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Abstract

In an economy which could not accommodate full employment of its labor force, it employs some but does not employ others. The bipartition of the labor force is random, and we characterize it by a probability distribution with equal employment opportunity. We value each employed individual by his marginal contribution to the production; we also value each unemployed individual by the potential marginal contribution he would make if the market hired him. We fully honor both the individual value and its national aggregate in our distribution of the net production to the unemployment welfare and the employment benefits. Using a balanced budget rule of taxation, we derive a fair, debt-free, and risk-free tax rate for any given unemployment rate. The tax rate minimizes both the asymptotic mean and variance of the underlying posterior unemployment rate; it also stimulates employment, boosts productivity, and mitigates income inequality. One could also apply the rate and valuation approach to areas other than the labor market. This framework is open to alternative identification strategies and other forms of equal opportunity.

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I. INTRODUCTION

The problem with which we are concerned relates to the following typical situation: consider an economy which could not achieve full employment of its labor force, and therefore some people are employed, and others are not. As the employed receives wages and employment benefits (e.g., pension, health insurance, social security, education allowances, paid vacation), should the unemployed receive some unemployment welfare? If yes, how much is fair? In a specific jurisdiction system, the term "unemployment welfare" here may also mean "unemployment benefits," "unemployment insurance," or even "unemployment compensation." In an advanced economy, the answer to the first question is likely YES. This paper answers the second question by justifying a fair share of unemployment welfare for the unemployed and deriving a fair tax rate for the employed. Fair unemployment welfare and a fair tax rate are among the most fundamental problems facing our society.

The fair-division problem arises in various real-world settings. For a motivating example, let us consider a k-out-of-n redundant system which has n identical components, any k of which being in good condition makes the system work properly. When valuing the importance of each component (either working or standby), one may intuitively claim that these components should be valued equally. A very similar situation is a simple majority voting where not all voters would vote for a proposal, and the proposal could be passed or failed. For another example, in the health insurance industry, not all of the policyholders are ill and use the insurance to cover their medical expenses. The question is how to fairly share the total medical expenses between the ill policyholders and the non-ill ones. In a labor market, we have a similar, but more complicated situation: on the one hand, the market could not employ all of its labor force even though everyone in the market would like to be employed; on the other hand, the participants in the market have heterogeneous performance in the production.

We have a few challenges to deal with when distributing the welfare and benefits, both generated by the employed. First, we believe that "fairness" is bound with the equality of employment opportunity, not with the equality of outcome, nor with the equality of productivity. Besides, we also believe that unemployment is not a fault of the unemployed, nor a fault of the labor market, but a self-adjustment mechanism toward efficiency of the market. Secondly, we attempt to apply a fiscal policy to the labor market which operates in the ever-changing economy and productivity. For it to be useful, the tax and division rule should be not only fair to all people but also be able to cope with the uncertainty and sustainability in generating and distributing the production. Ideally, it should balance the account of value generated and that of value paid in each employment contingency. Thirdly, a fair tax rate should depend only on observed data to avoid any excessively political bargaining and costly strategic voting. One major issue, however, is the non-observability of the heterogeneous-agent production function for all employment scenarios. Another data issue is the desynchronization between the unemployment rate and the tax rate: the former is high-frequency data while the latter has a lower frequency. Often in a yearly time-frame, policymakers determine the tax rate after observing most monthly unemployment rates.

Vast literature (e.g., Kornhauser [1995], Fleurbaey and Maniquet [2006]) from various directions has studied the fairness in taxation and unemployment payments. In particular, Shapley [1953] proposes an axiom of fairness to develop a fair-division method, called the Shapley value, which is widely used in distributing employment compensation and welfare (e.g., Moulin [2004]; Devicienti [2010]; Giorgi and Guandalini [2018]; Krawczyk and Platkowski [2018]). Beneath the pillars of the Shapley value and the Shapley axiom are two underlying assumptions: players' unanimous participation in the production, and perfect information about the production function. Hu [2002, 2006, 2018] relax the unanimity assumption and generalize the Shapley value using a few classes of probability distributions for the dichotomy or bipartition of the players. In particular, Hu [2018] proposes using a Beta-Binomial distribution to address the equality of opportunity. This current research capitalizes on the Beta-Binomial distributions as well. Furthermore, we do not assume the perfect information about the production function.

The advantage of our approach is twofold. On the one hand, we provide a game-theoretical micro-foundation for a fair-division solution to distribute the unemployment welfare and employment benefits. We can apply the solution to many similar situations without substantive alternations and extend the framework using other identification schemes rather than tax policy stability or unemployment rate minimization. On the other hand, the fair tax rate we provide is simple enough to be used in practice. It relies only on the unemployment rate and a reserved portion of production which is not for the individual use of the labor force. Meanwhile, the total unemployment welfare depends only on the observed unemployment rate and the observed production. We attempt to immunize our solution from any unnecessary randomness, hypotheticals, ambiguity, and latency. These include, but are not limited to, the competitive and cooperative features of the labor market, the complexity of the taxation system, the exact sizes of the labor market and time-varying unemployment population.

We organize the remainder of the paper as follows. Section II applies the framework of dichotomous valuation (or simply, "D-value") in Hu [2002, 2006, 2018] to value each person in the labor market, assuming equal employment opportunity. The two sides of D-value rely on two unknown parameters and are aggregated separately for the unemployed and the employed labor. Next, Section III formulates a set of fair divisions of the generated value using the aggregate components of D-value. In this section, we base the set of fair tax rates on two accounting identities for a balanced budget. In Section IV, by maximizing the stability of the unemployment rate or minimizing the expected posterior unemployment rate, we identify a specific relation between the fair tax rate and the unemployment rate. The special solution is robust under a few other criteria. Section V lists a few other applications, and Section VI suggests several ways to extend this framework. The account is self-contained, and the proofs are in the Appendix.

II. DICHOTOMOUS VALUATION

Before our formal discussion, let us introduce a few notations. For a general economy, we assume that its labor force consists of the employed labor and the unemployed labor, ignoring any part-time labor. Let $N = \{1, 2, \cdots, n\}$ denote the set of people in the labor force, indexed as $1, 2, ..., n$, and let $\mathbf{S} \subseteq N$ denote the random subset of the employed labor in N. For any subset T of N, let $|T|$ denote the cardinality of T; for notational simplicity, we often use n for $|N|$, t for $|T|$, and s for $|\mathbf{S}|$. Let us also write the employment rate as $\omega = \frac{s}{n}$, which is one minus the unemployment rate. Besides, we employ the vinculum (overbar) in naming a singleton set; for example, "$\bar{i}$" is for the set $\{i\}$. Also, we use "$\backslash$" for set subtraction, "$\cup$" for set union, and $\beta(\cdot, \cdot)$ for a Beta function. The Appendix defines the symbols Δ, $\Delta_1, \cdots, \Delta_9$ as shorthand notations.

I. Equality of Employment Opportunity

Equal employment opportunity is widely acknowledged and is the starting point for us to study welfare and fairness. In the United States, for example, equal employment opportunity has been enacted to prohibit federal contractors from discriminating against employees by race, sex, creed, religion, color, or national origin since President Lyndon Johnson signed Executive Order 11246 in 1965. In the literature, there are abundant qualitative descriptions, informal or formal, about equal opportunity (e.g., Friedman and Friedman [1990]; Roemer [1998]; Rawls [1999]). In this section, we introduce a quantitative and probabilistic version of equal opportunity whereby the employment opportunity is assumed equitable for all persons in the labor force.

