Asymmetric Budget Constraints in a First Price Auction

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3 February 2017

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August 2018

Abstract

I solve a first-price auction for two bidders with asymmetric budget distributions and known valuations for one object. I show that in any equilibrium, the expected utilities and bid distributions of both bidders are unique. If budgets are sufficiently low, the bidders will bid their entire budget in any equilibrium. For sufficiently high budgets, mass points in the equilibrium strategies arise. A less restrictive budget distribution could make both bidders strictly worse off. If the budget distribution of a bidder is dominated by the budget distribution of his opponent in the reverse-hazard-rate order, the weaker bidder will bid more aggressively than his stronger opponent. In contrast to existing results for symmetric budget distributions, with asymmetric budget distributions, a second-price auction can yield a strictly higher revenue than a first-price auction. Under an additional assumption, I derive the unique equilibrium utilities and bid distributions of both bidders in an all-pay auction.

Keywords Budget Constraints; First Price Auctions; Asymmetric Bidders.

JEL Classification: C72; D44; D82.

1 Introduction

Auctions are a widely used method of allocating objects, property rights and procurement contracts. If bidders in an auction are budget constrained, this will influence their bidding strategies, breaks the revenue equivalence of standard auctions, and lowers revenues. Budget

*For insightful discussions and suggestions, I am very thankful to Tim Frommeyer, Maciej Kotowski, Daniel Krähmer, Stephan Lauermann, Benny Moldovanu, Philip Reny, Maryam Saeedi, Dezsö Szalay, two very helpful anonymous referees and the participants of the 15th IIOC 2017, the Winter Meeting of the Econometric Society 2016, the World Congress of the Game Theory Society 2016, the World Congress of the Econometric Society 2015, the Conference on Economic Design 2015 and Roger Myerson and his Master Class participants of the 5th Lindau Meeting on Economic Sciences 2014.

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constraints can arise due to credit limits and imperfect capital markets, such that bidders' willingness to pay might exceed their ability to pay.

The existing research on standard auctions with budget constrained bidders concentrates on identical budget distributions. Yet, there are scenarios where bidders have asymmetric budget distributions. In a narrow market with a few major players, e.g., a telecommunications sector, bidders hold noisy information about the other bidders and their budgets. This information might stem from previous interactions or from publicly available information, such as annual budget reports. Moreover, the auctioneer can contribute to this asymmetry by revealing the identities of the participants before the auction via a participation register.

In the spectrum auction of the U.S. Federal Communications Commission, 30 bidders registered for the auction (Salant, 1997). Assessing the budget constraint of rival bidders was a major part of the preparation before the auction (Salant, 1997). GTE was one of the largest telecommunication firms in the U.S. Therefore, it would be reasonable to expect that the expectations of GTE about the budget of a smaller bidder, such as Poka Lambro, differed from the expectation of the smaller bidder about the resources of GTE.

The contribution of this paper is to solve the first-price auction for bidders with asymmetric budget distributions. I develop a solution technique that builds on an indirect utility approach by Che and Gale (1996). I provide a closed-form expression for the expected utilities and bid distributions of the bidders, which are unique in any equilibrium.[1]

In my model, two bidders are competing for one object in a first-price auction. Their valuations are common knowledge and might differ. Each bidder has a private budget constraint that is drawn independently from a bidder-specific distribution. Budget constraints are hard, that is, no bidder can bid above his budget.[2] First, budget constraints directly limit the ability to bid. Second, budgets have an indirect strategic effect: if the opponent is budget constrained, the necessary bid to outbid him might be lower than without a budget. Then, the constrained bidder anticipates this inference of his opponent and incorporates this into his bidding strategy, and so forth. The extent of these strategic effects varies with the asymmetry in budget distributions.

Che and Gale (1996) solve the first-price auction for bidders with identical budget distributions and the same common value for the object. Equilibrium utility in their model always equals some exogenous *lower bound* on utility. This lower bound is the highest utility a bidder can achieve if his opponent always bids his entire budget and, thus, minimizes the

[1] I do not restrict attention to symmetric equilibria, nor to monotonic bidding strategies.

[2] See, e.g., Zheng (2001) for a model with soft budget constraints where bidders can borrow.

probability of his opponent winning at any bid. They restrict attention to monotonic bidding strategies and symmetric equilibria, hence, mass points cannot arise in their setup.

In my model, I allow for asymmetric budget distributions and different values. I do not restrict attention to symmetric or monotonic equilibria. I determine the relationship between a lower bound utility and the actual equilibrium utility. This relationship is not an equality as in the symmetric setup in Che and Gale (1996). My approach rules out candidate equilibria utilities, until exactly one candidate utility is left.

Mass points arise in equilibrium. Bidders who would like to deviate and bid at the mass point or slightly above to increase their winning probability cannot afford such deviations due to their budget constraints. I show that each bidder places at most one mass point.

If the reverse hazard rate of both bidders is above a threshold, bidders bid the entire budget in any equilibrium. Then, equilibrium utility equals the lower bound utility, and bidding the entire budget on this interval constitutes a fix point.

If one reverse hazard rate drops below the threshold, equilibrium utilities either jump up due to a mass point of the opponent, or are constant on some indifference regions. In indifference regions, the opponent uses a bid distributions that makes a bidder indifferent between any bid in this interval. Equilibrium utilities can strictly exceed their lower bound.

I show that asymmetric budget distributions break the revenue dominance of the first-price auction over the second-price auction. For the special case of reverse hazard rate dominance in budget distributions, a weak bidder bids more aggressively than a strong bidder. This is in line with the literature on asymmetrically distributed valuations (Maskin and Riley, 2000), where the weaker bidder (with regards to the valuation distribution) bids more aggressively. Similarly, if budget distributions are identical, but one bidder values the object more, he bids more aggressively.

I find the necessary and sufficient conditions for a bidder to derive a higher utility than his opponent at every budget realization. I show that a first-price auction might allocate the object inefficiently, even if the bidder with the highest value for the object has also a higher budget realization than his opponent. Finally, I apply my technique to derive a closed-form equilibrium for an all-pay auction.

Related Literature: Che and Gale (1996, 1998, 2000) are amongst the first to analyze auctions with budget constrained bidders. In their seminal contributions, they derive the equilibrium for auctions with budget constraints and show that revenue equivalence no longer holds when bidders are symmetrically budget constrained. Research on budget constraints in standard 1-object auctions (see, e.g., Che and Gale, 1996, 1998; Kotowski, 2018; Kotowski and Li, 2014) considers symmetric budget distributions. However, literature on asymmetrically budget constrained bidders is scarce. Malakhov and Vohra (2008) derived the optimal auction with two bidders, where only one bidder is constrained and his identity is common knowledge. Some literature on multiple object auctions (see, e.g., Benoît and Krishna, 2001; Dobzinski et al., 2012) considers asymmetric budgets, however, it relies upon common knowledge of budget realizations. In this work, I merge the assumption of asymmetric budgets into a framework that allows for private budget realization.

The previous work that is closest to my framework is Che and Gale (1996). They considered many bidders with an identical commonly known valuation for the object. Budget realizations of the bidders in Che and Gale (1996) are private and independent draws from the same distribution. My model generalizes their model in two directions: first, in my model, budgets are drawn from asymmetric distributions. Second, the valuations for the object may differ between bidders. This allows me to capture the effect of valuation heterogeneity on the bidding strategies. In contrast to Che and Gale (1996), I do not restrict attention to symmetric and monotonic equilibria, but I impose log-concavity on the budget distribution and consider two bidders. I show in Section 4.1 that there exist no other asymmetric equilibrium utilities and bid distributions, aside from the symmetric equilibrium utility that Che and Gale (1996) found.

The analysis of this paper relates to asymmetric auctions, in which the valuations of bidders are drawn from non-identical distributions, and bidders do not have budget constraints (see the seminal contribution of Maskin and Riley, 2000). Analytical solutions exist for only a few particular distributions, e.g., Maskin and Riley (2000) and Kaplan and Zamir (2012) for uniform distributions, and Plum (1992) and Cheng (2006) for power distributions. Asymmetric auctions have been approached by perturbation analysis (see, e.g., Fibich and Gavious, 2003; Fibich et al., 2004; Lebrun, 2009). Nevertheless, even for two bidders with asymmetrically drawn valuations from the same support, no general closed-form solution is known. The first-price and second-price auctions no longer yield the same revenue under asymmetric value distributions, with the revenue ranking depending on the asymmetry of the value distributions (Maskin and Riley, 2000; Cantillon, 2008; Gavious and Minchuk, 2014).

If bidders are asymmetric not in valuations but in budgets, my results apply. In contrast to the literature on asymmetry in valuations, a closed-form solution exists for asymmetric budget distributions. Revenue can therefore be easily computed. A unique equilibrium utility and bid distribution exist under mild regularity conditions. This holds for all log-concave budget distribution functions with the same support, without assuming any stochastic dominance order.

The paper is organized as follows. Section 2 introduces the model. The characterization of the equilibrium in a first-price auction follows in Section 3, using a lower bound on the utility (Section 3.1). In Section 3.2, I determine a unique equilibrium utility by deriving four core lemmas. Section 3.3 establishes the uniqueness of the bid distributions, and Section 3.4 the existence of an equilibrium. Section 4 discusses the implications for symmetric bidders, bidding aggression, welfare, and efficiency. In Section 5, I extend my results to compare the revenue in a first-price and a second-price auction, analyze information disclosure about budget types, and solve an all-pay auction. I conclude in Section 6. All omitted proofs are in the Appendix.

2 Model

An auctioneer (she) sells one object with zero value for her in a first-price auction (FPA). She employs an equal tie-breaking rule and no reserve price. There are 2 risk-neutral bidders, indexed by $i \in \{1, 2\}$. Bidder i has a valuation v_i for the object. The valuation tuple $\{v_1, v_2\}$ is common knowledge for the bidders.

Each bidder (he) has a private budget w_i, which is drawn independently from a distribution with a continuous and differentiable cumulative distribution function $F_i(w)$ and probability density function $f_i(w)$. Both distribution functions $\{F_i(w)\}_{i=1,2}$ have common support $[\underline{w}, \overline{w}]$, are atom-less, are common knowledge, and have full support. Both bidders are budget constrained with non-zero probability, $\min\{v_1, v_2\} > \underline{w}$.[3]

Assumption 1. *$F_1(w)$ and $F_2(w)$ satisfy log-concavity on $(\underline{w}, \overline{w})$.*[4]

Due to Assumption 1, the reverse hazard rates (RHRs) $\frac{f_i(w)}{F_i(w)}$ are decreasing in w.

The bidding strategy of bidder i maps his budget realization w into a distribution over feasible bids $[0, w]$. Let b_i be a random variable denoting the placed bid of bidder i. Let $b_i(w)$ be a bid in the bidding support of bidder i with budget w. Bidders have hard[5] budget constraints: they cannot bid above their budget. A feasible bidding strategy satisfies $b_i(w) \leq w$ for all bids of any budget type w. If a bidder i wins the object by placing a feasible bid b_i, his utility is $v_i - b_i$.

[3]If one bidder is unconstrained, $v_i \leq \underline{w}$, the game effectively reduces to Bertrand competition.

[4]See Bagnoli and Bergstrom (2005) for many commonly used distributions that satisfy log-concavity.

[5]An equivalent formulation for my analysis is to impose fines on overbidding and to forbid renegotiation. E.g., see Footnote 2 in Che and Gale (1996).

Example 1. *Bidder* 1 *and* 2 *have the same valuation* $v := v_1 = v_2$ *for the object. Their budget distributions are* $F_1(w) = w^2$ *and* $F_2(w) = w$ *for* $w \in [0, 1]$.

In the above example, bidder 1 is stronger than bidder 2 in the sense of first oder stochastic dominance (FOSD). I use this example in the following to depict my solution technique.[6]

3 Equilibrium of the First Price Auction

Let $G_i(x) = \Pr(b_i \leq x)$ be the cumulative distribution function of bidder i's bid, that is, the probability of bidder i bidding below or equal to x. A *feasibility constraint* holds as a necessity of the hard budget constraints:

$$G_i(x) \geq F_i(x) \quad \forall x \in [0, \overline{w}]. \tag{1}$$

Every bidder with a budget below x bids weakly below x. Moreover, a bidder with a budget strictly above x might shade his bid down below x, yielding the weak inequality in the feasibility constraint in Equation 1. If bidders always bid their entire budget for any budget realization, the above feasibility constraint holds with equality at every x.

The expected utility of bidder i with a budget w, given bidder j's bid distribution, is

$$U_i(w) = \max_{0 \leq b_i \leq w} \{(v_i - b_i)[\Pr(b_j < b_i)] + \frac{1}{2}(v_i - b_i)\Pr(b_j = b_i)\}. \tag{2}$$

The second summand accounts for the equal tie-breaking rule. In my model, equilibrium strategies may contain mass points and the probability of a tie is therefore non-negligible. Due to the individual (potentially slack) budget constraints, and the discontinuities in the objective function in Equation 2 induced by atoms in the bid distributions, a classic differentiation approach with invertible bidding functions is not possible. I solve this problem via an indirect utility approach, using a lower bound on the equilibrium utility.

In the following, I derive a uniqueness result for the equilibrium utilities U_1 and U_2 and the bid distributions G_1 and G_2. An existence result follows in Section 4, where I show that there always exist an equilibrium in weakly monotonic pure strategies.

[6]Note that I do not impose any stochastic order between $F_1(w)$ and $F_2(w)$ in the general model.

3.1 Lower Bound

Consider the lowest feasible bound on the equilibrium utility of bidder i with budget w, called the *lower bound utility* $\underline{U}_i(w)$. This lower bound utility is achieved if the opponent j always plays the following naive strategy: bidding his entire budget and, hence, minimizing the winning probability of bidder i at any bid. Under this naive strategy of bidder j, bidder i's bid b_i wins with a probability $G_j(b_i) = F_j(b_i)$. Any bid placed by i wins with a weakly lower probability under the naive strategy than under any other feasible strategy for j.

Next, I characterize the properties of the lower bound $\underline{U}_i(w) = \max_{b_i \leq w}(v_i - b_i)F_j(b_i)$. Subsequently, I derive the equilibrium utility $U_i(w)$ from the lower bound utility $\underline{U}_i(w)$.

Lemma 1. *Let bidder j bid his entire budget, and F_j be log-concave. Then, the unique best response bid for bidder i with budget w is*

$$\arg\max_{b_i \leq w}(v_i - b_i)F_j(b_i) = \begin{cases} w & \text{if } w < m_i, \\ m_i & \text{if } w \geq m_i. \end{cases} \tag{3}$$

with some unique $m_i \in (\underline{w}, \overline{w}]$. The lower bound $\underline{U}_i(w)$ for $i \in \{1, 2\}$ is continuous, strictly increasing for $w < m_i$ and constant for $w \in [m_i, \overline{w}]$.

Bid m_i is the unconstrained best response of bidder i to a naive opponent bidding $G_j = F_j$. Either bidder i can afford to bid m_i ($w \geq m_i$), or he bids his entire budget to bid as close as possible to m_i ($b_i(w) = w$ if $w < m_i$). The resulting lower bound utility is

$$\underline{U}_i(w) = \max_{b_i \leq w}(v_i - b_i)F_j(b_i) = \begin{cases} (v_i - w)F_j(w) & \text{if } w < m_i, \\ (v_i - m_i)F_j(m_i) & \text{if } w \geq m_i, \end{cases} \tag{4}$$

The marginal utility of an increase in bid b_i is non-negative if the gain in the probability of winning offsets the higher payment in case of a win. This occurs if and only if

$$\frac{f_j(b_i)}{F_j(b_i)} \geq \frac{1}{v_i - b_i}, \tag{5}$$

where the RHR $\frac{f_j(b)}{F_j(b)}$ is monotonically decreasing by log-concavity, and the right hand side is strictly increasing in b_i. Inequality 5 holds with strict inequality for $b_i < m_i$, with equality for $b_i = m_i$, and does not hold for $b_i > m_i$. Any bid above m_i yields a strictly lower payoff than bidding m_i for bidder i. The unconstrained best response m_i to a naive strategy opponent is increasing in (and always below) v_i. If the value of the object is sufficiently high, bidding the entire budget may be the best response for every budget realization.

