 1, \varepsilon)) - \varepsilon[1 - (m - 1)/n]n\underline{q}^n_\lambda(m - 1, \varepsilon) \tag{36}$$

Since $\varepsilon < \theta/(m + \lambda - 1) = \delta_\lambda(m - 1)$, by Lemma A3(iii), we have

$$\lim_{n\to\infty} W^n_\lambda(m - 1, \varepsilon) = \Psi(c) - \varepsilon[Q(c) - (m - 1 + \lambda)G(c)/\theta]$$

Denote $\tau_\varepsilon \equiv 1-\varepsilon(m+\lambda-1)/\theta \in (0, 1)$. Since $G(c) = \varepsilon Q(c)$, we have $\lim_{n\to\infty} W^n_\lambda(m-1, \varepsilon) = \Psi(c) - \tau_\varepsilon G(c)$. By Lemma A3(i), $\lim_{n\to\infty} W^n_\lambda(n, 0) = \Psi(c) > \lim_{n\to\infty} W^n_\lambda(m - 1, \varepsilon)$, which proves part (III)(b). ∎

Proof of Proposition 4(I)-(II)

Proof of (I) We prove (I) by showing that for any FR policy with $k \leq n - 2$, there is an FR policy with $k = n - 1$ that gives a higher payoff to the innovator. Moreover, for $\varepsilon < \theta/(n + \lambda - 2)$, this payoff is strictly higher. We consider the following cases.

Case 1 $\varepsilon < \theta/(n+\lambda-2)$: In this case for any $k \leq n-2$, we have $\varepsilon < \theta/(k+\lambda)$ and by (7), $\varepsilon < \delta_\lambda(k)$. Hence for any policy $(k,\delta)_{FR}$ with $k \leq n-2$ and $\delta \in [0,\varepsilon]$, we have $\delta \leq \varepsilon < \delta_\lambda(k)$. Let $\widetilde{\delta} = k\delta/(n-1)$. By Lemma A4(iii), it follows that the policy $(n-1,\widetilde{\delta})_{FR}$ gives strictly higher payoff to the innovator than $(k,\delta)_{FR}$. This proves that for $\varepsilon < \theta/(n+\lambda-2)$, any optimal FR policy must have $k = n-1$ or n.

Case 2 $\varepsilon \geq \theta/(n+\lambda-2)$: Let $k \leq n-2$ and $\delta \in [0,\varepsilon]$. In this case ε may be more or less than $\theta/(k+\lambda)$ and so δ may be more or less than $\delta_\lambda(k)$. Consider a policy $(k,\delta)_{FR}$ such that $\delta < \delta_\lambda(k)$. Then as in Case 1, by taking $\widetilde{\delta} = k\delta/(n-1)$, we can show that $(n-1,\widetilde{\delta})_{FR}$ is superior to $(k,\delta)_{FR}$. Next consider a policy $(k,\delta)_{FR}$ such that $\delta \geq \delta_\lambda(k)$. Then by Lemma A4(i) we have $\Pi^n_\lambda(k,\delta)_{FR} = G(p^n_\lambda(k,\delta))$. Since $p^n_\lambda(k,\delta) \leq c < p_M(\varepsilon)$ (Lemma 1) and $G(p)$ is decreasing for $p < p_M(\varepsilon)$, we have $\Pi^n_\lambda(k,\delta)_{FR} \leq G(c)$. Consider the policy $(n-1,\delta_\lambda(n-1))_{FR}$. Since it results in price c (Lemma 1), by Lemma A4(i),

$$\Pi^n_\lambda(n-1,\delta_\lambda(n-1))_{FR} = G(p^n_\lambda(n-1,\delta_\lambda(n-1))) = G(c) \geq \Pi^n_\lambda(k,\delta)_{FR} \tag{37}$$

This shows that for any policy with $k \leq n-2$, there is a policy with $k = n-1$ which gives a higher payoff. So there is an optimal FR policy with $k = n-1$ or n. Since the inequality of (37) holds with equality if $\delta = \delta_\lambda(k)$, there might be optimal policies with $k \leq n-2$ as well. This completes the proof of part (I).

Proof of (II) Let $\varepsilon > \theta/(n+\lambda-1)$. Then by (7) we have $\varepsilon > \delta_\lambda(n-1)$.

(II)(a) Consider the policy $(n-1,\delta_\lambda(n-1))_{FR}$ (this policy has positive royalty $r = \varepsilon - \delta_\lambda(n-1)$). By Lemma 1, this policy results in Cournot price c. Then by Lemma A4(i), we have $\Pi^n_\lambda(n-1,\delta_\lambda(n-1))_{FR} = G(p^n_\lambda(n-1,\delta_\lambda(n-1)) = G(c)$. As this specific FR policy gives payoff $G(c)$ to I, under any optimal FR policy it must obtain at least $G(c)$.

(II)(b) Since $c < p_M(\varepsilon)$ and $G(p)$ is increasing for $p < p_M(\varepsilon)$, we have $G(p) < G(c)$ for $p < c$. Consider a policy $(k,\delta)_{FR}$ and denote $p^n_\lambda(k,\delta) = p$. By Lemma A4(i)-(ii), the payoff of I from this policy is bounded by $G(p)$. If $p < c$, this payoff would be lower than $G(c)$. Then by part (a) it follows that the resulting Cournot price under any optimal FR policy must be at least c.

(II)(c) Since the innovation is non-exact, $\exists\ 2-\lambda \leq m \leq n-1$ such that $\delta_\lambda(m) < \varepsilon < \delta_\lambda(m-1)$. A policy with zero royalty corresponds to $\delta = \varepsilon$. Consider a policy $(k,\varepsilon)_{FR}$ where $k \geq m$. Taking $\delta = \varepsilon$ in Lemma 1, we have $\varepsilon > \delta_\lambda(m) \geq \delta_\lambda(k)$. Therefore $p^n_\lambda(k,\delta) < c$ and by part (b), this policy cannot be optimal.

Next consider a policy $(k,\varepsilon)_{FR}$ where $k \leq m-1$ (so that $k \leq n-2$). Taking $\delta = \varepsilon$ in Lemma 1, we have $\varepsilon < \delta_\lambda(m-1) \leq \delta_\lambda(k)$. Let $\widetilde{\delta} = k\varepsilon/(n-1)$ and consider the policy $(n-1,\widetilde{\delta})_{FR}$. This policy has positive royalty (since $\widetilde{\delta} < \varepsilon$) and by Lemma A4(iii), this policy gives strictly higher payoff than $(k,\varepsilon)_{FR}$. This completes the proof of (c). ■

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