We assume three layers of uncertainty for the random subset $\mathbf{S}$ while maintaining the equality of employment opportunity. In the first layer, the employment size $|\mathbf{S}|$ follows a binomial distribution with parameters (n, p). When independence is assumed, p is the probability of any given person being employed. In the second layer, the unknown parameter p has a prior Beta distribution with hyperparameters (θ, ρ), where θ and ρ are to be determined by estimation or specification. Thus, the joint probability density of $p \in (0,1)$ and $|\mathbf{S}| = s$ is

$$\frac{p^{\theta-1}(1-p)^{\rho-1}}{\beta(\theta,\rho)} \begin{pmatrix} n \\ s \end{pmatrix} p^s(1-p)^{n-s} = \frac{n!}{s!(n-s)!} \frac{p^{\theta+s-1}(1-p)^{\rho+n-s-1}}{\beta(\theta,\rho)}. \quad (1)$$

Equation (1) implies the following marginal probability density for $|\mathbf{S}| = s$:

$$\begin{aligned} \mathbb{P}(|\mathbf{S}| = s) &= \int_0^1 \tfrac{n!}{s!(n-s)!} \tfrac{p^{\theta+s-1}(1-p)^{\rho+n-s-1}}{\beta(\theta,\rho)} \mathrm{d}p \\ &= \tfrac{n!}{s!(n-s)!} \tfrac{\beta(\theta+s,\rho+n-s)}{\beta(\theta,\rho)}, \qquad s = 0, 1, \cdots, n. \end{aligned} \quad (2)$$

In the third layer, for any given employment size s, all subsets of size s have the same probability to be $\mathbf{S}$. As there are $\frac{n!}{s!(n-s)!}$ subsets of size s in N, the probability of the employment scenario $\mathbf{S} = T$ is then

$$\mathbb{P}(\mathbf{S} = T) = \begin{cases} \frac{\mathbb{P}(|\mathbf{S}|=s)}{\frac{n!}{s!(n-s)!}} = \frac{\beta(\theta+s,\rho+n-s)}{\beta(\theta,\rho)}, & \text{if } t = s; \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Clearly, the equality of employment opportunity is already implied in the assumed triple-layered uncertainty.

The posterior density function of p given $|\mathbf{S}| = s$, calculated from (1) and (2), is given by

$$\frac{\frac{n!}{s!(n-s)!} \frac{p^{\theta+s-1}(1-p)^{\rho+n-s-1}}{\beta(\theta,\rho)}}{\frac{n!}{s!(n-s)!} \frac{\beta(\theta+s,\rho+n-s)}{\beta(\theta,\rho)}} = \frac{p^{\theta+s-1}(1-p)^{\rho+n-s-1}}{\beta(\theta+s,\rho+n-s)}.$$

The above formula establishes that the posterior employment rate follows a Beta distribution with parameters $(\theta+s, \rho+n-s)$. In the following, let us use $\tilde{p}_{n,\omega}$ to denote the posterior employment rate given the observance of $|\mathbf{S}| = n\omega$. In contrast, p is the unobservable prior employment rate, and ω is the observed employment rate.

II. Value of the Employed and the Unemployed

For any $T \subseteq N$, we use a heterogeneous-agent production function $v(T)$ to measure the aggregate productivity when $\mathbf{S} = T$. The value-added production function excludes the labor cost which compensates, in the form of after-tax wages, the time and efforts consumed by the employed labor. To isolate the added value by the labor force alone, we also exclude the cost of consumed physical and financial resources from the production function $v(T)$. Both the labor cost and the resources cost are exempt from taxation. Thus, without loss of generality, we may assume that $v(\emptyset) = 0$ for the empty set $\emptyset$. But, $v(T)$ is not necessarily increasing with T or with its size $|T|$.

Let us formally introduce the two components of the D-value. For any $i \in N$, to analyze its marginal effect on the value generating process, we consider two jointly exhaustive and mutually exclusive events:

- *Event 1*: $i \in \mathbf{S}$. Then, i's marginal effect is $v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})$, called *marginal gain*, in that it contributes $v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})$ to the production, due to its existence in $\mathbf{S}$. The expected marginal gain is

$$\gamma_i[v] \overset{\text{def}}{=} \mathbb{E}\left[v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})\right]. \tag{4}$$

  In the above, "def" is for definition and "$\mathbb{E}$" for expectation under the probability distribution specified by (3).

- *Event 2*: $i \notin \mathbf{S}$. Then, his marginal effect is $v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})$ in that $\mathbf{S}$ faces a *marginal loss* $v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})$, due to i's absence from the employed labor force $\mathbf{S}$. In other words, the person would increase the production by $v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})$ if the market included him in $\mathbf{S}$. The expected marginal loss is then

$$\lambda_i[v] \overset{\text{def}}{=} \mathbb{E}\left[v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})\right]. \tag{5}$$

We let $\gamma_i[v]$ be the employment benefits i receives when he is employed, and let $\lambda_i[v]$ be the unemployment welfare he receives when he is unemployed. Note that, even if i always remains employed, both $\mathbf{S}$ and $\mathbf{S} \setminus \bar{i}$ change daily, if not hourly; thus i's marginal gain is not a constant. Similarly, even if i remains unemployed for a while, $\mathbf{S}$, $\mathbf{S} \cup \bar{i}$, and i's marginal loss are not constant. To account for this uncertainty, we have already taken expectations in (4) and (5) when defining $\gamma_i[v]$ and $\lambda_i[v]$.

A few key points worth mentioning to help us understand the profit-sharing strategy. First, in addition to receiving employment benefits, the employed labor also receives after-tax wages, which compensate their human capital and human resources consumed in generating $v(\mathbf{S})$, and accumulated in the pre-employment era. Wages like physical or financial costs are not part of the generated value $v(\mathbf{S})$. The unemployed labor, however, only receives unemployment welfare. Secondly, if $i \in \mathbf{S}$, then $v(\mathbf{S} \backslash \bar{i})$ is not observable while we observe $v(\mathbf{S})$. Similarly, when $j \notin \mathbf{S}$, we cannot observe both $v(\mathbf{S} \cup \bar{j})$ and $v(\mathbf{S})$ at the same time. Thus, we need to transform the marginals into observable forms, such as those in Theorem 1. Thirdly, the aggregate employment benefits $\sum_{i \in \mathbf{S}} \left[v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})\right]$ is not necessarily equal to $v(\mathbf{S})$, the value collectively generated by $\mathbf{S}$. Thus, we distribute some of the surpluses $v(\mathbf{S}) - \sum_{i \in \mathbf{S}} \left[v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})\right]$ to the unemployed labor $N \setminus \mathbf{S}$. The distribution is not through personal giving but government taxation and unemployment payment systems. The distribution channel also appeals to us for the aggregate benefits and aggregate welfare at the national level.

Theorem 1. *The aggregate components of the D-value are*

$$\begin{aligned}
\sum_{i \in N} \gamma_i[v] &= \mathbb{E}\left[\sum_{i \in \mathbf{S}} \left(v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})\right)\right] \\
&= \tfrac{n\beta(\theta+n,\rho)}{\beta(\theta,\rho)} v(N) + \sum_{T \subseteq N: T \neq N} \tfrac{t(\theta+\rho-1)-n\theta}{\rho+n-t-1} \tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)} v(T), \\
\sum_{i \in N} \lambda_i[v] &= \mathbb{E}\left[\sum_{i \in N \setminus \mathbf{S}} \left(v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})\right)\right] \\
&= \sum_{T \subseteq N: T \neq \emptyset} \tfrac{t(\theta+\rho-1)-n(\theta-1)}{\theta+t-1} \tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)} v(T) - \tfrac{n\beta(\theta,\rho+n)}{\beta(\theta,\rho)} v(\emptyset).
\end{aligned} \tag{6}$$

III. ACCOUNTING IDENTITIES FOR A BALANCED BUDGET

By (3), the expected production is

$$\mathbb{E}\left[v(\mathbf{S})\right] = \sum_{T \subseteq N} \frac{\beta(\theta+t, \rho+n-t)}{\beta(\theta, \rho)} v(T). \tag{7}$$

Let us consider the particular employment scenario $\mathbf{S} = T$, which occurs with probability $\frac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}$ and which generates the value $v(T)$. After the production, we face a challenge of fairly dividing $v(T)$ between T and $N \setminus T$, who both have an entitlement to $v(T)$. Our division rule should fully respect the entitlement claim: each employed person receives his expected marginal gain, and each unemployed person receives his expected marginal loss. Besides, we should reserve a portion of $v(T)$ for the common good of the economy.