In what follows, I assume without loss of generality that $m_1 \leq m_2$.[7] Both lower bounds are strictly increasing for $w < m_1$. Figure 1 shows the lower bound utilities for Example 1 with $v = 1$. $\underline{U}_1(w)$ is strictly increasing for $w < m_1 = \frac{1}{2}$, and constant for higher budget realizations. $\underline{U}_2(w)$ is strictly increasing for $w < m_2 = \frac{2}{3}$.

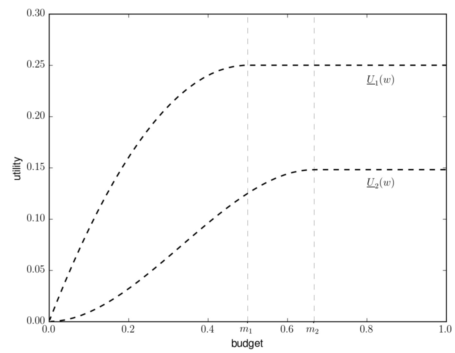

Figure 1: Lower bound utilities for Example 1 with $v = 1$.

The lower bound utility $\underline{U}_i(w)$ is a generalization of the lower bound expression in Che and Gale (1996). In contrast to their model, I allow for asymmetric budget distributions and different valuations, and impose the assumption of log-concavity on F_1 and F_2. They show that in the class of symmetric and monotonic equilibria, the lower bound binds, i.e., $U_i(w) = \underline{U}_i(w)$ for all $w \in [\underline{w}, \overline{w}]$.[8] For asymmetric bidders and valuations, I show that the lower bound does not generically bind and mass points may arise in equilibrium.

I differentiate between two cases:[9]

$(C1)$ $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$.

$(C2)$ $\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$.

Note the parallel to Bertrand competition with unconstrained bidders. If both bidders have unlimited budget and $v_i > v_j$, bidder i wins by bidding v_j and his opponent j randomizes in some non-empty interval below v_j in the equilibrium in undominated strategies (Blume,

[7]If $m_1 = m_2$, I label bidders such that $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$. This is without loss and guarantees that if there is exactly one mass point in equilibrium, it will be placed by bidder 1.

[8]In Section 4.1, I apply my findings to the setting of Che and Gale (1996) to find all (possibly asymmetric and non-monotonic) equilibrium utilities and bid distributions.

[9]This case distinction determines how many mass points arise in equilibrium.

2003).[10] The payoff of bidder i is $(v_i - v_j)$, and bidder j has zero payoff. The difference in payoffs is the right-hand side of (C1) and (C2). $\underline{U}_i(m_i)$ is the utility of an unconstrained bidder i (being able to afford the best response bid m_i) from bidding against a naive opponent who bids his entire budget. The left-hand side of (C1) and (C2) is the difference in utilities from bidding against a naive constrained opponent.

The two cases compare the net gain in utility: who gains more from bidding against a naive constrained opponent instead of an unconstrained opponent? This gain is higher for bidder 1 in (C1) and strictly higher for bidder 2 in (C2).

3.2 Equilibrium Utility

In the following, I derive four properties that any equilibrium satisfies. Together, these properties rule out all but a single candidate for the shape of the equilibrium utility.

Consider any given candidate equilibrium utility U_i. Bidding strategies may sometimes be inferred from the properties of U_i, as the next lemma shows.

Lemma 2. *Let $U_i(w)$ be strictly increasing on some open interval (w', w''). Then, bidder i with any budget realization $w \in (w', w'')$ always bids his entire budget, and $G_i(w) = F_i(w)$.*

This has been noted by Che and Gale (1996).[11] For completeness, I include the proof in the appendix. The same (feasible) bid b_i always yields the same utility to bidder i, irrespective of his budget realization: $(v_i - b_i)\Pr(b_j < b_i) + \frac{1}{2}(v_i - b_i)\Pr(b_j = b_i)$. This is because the valuation v_i and winning probability of bidder i do not depend on his budget. If bidder i achieves a strictly higher utility with a higher budget than a lower budget, then those bids placed with the higher budget must be unaffordable for him with a lower budget. Hence, if U_i is strictly increasing in bidder i's budget, bidder i bids his their entire budget: this is the only bid which cannot be mimicked by any lower budget type.

The following lemma shows that whenever the utility is strictly increasing, the lower bound utility binds.

Lemma 3. *Let $U_i(w)$ be strictly increasing for $w \in (w', w'')$. Then, for all $w \in (w', w'')$ the lower bound binds: $U_i(w) = \underline{U}_i(w)$.*

If the expected utility of bidder i in equilibrium is strictly increasing, by Lemma 2, bidder i exhausts his budget over this interval. Thus, bidder j is essentially facing a naive opponent

[10] If $v_1 = v_2$, both unconstrained bidders have zero payoff and the interpretation still applies.
[11] See Footnote 7 in the proof of Lemma 1 in Che and Gale (1996).

i within this interval who always bids his entire budget. Either the opponent j would not want to bid within this interval at all (e.g., if the interval is above m_j), or he would also want to exhaust his entire budget. The former leads to a contradiction, as bidder i would never want to be the only one bidding in an open interval. The latter leads to a scenario in which both bidders bid their entire budget, and by definition receive nothing more than their lower bound utility.

Put differently, for U_i to rise strictly above the lower bound, the bid distribution of the opponent must contain mass points. A smooth increase of U_i over an open interval is impossible according to Lemma 3, and the utility in Equation 2 is continuous, unless the opponent places mass points.

The next result shows that as long as both lower bound utilities are strictly increasing, bidding the entire budget is the unique best response correspondence.

Lemma 4. *In any equilibrium, bidders with a budget $w \in (\underline{w}, m_1)$ bid their entire budget. For all $w \in [\underline{w}, m_1)$, the lower bound binds: $U_i(w) = \underline{U}_i(w)$.*

Assume the opponent bids his entire budget on $(\underline{w}, m_1)$. Any bid below one's budget loses so much probability in terms of winning that it does not justify the gain from the lower payment in case of a win. This guarantees that both bidders bidding their entire budget on this interval are mutually best responses. Lemma 4 establishes that this is the unique best response correspondence in any equilibrium.[12]

The following properties further narrow down the set of candidate equilibria.

Lemma 5. *For $i \neq j$, the following holds in any equilibrium:*

1. *U_i has at most one discontinuity. If it arises, it occurs at m_j and $U_j(m_j) = \underline{U}_j(m_j)$.*
2. *U_1 is constant on (m_1, m_2) and $(m_2, \overline{w}]$; U_2 is constant on $(m_1, \overline{w}]$.*

For a sketch of the argument, let bidder i have a strictly increasing utility U_i on some interval above m_1. Then, by Lemma 4, the lower bound binds, $U_i = \underline{U}_i$. As $\underline{U_i}$ is constant above m_i, this rules out any strict increase in U_1 above m_1 for bidder 1, and in U_2 above m_2 for bidder 2. In the remaining case, bidder 2 has a strictly increasing utility U_2 on some interval within (m_1, m_2). Then, bidder 2 bids his entire budget and $F_2 = G_2$ on this interval by Lemma 2. Bidder 1 then essentially faces a naive opponent in this interval. But bidder

[12]Lemma 4 does not specify the bid of the lowest budget type $\underline{w}$, while it determines his utility $U_i(\underline{w}) = 0$. As a budget $\underline{w}$ is a zero-probability event, $b_i(\underline{w})$ has no impact on G_i.

1 would not want to bid above m_1 if his winning probability is the same as against a naive opponent; he could strictly do better by bidding m_1. Thus, U_2 cannot be strictly increasing.

I show that discontinuities can only be due to mass points of the opponent. Mass points can occur only at m_1 (in bidder 1's strategy) and m_2 (in bidder 2's strategy). For a sketch of the argument, consider bidder 1 placing a mass point at some bid $x > m_1$. For his opponent 2, a bid at the mass point yields lower utility than bidding infinitesimally above the mass point (due to a discrete jump in winning probability). However, this implies that bidder 2 with a budget above the mass point would never bid at or below the mass point. Furthermore, bidder 2 with a lower budget prefers to bid at the mass point, but cannot afford it. Hence, $G_2(x) = F_2(x)$. The mass point placing bidder 1 essentially faces a naive opponent 2 at the mass point, and could do strictly better by bidding m_1 instead of x.

The next result determines a unique equilibrium utility as a function of the lower bound, using Lemma 2, 3, 4 and 5 to derive the complete characterization.

Theorem 1. *Let (C1) hold. In any equilibrium, utilities are*

$$U_1(w) = \underline{U}_1(w) \text{ for all } w, \tag{6}$$

$$U_2(w) = \begin{cases} \underline{U}_2(w) \text{ if } w < m_1, \\ \frac{1}{2}\left(\underline{U}_2(m_1) + \underline{U}_1(m_1) - (v_1 - v_2)\right) \text{ if } w = m_1, \\ \underline{U}_1(m_1) - (v_1 - v_2) \text{ otherwise.} \end{cases} \tag{7}$$

Let (C2) hold. In any equilibrium, utilities are

$$U_1(w) = \begin{cases} \underline{U}_1(w) \text{ if } w < m_2, \\ \frac{1}{2}\left(\underline{U}_1(m_2) + \underline{U}_2(m_2) + (v_1 - v_2)\right) \text{ if } w = m_2, \\ \underline{U}_2(m_2) + (v_1 - v_2) \text{ otherwise.} \end{cases} \tag{8}$$

$$U_2(w) = \begin{cases} \underline{U}_2(w) \text{ if } w < m_1, \\ \frac{1}{2}\left(\underline{U}_2(m_1) + \underline{U}_2(m_2)\right) \text{ if } w = m_1, \\ \underline{U}_2(m_2) \text{ otherwise.} \end{cases} \tag{9}$$

A unique equilibrium utility can be recovered by computing the lower bound utilities $\underline{U}_1$ and $\underline{U}_2$, the best response bids to a naive opponent m_1 and m_2, and the difference $v_1 - v_2$.

The utility levels of the wealthiest bidders with budget $\overline{w}$ are exactly $v_1 - v_2$ apart, as they both have the same supremum bid that wins with a probability of one.[13] In order to achieve this distance $(v_1 - v_2)$ in the final utility levels, Lemma 5 allows the following variants: equilibrium utility U_i can jump only at m_j, but not increase continuously; a jump in U_i at m_j determines the utility of bidder $j \neq i$ to $\underline{U}_j(m_j)$ for an interval of higher budget levels. The above theorem shows that there is one unique way to allocate discontinuities such that the required utility difference $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2$ is satisfied. This involves exactly one discontinuity in bidder 2's utility in Case (C1), and two discontinuities in both bidders' utilities in Case (C2).

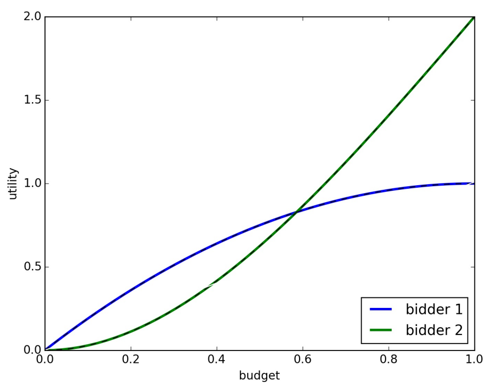

Figure 2: Case (C1) with $v_1 = 2$, $v_2 = 3$, and $m_1 = m_2 = \overline{w}$.

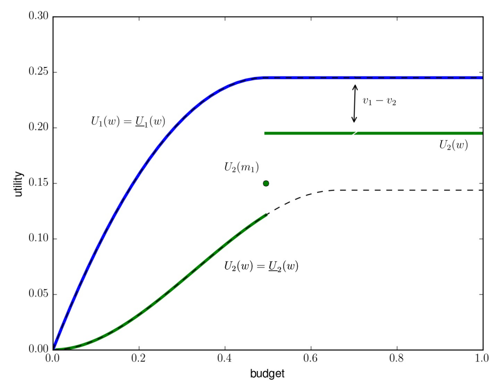

Figure 3: Case (C1) with $v_1 = 1$, $v_2 = 0.95$, and $\underline{U}_1 > \underline{U}_2$.

Figure 2 illustrates the case $m_1 = m_2 = \overline{w}$ for the budget distributions in Example 1, with $v_1 = 2$ and $v_2 = 3$. The object is valuable enough and both bidders are likely to have a sufficiently high budget (i.e., Inequality 5 holds at every budget level) such that both lower bounds are strictly increasing within the entire budget domain. Case (C1) applies, as $\underline{U}_i(m_i) = \underline{U}_i(\overline{w}) = v_i - 1$. The grey dashed lines are the lower bound utilities; they coincide with the equilibrium utility of bidder 1 (bidder 2), depicted by the blue (green) solid line. For all $w < 1 = \overline{w}$, by Lemma 4, it holds that $U_i = \underline{U}_i$. Bidder i with a budget $\overline{w} = 1$ bids his entire budget[14] and derives a payoff $U_i(1) = \underline{U}_i(1) = v_i - 1$.

Figure 3 illustrates Case (C1) for Example 1 with $v_1 = 1$ and $v_2 = 0.95$. A quick calculation reveals $m_1 = 0.5$ and $m_2 = 0.6\overline{3}$. The lower bound utilities of the bidders are depicted by the dashed lines, the equilibrium utilities by the solid lines. The lower bounds and the equilibrium utilities coincide for $w < m_1$. By Lemma 5, bidder 2's utility is constant on $(m_1, \overline{w}]$ and can only have a jump discontinuity at m_1. It holds that $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq$

[13] See Lemma A.1 in the Appendix for further details.

[14] Any lower bid yields a strictly lower payoff as it corresponds to the payoff of a x-budget type.

$v_1 - v_2 = 0.05 = U_1(1) - U_2(1)$. That is, for high enough budgets, the distance between the dashed gray lines is larger than the necessary distance between the solid green and blue lines in equilibrium. Thus, at least one utility has to lie strictly above the lower bound to achieve the necessary distance. Bidder 2's utility jumps at m_1 by such a magnitude to its final level such that the utility difference between the equilibrium amounts to $v_1 - v_2 = 0.05$. This rules out a discontinuity in bidder 1's utility. Otherwise, by Lemma 5, it would hold that $U_2(w \geq m_2) = \underline{U}_2(m_2)$. Then, the distance to the highest utility of bidder 1 (which is bounded below by $\underline{U}_1$) is strictly larger than the required $v_1 - v_2$.

Figure 4 shows another example for Case (C1) with $\underline{U}_1 < \underline{U}_2$. Bidder 1 achieves exactly his lower bound utility for every budget level and bidder 2's utility U_2 jumps at m_1 above the lower bound $\underline{U}_2$ to achieve the required distance $(v_1 - v_2)$.

If $m_1 = m_2$ and $\underline{U}_1(m_1) - \underline{U}_2(m_2) = v_1 - v_2$, then Case (C1) applies. The lower bounds bind for every budget, and there are no discontinuities in the utilities: any mass point would distort the utility difference of the bidders away from the correct distance $v_1 - v_2$.[15]

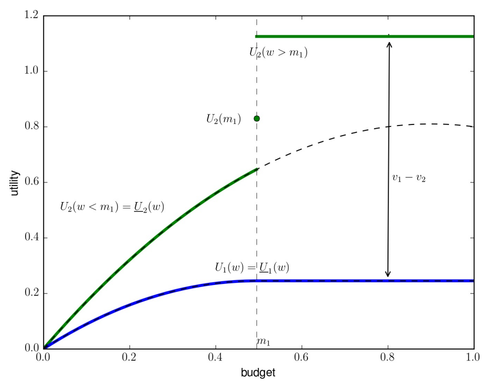

Figure 4: Case (C1) with $\underline{U}_1 < \underline{U}_2$.

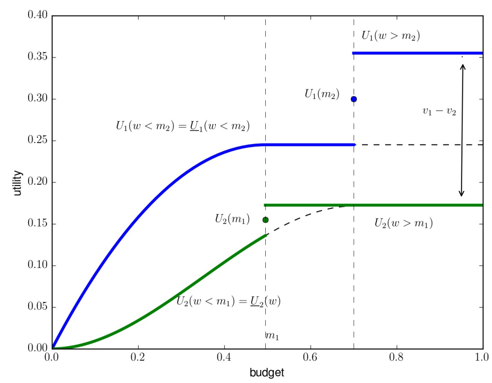

Figure 5: Case (C2) for $\underline{U}_1 > \underline{U}_2$.

The utilities in Case (C2) in Theorem 1 are outlined in Figure 5. In this case, the difference between the equilibrium utilities $(v_1 - v_2)$ is larger than $\underline{U}_1(m_1) - \underline{U}_2(m_2)$. This cannot be achieved by a mass point of bidder 1 alone, as this would pin down the utility of bidder 1 to $U_1(w \geq m_1) = \underline{U}_1(m_1)$ by Lemma 5. Bidder 2 has to place a mass point at bid m_2 to increase the difference in the equilibrium utilities to the required level.

In summary, if $m_1 = \overline{w}$, Lemma 4 characterizes effectively the entire equilibrium utility and no mass points arise. If $m_1 < \overline{w}$ and Case (C1) holds, then bidder 1 places a mass point of such magnitude to elevate bidder 2's utility to its final level ($v_1 - v_2$ below his own utility). In Case (C2), bidders 1 and 2 place mass points at distinct bids to achieve the

[15] This is the case in the symmetric setup with $v_1 = v_2$ and $F_1 = F_2$ in Che and Gale (1996).

required distance $v_1 - v_2$.

3.3 Equilibrium Bid Distributions

The next result shows the unique bid distributions and supremum bids in any equilibrium.

Theorem 2. *In any equilibrium, the supremum bid of both bidders is*

$$\bar{b} = \begin{cases} v_1 - (v_1 - m_1)F_2(m_1) & \textit{if } (C1), \\ v_2 - (v_2 - m_2)F_1(m_2) & \textit{if } (C2), \end{cases} \tag{10}$$

and the cumulative bid distributions satisfy

$$G_1(b) = \begin{cases} F_1(b) & \textit{if } b < m_1, \\ \frac{v_2 - \bar{b}}{v_2 - b} & \textit{if } b \in [m_1, \bar{b}], \end{cases} \qquad G_2(b) = \begin{cases} F_2(b) & \textit{if } b < m_1, \\ \frac{(v_1 - m_1)F_2(m_1)}{v_1 - b} & \textit{if } b \in [m_1, m_2), \\ \frac{v_1 - \bar{b}}{v_1 - b} & \textit{if } b \in [m_2, \bar{b}]. \end{cases} \tag{11}$$

Bid distributions are unique in equilibrium, should it exist. They are invariant to the degree of asymmetry, as long as both lower bound utilities are strictly increasing. $F_1(w)$ and $F_2(w)$ can be identical, stochastically ordered (e.g., FOSD); bidding the entire budget is always the unique equilibrium candidate for budgets below m_1. Above m_1, bidders place bids on a non-empty interval to make their opponent indifferent. For example, in Case (C1), both bidders allocate their bidding mass on $\left(m_1, \bar{b}\right)$ in such a way that any bid in this interval yields the same expected payoff to the opponent.

The equilibrium bid distributions might require mass points. This is summarized in the following result which immediately follows from the bid distributions in Theorem 2.

Corollary 1. *Each bidder has at most one mass point. Bidder 1 has a mass point at m_1 if and only if $m_1 < m_2$. Bidder 2 has a mass point at m_2 if and only if (C2) holds.*

3.4 Equilibrium Existence

The following result establishes existence of an equilibrium. I derive pure strategy weakly monotonic bidding functions that are feasible and optimal for the bidders.

Theorem 3. *A pure strategy weakly monotonic equilibrium exists in the FPA.*

The proof is by construction. If (C1) holds, the following weakly monotonic bidding functions constitute an equilibrium. (The bidding functions for (C2) are in the appendix.)

$$b_1(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ m_1 & \text{if} \quad w \in [m_1, F_1^{-1}(\frac{v_2 - \bar{b}}{v_2 - m_1})], \\ v_2 - \frac{v_2 - \bar{b}}{F_1(w)} & \text{otherwise.} \end{cases} \tag{12}$$

$$b_2(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ v_1 - \frac{v_1 - \bar{b}}{F_2(w)} & \text{otherwise.} \end{cases} \tag{13}$$

It is straightforward to show that these bidding functions b_1 and b_2 are feasible (i.e., $b_i(w) \leq w$) and aggregate into the bid distributions G_1 and G_2 in Theorem 2. They are also optimal for the bidders. Bidder i with a budget in $[\underline{w}, m_1)$ would prefer to bid more than his budget, but cannot afford it; bidding below his budget yields a strictly lower payoff. On $(m_1, \bar{b}]$, both bidders are bidding in such a way as to make their opponent indifferent between any bid in this interval. A mass point of bidder 1 at m_1 can be sustained in equilibrium. Bidder 2, who bids at or slightly below the mass point m_1 of his opponent, would want to increase his bid and get a jump in winning probability. However, bidder 2 cannot afford this upward deviation as he is already bidding his entire budget at and below the mass point.

Numerical Example. Consider the budget distributions in Example 1 and let $v = 1$. Then, $m_1 = \frac{1}{2}$, $m_2 = \frac{2}{3}$ and (C1) applies. The lower bound utilities are depicted in Figure 1. The following pure strategy bidding functions constitute a weakly monotonic equilibrium:

$$b_1(w) = \begin{cases} w & \text{if} \quad w < \frac{1}{2}, \\ \frac{1}{2} & \text{if} \quad w \in [\frac{1}{2}, \frac{1}{\sqrt{2}}], \\ 1 - \frac{1}{4w^2} & \text{otherwise,} \end{cases} \qquad \text{and} \qquad b_2(w) = \begin{cases} w & \text{if} \quad w < \frac{1}{2}, \\ 1 - \frac{1}{4w} & \text{otherwise.} \end{cases}$$

Figure 6 illustrates these two bidding functions. The blue dashed (green solid) line is the bidding function of the strong bidder 1 (weak bidder 2). Bidders place their entire budget if $w < \frac{1}{2}$. Bidder 1 places a mass point on $m_1 = \frac{1}{2}$. The highest bid $\bar{b} = \frac{3}{4}$ wins with a probability of one and yields the same payoff $v - \bar{b} = \frac{1}{4}$ to both bidders.

Figure 7 shows the corresponding equilibrium utility of bidder 1 (blue line) and bidder 2 (green line). Utility is strictly increasing below $m_1 = \frac{1}{2}$. Bidder 1's mass point at m_1 raises bidder 2's utility to the same level as his own for budgets above $\frac{1}{2}$. Bidder 2, with a budget at the mass point or slightly below, cannot deviate upwards as his budget constraint binds.

It is important to note that the constancy in equilibrium utility does not correspond to constancy in bids. The bidding function of bidder 2 makes bidder 1 indifferent between all bids in $[\frac{1}{2}, \frac{3}{4}]$, including his mass point m_1. Similarly, bidder 2 is indifferent between any bid between $(\frac{1}{2}, \frac{3}{4}]$ due to bidder 1's bid distribution.

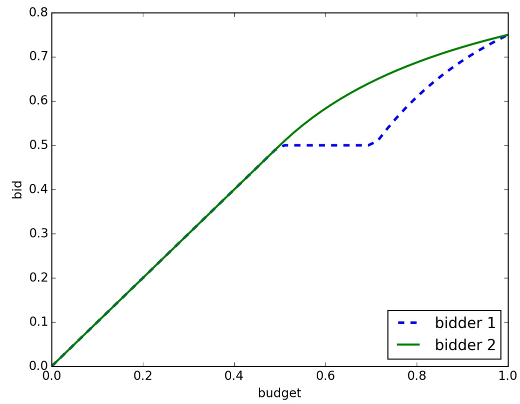

Figure 6: Bidding functions for $v = 1$.

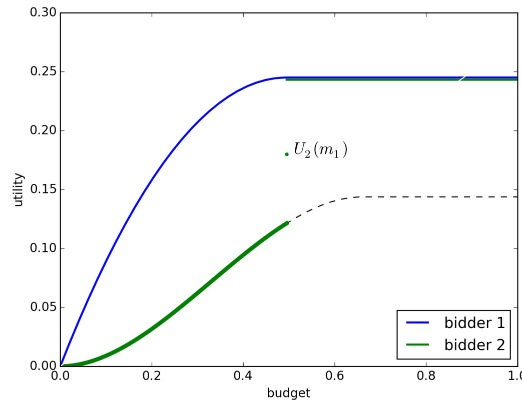

Figure 7: Equilibrium utilities for $v = 1$.

4 Discussion of the Results

4.1 Symmetric Bidders

In this section, I derive the equilibria for symmetric bidders. Let $F(w) := F_1(w) = F_2(w)$ log-concave and $v := v_1 = v_2$. Both bidders have the same lower bound utility $\underline{U}(w) := \underline{U}_1(w) = \underline{U}_2(w)$ and $m := m_1 = m_2$, and (C1) therefore holds. If the opponent always bids his entire budget, both bidders prefer to bid m (or their full budget if they cannot afford m).

Che and Gale (1996) solve for a symmetric and monotonic equilibrium.[16] They show that the lower bound always binds, $U_i(w) = \underline{U}(w)$ for all w. I show that in any (possibly asymmetric and non-monotonic) equilibrium the following holds:

Corollary 2. *Let $v := v_1 = v_2$ and $F(w) := F_1(w) = F_2(w)$ satisfy Assumption 1. Any equilibrium has the following properties:*

1. *lower bounds utilities bind, i.e., for all w, $U_i(w) = \underline{U}(w)$,*

[16] Che and Gale (1998) allow for $n \geq 2$ bidders and do not impose log-concavity on the distribution $F(w)$.

2. *bid distributions are* $G_i(b) = \begin{cases} F(b) & \text{if } b < m, \\ \frac{(v-m)F(m)}{v-b} & \text{if } b \in [m, v-(v-m)F(m)]. \end{cases}$

This is a direct application of Theorems 1 and 2 for Case (C1), the proof is therefore omitted. By Corollary 1, no bidder can place a mass point. Figure 8 shows an example with $v = 1$ and $F(w) = w$ for $w \in [0, 1]$. The equilibrium utility coincides for both bidders (blue line) and equals the lower bound utility, which is strictly increasing below $m = 0.5$.

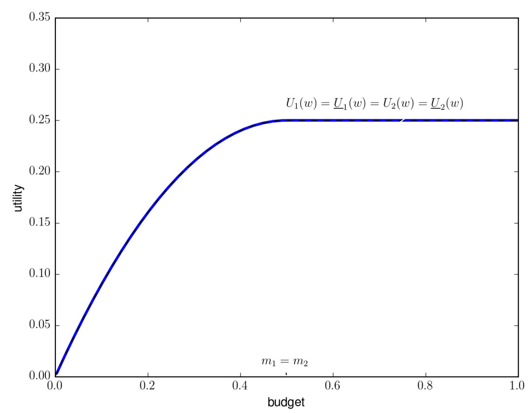

Figure 8: Equilibrium utilities with $v = 1$ and $F(w) = w$.

Che and Gale (1998) derive a symmetric equilibrium in pure strategies (see Lemma 1 in Che and Gale (1998)). For the case of two bidders with log-concave budget distributions, these strategies correspond with the bid functions in Equations 12 and 13 in this paper and aggregate into the bid distributions in Statement 2 of Corollary 2.

What other equilibria can exist within this framework? First, consider $m = \overline{w}$. Then, by Lemma 4, bidders with $w \in (\underline{w}, \overline{w})$ bid their entire budget in any equilibrium. The bidder with the highest budget $\overline{w}$ bids his entire budget as any lower bid yields a strictly lower payoff. The bidder with the lowest budget $\overline{w}$ bids anything weakly lower than his budget. Hence, if $m = \overline{w}$ there exists a unique[17] equilibrium where bidders exhaust their budgets.[18]

Second, let $m < \overline{w}$. For $w < m$, Lemma 4 also pins down the behavior in any equilibrium, $b_i(w) = w$. For $w \geq m$, there exists a multiplicity of other (asymmetric) bidding functions in the indifference regions above m, aside from the symmetric pure strategy monotonic equilibrium in Equations 12 and 13. These equilibrium bidding functions aggregate into the

[17]It is unique up to the behavior of the lowest budget bidder, who will lose for any feasible bid.

[18]This is also the equilibrium that Che and Gale (1998) describe in their Lemma 1.

same bid distributions $G_i(b)$ in Statement 2 in Corollary 2. Bidders with a budget above m can play a variety of mixed strategy combinations, as they are indifferent between any bid in the bidding support $[m, \bar{b}]$. A bidder cares only about the aggregated bid distribution of his opponent, not about which specific bidding function it stems from.[19]

Overall, lower bounds bind $U_i(w) = \underline{U}(w)$ and the bid distributions $G_i(b)$ are unique in any equilibrium in this symmetric framework. If $m < \overline{w}$, there might also exist a variety of asymmetric and mixed strategy bidding functions apart from the symmetric pure strategy equilibrium that Che and Gale (1998) found.

4.2 Bidding Aggression

The monotonic pure strategy bidding functions in Section 3.4 allow a direct comparison in bidding behavior. For example, which bidder bids more aggressively if both have the same budget? When comparing bidding aggression in my model, there are two channels of interest. First, how does bidding aggression depend on the budget distribution? Second, how does bidding aggression depend on the valuation for the object?

As Lemma 4 shows, bidders with budget realization in $[\underline{w}, m_1)$ always bid their entire budget and are equally aggressive, irrespectively of any order statistic assumption on their budget distributions. If $m_1 = \overline{w}$, under any order statistic, both bidders bid equally aggressive on the entire budget support in my framework.

Definition 1. *F_i dominates F_j in terms of RHRs ($F_i \geq_{RHR} F_j$) if for all $x \in (\underline{w}, \overline{w})$,*

$$\frac{f_i(x)}{F_i(x)} \geq \frac{f_j(x)}{F_j(x)}.$$

In the next result, I assume that both bidders have the same valuation v for the object, and one bidder dominates his opponent in terms of RHR in the budget distribution. This allows me to elicit the differences in bidding aggression that are only due to differences in the budget distributions and not to heterogeneous valuations.

[19]For example, with $v = 1$ and $F(w) = w$, one bidder could bid according to the following non-monotonic feasible bidding function, which aggregates into the required G_i:

$$b_i(w) = \begin{cases} w & \text{if } w \in [0, \frac{3}{4}], \\ \frac{1}{8}\left(\sqrt{16w^2 + 8w - 15} - 4w + 7\right) & \text{if } w \in (\frac{3}{4}, 1]. \end{cases}$$

Proposition 1. *Let $v := v_1 = v_2$ and $F_i \geq_{RHR} F_j$. Then, the dominant bidder bids less aggressively, $b_i(w) \leq b_j(w)$.*

Proof. As bidder i RHR-dominates bidder j, it holds that $i = 1$ because $m_1 \leq m_2$. This is because the RHR-condition of the dominant bidder i is stricter than the RHR-condition of the dominated bidder. As RHR-dominance implies FOSD, it holds that $\underline{U}_1(m_1) \geq \underline{U}_2(m_2)$ and Case (C1). The highest bid is $\bar{b} = v - (v - m_1)F_2(m_1)$ and it can be easily checked that the bidding strategies in Equations 12 and 13 imply $b_1(w) \leq b_2(w)$ for all $w \in [\underline{w}, \overline{w}]$. □

Maskin and Riley (2000) target a related question for asymmetrically distributed valuations and bidders with unconstrained liquidity. They consider a variant of the RHR-dominance on valuation distributions and show that if both bidders have the same valuation, the RHR-dominated bidder bids more aggressively. This is in line with the findings of this paper: the weaker bidder in the sense of RHR on budgets bids more aggressively.

Proposition 1 compared bidding behavior for bidders with equal valuations and distinct bid distributions. Next, I compare bidders with identical budget distributions $F(w) := F_1(w) = F_2(w)$, but different valuations $v_1 \neq v_2$.

Let $v_i > v_j$. Then, the RHR-condition is satisfied for bidder i whenever the RHR-condition is satisfied for bidder j, because $\frac{1}{v_i - w} < \frac{1}{v_j - w}$ for all $w \in (\underline{w}, \min\{v_j, \overline{w}\})$. Therefore, it holds that $j = 1$ and $v_2 > v_1$. An example is depicted in Figure 4. Due to the higher valuation, the lower bound of bidder 2 is always above the lower bound of bidder 1.