I. A Balanced Budget Rule

As noted, we purposely divide the generated value $v(T)$ into three components. The first one is for the employment benefits. We compare the coefficients of $v(T)$ in (7) and that of $\mathbb{E}\left[\sum_{i\in\mathbf{S}}\big(v(\mathbf{S})-v(\mathbf{S}\setminus\bar{i})\big)\right]$ in (6), ignoring the probability density for the scenario; the employed labor force T should retain $\frac{t(\theta+\rho-1)-n\theta}{\rho+n-t-1}v(T)$ as their employment benefits. The rest, $\left[1-\frac{t(\theta+\rho-1)-n\theta}{\rho+n-t-1}\right]v(T)$, pays to the government as "tax", which also includes the profits retained with the firms. Besides, we assume both $\lambda_i[v]$ and $\gamma_i[v]$ are tax-exempt in order to avoid any double taxation. Thus, we define the "tax rate" τ as

$$\tau \stackrel{\text{def}}{=} 1-\frac{s(\theta+\rho-1)-n\theta}{\rho+n-s-1}, \qquad s=1,\cdots,n-1. \tag{8}$$

For the time being, the rate relies on the labor market size n and the employment rate $\omega=\frac{|\mathbf{S}|}{n}$, but not on the production $v(\mathbf{S})$. In this definition, we exclude two extreme but unlikely cases at $\omega=0$ and $\omega=1$. Secondly, we compare the coefficients of $v(T)$ in (7) and that of $\mathbb{E}\left[\sum_{i\in N\setminus\mathbf{S}}\big(v(\mathbf{S}\cup\bar{i})-v(\mathbf{S})\big)\right]$ in (6); the unemployed labor should claim $\frac{t(\theta+\rho-1)-n(\theta-1)}{\theta+t-1}v(T)$ from the tax revenue $\tau v(T)$ as their unemployment welfare. Thirdly, we assume that a reserved proportion, δ, of $v(T)$ is not individually and not directly distributed to the labor force. As a consequence, the tax rate τ includes both the reserve ratio δ and the proportion to the unemployment welfare, i.e.,

$$\tau \equiv \delta+\frac{s(\theta+\rho-1)-n(\theta-1)}{\theta+s-1}. \tag{9}$$

The reserve $\delta v(T)$ is meant to serve the general public's interests and to have broad appeal, rather than the individual needs. More specifically, $\delta v(T)$ includes, but is not limited to, the payments to the population who are not in the labor force, to the corporate equity earnings not used for employment benefits, to the public administration and national defense, to the public welfare, to the tax deficit from the past, to the future development, and so on. By admitting the corporate earnings to the reserve $\delta v(T)$, we have deliberately ignored firm-specific re-distribution processes of the net corporate earnings, and have purposely avoided the associated corporate income taxation. In practice, government expenditure (including that in all levels of government but excluding all unemployment welfare payments) and corporate earnings are two significant components of the reserve $\delta v(T)$. Thus, a government could implement a countercyclical fiscal policy by adjusting its spending level in $\delta v(T)$ such that it negatively responds to the change in the ratio between the corporate earnings to the production. Also, the above tax rate τ automatically reacts positively to the change of the corporate earning ratio, other things remaining constant.

Put in another way, the balanced budget rule implied in (8) and (9) forbids any borrowing between different labor market scenarios or forbids any inter-temporal borrowing. Thus, this sustainable taxation policy meets the needs of the present market scenario without compromising the ability of future market scenarios to meet their own needs. In practice, however, it is challenging, if not impossible, to enforce or enact the balanced budget rule at the labor market scenario level. In the United States, for example, the employment rate ω changes daily and is recorded monthly by the Bureau of Labor Statistics; as a policy variable, the tax rate τ changes yearly. Striving for the balanced budget at the maximum degree, one could minimize the variance of the employment rate ω within a yearly time frame. In a perfect situation (e.g., Theorem 2), the employment rate closely follows a degenerate distribution and remains almost constant within the year. We discuss the variance minimization in the next section. In sum, the above account reflects how the tax is collected and distributed on the national level, whereby production and exhaustive distribution of the production co-occur. In contrast, a household could still maximize its utility through inter-temporal borrowing, saving, lending, and consuming.

II. Sets of Feasible Solutions

Although the terms "tax rate" and "tax rule" have often been used loosely and interchangeably about τ, we must distinguish between them to have an accurate discussion. As a tax rule or tax policy, τ is a function of $\omega \in (0,1)$, subject to the equality of employment opportunity and the balance of budget as specified in (8) and (9). In contrast, as a tax rate, τ is merely the value of the function at a specific $\omega \in (0,1)$.

For a given triple of (n, ω, δ), in the system of two equations (8) and (9), there are three indeterminates (θ, ρ, τ). Let $\Omega_{n,\omega,\delta}$ denote the set of all feasible combinations of (θ, ρ, τ) which satisfy (8) and (9):

$$\Omega_{n,\omega,\delta} \stackrel{\text{def}}{=} \left\{ (\theta, \rho, \tau) \;\middle|\; \begin{array}{l} \tau = 1 - \frac{n\omega(\theta+\rho-1)-n\theta}{\rho+n-n\omega-1}, \\ \tau = \delta + \frac{n\omega(\theta+\rho-1)-n(\theta-1)}{\theta+n\omega-1}, \\ 0 \le \tau \le 1,\ \theta > 0,\ \rho > 0. \end{array} \right\}.$$

From now on, a "fair tax rate" could mean a solution of τ in $\Omega_{n,\omega,\delta}$; or it could be a limit of tax rates which satisfy (8) and (9).

In general, for a reasonable δ, a finite n, and a $\omega \in (0,1)$, there could be infinitely many solutions in $\Omega_{n,\omega,\delta}$; in this case, we need one more restriction to solve (θ, ρ, τ) uniquely. To do so, one could capitalize on the statistical relation between ω and (θ, ρ): the prior mean and mode of ω are $\frac{\theta}{\theta+\rho}$ and $\frac{\theta-1}{\theta+\rho-2}$, respectively (e.g., Johnson, Kotz and Balakrishnan [1995, Chapter 21]); the prior median of ω is very close to $\frac{\theta-\frac{1}{3}}{\theta+\rho-\frac{2}{3}}$ (e.g., Kerman [2011]). Alongside this direction, for example, we could set $\frac{\theta}{\theta+\rho}$ to be the historical average of ω or the yearly average of ω in the previous year. Alternatively, we could set $\frac{\theta}{\theta+\rho}$, $\frac{\theta-1}{\theta+\rho-2}$, or $\frac{\theta-\frac{1}{3}}{\theta+\rho-\frac{2}{3}}$ to be a target employment rate or the natural employment rate. However, this type of identification schemes requires additional input, e.g., the historical average of ω, or a target employment rate, or the natural employment rate.

Furthermore, to derive a unique solution from $\Omega_{n,\omega,\delta}$ or its boundary, we should base our deviation on observed data only. Our first concern is the size of the labor market. Even though the size n is not random, it is likely a time-varying latent variable in that there is no clear cut between entry to and exit from the labor market; many unemployed persons may not be actively seeking new positions. In practice, n changes daily whereas τ most likely changes yearly. No matter how it changes and how much latent it is, we are confident that n is a large number in the general economy. Thus, in contrast to (8) and (9), we seek a fair tax rule which is valid for all large n's but is not specific to a particular n. As a result, the tax rule we are targeting should not involve n, and we may write it as $\tau(\omega) : (0, 1) \to (0, 1)$.

To visualize a solution set $\Omega_{n,\omega,\delta}$, let us represent (θ, ρ) in terms of $(n, \delta, \tau, \omega)$, as stated in Lemma 1 in the Appendix. Figure I (a−b) plots the feasible solution sets for $n = 10000$, $\delta = .1$, and any $\omega \in (0, 1)$. Note that there is a straight line which dramatically raises both θ and ρ. From Figure I (c−d), we observe that both θ and ρ drop sharply on the other side of the straight line. As a matter of fact, any point on the other side of the straight line does not represent a fair solution, owing to the positivity requirement of θ and ρ. Actually, the straight line has the tax rule $\tau(\omega) = 1 - \omega + \delta\omega$. This linear relation between τ and ω can be seen from Lemma 1 in the Appendix: when $\Delta \equiv \omega + \tau - \delta\omega - 1 > 0$, both $\theta = \frac{n^2\omega\Delta_1 + n\Delta_2 + \Delta_3}{n\Delta + \Delta_3}$ and $\rho = \frac{n^2(1-\omega)\Delta_1 + n\Delta_4 + \Delta_3}{n\Delta + \Delta_3}$ increase to $+\infty$ as $n \to \infty$, due to the positivity of $\Delta_1 \equiv \delta\omega - \omega - \tau + 2 > 0$. On the other hand, when $\Delta < 0$, both θ and ρ decrease to $-\infty$ as $n \to \infty$. Thus, the singular line is expressed as $\Delta = 0$. Furthermore, for any fair tax rule on the side of $\Delta > 0$, the posterior employment rate $\tilde{p}_{n,\omega}$ has a degenerated limit distribution, as claimed in Theorem 2. The next section answers a related question: which $\tau(\omega)$ makes the distribution convergence fastest. The answer happens to be the solution on the singular line.