Proposition 2. *Let $v_i > v_j$ and $F_i(w) = F_j(w)$. Then, $b_i(w) \geq b_j(w)$ for all $w \in [\underline{w}, \overline{w}]$.*

Proof. Let $v_2 > v_1$ and $F(w) := F_1(w) = F_2(w)$. It holds that $\underline{U}_1(m_1) = \underline{U}_1(m_2) \geq (v_1 - m_2)F(m_2)$, and hence

$$\begin{aligned}
\underline{U}_2(m_2) - \underline{U}_1(m_1) &= \underline{U}_2(m_2) - \underline{U}_1(m_2) \\
&\leq (v_2 - m_2)F(m_2) - (v_1 - m_2)F(m_2) \\
&= (v_2 - v_1)F(m_2) \leq v_2 - v_1.
\end{aligned}$$

Thus, Case (C1) applies. Using the pure monotonic bidding strategies in Equations 12 and 13, it immediately follows that $b_2(w) \geq b_1(w)$ for $w \in [\underline{w}, F^{-1}\left(\frac{v_2 - \bar{b}}{v_2 - m_1}\right)]$. For higher w, by assumption $v_1 < v_2$, and it holds that $b_1(w) = v_2 - \frac{v_2 - \bar{b}}{F(w)} \leq v_1 - \frac{v_1 - \bar{b}}{F(w)} = b_2(w)$. □

As before, bidders exhaust their entire budget below m_1. For any higher budget, the bidder who values the object more bids more aggressively.

4.3 Bidder Welfare

When does a bidder have a higher utility level than his opponent at any budget realization? The following result provides necessary and sufficient conditions.

Proposition 3. *The following statements are equivalent:*

1. *For all w, $U_i(w) \geq U_j(w)$.*

2. *For all w, $\underline{U}_i(w) \geq \underline{U}_j(w)$ and $v_i \geq v_j$.*

It is instructive to consider when the primitives of the model $\{v_1, v_2, F_1, F_2\}$ translate into a higher lower bound utility $\underline{U}_i$ at every budget level. If $v_1 = v_2$, FOSD is a sufficient, but not a necessary condition for $\underline{U}_i \geq \underline{U}_j$. Let $m_1 < \overline{w}$. Below m_1, the object is sufficiently valuable and the opponent has a sufficiently high RHR to justify always exhausting their budget. In this case, the condition $\underline{U}_i(w) \geq \underline{U}_j(w)$ is equivalent to $F_i(w) \leq F_j(w)$. For higher budget realizations than m_1, the condition $\underline{U}_i(w) \geq \underline{U}_j(w)$ is less strict than FOSD. This stresses that the precise shape of the budget distribution matters only for sufficiently low budget realizations for which Inequality 5 holds.

Below are the two special cases from Section 4.2 that are sufficient for one bidder to derive a higher utility than his opponent at every budget realization.

Observation 1. *Let $v_i = v_j$, and $F_i \geq_{RHR} F_j$. Then, $i = 1$, (C1) holds, and $\underline{U}_1(w) \geq \underline{U}_2(w)$ for all w.*

Let $v_i > v_j$, and $F_i = F_j$. Then, $i = 2$, (C2) holds, and $\underline{U}_2(w) \geq \underline{U}_1(w)$ for all w.

If a bidder has either a higher valuation or a higher budget distribution (in the sense of the RHR-dominance) than his opponent, he enjoys a higher lower bound utility: at every budget level, he is either more likely to win (RHR-dominance) or values the event of winning more (higher valuation) when bidding against a naive opponent.

A related question is how a change in the budget distribution impacts upon the utility of a bidder. Is a bidder better off with a higher (in terms of RHRs) budget distribution F'_k instead of F_k?[20] It turns out that the opposite is the case: both bidders are harmed by a stronger budget distribution.

As in Equation 2, let $U_i(w)$ be the equilibrium utility of bidder i with F_k, and $U'_i(w)$ with F'_k. Let $\overline{b}$ be the supremum bid with distribution F_k, and $\overline{b}'$ with distribution F'_k.

[20] My findings rely on both bidders knowing the budget distribution F_k or F'_k of bidder k.

Proposition 4. *Let $F'_k \geq_{RHR} F_k$ for some $k \in \{1, 2\}$ and fix $\{v_1, v_2, F_{j \neq k}\}$. Then, $\bar{b}' \geq \bar{b}$. For both $i \in \{1, 2\}$ it holds that G'_i is FOSD over G_i and for every w, $U'_i(w) \leq U_i(w)$. Both bidders can be strictly worse off under F'_k than under F_k.*

If, ceteris paribus, a bidder enjoys a stronger budget distribution, the bidding interval is stretched toward a higher supremum bid. The winning probability of any bid is lower for both bidders, and both bidders are worse off. The inequalities can be strict for both bidders, such that both bidders are strictly harmed by an increase in one's budget distribution at every budget level.[21] Overall, a more slack budget constraint in the sense of RHR-dominance might be strictly worse for a bidder: although the budget distribution places more mass on higher budget realizations (and, thus, possibly higher surplus), this effect might not fully compensate his higher bids and his lower winning probability at every budget level.

The private binding budget constraints shield bidders from overbidding each other as in Bertrand competition, until at least one surplus is zero (Blume, 2003). A less restrictive budget distribution strengthens a bidder's competitive position in comparison to his opponent, who reacts by bidding higher. A less restrictive budget distribution also brings a bidder closer to the unrestricted Bertrand game with zero payoff, as it relaxes his ability to pay up to his valuation.

4.4 Efficiency

If $v_1 \neq v_2$, is the winner of the FPA the bidder with the highest value for the object? It is straightforward to see that this is not the case: a bidder with the lowest budget $\underline{w}$ always loses irrespective of his valuation.

A weaker requirement on efficiency is the following: does a bidder i who has a higher valuation $v_i > v_j$ *and* a higher budget realization $w_i > w_j$ win? In the following, I show that this weaker statement is also not true in general, but can hold under additional assumptions on the budget distributions F_1 and F_2.

For example, let $m_1 = \overline{w}$. Then, by the weakly monotonic strategies in Theorem 3 bidders bid their entire budget $b_i(w) = w$. If a bidder with the highest value has a strictly higher budget than his opponent, he wins with probability one.

Next, let $m_1 < \overline{w}$ and consider bidders with equal budget distribution $F := F_1 = F_2$ who play the pure strategy weakly monotonic equilibrium with the bidding functions in Section 3.4. In Section 4.2, I established that a common budget distribution F and $v_i \geq v_j$ translates

[21] This requires a stronger version of RHR-dominance. See the proof in the appendix for further details.

into $i = 2$ (i.e., $v_2 \geq v_1$) and Case (C1). Proposition 2 establishes that $b_2(w) \geq b_1(w)$. Note that in Case (C1) only bidder 1 can place a mass point, while bidder 2 follows a strictly increasing pure strategy. Hence, for $w_2 > w_1$, it holds that $b_1(w_1) \leq b_2(w_1) < b_2(w_2)$: bidder 2 with the higher valuation wins the auction if he has a higher budget than his opponent.

The finding that the highest valuation bidder wins if he has a higher budget cannot be extended for arbitrary distributions. For example, let $v_1 = 1.2$, $v_2 = 1$ and the budget distributions stem from Example 1. A quick computation reveals that Case (C1) holds, as $m_1 = 3/5$, $m_2 = 2/3$, $\underline{U}_1(m_1) = 9/25$ and $\underline{U}_2(m_2) = 4/27$. Then, bidder 1 with a budget in $[0.6, \sqrt{0.4}]$ bids at the mass point on 0.6 and loses against the lower value opponent who has a budget above m_1. Hence, although $v_1 > v_2$ and $w_1 > w_2$, bidder 1 loses with a probability of one for all $w_1 \in (0.6, \sqrt{0.4})$ if $w_2 > m_1$. The stronger bidder bids less aggressively and admits a mass point. This is particularly inefficient if the stronger bidder has a lower valuation.

5 Extensions

5.1 Revenue Comparison

Revenue equivalence between standard auctions does not hold when bidders are budget constrained, as noted by Che and Gale (1996, 1998, 2006).[22] Consider a framework where values are common knowledge and identical, i.e., $v_1 = v_2$, and budgets are drawn from an identical distribution, $F(w) := F_1(w) = F_2(w)$. Che and Gale (1996) proved that in this framework, the FPA dominates the second-price auction (SPA) with regards to revenue.

I show that this revenue ranking does not hold under asymmetric budget distributions $F_1 \neq F_2$: the SPA can yield strictly higher revenue than the FPA for sufficiently asymmetric bidders. First, consider bidding strategies in a SPA without reservation values.

Proposition 5. *Let $v := v_1 = v_2$ be public information, and budgets distributions governed by the distribution functions $F_1(w)$ and $F_2(w)$. In a SPA without a reservation price, it is a weakly dominant strategy to bid $b_i(w) = \min\{v, w\}$, $\forall i \in \{1, 2\}, \forall w \in [\underline{w}, \overline{w}]$.*

Proof. Consider bidder i, who has a budget w and whose opponent bids some b_j. Let $v \leq w$. Then, the classical argument of the SPA applies: bidding less ($b_i < v$) potentially loses the auction and forgoes a positive payoff, and changes nothing in case of a win. Bidding higher ($b_i > v$) only changes the outcome if it results in purchasing the object for more than v,

[22] These three papers show for different value-budget type spaces that the FPA dominates the SPA in terms of expected revenue.

and, thus, a negative payoff. Let $w < v$. Then, bidding higher than w is infeasible. The only possible deviation is downward, $b_i < w$. However, this is not profitable, because it only changes the outcome if $w > b_j \geq b_i$. In this case bidder i loses, while bidding $b_i = w$ would have resulted in strictly positive payoff $v - b_j$. $\square$

Whenever both bidders have a budget above v, the auctioneer gets a payment of the full object value v. Whenever at least one bidder has a budget below the object value, the payoff of the seller is the lowest of the two budgets.

Let $x := \min\{v, \overline{w}\}$ be the highest possible bid under any budget realization. The bidding strategies in Proposition 5 result in the following expected revenue for the designer in the SPA, Π^{SPA}, where the last equality follows by applying integration by parts.

$$\begin{aligned}\Pi^{SPA} &= \int_{\underline{w}}^{x} w \left(f_2(w)(1 - F_1(w)) + f_1(w)(1 - F_2(w))\right) dw + x(1 - F_1(x))(1 - F_2(x)) \\ &= \underline{w} + \int_{\underline{w}}^{x} (1 - F_1(w))(1 - F_2(w)) dw. \end{aligned} \tag{14}$$

Now, consider the revenue in a FPA. The bidders share the same valuation v, the auctioneer's valuation is zero, and the object is always sold. Hence, the auction outcome is efficient and the total generated surplus is v. The revenue of the seller is the object value v minus the expected utilities of the bidders. That is,

$$\Pi^{FPA} = v - \int_{\underline{w}}^{\overline{w}} U_1(w) f_1(w) dw - \int_{\underline{w}}^{\overline{w}} U_2(w) f_2(w) dw. \tag{15}$$

The next proposition shows that the revenue ranking $\Pi^{FPA} \geq \Pi^{SPA}$ of the symmetric framework in Che and Gale (1996) does not extend to asymmetric budget distributions.

Proposition 6. *Let $v := v_1 = v_2$, and budgets be drawn with log-concave distribution functions $F_1(w)$ and $F_2(w)$. Then, the SPA can yield a strictly higher revenue than the FPA.*

Proof. Let $w \in [0, 1]$, $F_1(w) = w^9$, $F_2(w) = w^{\frac{1}{9}}$, and both bidders have valuation $v = 0.2$. Then, $m_1 = \frac{1}{50}$, and $m_2 = \frac{9}{50}$. Plugging this into Equation 14 yields an expected revenue in the SPA of

$$\Pi^{SPA} \approx 0.05.$$

Next, consider the FPA. The ex-ante utilities of the bidders can be computed from the equilibrium utilities for Case (C1) in Equations 12 and 13.

$$EU_1 = \int_{\underline{w}}^{m_1} (v - w)F_2(w)f_1(w)dw + (v - m_1)F_2(m_1)(1 - F_1(m_1)) \approx 0.117,$$
$$EU_2 = \int_{\underline{w}}^{m_1} (v - w)F_1(w)f_2(w)dw + (v - m_1)F_2(m_1)(1 - F_2(m_1)) \approx 0.041$$

Plugging this into Equation 15 yield $\Pi^{FPA} \approx 0.042$. Thus, unlike in the symmetric setup with identical budget distributions, the SPA can yield a strictly higher revenue than the FPA, $\Pi^{SPA} > \Pi^{FPA}$. □

In the literature on standard auctions without budget constraints, asymmetrically distributed valuations break revenue equivalence between standard auctions (Maskin and Riley, 2000). A revenue ranking between standard auctions remains a subject of research, as no general revenue ranking can be established. For some particular distributions, revenue in a FPA is higher than in a SPA (see, e.g., Maskin and Riley, 2000). This ranking does not always hold, as Gavious and Minchuk (2014) show that revenue in a SPA can be higher than that in a FPA under asymmetry.

With asymmetric budget constraints and common valuations, I showed that the revenue ranking $\Pi^{FPA} \geq \Pi^{SPA}$ no longer holds. It remains an open question as to under which conditions the FPA yields a higher revenue than the SPA in a framework with asymmetric budget constraints. Yet, finding a revenue ranking in this framework for particular asymmetric budget constraints might turn out more practical than for asymmetric valuations, as this paper provides a closed form expression for revenue and bidding behavior.

5.2 Information Disclosure

In the following, the auctioneer has the choice whether to disclose the identities of the bidders, e.g., by publishing a participation register. For this section, I assume that both bidders value the object equally, $v \coloneqq v_1 = v_2$.

Giving up the anonymity of the bidders is a relevant strategic decision for the designer. If the auctioneer discloses nothing, bidders are ex-ante symmetric in the sense that their distribution is drawn from the same prior distribution. If the auctioneer publishes a public participation register, bidders can look up annual budget reports and make inferences about the budget distributions of the opponents. I show that with ex-ante symmetric bidders, the auctioneer can never gain by disclosing noisy information about the budgets.

Let S be the finite set of budget type distributions, with each $s \in S$ corresponding to a log-concave budget distribution function $F_s(w)$ on equal support $[\underline{w}, \overline{w}]$. The term *type* in this section refers to the type s of the budget distribution function, not the budget realization w. The budget distribution types s_1 and s_2 of bidders 1 and 2 are drawn independently and identically, with a probability $p_s > 0$ for type $s \in S$, with $\sum_{s \in S} p_s = 1$. Let the expected budget distribution $F(w) \coloneqq \sum_{s \in S} p_s F_s(w)$ also satisfy log-concavity.

Before the start of the auction, the auctioneer commits whether she wants to publish a participation register. Then, bidders arrive and budget types $F_i \in \{F_s\}_{s \in S}$ are drawn for $i = \{1, 2\}$. Bidders know their own type, but not the type of their opponent. The auctioneer observes both types and publicly announces the types, if she committed to do so. Then, budgets are drawn and observed only by the respective bidder. Finally, a FPA takes place.

Proposition 7. *Revenue is weakly decreasing, if the auctioneer discloses budget-type information about ex-ante symmetric bidders.*

The total surplus generated by the auction equals v, and consists of the auctioneer's revenue and the bidders' expected utilities. Hence, a higher expected utility for the bidders corresponds to a lower payoff for the auctioneer. Under no disclosure (of budget types s_1 and s_2), the bidders have identical expectations about their opponent's budget distribution. In this symmetric case, the lower bound on the equilibrium utility binds for every budget w.[23] Under disclosure of budget types, the lower bound utility is weakly higher, as bidders can condition their best response bid upon their opponent's budget type. Furthermore, as I showed in Theorem 1, under asymmetry, a bidder can achieve an equilibrium utility strictly above his lower bound utility. Thus, under information disclosure, bidders are better off than under no disclosure, which leaves a smaller share of the total surplus for the auctioneer.

In many auction houses, such as Sotheby's, bidding is anonymous: bidders take part in an auction, before knowing who their opponents will be. Moreover, during the auction, bidders remain anonymous by placing bids via phone or by raising one's auction paddle. For narrow markets such as the telecommunications sector, while usually participants are announced before the start of the auction, this in fact might not constitute a strategic decision of the auction designer, but rather a peculiarity of the respective market: anonymity might not be implementable in such a narrow market with few constantly interacting participants.

In this section, I analyzed a very specific information disclosure rule: the auctioneer has the choice as to whether she wants bidders to remain symmetric, or reveal noisy information

[23] Che and Gale (1996) showed this for a symmetric monotonic equilibrium. In Corollary 2, I established that the lower bound binds in any equilibrium.

about the budget distributions. However, this noisy information is exogenously given and the auctioneer cannot modify its precision or send private and potentially correlated signals. Future research could endogenize the information structure further by allowing the auctioneer to design the signal precision, as in Bergemann and Pesendorfer (2007), in line with the expanding literature on Bayesian persuasion (see, e.g., Kamenica and Gentzkow, 2011). Proposition 7 still holds, if the type space S is not given, but designed by the auctioneer. However, any type in S has to satisfy log-concavity and have a strictly positive density on the same support as all other types: $[\underline{w}, \overline{w}]$. Enabling the designer to create types with different support (e.g., by allowing a monotone partition[24] into a low-budget and a high-budget interval) might yield further insights about the optimal information disclosure policy.

5.3 All-Pay Auction

In this section, I apply my results to the all-pay auction. Similar to the FPA, the lower bound of bidder i with budget w, who faces a naive j-opponent bidding his entire budget, is

$$\underline{U}_i^a(w) \coloneqq \max_{0 \leq b_i \leq w} v_i F_j(b_i) - b_i.$$

For the FPA, Assumption 1 is sufficient to guarantee that the lower bound utility $\underline{U}_i$ is strictly increasing in a bid below some $m_i > \underline{w}$, and constant thereafter. For the all-pay auction, Assumption 1 is not sufficient to guarantee these properties of the lower bound utility $\underline{U}_i^a$. For example, with $v \leq 1$ and $F_2(w) = w$ for $w \in [0, 1]$, the lower bound utility of bidder 1 is $\underline{U}_1^a(w) = 0$ for all w. To employ similar tools to the ones developed in previous sections, I impose the following assumption.

Assumption 2. *It holds that $\underline{w} = 0$. For any $i \in \{1, 2\}$, $v_i F_j(b) - b$ has a unique global maximum at $m_i^a > \underline{w}$, and is strictly increasing in b below m_i^a.*

For example, $v_i \geq 1$ and $F_i(b)$ for $i = 1, 2$ strictly concave satisfies Assumption 2. The following result sums up the equilibrium bid distribution in an all-pay auction.

Theorem 4. *Let Assumption 2 hold. The all-pay auction has an equilibrium.*

In any equilibrium, the supremum bid for both bidders is

$$\overline{b}^a = \begin{cases} v_1(1 - F_2(m_1^a)) + m_1^a & \text{if} \quad \underline{U}_1^a(m_1^a) - \underline{U}_2^a(m_2^a) \geq v_1 - v_2, \\ v_2(1 - F_1(m_2^a)) + m_2^a & \text{if} \quad \underline{U}_1^a(m_1^a) - \underline{U}_2^a(m_2^a) < v_1 - v_2. \end{cases} \tag{16}$$

[24]See, e.g., Bergemann and Pesendorfer (2007) for disclosing information about valuations, not budgets, in auctions, where monotone partitions arise as part of the optimal disclosure policy.

In any equilibrium, equilibrium utilities are unique and the bid distributions satisfy

$$G_1^a(b) = \begin{cases} F_1(b) & \textit{if } b \in [0, m_1^a), \\ \frac{v_2 - \bar{b}^a + b}{v_2} & \textit{if } b \in [m_1^a, \bar{b}^a], \end{cases} \qquad G_2^a(b) = \begin{cases} F_2(b) & \textit{if } b \in [0, m_1^a), \\ F_2(m_1^a) + \frac{b - m_1^a}{v_1} & \textit{if } b \in [m_1^a, m_2^a), \\ \frac{v_1 - \bar{b}^a + b}{v_1} & \textit{if } b \in [m_2^a, \bar{b}^a]. \end{cases} \tag{17}$$

Note that bidder 1 bids with a uniform distribution above m_1^a, and bidder 2 also bids uniformly on (m_1^a, m_2^a) and $(m_2^a, \bar{b}^a]$. To make the opponent indifferent between any bid in some interval in an all-pay auction, uniform bidding is required.[25] Similar to the FPA, equilibrium utilities in the all-pay auction are unique in any equilibrium. They are given also by Theorem 1, after substituting $\underline{U}_i^a$ for $\underline{U}_i$, and m_i^a for m_i. In the proof in the Appendix, I construct weakly monotonic pure strategies in Equations 34 and 35 that establish the existence of an equilibrium.

The uniqueness result of equilibrium utility and bid distributions in the all-pay auction holds if the lower bound is strictly increasing for sufficiently low budget levels, and constant thereafter. In the FPA, the log-concavity of the budget distributions is sufficient to guarantee that the lower bound utility has these properties. In the all-pay auction, the log-concavity of the budget distributions does not imply this shape of the lower bound, and imposing Assumption 2 is with loss of generality.[26]

6 Concluding Remarks

In this work, I have derived equilibrium utilities and bid distributions for two bidders with asymmetric budget distributions, who compete for one object in a FPA. I allow for any asymmetry in the budget distributions, as long as they satisfy log-concavity and common full support.

Che and Gale (1996) showed that in a symmetric equilibrium with identically distributed budgets, the equilibrium utilities of the bidders equal a lower bound on utility. I have

[25] This has been noted in the context of all-pay auctions with complete information and no budget constraints. See, e.g., Amann and Leininger (1996); Baye et al. (1996).

[26] For arbitrary budget distributions F_1 and F_2, $\underline{U}_i^a$ can be zero for every budget level or have multiple flat intervals. Hence, an equilibrium might involve multiple mass points where the lower bound alternates between flat intervals and increasing intervals. While some results and methods from this paper are applicable to such a scenario, a closed-form characterization and the uniqueness of an equilibrium utility remains an open questions.

extended the framework of Che and Gale (1996) in two directions: I allow for different valuations for the object, and introduce asymmetric budget constraints. In this framework, the lower bound does not necessarily bind. However, the equilibrium utilities in a FPA can still be recovered from the lower bound. I also applied this technique to the all-pay auction to derive the unique equilibrium shape and bid distributions.

My approach unravels the equilibrium via eliminating candidate equilibria shapes. As long as both RHR are sufficiently high, equilibrium strategies are invariant to the degree of asymmetry and the lower bounds bind. If a RHR drops below a threshold, mass points can arise and bidders achieve a utility strictly above their lower bound. I show that there remains only one potential shape for the equilibrium utility and bid distributions after this elimination, and construct weakly monotonic bidding strategies to establish existence.

There exists a parallel between my framework and bidders with asymmetrically distributed valuations with no budget constraints. If bidders are asymmetric in the valuation dimension and have no budget constraints, bidding behavior in a FPA beyond a differential equation is complicated to derive without additional assumptions[27] (see, e.g., Maskin and Riley, 2000). If bidders have asymmetric budget distributions and common knowledge valuations, the auction game is solvable in closed-form under weak assumptions (log-concavity and full common support) as my results show. I impose no stochastic order or particular distribution on the budget distributions. Maskin and Riley (2000) show that a weaker bidder bids more aggressively under the additional assumption of RHR-dominance on valuations if there are no budget constraints. In my model, under RHR-dominance on budget distributions, the weaker bidder with respect to the budget also bids more aggressively than his stronger opponent.

Mass points can be part of an equilibrium because budget constraints are hard, and bidders cannot outbid their budget. Due to the tie-breaking rule, bidding below a mass point of the opponent yields a strictly lower utility than bidding exactly at a mass point. Furthermore, bidding at a mass point of the opponent yields strictly lower utility than bidding above a mass point. The incentives to increase one's budget are particularly strong around mass points. For example, if a bidder with a budget slightly below a mass point could borrow to increase his budget, he could derive a discrete jump in surplus by bidding at the mass point. This might influence the initial budget distribution if the budget is determined endogenously before the start of the auction.[28] Finding an equilibrium with asymmetric

[27]For example, RHR-dominance or uniform or power law value distributions.

[28]I am grateful to an anonymous referee for pointing this out.

budget distributions and allowing bidders to borrow (see, e.g., Zheng, 2001, for soft budgets with a borrowing market) might be an interesting question for future research.

A Appendix

Proof of Lemma 1. Let $m_i \in \arg\max(v_i - b)F_j(b)$ be a best reply bid of an unconstrained bidder who faces a naive opponent. A bid at or below the lowest budget $\underline{w}$ never wins and yields utility zero. A bid $b > \underline{w}$ such that $b < v_i$ yields a strictly positive expected payoff. Hence, $m_i > \underline{w}$.

The derivative of the expected payoff with respect to b is $(v_i - b_i)f_j(b) - F_j(b)$. This is positive whenever the following inequality (Inequality 5) is satisfied.

$$\frac{f_j(b)}{F_j(b)} \geq \frac{1}{v_i - b}.$$

The RHR on the left-hand side is decreasing in b by Assumption 1; the right-hand side is strictly increasing in b for $b < v_i$. If there exists an m_i such that $\frac{f_j(m_i)}{F_j(m_i)} = \frac{1}{v_i - m_i}$, it is unique. Then, for all $w < m_i$, the RHR inequality is strict and the lower bound is strictly increasing. For $w > m_i$, the inequality does not hold and the lower bound is constant at $(v_i - m_i)F_j(m_i)$. Any bid $b > m_i$ yields a strictly lower expected utility than bidding m_i.

The above inequality might also hold strictly for all w. In this case, $m_i = \overline{w}$ as the opponent never bids above $\overline{w}$. The lower bound strictly increases over the entire domain. $\square$

Proof of Lemma 2. By contradiction, let bidder i with a budget $\tilde{w} \in (w', w'')$ place a bid $\tilde{b} < \tilde{w}$ below his budget. The expected utility of budget level $\tilde{w}$ is then

$$U_i(\tilde{w}) = (v_i - \tilde{b}) \Pr(b_j < \tilde{b}) + \frac{1}{2}(v_i - \tilde{b}) \Pr(b_j = \tilde{b}). \tag{18}$$

Take a bidder i with a lower budget $(\tilde{w} - \epsilon) \in (w', w'')$ who can afford the bid $\tilde{b}$, i.e., choose $\epsilon > 0$ such that $\tilde{b} \leq \tilde{w} - \epsilon$. Such a bidder can mimic the $\tilde{w}$-budget type: bid $\tilde{b}$ and obtain the same utility as in Equation 18. This contradicts $U_i(\tilde{w}) > U_i(\tilde{w} - \epsilon)$ if U_i is strictly increasing on (w', w''). Hence, $b_i(w) = w$ for $w \in (w', w'')$.

A bidder i with a budget $w \leq w'$ cannot afford a bid above w'; bidder i with a budget above w'' bids at least w'' as any lower bid yields lower payoff as U_i is strictly increasing on (w', w''). Hence, $G_i(w) = F_i(w)$ for $w \in (w', w'')$. $\square$

Proof of Lemma 3. By contradiction, let U_i strictly increase on (w', w'') and there exists a budget level $w' \in (w', w'')$ such that $U_i(w') > \underline{U}_i(w')$.

As U_i is strictly increasing, by Lemma 2 this implies that bidder i with budget realization in (w', w'') bids his entire budget $b_i(w) = w$ and

$$G_i(w) = F_i(w) \text{ for all } w \in (w', w''). \tag{19}$$

Next, I show that the bidding distribution function G_j of bidder j does not place sufficient probability on w' to elevate bidder i's utility strictly above the lower bound $\underline{U}_i(w')$.

Case 1: $m_j < w''$. A bid in $\tilde{b}$ in the interval (m_j, w'') yields utility $(v_j - \tilde{b})F_i(\tilde{b})$ to bidder j, due to the observation in Equation 19. In this interval, it is equivalent to bidding against a naive opponent. Bidding $\tilde{b}$ yields a utility strictly below the lower bound, as bidder j could be strictly better off by bidding m_j instead of $\tilde{b}$:

$$(v_j - \tilde{b})F_i(\tilde{b}) < (v_j - m_j)F_i(m_j) = \underline{U}_j(w \geq m_j).$$

Hence, bidder j never bids within the interval (m_j, w''). However, in this case, U_i cannot be strictly increasing on $w \in (m_j, w'')$, leading to a contradiction.[29]

Case 2: $m_j \geq w''$. The utility of bidder j is strictly increasing in a bid on this interval, as it coincides with the lower bound $\underline{U}_j(w) = (v_j - w)F_i(w)$, which is strictly increasing below m_j. Hence, if bidding in this interval, bidder j would choose the highest feasible bid for him, $b_j(w) = w$.

Either bidder j with budget $w \in (w', w'')$ bids $b_j(w) = w$, or he places another feasible bid $b' \leq w'$ if it yields a strictly higher utility than the lower bound utility at w. However, if such a b' exists, bidder j with a budget in (w', w) never places a bid in this interval. This contradicts that U_i is strictly increasing on $(w'a, w)$. If such a b' does not exist, it holds that $G_j(w) = F_j(w)$ for all $w \in (w', w'')$. However, this is not sufficient to elevate the utility of bidder i with budget w' above the lower bound:

$$\begin{aligned} U_i(w') &= (v_i - w') \Pr(b_j < w') + \frac{1}{2}(v_i - w') \Pr(b_j = w') \\ &= (v_i - w')F_j(w') = \underline{U}_i(w'). \end{aligned}$$

which yields a contradiction. $\square$

[29]Slightly decreasing bidder i's bid yields the same positive winning probability for a lower payment.

Proof of Lemma 4. First, I establish that for the lowest possible budget $\underline{w}$ it always holds that the lower bound binds: $\underline{U}_i(\underline{w}) = U_i(\underline{w})$. Assume by contradiction that for some $i \in \{1, 2\}$, it holds that $\underline{U}_i(\underline{w}) = 0 < U_i(\underline{w})$. Because bidder i with any budget realization has a strictly positive utility,[30] any bid wins with a strictly positive probability. Let a bid $\underline{b}_i$ be the infimum bid in the entire bidding support of the i-bidder with any budget realization. Note that $\underline{b}_i \leq \underline{w}$. For $\underline{b}_i$ to win with positive probability, this requires bidder j to (i) place a mass point at $\underline{b}_i$, or (ii) bid with a strictly positive probability below $\underline{b}_i$. The latter scenario (ii) is impossible in equilibrium, as it requires some bidder j with a budget above $\underline{w}$ to bid below $\underline{b}_i$. Each such bid of j strictly below $\underline{b}_i$ results in an expected utility of zero (it never wins), which is below the lower bound $\underline{U}_j$ and, thus, impossible. The former scenario (i) (a mass point at $\underline{b}_i$) cannot be part of an equilibrium, because bidder i would always want to slightly outbid the mass point if feasible to gain a discrete jump in winning probability. This implies that $F_i(\underline{b}_i) = G_i(\underline{b}_i) = 0$. The positive mass of the j-bidders who bid at the mass point derive zero utility, which is strictly below the lower bound $\underline{U}_j$ for some of them. This yields a contradiction, and hence, $\underline{U}_i(\underline{w}) = U_i(\underline{w}) = 0$.

Next, I prove the theorem by contradiction for $w \in (\underline{w}, m_1)$. Assume $U_i(w) > \underline{U}_i(w)$ for some $w \in (\underline{w}, m_1)$. Both $\underline{U}_i(w)$ for $i \in \{1, 2\}$ are strictly increasing on $[\underline{w}, m_1)$. Let $x > w$ be the budget within $(\underline{w}, m_1)$, for which the strictly monotonic lower bound $\underline{U}_i(.)$ catches up and reaches the same value, i.e., $U_i(w) = \underline{U}_i(x)$. If such x does not exists, take m_1. That is, equilibrium utility is strictly above the lower bound $U_i > \underline{U}_i$, on at least the non-empty interval $[w, x)$.