Theorem 2. *For any fair tax rule $\tau(\omega) : (0, 1) \to (1 - \omega + \delta\omega, 1)$, as $n \to \infty$, $\tilde{p}_{n,\omega}$ converges in distribution to the degenerate distribution with mass at ω.*

IV. A RISK-FREE TAX RATE

In this section, we derive the limit fair tax rule $\tau(\omega) = 1 - \omega + \delta\omega$ from a few different angles and study its properties. At a minimum, a good tax rule should not discourage employment incentives and productivity, such as detailed in Theorem 5 and 8. On the other end, we expect a good rule to be robust, i.e., optimal under multiple criteria. We study six criteria, any of which uniquely identifies the solution. Theorem 3, 4, 5, 6, 7, and 9 cover the robustness.

We should heavily utilize the observed market behavior. Though an efficient labor market stimulates productivity $v(\mathbf{S})$, a higher employment rate does not mean higher productivity and vice versa — the production function $v(\mathbf{S})$ does not necessarily increase with the employment size $|\mathbf{S}|$. Acemoglu and Shimer [2000] find that a moderate level of unemployment could boost productivity by improving the quality of jobs. Indeed, a moderate unemployment rate fosters peer pressure in producing $v(\mathbf{S})$, allows workers to move on from declining firms, and enables rising companies and the economy as a whole to optimally respond to external shocks. Therefore, our tax rule in this section is not merely targeting a higher expectation of employment rate, but grounds its assumption in the observed market behavior, and lets the market itself respond with a higher employment rate.

Besides, from a statistical viewpoint, our tax rule relies on a realization of p, not on the uncertainty of the realized value. A natural way to ignore the uncertainty is to study the posterior $\tilde{p}_{n,\omega}$, where both s and ω are no longer random. As Theorem 2 shows, ω is informative and indicative in revealing the central tendency of the posterior labor market, delineated by $\tilde{p}_{n,\omega}$. Asymptotic dispersion of $\tilde{p}_{n,\omega}$ and the market's response to the tax rule $\tau(\omega)$ are among other essential ingredients in the complete profile of $\tilde{p}_{n,\omega}$. As a result, we could set optimal criteria to minimize the dispersion measures or maximize the expected market response.

I. Minimum Asymptotic Posterior Variance

Tax rate stability creates the right environment for a balance of payments, reduces the uncertainty of the labor market, and creates confidence in technology and human capital investment. As a function of ω, τ migrates the risk from the unemployment rate to the tax rate; with the absence of other exogenous shocks, indeed, stability in the tax rate is equivalent to stability in the unemployment rate.

When the variance of $\tilde{p}_{n,\omega}$ measures its stability, Theorem 3 states that the tax rule $\tau(\omega) = 1 - \omega + \delta\omega$ minimizes the asymptotic variance of $\tilde{p}_{n,\omega}$. Furthermore, it is also the limit of variance-minimizing rules in finite labor markets. We add the restriction "$\theta, \rho \geq \frac{1}{n}$" in the finite labor markets to ensure the positivity of the hyperparameters (θ, ρ).

Theorem 3. *As $n \to \infty$, the tax rule $\tau(\omega) = 1 - \omega + \delta\omega$ minimizes the limit variance of $\sqrt{n}\tilde{p}_{n,\omega}$, i.e., $\underset{\tau}{\operatorname{argmin}} \lim_{n\to\infty} \operatorname{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}\right) = 1 - \omega + \delta\omega$. Besides, $\lim_{n\to\infty} \underset{\tau}{\operatorname{argmin}} \operatorname{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}|\theta, \rho \geq \frac{1}{n}\right) = 1 - \omega + \delta\omega$. Furthermore, the limit minimum variance is zero.*

We provide a few comments to help clarify any potential misunderstandings about the theorem. First, a stable $\sqrt{n}\tilde{p}_{n,\omega}$ is achieved at $n = \infty$ where the limit variance is zero. However, $\tilde{p}_{n,\omega}$ is still exposed to exogenous shocks, such as those studied in Pissarides [1992] and Blanchard [2000]. Secondly, it is worth emphasizing that $1-\omega+\delta\omega$ is the limit tax rate as $n \to \infty$. For a large but finite n, a small positive number could be added to $1-\omega+\delta\omega$ to ensure the positivity of θ and ρ. That small positive number, however, is negligible; thus, we can practically use the rule $\tau(\omega) = 1 - \omega + \delta\omega$ without any addition. Moreover, a higher-order approximation $\tau = 1 - \omega + \delta\omega + \frac{\omega(1-\omega)(1-\delta)^2}{n}$ could be a good alternative to $\tau = 1 - \omega + \delta\omega$ (see the proof of Theorem 3). Thirdly, with a zero or near zero variance in the unemployment rate, labor mobility means one layoff and one new hire should coincide to ensure the total employment size $|\mathbf{S}|$ remains unchanged. It also means that the sizes of employment and labor market change proportionally so that their ratio remains unchanged. Lastly, though the posterior distribution is skewed, the tax rule minimizes both the overall risk and the one-sided risk of $\sqrt{n}\tilde{p}_{n,\omega}$, as stated in Theorem 4. In particular, a policymaker's concern is on the downside risk.

Theorem 4. *As $n \to \infty$, the tax rule $\tau(\omega) = 1 - \omega + \delta\omega$ minimizes both the limit lower semivariance and the limit upper semivariance of $\sqrt{n}\tilde{p}_{n,\omega}$.*

II. Consistency and Robustness

The above fair tax rule captures several striking features of the labor market. In the first place, it is the best response to the market to stimulate the employment within the framework of (8) and (9). In the second place, we can derive it by minimizing statistical dispersion measures other than the posterior variance or semivariance. Meanwhile, it helps mitigate the income inequality.

The rule $\tau(\omega) = 1 - \omega + \delta\omega$ is an effective taxation strategy to maximally boost employment without breaking the opportunity equality and budget balance. For an economic policymaker, one primary concern is the forward-looking employment profile $\tilde{p}_{n,\omega}$. By Theorem 2, the mean and median of $\tilde{p}_{n,\omega}$ converge to ω as $n \to \infty$ for any fair $\tau(\omega) \in (1 - \omega + \delta\omega, 1)$. When n is finitely large, they counteract an increasing τ (cf Theorem 5). Consequently, to maximize the posterior mean and median, we should reduce the tax rate τ to its minimum while still maintaining the conditions $\tau(\omega) \in (1 - \omega + \delta\omega, 1)$, $\theta > 0$ and $\rho > 0$. Thus, the limit of fair tax rates which maximize the mean or median of $\tilde{p}_{n,\omega}$ would be $1-\omega+\delta\omega$. As a remark, the condition $\omega > .5$ in Theorem 5 is satisfied in a general economy.

Theorem 5. *Assume a fair tax rule $\tau(\omega) : (0, 1) \to (1 - \omega + \delta\omega, 1)$. For any $\omega \in (.5, 1)$ and a finitely large n, the mean and median of $\tilde{p}_{n,\omega}$ react negatively to an increasing tax rule $\tau(\omega)$. As a consequence,*

$$\begin{cases} \lim\limits_{n\to\infty} \underset{\tau}{\operatorname{argmax}} \operatorname{MEAN}\left(\tilde{p}_{n,\omega}|\theta, \rho \geq \frac{1}{n}\right) & = \quad 1 - \omega + \delta\omega, \\ \lim\limits_{n\to\infty} \underset{\tau}{\operatorname{argmax}} \operatorname{MEDIAN}\left(\tilde{p}_{n,\omega}|\theta, \rho \geq \frac{1}{n}\right) & = \quad 1 - \omega + \delta\omega, \end{cases}$$

for any $\omega \in (.5, 1)$.