As U_i is a monotonic function, it can have only countable jump discontinuities on (w, x). Hence, U_i has to be either (i) continuous and strictly increasing, or (ii) constant on some subinterval within $[w, x)$. A strict increase in (i) is ruled out by Lemma 3: U_i cannot be both strictly increasing and strictly above the lower bound $\underline{U}_i$. I show in the following that the latter yields a contradiction as well, if both $\underline{U}_i(w)$ are strictly increasing.

Let $U_i(w) = U > \underline{U}_i(w)$ be constant on some intervals within (w, x). Define $z_i = inf\{w : U_i(w) = U\}$ as the lowest budget, above which bidder i achieves a payoff equal to U. First, let $z_i = \underline{w}$. This is ruled out by the first paragraph of this proof that established that $U_i(\underline{w}) = \underline{U}_i(\underline{w}) = 0$.

Second, let $z_i > \underline{w}$. U_i is a monotonic function, and a continuous strict increase on an open interval is ruled out by Lemma 3. Hence, bidder j has a mass point at z_i, as by Lemma 3 any increase above the lower bound in the interior is due to a mass point. This implies that

[30] This is because $U_i(w) \geq U_i(\underline{w}) > 0$ for all w, where the second inequality is assumed by contradiction.

z_i is indeed the infimum, not the minimum. Bidder i-types with a budget above z_i always bid above z_i to extract the additional winning probability from avoiding the tie-breaking rule; therefore, $F_i(z_i) = G_i(z_i)$. However, this yields a contradiction for the utility of bidder j: a mass point of j at z_i implies that there is a continuum of bidder j with a budget above z_i who can at most achieve $(v_j - z_i)F_j(z_i) = \underline{U}_j(z_i)$; otherwise they would have a profitable deviation by bidding above z_i instead of sticking to the mass point. Bidding z_i yields an expected payoff of $(v_j - z_i)G_i(z_i) = (v_j - z_i)F_i(z_i)$, which equals the lower bound $\underline{U}_j(z_i)$. Yet, the lower bound is strictly increasing around z_i: if bidder j-types had a constant utility to establish the mass point for higher budget realizations than z_i, their utility would fall strictly below the lower bound on utility, which is impossible.

The previous argument established that $U_i(w) = \underline{U}_i(w)$ below m_1. The second part of the Lemma follows from Lemma 3: as $\underline{U}_i(w)$ is strictly increasing below m_1, all bidders bid their entire budget on $(\underline{w}, m_1)$. □

Proof of Lemma 5. First, I introduce two auxiliary results in Lemma A.1 and Lemma A.2. Let $\bar{b}_i$ be the supremum.

Lemma A.1. *Let $m_1 < \overline{w}$. Then, the following holds in any equilibrium:*

1. *the supremum bids of the bidders coincide, $\bar{b} := \bar{b}_1 = \bar{b}_2$,*
2. *there is no mass point at $\bar{b}$,*
3. *$m_1 < \bar{b} < min(v_1, v_2)$.*

Proof. For the first statement of the lemma, assume by contradiction that $\bar{b}_1 < \bar{b}_2$. Any bid of bidder 2, denoted b_2, in the interval $(\bar{b}_1, \bar{b}_2]$ wins with a probability of one and yields a utility of $(v_2 - b_2)$. For any bid in this open interval, there exists a profitable deviation to shade the bid down by some $\epsilon > 0$ small enough such that $b_2 - \epsilon > \bar{b}_1$. This deviation still wins the auction with certainty, however, for a strictly lower payment.

Let by contradiction $\bar{b} \geq min(v_1, v_2)$. If a bidder bids his full valuation v_i or above, this yields a utility of 0 or below. However, as $v_i > \underline{w}$, the lower bound utility of a bidder with some budget $w' \geq v_i$ is strictly positive, which yields a contradiction. Hence, $\bar{b} < min(v_1, v_2)$.

Next, consider the second statement of the lemma. A mass point at $\bar{b} = \overline{w}$ is infeasible, as only bidders with budget realization $\overline{w}$ (which is a zero probability event) can afford the highest bid. Now, assume $\bar{b} < \overline{w}$. Let bidder i have a mass point at $\bar{b} < min(v_1, v_2)$. Then, bidder j with a budget above $\bar{b}$ has a profitable deviation: slightly outbidding his supremum bid $\bar{b}$. Bidding $\bar{b} + \epsilon$ for some $\epsilon > 0$ sufficiently small yields a jump in winning probability, by avoiding the tie-breaking at $\bar{b}$ for an infinitesimally lower payment.

Finally, I show that $\bar{b} > m_1$. By Lemma 4, it holds that $\bar{b} \geq m_1$ and $F_i(w) = G_i(w)$ for $w < m_1$. Let by contradiction $\bar{b} = m_1$. Since $m_1 < \overline{w}$ holds by assumption, there is a positive mass of bidders with budget in $[m_1, \overline{w}]$ who bid at m_1. This contradicts that there is no mass point at $\bar{b}$. $\square$

The next result shows that every non-empty open interval within $[m_1, \bar{b}]$ contains strictly positive bidding probability mass.

Lemma A.2. *For any pair $x, y \in [m_1, \bar{b}]$ with $x < y$, it holds that $G_i(x) < G_i(y)$ for any i.*

Proof. Assume by contradiction that there exist $x < y$ in $[m_1, \bar{b}]$, such that $G_i' := G_i(x) = G_i(y)$.[31] Let $\alpha := \inf\{w : G_i(w) = G_i'\}$ and $\beta := \sup\{w : G_i(w) = G_i'\}$. By Lemma 4, $\alpha \geq m_1$, and $\beta < \bar{b}$.

Opponent $j \neq i$ also places zero bidding mass on (α, β), as lowering j's bid in this range yields a strictly lower payment for the same winning probability. Let $G_j' := G_j(b \in (\alpha, \beta))$.

There cannot be a mass point at β. If both bidders place a mass point at β, both have a strictly profitable deviation to slightly outbid the mass point. If only one bidder places a mass point at β, he has a strictly profitable deviation to decrease his bid; without an atom of his opponent at β, a decrease in the bid slightly below β implies a strictly lower payment for the same winning probability. Hence, $\Pr(b_i = \beta) = \Pr(b_j = \beta) = 0$. Furthermore, by the properties of a cumulative distribution function, it holds that $\lim_{b \searrow \beta} \Pr(b_j = b) = 0$.

Bidder i's payoff of bidding $b \searrow \beta$ is $\lim_{b \searrow \beta} \left[(v_i - b)\Pr(b_j < b) + \frac{1}{2}(v_i - b)\Pr(b_j = b)\right]$ $= (v_i - \beta)G_j'$. A bid of α yields $(v_i - \alpha)G_j'$ (if there is no mass point at a), which is strictly higher. Hence, a positive bidder mass bidding sufficiently close to β could do strictly better by shading their bid to α, and pay strictly less for a negligible loss in winning probability. As a downward deviation is feasible with budget constraints, this is a contradiction. $\square$

Assume by contradiction that there exist w' and w'' with $w' < w''$ such that $U_i(w') < U_i(w'')$ in one of the three intervals in Statement 2 in Lemma 5. U_i is a monotonic function. It is only possible to have $U_i(w') < U_i(w'')$ if at least one of the following statements hold:

(a) U_i increases continuously and strictly on some open interval in $[w', w'']$.

(b) U_i has a jump discontinuity in $[w', w'']$.[32]

[31] As G_i is a cumulative distribution function, $G_i(x) > G_i(y)$ is impossible.
[32] All discontinuities of U_i are jump discontinuities, as U_i is monotonic.

First, I show that (a) cannot hold in any equilibrium. Let U_i be strictly and continuously increasing on some $(a, b) \subseteq (w', w'')$.

Let $i = 1$. By Lemma 2, $U_1 = \underline{U}_1$ on (a, b). However, $\underline{U}_1$ is constant above m_1, which yields a contradiction. Next, let $i = 2$. If $(w', w'') \subset (m_2, \overline{w}]$, the same argument applies: $U_2 = \underline{U}_2$ on (a, b). However, $\underline{U}_2$ is constant above m_2, yielding a contradiction. If $(w', w'') \subset (m_1, m_2)$, by Lemma 2, bidder 2 bids with $G_2 = F_2$ on (a, b). This implies that bidder 1 would not want to place any bid in the interval (a, b): a bid x would yield utility $(v_1 - x)F_2(x)$, which is strictly smaller than the surplus from bidding $m_1 < a$, which is at least equal to $(v_1 - m_1)F_2(m_1)$. As bidder 1 never places a bid in (a, b), bidder 2 (who places a bid in this interval) has a strictly profitable deviation: decrease one's bid, win with the same (strictly positive) winning probability, but pay less in case of a win.

Second, I show that (b) is also not possible in equilibrium. Let the jump discontinuity occur at $x \in [w', w'']$. A jump discontinuity in U_i at x can only occur if G_j contains an atom at which a bidder i with budget x is bidding.[33] This follows from Lemma A.2, as without atoms the payoff of a bid $b \in [\underline{w}, \overline{b}]$ is $(v_i - b)G_j(b)$ (and also the maximum of this in Equation 18), which is a continuous function.

Then, a bid slightly above the atom yields a strictly higher utility to bidder i than bidding at the mass point. This is because by infinitesimally outbidding the mass point, bidder i gets a discrete jump in winning probability for a marginally higher payment, and the mass point lies strictly below $\min\{v_1, v_2\}$ by Lemma A.1. Any bidder i with a budget strictly above x would not bid at the mass point, but slightly above x or even higher. Therefore, $F_i(x) = G_i(x)$. However, bidder j then placing the mass point at x has a strictly profitable deviation: bidding m_j instead of the mass point and getting a payoff of at least $(v_j - m_j)F_i(m_i)$. This is strictly larger than the payoff from bidding at the mass point, $(v_j - x)F_i(x)$ for $x > m_j$.

As (a) or (b) lead to a contradiction, this establishes Statement 2 of the lemma. Furthermore, from the last paragraph, any discontinuity of U_i at $x > m_j$ yields a strictly profitable deviation for bidder j. Any discontinuity of U_i below m_j is ruled out by Lemma 4. This leaves one potential discontinuity at m_j, proving Statement 1 of the lemma.

A discontinuity in U_j at m_i is due to a mass point of bidder i at m_i, due to Lemma A.2. If bidder i places an atom at m_i, bidder j's utility has a jump discontinuity at m_i: bidder j has a strictly higher surplus from bidding slightly above the mass point than his surplus from

[33] If the mass point were below, bidder i with a lower budget could afford it and obtain the same utility as budget type x, which contradicts the existence of a jump discontinuity in U_i at budget realization x.

bidding exactly at the mass point, because he can avoid the tie-breaking rule. Therefore, bidder j prefers to bid above m_i whenever he can afford it, and $F_j(m_i) = G_j(m_i)$. Hence, bidder i's utility from bidding at the atom is $U_i(m_i) = (v_i - m_i)G_j(m_i) = \underline{U}_i(m_i)$. $\square$

Proof of Theorem 1. First, let $m_1 = \overline{w}$, and, thus, $m_2 = \overline{w}$. Then, Lemma 4 pins down the utilities for $w < \overline{w}$ and the supremum bid $\bar{b} = \overline{w}$. As by Lemma A.1 there is no mass point at $\bar{b}$, it also holds that $U_i(\overline{w}) = \underline{U}_i(\overline{w}) = (v_i - \bar{b})$.

Next, assume $m_1 < \overline{w}$ and let (m_1, m_2) be non-empty. Then, the equilibrium utility of bidder 2 is discontinuous at m_1 and requires a mass point of bidder 1. This is because by Lemma 5, $U_2(w)$ is constant for budget realizations $w \in (m_1, \overline{w}]$. Note that $U_2(w) \geq \underline{U}_2(m_2)$ for $w \in (m_1, \overline{w}]$, with $\underline{U}_2(m_2)$ being the highest value for the lower bound, which the equilibrium utility cannot undercut. Moreover, $\underline{U}_2(m_2) > \underline{U}_2(m_1)$ strictly, as the lower bound is strictly increasing below m_2 due to the log-concavity assumption. By Theorem 4, the lower bound binds, $\underline{U}_2(w) = U_2(w)$ for $w \in [\underline{w}, m_1)$. Approaching the utility of bidder 2 from both sides at m_1 shows the discontinuity and, therefore, a mass point of bidder 1: $\lim_{w \nearrow m_1} U_2(w) = \underline{U}_2(m_1) < \underline{U}_2(m_2) \leq \lim_{w \searrow m_1} U_2(w)$. Bidder 1 has to have a mass point at m_1 in his bid distribution function to enable this jump in the expected utility of bidder 2 to achieve a utility of at least the lower bound.

When does bidder 2 place a mass point at m_2? Both bidders share the same supremum bid. The payoff from bidding $\bar{b}$ is $v_i - \bar{b}$, as it wins with a probability of one, as there is no mass point at it.[34] As the payoff is non-decreasing in the bid, the highest budget type $\overline{w}$ has the same utility as bidding at the supremum. Hence, the difference in surplus of the highest budget types always satisfies

$$U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2.$$

If bidder 2 does not place a mass point, bidder 1's utility is constant at $U_1(\overline{w}) = \underline{U}_1(m_1)$ by Lemma 5. Furthermore, by Lemma 5, U_1 is constant on (m_1, m_2) and $(m_2, \overline{w}]$. It is equal to the lower bound on (m_1, m_2) as bidder 1 places a mass point at m_1. Thus, unless there is a jump discontinuity at m_2, bidder 1's utility is equal to his lower bound for all w.

Let (C1) holds, i.e., $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$. If bidder 2 were to place a mass point, it would hold that $U_1(\overline{w}) > \underline{U}_1(m_1)$ and $U_2(\overline{w}) = \underline{U}_2(m_2)$. However, then $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2 > \underline{U}_1(m_1) - \underline{U}_2(m_2)$, which is a contradiction.

[34]If no bidder bids at the supremum, approaching the supremum bid in a small enough neighbourhood yields the same argument.

As bidder 2 does not place a mass point, it holds that $U_1(\overline{w}) = \underline{U}_1(m_1)$ and $U_2 =: U_2(w > m_1)$ being constant. Thus,

$$U_1(\overline{w}) - U_2(\overline{w}) = \underline{U}_1(m_1) - U_2 = v_1 - v_2.$$

This pins down the utility level for bidder 2 with a budget above m_1. Due to the equal tie-breaking rule, bidder 2's utility at m_2 is exactly the average of the left and right hand side limit of the utility. We have $U_1(w) = \underline{U}_1(w)$ for all w, and the expression for U_2 in the theorem follows.

Let (C2) holds, i.e., $\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$. If bidder 2 does not place a mass point, it holds that $U_1(\overline{w}) = \underline{U}_1(m_1)$ and $U_2(\overline{w}) \geq \underline{U}_2(m_2)$. In this case, $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2 \leq \underline{U}_1(m_1) - \underline{U}_2(m_2)$, yielding again a contradiction. Hence, bidder 2 places a mass point at m_2. Note that U_2 is constant on (m_1, m_2) and $U_2(m_2) = \underline{U}_2(m_2)$. Furthermore, $U_2(\overline{w}) = \underline{U}_2(m_2)$ and the constant utility of bidder 1 has to satisfy $U_1(\overline{w}) - \underline{U}_2(m_2) = v_1 - v_2$. This pins down a unique equilibrium utility U_1 and U_2.

Finally, consider $m_1 = m_2 < \overline{w}$. By Footnote 8, bidders are labeled such that $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$. Let $\underline{U}_1(m_1) - \underline{U}_2(m_2) > v_1 - v_2$. Then, by the same arguments as above, bidder 1 places a mass point at m_1 which leads to a jump discontinuity in U_2 at m_1. This pins down the utility of bidder 1 to $U_2 = \underline{U}_2$ for all $w \geq m_1$, and raises the equilibrium utility of bidder 2 via a jump discontinuity to the required level such that $U_2(w > m_2) = \underline{U}_1(m_1) - (v_1 - v_2)$. Let $\underline{U}_1(m_1) - \underline{U}_2(m_2) = v_1 - v_2$. Then, no mass point can be sustained, as it would distort utilities away from their final levels.

Plugging in the expression for the lower bounds completes the proof of Theorem 1.

□

Proof of Theorem 2. By Lemma A.1, there is no mass point at the highest bid. For simplicity, let $\overline{b}$ also be the maximum bid.[35] A bid at the maximum $\overline{b}$ yields payoff $v_i - \overline{b}$, as it wins with certainty. As surplus has to be weakly increasing in the bid, the payoff from the highest bid corresponds to the payoff of the highest budget type, $U_i(\overline{w})$. In Case (C1), $U_1(\overline{w}) = \underline{U}_1(m_1) = (v_1 - m_1)F_2(m_1) = v_1 - \overline{b}$. In Case (C2), $U_2(\overline{w}) = \underline{U}_2(m_2) = (v_2 - m_2)F_1(m_2) = v_2 - \overline{b}$. Solving for $\overline{b}$ in both cases yields the expression for the supremum bid in the theorem.

Next, I derive the bid distributions G_1 and G_2. Let $m_1 = \overline{w}$. Then, the bid distributions

[35]If not, a similar argument holds by approaching the supremum bid from below, such that the probability of a win converges to one. For clarity of the exposition, I omit this case.

in the theorem follow from Lemma 4. For the following, assume that $m_1 < \overline{w}$. Below m_1, bid distributions coincide with the budget distributions $G_i = F_i$, by Lemma 4. By Lemma A.2 and 5, the bidding support coincides for both bidders and has no empty intervals in $(\underline{w}, \bar{b})$. The following identifies where mass points can occur.

Let bidder j place a mass point at $x \in (m_1, \bar{b})$. By Lemma A.2 there is a continuum of bidder i bidding in the left- or right-neighborhood of x. Due to a discrete jump in winning probability at and above x, these bidders i bidding below the mass point have a strictly lower utility than at the mass point, which in turn enjoy a lower utility than those bidding slightly above the mass point. To rule out profitable deviations, this is only possible if $U_i(w < x) < U_i(x) < U_i(w > x)$, as bidder i with a budget of exactly (above) x would not want to bid below (at) x.

Case (C1): Bidder 2 does not place any mass point, as $U_1(w \geq m_1) = \underline{U}_1(m_1)$. Without mass points of the opponent and with full bidding support below $\bar{b}$ (by Lemma A.2), the following needs to be satisfied. For all $b \in [m_1, \bar{b}]$, it holds that

$$(v_1 - b)G_2(b) = (v_1 - m_1)F_2(m_1).$$

Solving for $G_2(b)$ yields the bid distribution G_2 above m_1 in Theorem 2 that has to be satisfied in any equilibrium.[36]

Similarly, bidder 1 cannot place a mass point above m_1, as this would contradict that U_2 is constant above m_1 (which is required by Theorem 1). Any bid on $(m_1, \bar{b})$ yields the same utility

$$(v_2 - b)G_1(b) = (v_1 - m_1)F_2(m_1) - (v_1 - v_2).$$

Solving for $G_1(b)$ yields the result for Case (C1).

Case (C2): Bidder 1 has no mass point above m_1, as this would contradict $U_2(w > m_1) = \underline{U}_2(m_2)$. Solving the following expression for G_1 yields the result for bidder 1:

$$(v_2 - b)G_1(b) = (v_2 - m_2)F_1(m_2).$$

By the same argument, due to the constancy of the equilibrium utility U_1, bidder 2 cannot place a mass point on $[m_1, m_2)$ and $(m_2, \bar{b}]$. Bidder 1, with a budget strictly below m_2, bids with full support in $[m_1, m_2)$, and the bid distribution of their opponent yields the

[36] The bid distributions below m_1 are given by Lemma 4.

indifference condition for these bids

$$(v_1 - b)G_2(b) = (v_1 - m_1)F_2(m_1).$$

Bidder 1, with a budget above m_2, bids above m_2, and their indifference condition satisfies

$$(v_1 - b)G_2(b) = (v_2 - m_2)F_1(m_2) + (v_1 - v_2).$$

Solving both indifference conditions for $G_2(b)$ yields the bid distribution in Theorem 2. $\square$

Proof of Theorem 3. The proof is by construction. If (C1) holds, let the bidding functions be as in Equations 12 and 13. If (C2) holds, let the bidding functions of bidder 1 be Equation 12 and the bidding function of bidder 2 be

$$b_2(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ v_1 - \frac{(v_1-m_1)F_2(m_1)}{F_2(w)} & \text{if} \quad w \in [m_1, F_2^{-1}(\frac{(v_1-m_1)F_2(m_1)}{v_1-m_2})), \\ m_2 & \text{if} \quad w \in [F_2^{-1}(\frac{(v_1-m_1)F_2(m_1)}{v_1-m_2}), F_2^{-1}(\frac{v_1-\bar{b}}{v_1-m_2})], \\ v_1 - \frac{v_1-\bar{b}}{F_2(w)} & \text{otherwise.} \end{cases} \tag{20}$$

First, I show that these bidding functions aggregate into the *bid distributions* $\{G_i\}_{i=1,2}$ in Theorem 2. For $w < m_1$, $b_i(w) = w$ and hence $G_i(b) = F_i(b)$ is satisfied. Let $w \geq m_1$. For a bid in $[m_1, \bar{b}]$, let $w_i(b) = \sup\{w : b_i(w) = b\}$ be the highest budget realization of a bidder i who bids b. Note that $F_i(w_i(b)) = G_i(b)$. Consider bidder 2 in Equation 13 for (C1) with a budget $w \geq m_1$. His bidding function $b_2(w)$ can be rewritten using the inverse bidding function $w_2(b)$: $b = v_1 - \frac{v_1-\bar{b}}{G_2(b)}$. Solving this expression for $G_2(b)$ yields the bid distribution in Theorem 2. The same approach applied to bidder 1 in Equation 12 and to bidder 2 in Equation 20 for (C2) yields the required bid distributions.

Next, I show the *feasibility* of the bidding functions, i.e., for all w and i it holds that $b_i(w) \leq w$. For any bid equal to or below m_1, feasibility is trivially satisfied. It is left to show that 1. $v_1 - \frac{(v_1-m_1)F_2(m_1)}{F_2(w)} \leq w$; 2. $v_i - \frac{v_i-\bar{b}}{F_j(w)} \leq w$; 3. $F_2^{-1}(\frac{(v_1-m_1)F_2(m_1)}{v_1-m_2}) \geq m_2$. Rewrite Inequality 1 as $(v_1 - w)F_2(w) \leq (v_1 - m_1)F_2(m_1)$, which holds by Lemma 1. Rewrite Inequality 2 as $(v_i - w)F_j(w) \leq v_i - \bar{b}$. This is true since for $w \geq m_i$, it holds that $(v_i - w)F_j(w) \leq \underline{U}_i(\overline{w}) \leq U_i(\overline{w}) = v_i - \bar{b}$. For Inequality 3, applying F_2 to both sides yields $(v_1 - m_1)F_2(m_1) \geq (v_1 - m_2)F_2(m_2)$. This holds by Lemma 1, hence, establishing feasibility.

Finally, I show *optimality*. Let $w < m_1$. Any bid $b < w$ yields strictly lower utility than

$b_i(w) = w$.[37] Any higher bid $b > w$ is unfeasible. Hence, a bidder with a budget $w < m_1$ has no strictly profitable deviation. Let $w \geq m_1$ and (C1) hold. Any bid of bidder 1 in the interval $[m_1, \bar{b}]$ yields constant utility to bidder 1. It is straightforward to show that any bid above $\bar{b}$ or below m_1 yields a strictly lower utility, and there is no profitable deviation. Bidder 2 with budget m_1 has a higher utility from bidding exactly at m_1 due to a mass point of bidder 1 than from any lower bid, and cannot afford bidding higher than m_1. Bidder 2 with budget $w > m_1$ is indifferent between any bid on $(m_1, \bar{b}]$, and strictly loses from any deviation outside of this interval. Optimality in (C2) can be establishes by the same technique. $\square$

Proof of Proposition 3. Proof of 2. ⇒ 1. For $w < m_1$, by Lemma 3, it holds that $U_i(w) = \underline{U}_i(w) \geq \underline{U}_j(w) = U_j(w)$. For $w \geq m_1$ the following case distinction establishes the result, utilizing the equilibrium utility in Corollary 1.

Let Case (C1) in Corollary 1 hold. If $\underline{U}_1(w) \geq \underline{U}_2(w)$, and $v_1 \geq v_2$, then

$$U_2(w \leq \overline{w}) \leq U_2(\overline{w}) = \underbrace{(v_2 - v_1)}_{\geq 0} + (v_1 - m_1)F_2(m_1) \tag{21}$$

$$\leq (v_1 - m_1)F_2(m_1) = U_1(m_1) \leq U_1(w \geq m_1). \tag{22}$$

If $\underline{U}_2(w) \geq \underline{U}_1(w)$ and $v_2 \geq v_1$ in Case (C1), then

$$U_1(w \leq \overline{w}) \leq U_1(w = \overline{w}) = (v_1 - m_1)F_2(m_1) \tag{23}$$

$$\leq \frac{1}{2}\left[\underbrace{(v_1 - m_1)F_2(m_1)}_{\underline{U}_1(m_1)} + \underbrace{(v_2 - m_1)F_1(m_1)}_{\underline{U}_2(m_1)} + \underbrace{v_2 - v_1}_{\geq 0}\right] \tag{24}$$

$$= U_2(m_1) \leq U_2(w \geq m_1). \tag{25}$$

Next, consider Case (C2).

Let $\underline{U}_1(w) \geq \underline{U}_2(w)$, and $v_1 \geq v_2$. Note that $(v_1 - m_1)F_2(m_1) = \underline{U}_1(m_1) = \underline{U}_2(m_2)$, and $U_2(\overline{w}) = (v_2 - m_2)F_1(m_2) = \underline{U}_2(m_2)$. Hence,

$$U_1(w \geq m_1) \geq U_1(m_1) = (v_1 - v_2) + \underline{U}_1(m_2) \geq \underline{U}_2(m_2) = U_2(\overline{w}) \geq U_2(w \leq \overline{w}). \tag{26}$$

[37] This is because for $w < m_1$, $G_i = F_i$ and $(v_i - w)F_j(w)$ increases in w by Lemma 1.

Finally, let $\underline{U}_2(w) \geq \underline{U}_1(w)$, and $v_2 \geq v_1$. For $w > m_1$, it holds that

$$U_2(w > m_1) = (v_2 - m_2)F_1(m_2) \geq (v_2 - m_2)F_1(m_2) + (v_1 - v_2) = U_1(\overline{w}) \geq U_1(w \leq \overline{w}).$$

At m_1, it holds that $U_1(m_1) = \underline{U}_1(m_1) \leq \underline{U}_2(m_1) \leq U_2(m_1)$, which establishes 2. $\Rightarrow$ 1.

Proof of 1. $\Rightarrow$ 2.

First, I show that $v_i \geq v_j$ is a necessary condition for $U_i \geq U_j$ at every budget level. Let by contradiction $v_i < v_j$ and $U_i \geq U_j$ for all w. The supremum bid $\overline{b}$ corresponds to the highest utility level and wins with probability one. As $v_i < v_j$, $U_i(\overline{w}) = v_i - \overline{b} < v_j - \overline{b} = U_j(\overline{w})$ which contradicts $U_i \geq U_j$.

It is left to show that $U_i \geq U_j$ implies $\underline{U}_i \geq \underline{U}_j$ at every w.

Assume by contradiction that there exists a $\tilde{w}$ such that $\underline{U}_i(\tilde{w}) \geq \underline{U}_j(\tilde{w})$, and for all w, it holds that $U_i \geq U_j$. If $\tilde{w} < m_1$, there is an immediate contradiction by Lemma 3, as the lower bound and the equilibrium utility coincide. Hence, it holds that $\underline{U}_i \geq \underline{U}_j$ for all $w < m_1$.

Next, consider $\tilde{w} \geq m_1$. First, let $i = 2$ such that for all w, $U_2 \geq U_1$. $\underline{U}_1$ is constant above m_1, and hence $\underline{U}_1(m_1) = \underline{U}_1(\tilde{w})$. As established above, $\underline{U}_2(w) \geq \underline{U}_1(w)$ for $w < m_1$. By continuity of $\underline{U}_i$ and $\underline{U}_j$, it also holds that $\underline{U}_2(m_1) \geq \underline{U}_1(m_1)$. Then, it has to hold that

$$\forall w \geq m_1, \qquad \underline{U}_2(w) \geq \underline{U}_2(m_1) \geq \underline{U}_1(m_1) = \underline{U}_1(w).$$

Therefore, a budget level of $\tilde{w}$ such that $\underline{U}_2(\tilde{w}) < \underline{U}_1(\tilde{w})$ cannot exist by the properties of the lower bound, yielding the contradiction.

Second, let $i = 1$ such that for all w, $U_1 \geq U_2$. Let there exist a $\tilde{w} \geq m_1$ such that $\underline{U}_1(\tilde{w}) < \underline{U}_2(\tilde{w})$. Then, it holds that $\underline{U}_1(m_1) = \underline{U}_1(\tilde{w})$ and $\underline{U}_2(m_2) \geq \underline{U}_2(\tilde{w})$. Furthermore, as established above, $v_1 \geq v_2$. Thus, Case (C2) in Corollary 1 holds. In this case, take any $w' \in (m_1, m_2)$[38]. By Corollary 1, this yields a contradiction to $U_1 \geq U_2$:

$$U_1(w) = \underline{U}_1(m_1) = \underline{U}_1(\tilde{w}) < \underline{U}_2(\tilde{w}) \leq \underline{U}_2(m_2) = U_2(w). \tag{27}$$

□

Proof of Proposition 4. Let $k = 2$, that is, $F_2' \geq_{RHR} F_2$. The proof for $k = 1$ with $F_1' \geq_{RHR} F_1$ works accordingly and is therefore omitted.

[38]Note that the interval (m_1, m_2) is nonempty if such a $\tilde{w}$ exists and it holds that $\underline{U}_1 \geq \underline{U}_2$ for $w > m_1$

For bidder 2, m_2 and $\underline{U}_2$ are not affected by his own budget distribution. For bidder 1, let $m_1 = \arg\max\{(v_1 - b)F_2(b)\}$ and $m_1' = \arg\max\{(v_1 - b)F_2'(b)\}$. Note that due to RHR-dominance, it holds that $m_1 \leq m_1'$. Furthermore, due to FOSD it holds that $\underline{U}_1(w) \geq \underline{U}_1'(w)$ for all w, and as the lower bound is non-decreasing it also holds that $\underline{U}_1(m_1) \geq \underline{U}_1'(m_1')$.

If $m_1' > m_2$, the labels of the bidders as bidder 1 and 2 are reversed with F_2 versus F_2'.[39] Let $i \in \{1, 2\}$ refer to the bidder identities with F_2, and $\tilde{i} \in \{\tilde{1}, \tilde{2}\}$ with F_2'.

Proof of $\bar{b}' \geq \bar{b}$: First, let the identities of the bidders be identical with both budget distributions, i.e., $i = \tilde{i}$. Then, there are three possibilities: 1. (C1) holds with both budget distributions, 2. (C2) holds with both budget distributions, 3. (C1) holds under F_2 and (C2) holds under F_2'.[40] Consider the equation for $\bar{b}$ and $\bar{b}'$ in Theorem 2. If 1., $\bar{b}' \geq \bar{b}$ by FOSD and $m_1' \geq m_1$. If 2., $\bar{b}' = \bar{b}$. If 3., it holds that $\bar{b}' - \bar{b} = \underline{U}_1(m_1) - \underline{U}_2(m_2) - (v_1 - v_2) \geq 0$, where the last inequality follows from (C1) under F_2.

Second, let the identities of the bidders change (that is, such that $\tilde{1} = 2$ and $\tilde{2} = 1$) because $m_1 \leq m_2 < m_1'$. If with both budget distributions (C1) applies, then $v_2 = v_{\tilde{1}}$ and $F_1 = F_{\tilde{2}}$ and hence, $\bar{b}' - \bar{b} = v_1 - (v_1 - m_1)F_2(m_1) - v_2 - (v_2 - m_2)F_1(m_2) \geq 0$. If (C2) holds in both cases, it holds that $\bar{b}' - \bar{b} = v_1 - v_2 - [\underline{U}_1'(m_1') - \underline{U}_2(m_2)] \geq v_1 - v_2 - [\underline{U}_1(m_1) - \underline{U}_2(m_2)] \geq 0$. If (C1) holds with F_2, and $(C2)$ holds with F_2'[41], then $\bar{b}' - \bar{b} = v_1 - \underline{U}_1'(m_1) - v_1 + \underline{U}_1(m_1) \geq 0$.