For a Beta distribution, especially with large parameters, the mean absolute deviation (thereafter, MAD) is a more robust measure of statistical dispersion than the variance. We consider two types of MAD. First, the MAD around the mean for the posterior $\tilde{p}_{n,\omega}$ is (e.g., Gupta and Nadarajah [2004, page 37]):

$$\mathbb{E}\left[|\tilde{p}_{n,\omega} - \mathbb{E}(\tilde{p}_{n,\omega})|\right] = \frac{2(\theta+s)^{\theta+s}(\rho+n-s)^{\rho+n-s}}{\beta(\theta+s,\rho+n-s)(\theta+\rho+n)^{\theta+\rho+n}}. \tag{10}$$

Second, the MAD around the median (say, m) is (e.g., Gupta and Nadarajah [2004, page 38]):

$$\mathbb{E}\left[|\tilde{p}_{n,\omega} - m|\right] = \frac{2m^{\theta+s}(1-m)^{\rho+n-s}}{(\theta+\rho+n)\beta(\theta+s,\rho+n-s)} \tag{11}$$

In the next two theorems, we identify the same tax rule by minimizing the asymptotic MADs.

Theorem 6. *As $n \to \infty$, the tax rule $\tau(\omega) = 1 - \omega + \delta\omega$ minimizes the MAD of $n\tilde{p}_{n,\omega}$ around the mean, i.e.* $\lim\limits_{n\to\infty} \operatorname*{argmin}\limits_{\tau} \mathbb{E}\left[n\,|\tilde{p}_{n,\omega} - \mathbb{E}(\tilde{p}_{n,\omega})| \mid \theta, \rho \geq \frac{1}{n}\right] = 1 - \omega + \delta\omega$.

Theorem 7. *As $n \to \infty$, the tax rule $\tau(\omega) = 1 - \omega + \delta\omega$ minimizes the MAD of $n^{\frac{3}{2}}\tilde{p}_{n,\omega}$ around the median, i.e.,* $\lim\limits_{n\to\infty} \operatorname*{argmin}\limits_{\tau} \mathbb{E}\left[n^{\frac{3}{2}}|\tilde{p}_{n,\omega} - m| \mid \theta, \rho \geq \frac{1}{n}\right] = 1 - \omega + \delta\omega$.

III. Equality of Outcome with Symmetric Production Functions

For any $i, j \in N$ with $i \neq j$, we say i *uniformly outperforms* j in v if

- $v(T \cup \bar{i}) - v(T) \geq v(T \cup \bar{j}) - v(T)$ for any $T \subseteq N \setminus \bar{i} \setminus \bar{j}$; and
- $v(T) - v(T \setminus \bar{i}) \geq v(T) - v(T \setminus \bar{j})$ for any $T \subseteq N$ with $i, j \in T$.

With these two conditions, i has higher productivity than j in all comparable employment contingencies (either both employed or both unemployed). As productivity is highly valued in (4) and (5), j should receive less employment benefits and less unemployment welfare than i does. This is formally claimed in Theorem 8. The theorem does not require the Beta-Binomial distribution, as long as i and j have the same chance of being employed. It is valid for all fair tax rules, including the specific one $\tau(\omega) = 1 - \omega + \delta\omega$.

Theorem 8. *If $i \in N$ uniformly outperforms $j \in N$ in v and they have equal employment opportunity, then $\gamma_i[v] \geq \gamma_j[v]$ and $\lambda_i[v] \geq \lambda_j[v]$.*

We say $i, j \in N$ are *symmetric* in the production function v if they uniformly outperform each other. By Theorem 8, clearly, $\lambda_i[v] = \lambda_j[v]$ and $\gamma_i[v] = \gamma_j[v]$ if i and j are symmetric in v and they have equal employment opportunity. In other words, they should receive the same amount of unemployment welfare if both are unemployed; they should also receive the same amount of employment benefits if both are employed.

Without any further analysis of or any prior knowledge about the production function, symmetry among the unemployed (or the employed) could be a reasonable *a priori* assumption to distribute the unemployment welfare (or employment benefits). For example, the Cobb-Douglas production function is symmetric among all employed persons in the labor market. Besides, if we assume symmetry among the employed labor and also assume symmetry among the unemployed labor, then the tax rule $\tau(\omega) = 1 - \omega + \delta\omega$ eliminates the income inequality when the income excludes the after-tax wages which pay the employed for their human resource cost. Theorem 9 affirms this equality of outcome, where an employed individual and an unemployed one may not be symmetric in production. Also, the theorem does not restrict the size of n. In the k-out-of-n redundant system mentioned in Section I, accordingly, the n components (either working or standby) are equally important if they have equal quality.

Theorem 9. *If all employed individuals are symmetric and all unemployed individuals are also symmetric in v, then $\tau(\omega) = 1 - \omega + \delta\omega$ if and only if an unemployed person's unemployment welfare equals an employee's employment benefits.*

V. OTHER APPLICATIONS

We could apply our framework to a wide range of areas where there are precisely two types of individuals. In this section, we analyze three applications other than labor markets.

I. Voting Power

In a voting game (e.g., Shapley [1962]), $v : 2^N \to \{0, 1\}$ is a monotonically increasing set function. Let $\mathbf{S}$ denote the random subset of voters who vote for the proposal. The proposal passes when $v(\mathbf{S}) = 1$; otherwise, it fails when $v(\mathbf{S}) = 0$. However, v should not mean "production" or alike.

No matter the outcome, Hu [2006] describes $\gamma_i[v]$ as i's probability of turning a failed result to a passed one, and $\lambda_i[v]$ the probability of turning a passed result to being failed. Thus, the sum of $\lambda_i[v]$ and $\gamma_i[v]$ quantifies his power in the game.

The ratio δ plays a role in some settings. Let us consider, for example, 10% of the voters approve just a proposal before a referendum voting on the proposal, and assume the number of other voters' support ballots follows a Beta-Binomial distribution.

Many voting games are symmetric. In these cases, the equality of outcome becomes egalitarian of power.

II. Health Insurance

Health insurance has two types of policyholders: some are ill and use the insurance to cover their medical expenses; others are healthy and don't use the insurance. Let $\mathbf{S}$ denote the random set of ill policyholders, $v(\mathbf{S})$ be the total medical expenses with copays deducted, and $\tilde{\tilde{\delta}} v(\mathbf{S})$ be the surcharge paid to the insurance companies. Let $\delta = -\tilde{\tilde{\delta}}$. Then the total expenses except for the copays, $(1 - \delta)v(\mathbf{S})$, are billed to all insurance policyholders.

If $\tau = 1 - \omega + \delta\omega$ and v is symmetric among the two types of policyholders, respectively, then by the equality of outcome, the cost to buy the insurance policy would be $\frac{(1-\delta)\mathbb{E}[v(\mathbf{S})]}{n}$ per policyholder. We take expectation on $\frac{(1-\delta)v(\mathbf{S})}{n}$ because the policyholders pay it upfront. On the contrary, the unemployment welfare and employment benefits payment coincide with the production.

In this example, patients pay the predetermined copays. In the labor market studied in Section II-IV, the after-tax wages are exempt from the distribution of the production function; it, however, have an indirect effect on τ through the corporate earning ratio. In the next example, we derive payments such as copays and wages, using the equality of outcome.

III. Highway Toll

The I-66 highway inside the Capital Beltway of the Washington metropolitan area has enforced a dynamic toll rule during rush hours: a carpool driver pays no toll, but a solo driver pays a dynamic toll fee, say $\xi(n, \omega)$. Here n is the number of cars in the segment of the highway, and ω is the percentage of solo drivers in the traffic.

We let $g(n)$ be a carpool driver's cost in the traffic when the traffic volume is n cars. It is likely a nonlinear increasing function of n. An excellent choice of $g(n)$ is the expected driving time (in hours) multiplied by the average hourly pay rate, plus expenses on gas and vehicle depreciation. Also, let $\mathbf{S}$ denote the random set of solo drivers. Then, $v(\mathbf{S}) = ng(n) - n(1-\omega)g\left(n(1-\omega)\right) - n\omega\xi(n, \omega)$ is the total cost of over-traffic generated by the solo drivers, with toll fees deducted.

The production function v is symmetric among all solo drivers and also symmetric among all carpool drivers. By the equality of outcome, each driver shares the same cost $\frac{v(\mathbf{S})}{n}$. As a carpool driver pays no toll, his shared cost should exactly offset the extra cost caused by the solo drivers, which is $g(n) - g\left(n(1-\omega)\right)$. Finally, the equation $\frac{v(\mathbf{S})}{n} = g(n) - g\left(n(1-\omega)\right)$ implies $\xi(n, \omega) = g\left(n(1-\omega)\right)$. An administration surcharge δ may apply.

VI. CONCLUSIONS

In this paper, we study a fair-division solution to allocate the unemployment welfare in an economy, where the heterogeneous-agent production function is almost unknown. To address the inequality issue in the welfare, we interpret "fairness" as equal employment opportunity and model it by a Beta-Binomial probability distribution. Our "sustainability" is meant to be free of debt and free of surplus in the taxation budget. To justify the value of the unemployed labor, we capitalize on the D-value concept in Hu [2018]. The D-value specifies how much of the net production to be retained with the employed labor, and what portion to be distributed to the unemployed. Both are subject to two unknown hyperparameters. Finally, we postulate the labor market is static and identify a sustainable tax policy by minimizing the asymptotic variance of the posterior employment rate. The policy can also be uniquely determined by minimizing the asymptotic posterior mean or median of the unemployment rate, or minimizing the downside risk of the posterior employment rate, or minimizing the posterior mean absolute deviation about the mean or median. Surprisingly, the tax rule is not only simple enough for practical use but also motivates the unemployed to seek employment and the employed to improve productivity.

One could easily extend this framework in several ways. One way is to re-specify the probability distribution of equal employment opportunity, for example, by any of the following re-specifications. First, we can replace the two-parameter Beta distribution with a four-parameter one or a Beta rectangular distribution. Secondly, we can let θ and ρ be some functions of other unknown parameters. Thirdly, we could substitute the Beta-Binomial distribution with a Dirichlet-Multinomial distribution or a Beta-Geometric distribution. Fourthly, we could randomize p without involving (θ, ρ), by generating two independent three-parameter Gamma random variables X and Y. Then, the ratio $\frac{X}{X+Y}$ is a Beta random variable. In any of the four cases, however, we need further identification restrictions to figure out a $\tau(\omega)$ precisely.

From other angles, we could apply other identification schemes or other objective functions to find a unique fair tax rule. From a statistical viewpoint, one could try the maximum likelihood estimation for $\tilde{p}_{n,\omega}$, or minimize the ex-ante risk of ω, or apply the statistical methods mentioned in Section III. From an economic viewpoint, one could minimize the Gini coefficient of the Beta distribution of $\tilde{p}_{n,\omega}$. From a strategic game-theoretical viewpoint, one could seek a bargaining solution from the feasible solution set $\Omega_{n,\omega,\delta}$, which may be particularly useful when n is small. Finally, a policymaker could treat the reserve ration δ as endogenous, for example, letting it be a function of ω.

The simple static model, however, ignores several important aspects of a real labor market. First, it does not capture the dynamic features of the income inequality, nor its rational response to the tax rule. Secondly, while preventing the fungibility of borrowing funds from the future reduces the risk that a government administration piles up its national debt, it impairs that administration's ability (especially, monetary policy) to intervene in the economy. The government, however, can still moderately stimulate the economy during a recession by adjusting the reserve ratio δ. Thirdly, the postulation of labor market efficiency does neglect the recent development of the incomplete-market theory (e.g., Magill and Quinzii [1996]). Fourthly, a multi-criteria objective function may be a viable alternative to the minimum-variance one, especially when there is a high unemployment rate $1-\omega$ or a large δ. Lastly, a single tax rule $\tau(\omega)$ could have overly simplified the complexity of the taxation system, which is also affected by other determinants. These are just a few challenges our framework introduces, which require further development.

In summary, the fair and sustainable tax policy studied here has a solid theoretical underpinning, together with simplicity in practical use, consistency with productivity and employment incentives, and robustness to similar objectives. When applying this framework to a real fair-division problem, one should also consider benefits of alternative probability distributions for equal opportunity, alternative objective functions, alternative restrictions, and dynamic thinking.

INTERNATIONAL MONETARY FUND

APPENDIX

Our focus in this paper is to study the relationship between the fair tax rate τ and the employment rate ω when the labor market size n is large. To analyze the limit behavior of τ and ω for a large n, we only need the relevant asymptotic approximations. We say two functions $f(n) = O\left(g(n)\right)$ if $\limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| < \infty$; and we say $f(n) \approx g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. For simplicity, let us denote the following shorthands:

$$
\begin{aligned}
\Delta &\equiv \omega + \tau - \delta\omega - 1, \\
\Delta_1 &\equiv \delta\omega - \omega - \tau + 2 \equiv 1 - \Delta, \\
\Delta_2 &\equiv \delta\omega\tau - 2\delta\omega - \omega\tau^2 + 2\omega\tau + \tau - 1, \\
\Delta_3 &\equiv (1-\tau)(\delta-\tau) = \delta - \tau - \delta\tau + \tau^2, \\
\Delta_4 &\equiv -\delta\omega\tau + \delta\omega + \delta\tau - 2\delta + \omega\tau^2 - 2\omega\tau + \omega - \tau^2 + 3\tau - 1, \\
\Delta_5 &\equiv -\delta\omega + \delta\tau - 2\delta + \omega - \tau^2 + 4\tau - 2 \equiv \Delta_2 + \Delta_4, \\
\Delta_6 &\equiv -\omega\delta + \omega\tau + \tau - 1 \equiv \Delta_2 + \omega\Delta_3, \\
\Delta_7 &\equiv -\delta - \omega\tau + 2\tau + \omega - 1 \equiv \Delta_4 + (1-\omega)\Delta_3, \\
\Delta_8 &\equiv -\delta\omega - \delta + \omega + 3\tau - 2 \equiv \Delta_3 + \Delta_5, \\
\Delta_9 &\equiv -2\delta\omega - \delta + 2\omega + 4\tau - 3 \equiv \Delta + \Delta_8.
\end{aligned}
$$

The following lemma, to be used in other proofs, re-writes (8) and (9).

Lemma 1. *In terms of $(n, \delta, \tau, \omega)$, we can solve (θ, ρ) from (8) and (9) as*

$$
\begin{cases}
\theta = \frac{n^2\omega\Delta_1 + n\Delta_2 + \Delta_3}{n\Delta + \Delta_3}, \\
\rho = \frac{n^2(1-\omega)\Delta_1 + n\Delta_4 + \Delta_3}{n\Delta + \Delta_3}.
\end{cases}
\tag{12}
$$

A1. Proof of Theorem 1

In this proof, we use the following relation about Beta functions:

$$
\begin{aligned}
\beta(x-1, y+1) &= \tfrac{y}{x-1}\beta(x,y), \quad x > 1, y > 0; \\
\beta(x+1, y-1) &= \tfrac{x}{y-1}\beta(x,y), \quad x > 0, y > 1.
\end{aligned}
$$

First, the expected aggregate marginal gain and loss are:

$$\begin{aligned}
\mathbb{E}\left[\sum_{i\in\mathbf{S}}\big(v(\mathbf{S})-v(\mathbf{S}\setminus\bar{i})\big)\right] &= \sum_{T\subseteq N}\mathbb{P}(\mathbf{S}=T)\sum_{i\in T}\big[v(T)-v(T\setminus\bar{i})\big]\\
&= \sum_{i\in N}\sum_{T\subseteq N:i\in T}\mathbb{P}(\mathbf{S}=T)\big[v(T)-v(T\setminus\bar{i})\big]\\
&= \sum_{i\in N}\gamma_i[v],\\
\mathbb{E}\left[\sum_{i\in N\setminus\mathbf{S}}\big(v(\mathbf{S}\cup\bar{i})-v(\mathbf{S})\big)\right] &= \sum_{T\subseteq N}\mathbb{P}(\mathbf{S}=T)\sum_{i\in N\setminus T}\big[v(T\cup\bar{i})-v(T)\big]\\
&= \sum_{i\in N}\sum_{T\subseteq N:i\notin T}\mathbb{P}(\mathbf{S}=T)\big[v(T\cup\bar{i})-v(T)\big]\\
&= \sum_{i\in N}\lambda_i[v].
\end{aligned}$$