Proof of G_i being FOSD over G_i: First, let the labels of the bidders be the same with both distributions, $\tilde{1} = 1$ and $\tilde{2} = 2$.

The bid distribution of bidder 1 with F_2 is in Equation 11 in Theorem 4. With F_2', it becomes $G_1'(b) = \begin{cases} F_1(b) \text{ if } b < m_1', \\ \frac{v_2 - \bar{b}'}{v_2 - b} \text{ if } b \in [m_1', \bar{b}']. \end{cases}$ Below m_1, $G_1' = G_1$. and above m_1', it is immediate that $G_1' \leq G_1$ as $\bar{b}' \geq \bar{b}$. For $b \in [m_1, m_1')$, it holds that $v_2 - \bar{b} = U_2(\overline{w}) \geq \underline{U}_2(m_1) \geq (v_2 - b)F_1(b)$. Thus, $\frac{v_2 - \bar{b}}{v_2 - b} \geq F_1(b)$ and hence, $G_1' \leq G_1$ for all b.

The bid distribution of bidder 2 with F_2' is again in Equation 11 in Theorem 4. Substituting F_2' for F_2, m_1' for m_1, and $\bar{b}'$ for $\bar{b}$, yields the new bid distribution under F_2'. As before, it is apparent that $G_2' \leq G_2$ for bids below m_1 or above m_2. For $b \in [m_1, m_1')$ and $b \in [m_1', m_2)$, the following inequality establishes that $G_2' \leq G_2$,

$$(v_1 - m_1)F_2(m_1) \geq (v_1 - m_1')F_2'(m_1') \geq (v_1 - b)F_2'(b).$$

Finally, let the labels of the bidders change such that $\tilde{1} = 2$ and $\tilde{2} = 1$. Consider bidder

[39] See the normalization in Section 3.1.

[40] It cant be the other way as $\underline{U}_1(m_1) \geq \underline{U}_1(m_1')$.

[41] Note that it is impossible to have (C2) with F_2 and (C1) with F_2'.

1. His bid distribution with F_2 and F_2' can be recovered from Equation 11. With F_2', his bid distribution is

$$G_{\tilde{2}}(b) = G_1'(b) = \begin{cases} F_{\tilde{2}}(b) = F_1(b) \text{ if } b < m_{\tilde{1}} = m_2, \\ \frac{v_{\tilde{1}} - m_{\tilde{1}} F_{\tilde{2}}(m_{\tilde{1}})}{v_{\tilde{1}} - b} = \frac{(v_2 - m_2) F_1(m_2)}{v_2 - b} \text{ if } b \in [m_{\tilde{1}}, m_{\tilde{2}}) = [m_2, m_1'), \\ \frac{v_{\tilde{1}} - \overline{b}'}{v_{\tilde{1}} - b} = \frac{v_1 - \overline{b}'}{v_2 - b} \text{ if } b \in [m_{\tilde{2}}, \overline{b}'] = [m_1', \overline{b}']. \end{cases} \tag{28}$$

Below m_1 and above m_1', it immediately holds that $G_1' \leq G_1$ using $\overline{b}' \geq \overline{b}$. For $b \in [m_1, m_1')$, it holds that $v_2 - \overline{b} = U_2(\overline{w}) \geq (v_2 - m_2)F_1(m_2) \geq (v_2 - b)F_1(b)$. This establishes the inequality for the entire interval, and $G_1' \leq G_1$. For bidder 2, the same approach establishes that $G_2' \leq G_2$ when identities change.

Note that the utility at every budget level is also lower with F_2' than with F_2. Without loss, consider the utility of bidder 1 with budget w in Equation 18. The opponent bids either with G_2' or G_2. That is, any bid yields a weakly lower winning probability with G_2' than with G_2, while the surplus of a win $(v_1 - b)$ remains the same.

Both bidders strictly worse off: I show by example that both bidders can be strictly worse off under F_2' than under F_2. Let $v_1 = v_2 = 1$, and the budget distribution functions for $w \in [0, 1]$ be $F_1(w) = w^2$, $F_2(w) = w$, and $F_2(w) = w^2$. Then, it holds that $m_2 = m_1' = \frac{2}{3}$ and $m_1 = \frac{1}{2}$. The equilibrium utilities at every budget realization of both bidders can be computed using Equations 12 and 13 for (C1). The ex-ante expected utilities are

$$EU_1' = EU_2' = \int_0^1 U_1(w) f_1(w) dw = \int_0^{\frac{2}{3}} (1 - w) w^2 2w dw + (1 - \frac{2}{3}) \left(\frac{2}{3}\right)^2 \frac{5}{9} \approx 0.13, \tag{29}$$

$$EU_1 = \int_0^1 U_1(w) f_1(w) dw = \int_0^{\frac{1}{2}} (1 - w) w 2w dw + (1 - \frac{1}{2}) \frac{1}{2} \frac{1}{2} \approx 0.24, \tag{30}$$

$$EU_2 = \int_0^1 U_2(w) f_2(w) dw = \int_0^{\frac{1}{2}} (1 - w) w^2 dw + (1 - \frac{1}{2}) \frac{1}{2} \frac{1}{2} \approx 0.15. \tag{31}$$

□

Proof of Proposition 7. Let $s_1 \in S$ and $s_2 \in S$ be the budget type realization of bidders 1 and 2.

First, consider the disclosure regime in which both budget distributions s_1 and s_2 are public. Let $U_i^D(w; s_i, s_j)$ be bidder i's equilibrium utility with budget w if the two budget types are s_i and s_j. Equilibrium utility is above the lower bound utility in which the opponent always bids his entire budget,

$$U_i^D(w;s_i,s_j) \geq \underline{U}_i^D(w;s_i,s_j) := \max_{b\leq w}(v-b)F_{s_j}(b).$$

Second, consider the no-disclosure regime in which both budget distributions s_1 and s_2 are private information. Let $\underline{U}_i^{ND}(w;s_i)$ be bidder i's lower bound utility with budget w who knows only his own budget type s_i, where his opponent with unknown budget type always bids his entire budget,

$$\underline{U}_i^{ND}(w;s_i) = \max_{b\leq w}(v-b)F(b),$$

where the expected budget distribution of the opponent is $F(b) = \sum_{s_j\in S} p_{s_j}F_{s_j}(b)$. Note that the lower bound at a budget w does not depend on a bidder's own budget type s_i. This is because his own budget distribution F_{s_i} has no impact on his best response bids given w and the bidding behavior of the opponent, as he bids after learning w.[42]

Both bidders share the same lower bound utility for any fixed budget realization w, irrespective of the own budget type, $\underline{U}_i^{ND}(w;s_i) = \underline{U}_j^{ND}(w;s_j)$ for any $s_i, s_j \in S$. Thus, the analysis of the symmetric framework in Corollary 2 in Section 4.1 applies. In any equilibrium, the lower bound and equilibrium utility at all w coincide in the no-disclosure regime,

$$U_i^{ND}(w;s_i) = \underline{U}_i^{ND}(w;s_i), \text{ for all } w, i, s_i.$$

Finally, I establish that at every w for any budget type s_i, equilibrium utility in the no-disclosure regime is weakly lower than in the disclosure regime.

$$\begin{aligned} U_i^{ND}(w;s_i) = \underline{U}_i^{ND}(w;s_i) &= \max_{b\leq w}(v-b)\sum_{s_j\in S} p_{s_j}F_{s_j}(b) \\ &\leq \sum_{s_j\in S} p_{s_j}\max_{b\leq w}(v-b)F_{s_j}(b) \\ &= \sum_{s_j\in S} p_{s_j}\underline{U}_i^D(w;s_i,s_j) \\ &\leq \sum_{s_j\in S} p_{s_j}U_i^D(w;s_i,s_j). \end{aligned}$$

The revenue of the designer is the total surplus generated, v, minus the expected utilities of both bidders (see Equation 15). Taking expectations over bidder i's budget types s_i and

[42]The knowledge of one's own budget type s_i has no impact on the best response bids, given one's own budget realization w and the bidding behavior of his opponent j.

budget realizations w, the ex-ante expected utility of a bidder under no disclosure is lower than under the disclosure regime. Hence, a lower expected utility for the bidders in the no-disclosure regime corresponds to a higher revenue for the seller. $\square$

Proof of Theorem 4. The proof is analogous to the steps for the FPA. Let $\bar{b}^a$ be the supremum bid in the all-pay auction, and define the two cases as for the FPA:

$(C1)$ if $\underline{U}_1^a(m_1) - \underline{U}_2^a(m_2) \geq v_1 - v_2$,

$(C2)$ if $\underline{U}_1^a(m_1) - \underline{U}_2^a(m_2) < v_1 - v_2$.

Under Assumption 2, the four core Lemmas 2, 3, 4 and 5 also hold for the all-pay auction, after substituting $\underline{U}_i^a$ for $\underline{U}_i$, m_i^a for m_i, and $\bar{b}^a$ for $\bar{b}$. The two auxiliary Lemmas A.1 and A.2 in the Appendix, and Theorem 1 also hold after substituting the notation. This can be checked using exactly the same steps of the proofs as for the FPA (therefore, I omit the proofs). The only difference is to use $\underline{U}_i^a$ instead of $\underline{U}_i$, and to apply the payment rule of the all-pay auction (always forgo one's bid) instead of that for the FPA (pay only if win). Hereby, the assumption $\underline{w} = 0$ guarantees that the utility of bidding the lowest budget $\underline{w} = 0$ is zero. As Theorem 1 holds for the all-pay auction, equilibrium utilities are uniquely determined.

Supremum bids: Next, I derive the supremum bids of the bidders. As Lemma A.1 also holds for the all-pay auction, there is no mass point at $\bar{b}^a$. Thus, bidding at the supremum yields the highest possible utility $v_i - \bar{b}^a$. Furthermore, as Theorem 1 also holds for the all-pay auction, it holds that

$$U_1^a(\overline{w}) = \underline{U}_1^a(m_1^a) = v_1 F_2(m_1^a) - m_1^a = v_1 - \bar{b}^a \text{ if } (C1),$$
$$U_2^a(\overline{w}) = \underline{U}_2^a(m_2^a) = v_2 F_1(m_2^a) - m_2^a = v_2 - \bar{b}^a \text{ if } (C2).$$

Solving both equations for $\bar{b}^a$ yields the supremum bids in the theorem.

Unique bid distributions: As Lemma 4 also holds for the all-pay auction, it holds that $G_i^a(b) = F_i(b)$ for $b < m_1^a$. In the following, consider bids $b \geq m_1^a$. Any mass point of bidder j at $x \in [m_1^a, \bar{b}^a]$ leads to $U_i^a(w < x) < U_i^a(x) < U_i^a(w > x)$. This holds as the bidding support has no holes due to Lemma A.2 for the all-pay auction, and bidders profit from the tie-breaking rule if they can afford it. Due to the constancy of utility in Lemma 5, bidder 1 can only have a mass point at m_1^a, and bidder 2 at m_1^a or m_2^a.

First, let (C1) hold. By Theorem 1, $U_1^a(w \geq m_1^a) = \underline{U}_1^a(w)$. Any mass point of bidder 2 would elevate the utility of bidder 1 strictly above the required level, and is therefore impossible. For any bid above m_1^a to yield a constant utility, the following inequality has to be satisfied for $b \in [m_1^a, \overline{b}^a]$,

$$v_1 G_2^a(b) - b = v_1 F_2(m_1^a) - m_1^a.$$

Solving this for $G_2^a(b)$ yields the bid distribution in the theorem. Similarly, the constancy of the utility of bidder 2 at $U_2^a(w > m_1^a) = \underline{U}_1^a(m_1^a) - (v_1 - v_2)$ requires that

$$v_2 G_1^a(b) - b = v_1 F_2(m_1^a) - m_1^a - (v_1 - v_2).$$

Solving this for G_1^a yields the bid distribution in the theorem.

Finally, consider (C2). For the constant utility $U_2^a(w > m_1^a) = \underline{U}_2^a(m_2^a)$, it has to hold that

$$v_2 G_1^a(b) - b = v_2 F_1(m_2^a) - m_2^a.$$

Solving this for $G_1^a(b)$ yields the result for bidder 1.

Bidder 1's utility is constant on $U_1^a(w) = \underline{U}_1^a(m_1^a)$ for $w \in [m_1^a, m_2^a)$, and $U_1^a(w) = \underline{U}_2^a(m_2^a) + (v_1 - v_2)$ for $w \in (m_2^a, \overline{b}^a]$. Therefore,

$$v_1 G_2^a(b) - b = v_1 F_2(m_1^a) - m_1^a \quad \text{for } w \in [m_1^a, m_2^a), \tag{32}$$

$$v_1 G_2^a(b) - b = v_2 F_1(m_2^a) - m_2^a + (v_1 - v_2) \quad \text{for } w \in (m_2^a, \overline{b}^a]. \tag{33}$$

Solving for G_2^a yields the bid distribution in the theorem.

The remaining bid distributions G_1^a and G_2^a at m_1^a and m_2^a can be computed by taking the right-sided limit of G_i^a at m_1^a and m_2^a (as a cumulative distribution function is right-continuous).

Equilibrium existence follows via the following weakly monotonic pure strategies,

$$b_1^a(w) = \begin{cases} w & \text{if} \quad w \in [0, m_1^a), \\ m_1^a & \text{if} \quad w \in [m_1^a, F_1^{-1}(\frac{v_2 - \overline{b}^a + m_1^a}{v_2})], \\ \overline{b}^a - v_2(1 - F_1(w)) & \text{otherwise.} \end{cases} \tag{34}$$

$$b_2^a(w) = \begin{cases} w & \text{if} \quad w \in [0, m_1^a), \\ v_1[F_2(w) - F_2(m_1^a)] + m_1^a & \text{if } w \in [m_1^a, F_2^{-1}\left(F_2(m_1^a) + \frac{m_2^a - m_1^a}{v_1}\right)), \\ m_2^a & \text{if } w \in [F_2^{-1}\left(F_2(m_1^a) + \frac{m_2^a - m_1^a}{v_1}\right), F_2^{-1}(\frac{v_1 - \overline{b}^a + m_2^a}{v_1})), \\ \overline{b}^a - v_1(1 - F_2(w)) & \text{otherwise.} \end{cases} \tag{35}$$

As in the proof for the FPA, it can be easily checked that these bidding functions satisfy the bid distributions in Theorem 4, and lead to the equilibrium utilities in Theorem 1 for the all-pay auction.

Optimality is satisfied: bidders with budget strictly below m_1^a (and bidder 2 with budget m_1^a) prefer to bid higher, but cannot afford it. If (C1) holds, all the remaining budget types have no strictly profitable deviation, as any bid yields the same or lower utility. If (C2) holds, all the remaining bidder 2 have the same equilibrium utility and can only do worse by bidding differently. The remaining bidder 1 types have no profitable deviation: note that $b_1^a(m_2) = m_2$. Thus, bidders with a budget below or at m_2 cannot afford the strictly profitable deviation to bid above m_2, while bidders with a budget above m_2 achieve the highest possible utility with bids above m_2 and have no strictly profitable deviation.

Finally, I establish feasibility. For bidder 1, $b_1^a(w) \leq w$ follows immediately for $w \leq F_1^{-1}(\frac{v_2 - \overline{b}^a + m_1^a}{v_2})$. For higher budgets,

$$w - b_1^a(w) = w - \left(\overline{b}^a - v_2(1 - F_1(w))\right) = (v_2 - \overline{b}^a) - (v_2 F_1(w) - w) \geq U_2^a(\overline{w}) - \underline{U}_2^a(\overline{w}) \geq 0.$$

For bidder 2, the same argument holds for $w \leq m_1^a$ and $w \geq F_2^{-1}(\frac{v_1 - \overline{b}^a + m_2^a}{v_1})$. Feasibility also holds for the remaining budget realizations, as

$$w - (v_1 [F_2(w) - F_2(m_1^a)] + m_1^a) = (v_1 F_2(m_1^a) - m_1^a) - (v_1 F_2(w) - w) \geq 0, \tag{36}$$

where the last inequality holds by Assumption 2. □

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