Next, by (3), (4) and (5), we re-write the expected marginal gain and loss as:

$$\begin{aligned}
\gamma_i[v] &= \sum_{T\subseteq N:T\ni i}\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}[v(T)-v(T\setminus\bar{i})]\\
&\overset{Z=T\setminus\bar{i}}{=} \sum_{T\subseteq N:T\ni i}\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)-\sum_{Z\subseteq N\setminus\bar{i}}\tfrac{\beta(\theta+|Z|+1,\rho+n-|Z|-1)}{\beta(\theta,\rho)}v(Z),\\
\lambda_i[v] &= \sum_{Z\subseteq N\setminus\bar{i}}\tfrac{\beta(\theta+|Z|,\rho+n-|Z|)}{\beta(\theta,\rho)}[v(Z\cup\bar{i})-v(Z)]\\
&\overset{T=Z\cup\bar{i}}{=} \sum_{T\subseteq N:T\ni i}\tfrac{\beta(\theta+t-1,\rho+n-t+1)}{\beta(\theta,\rho)}v(T)-\sum_{Z\subseteq N\setminus\bar{i}}\tfrac{\beta(\theta+|Z|,\rho+n-|Z|)}{\beta(\theta,\rho)}v(Z).
\end{aligned}\tag{13}$$

By (13), the aggregate value of the employed labor $\sum_{i\in N}\gamma_i[v]$ is

$$\begin{aligned}
&\sum_{i\in N}\sum_{T\subseteq N:T\ni i}\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)-\sum_{i\in N}\sum_{Z\subseteq N\setminus\bar{i}}\tfrac{\beta(\theta+|Z|+1,\rho+n-|Z|-1)}{\beta(\theta,\rho)}v(Z)\\
\overset{T=Z}{=}\; &\sum_{T\subseteq N:T\neq\emptyset}\sum_{i\in T}\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)-\sum_{T\subseteq N:T\neq N}\sum_{i\in N\setminus T}\tfrac{\beta(\theta+t+1,\rho+n-t-1)}{\beta(\theta,\rho)}v(T)\\
=\; &\sum_{T\subseteq N:T\neq\emptyset}\tfrac{t\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)-\sum_{T\subseteq N:T\neq N}\tfrac{(n-t)\beta(\theta+t+1,\rho+n-t-1)}{\beta(\theta,\rho)}v(T)\\
=\; &\tfrac{n\beta(\theta+n,\rho)}{\beta(\theta,\rho)}v(N)-\tfrac{n\beta(\theta+1,\rho+n-1)}{\beta(\theta,\rho)}v(\emptyset)\\
&+\sum_{T\subseteq N:T\neq\emptyset,T\neq N}\tfrac{t\beta(\theta+t,\rho+n-t)-(n-t)\beta(\theta+t+1,\rho+n-t-1)}{\beta(\theta,\rho)}v(T)\\
=\; &\tfrac{n\beta(\theta+n,\rho)}{\beta(\theta,\rho)}v(N)-\tfrac{n\theta}{\rho+n-1}\tfrac{\beta(\theta,\rho+n)}{\beta(\theta,\rho)}v(\emptyset)\\
&+\sum_{T\subseteq N:T\neq N,T\neq\emptyset}\left[t-\tfrac{(n-t)(\theta+t)}{\rho+n-t-1}\right]\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)\\
=\; &\tfrac{n\beta(\theta+n,\rho)}{\beta(\theta,\rho)}v(N)+\sum_{T\subseteq N:T\neq N}\tfrac{t(\theta+\rho-1)-n\theta}{\rho+n-t-1}\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T).
\end{aligned}$$

Also by (13), the aggregate value of the unemployed labor $\sum_{i\in N}\lambda_i[v]$ is

$$\begin{aligned}
& \sum_{i\in N}\sum_{T\subseteq N:T\ni i}\tfrac{\beta(\theta+t-1,\rho+n-t+1)}{\beta(\theta,\rho)}v(T)-\sum_{i\in N}\sum_{Z\subseteq N\setminus\bar{i}}\tfrac{\beta(\theta+|Z|,\rho+n-|Z|)}{\beta(\theta,\rho)}v(Z)\\
\overset{T=Z}{=} & \sum_{T\subseteq N:T\neq\emptyset}\sum_{i\in T}\tfrac{\beta(\theta+t-1,\rho+n-t+1)}{\beta(\theta,\rho)}v(T)-\sum_{T\subseteq N:T\neq N}\sum_{i\in N\setminus T}\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)\\
= & \sum_{T\subseteq N:T\neq\emptyset}\tfrac{t\beta(\theta+t-1,\rho+n-t+1)}{\beta(\theta,\rho)}v(T)-\sum_{T\subseteq N:T\neq N}\tfrac{(n-t)\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)\\
= & \tfrac{n\beta(\theta+n-1,\rho+1)}{\beta(\theta,\rho)}v(N)-\tfrac{n\beta(\theta,\rho+n)}{\beta(\theta,\rho)}v(\emptyset)\\
& +\sum_{T\subseteq N:T\neq\emptyset,T\neq N}\tfrac{t\beta(\theta+t-1,\rho+n-t+1)-(n-t)\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)\\
= & \tfrac{n\rho}{\theta+n-1}\tfrac{\beta(\theta+n,\rho)}{\beta(\theta,\rho)}v(N)-\tfrac{n\beta(\theta,\rho+n)}{\beta(\theta,\rho)}v(\emptyset)\\
& +\sum_{T\subseteq N:T\neq\emptyset,T\neq N}\left[\tfrac{t(\rho+n-t)}{\theta+t-1}-(n-t)\right]\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)\\
= & \sum_{T\subseteq N:T\neq\emptyset}\tfrac{t(\theta+\rho-1)-n(\theta-1)}{\theta+t-1}\tfrac{\beta(\theta+t,\rho+n-t)}{\beta(\theta,\rho)}v(T)-\tfrac{n\beta(\theta,\rho+n)}{\beta(\theta,\rho)}v(\emptyset).
\end{aligned}$$

A2. Proof of Lemma 1

We can re-write (8) and (9) as a linear system of unknowns (θ,ρ):

$$\begin{cases}(1-\tau)(\rho+n-s-1) = s(\theta+\rho-1)-n\theta,\\ (\tau-\delta)(\theta+s-1) = s(\theta+\rho-1)-n(\theta-1).\end{cases}$$

As a consequence, the symbolic solution of (θ,ρ) is unique. Let us assume (12) and verify that (12) satisfies both (8) and (9) by the following identities, some of which are used in other proofs:

$$\begin{aligned}
\theta+s &= \tfrac{n^2\omega\Delta_1+n\Delta_2+\Delta_3}{n\Delta+\Delta_3}+\tfrac{n\omega(n\Delta+\Delta_3)}{n\Delta+\Delta_3}=\tfrac{n^2\omega+n\Delta_6+\Delta_3}{n\Delta+\Delta_3},\\
\theta+s-1 &= \tfrac{n^2\omega+n\Delta_6+\Delta_3}{n\Delta+\Delta_3}-\tfrac{n\Delta+\Delta_3}{n\Delta+\Delta_3}=\tfrac{n^2\omega+n(\Delta_6-\Delta)}{n\Delta+\Delta_3}=\tfrac{n^2\omega+n\omega(\tau-1)}{n\Delta+\Delta_3},\\
\theta+\rho &= \tfrac{n^2\omega\Delta_1+n\Delta_2+\Delta_3}{n\Delta+\Delta_3}+\tfrac{n^2(1-\omega)\Delta_1+n\Delta_4+\Delta_3}{n\Delta+\Delta_3}=\tfrac{n^2\Delta_1+n\Delta_5+2\Delta_3}{n\Delta+\Delta_3},\\
\theta+\rho-1 &= \tfrac{n^2\Delta_1+n\Delta_5+2\Delta_3}{n\Delta+\Delta_3}-\tfrac{n\Delta+\Delta_3}{n\Delta+\Delta_3}=\tfrac{n^2\Delta_1+n(\delta\tau-2\delta-\tau^2+3\tau-1)+\Delta_3}{n\Delta+\Delta_3},\\
\theta+\rho+n &= \tfrac{n^2\Delta_1+n\Delta_5+2\Delta_3}{n\Delta+\Delta_3}+\tfrac{n(n\Delta+\Delta_3)}{n\Delta+\Delta_3}=\tfrac{n^2+n\Delta_8+2\Delta_3}{n\Delta+\Delta_3},\\
\theta+\rho+n+1 &= \tfrac{n^2+n\Delta_8+2\Delta_3}{n\Delta+\Delta_3}+\tfrac{n\Delta+\Delta_3}{n\Delta+\Delta_3}=\tfrac{n^2+n\Delta_9+3\Delta_3}{n\Delta+\Delta_3},\\
\theta+\rho+n-1 &= \tfrac{n^2+n\Delta_8+2\Delta_3}{n\Delta+\Delta_3}-\tfrac{n\Delta+\Delta_3}{n\Delta+\Delta_3}=\tfrac{n^2+n(\Delta_8-\Delta)+\Delta_3}{n\Delta+\Delta_3},\\
\theta+\rho+n-2 &= \tfrac{n^2+n\Delta_8+2\Delta_3}{n\Delta+\Delta_3}-\tfrac{2(n\Delta+\Delta_3)}{n\Delta+\Delta_3}=\tfrac{n^2+n(\Delta_8-2\Delta)}{n\Delta+\Delta_3},\\
\rho+n-s &= \tfrac{n^2(1-\omega)\Delta_1+n\Delta_4+\Delta_3}{n\Delta+\Delta_3}+\tfrac{n(1-\omega)(n\Delta+\Delta_3)}{n\Delta+\Delta_3}=\tfrac{n^2(1-\omega)+n\Delta_7+\Delta_3}{n\Delta+\Delta_3},\\
\rho+n-s-1 &= \tfrac{n^2(1-\omega)+n\Delta_7+\Delta_3}{n\Delta+\Delta_3}-\tfrac{n\Delta+\Delta_3}{n\Delta+\Delta_3}=\tfrac{n^2(1-\omega)+n(1-\omega)(\tau-\delta)}{n\Delta+\Delta_3}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
s(\theta+\rho-1)-n\theta &= \tfrac{n\omega[n^2\Delta_1+n(\delta\tau-2\delta-\tau^2+3\tau-1)+\Delta_3]}{n\Delta+\Delta_3} \\
&\quad -\tfrac{n(n^2\omega\Delta_1+n\Delta_2+\Delta_3)}{n\Delta+\Delta_3} \\
&= \tfrac{n^2[\omega(\delta\tau-2\delta-\tau^2+3\tau-1)-\Delta_2]+n(\omega-1)\Delta_3}{n\Delta+\Delta_3} \\
&= \tfrac{n^2(1-\omega)(1-\tau)+n(\omega-1)\Delta_3}{n\Delta+\Delta_3}, \\
s(\theta+\rho-1)-n(\theta-1) &= [s(\theta+\rho-1)-n\theta]+n \\
&= \tfrac{n^2(1-\omega)(1-\tau)+n(\omega-1)\Delta_3}{n\Delta+\Delta_3}+\tfrac{n(n\Delta+\Delta_3)}{n\Delta+\Delta_3} \\
&= \tfrac{n^2\omega(\tau-\delta)+n\omega\Delta_3}{n\Delta+\Delta_3}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tfrac{s(\theta+\rho-1)-n\theta}{\rho+n-s-1} &= \tfrac{n^2(1-\omega)(1-\tau)+n(\omega-1)\Delta_3}{n^2(1-\omega)+n(1-\omega)(\tau-\delta)} &&= 1-\tau, \\
\tfrac{s(\theta+\rho-1)-n(\theta-1)}{\theta+s-1} &= \tfrac{n^2\omega(\tau-\delta)+n\omega\Delta_3}{n^2\omega+n\omega(\tau-1)} &&= \tau-\delta,
\end{aligned}
$$

which are equivalent to (8) and (9), respectively.

A3. Proof of Theorem 2

For any integer $z \geq 0$, by the proof of Lemma 1,

$$
\begin{aligned}
\tfrac{\theta+s+z}{\theta+\rho+n+z} &= \frac{\frac{n^2\omega+n\Delta_6+\Delta_3}{n\Delta+\Delta_3}+\frac{z(n\Delta+\Delta_3)}{n\Delta+\Delta_3}}{\frac{n^2+n\Delta_8+2\Delta_3}{n\Delta+\Delta_3}+\frac{z(n\Delta+\Delta_3)}{n\Delta+\Delta_3}} \\
&= \tfrac{n^2\omega+n(\Delta_6+z\Delta)+(1+z)\Delta_3}{n^2+n(\Delta_8+z\Delta)+(2+z)\Delta_3} \\
&\to \omega, \quad \text{as } n\to\infty.
\end{aligned}
$$

The characteristic function of $\tilde{p}_{n,\omega}$ (e.g., Johnson et al. [1995, Chapter 21]) is

$$
\mathbb{E}\left[e^{i\eta\tilde{p}_{n,\omega}}\right] = 1+\sum_{k=1}^{\infty}\frac{(i\eta)^k}{k!}\prod_{z=0}^{k-1}\frac{\theta+s+z}{\theta+\rho+n+z}.
$$

We let $n\to\infty$,

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{E}[e^{i\eta\tilde{p}_{n,\omega}}] &= 1+\sum_{k=1}^{\infty}\tfrac{(i\eta)^k}{k!}\lim_{n\to\infty}\prod_{z=0}^{k-1}\tfrac{\theta+s+z}{\theta+\rho+n+z} \\
&= 1+\sum_{k=1}^{\infty}\tfrac{(i\eta\omega)^k}{k!} \\
&= e^{i\eta\omega}.
\end{aligned}
$$

Therefore, as $n\to\infty$, $\tilde{p}_{n,\omega}$ converges in distribution to the degenerate distribution with mass at ω, which has the characteristic function $e^{i\eta\omega}$.

A4. Proof of Theorem 3

As $\tilde{p}_{n,\omega}$ has a Beta distribution with parameters $(\theta + s, \rho + n - s)$ (see Section II), its variance is $\frac{(\theta+s)(\rho+n-s)}{(\theta+\rho+n)^2(\theta+\rho+n+1)}$ (e.g., Gupta and Nadarajah [2004, page 35]). By the proof of Lemma 1, the variance of $\sqrt{n}\tilde{p}_{n,\omega}$ is

$$
\begin{aligned}
& \frac{n\frac{n^2\omega+n\Delta_6+\Delta_3}{n\Delta+\Delta_3}\frac{n^2(1-\omega)+n\Delta_7+\Delta_3}{n\Delta+\Delta_3}}{\left(\frac{n^2+n\Delta_8+2\Delta_3}{n\Delta+\Delta_3}\right)^2\frac{n^2+n\Delta_9+3\Delta_3}{n\Delta+\Delta_3}} \\
= & \frac{n(n\Delta+\Delta_3)(n^2\omega+n\Delta_6+\Delta_3)\left[n^2(1-\omega)+n\Delta_7+\Delta_3\right]}{\left(n^2+n\Delta_8+2\Delta_3\right)^2\left(n^2+n\Delta_9+3\Delta_3\right)} \\
= & \frac{\left(\Delta+\frac{\Delta_3}{n}\right)\left[\omega\left(1+\frac{\Delta_6}{n\omega}+\frac{\Delta_3}{n^2\omega}\right)\right]\left[(1-\omega)\left(1+\frac{\Delta_7}{n(1-\omega)}+\frac{\Delta_3}{n^2(1-\omega)}\right)\right]}{\left(1+\frac{\Delta_8}{n}+\frac{2\Delta_3}{n^2}\right)^2\left(1+\frac{\Delta_9}{n}+\frac{3\Delta_3}{n^2}\right)} \qquad (14) \\
= & \omega(1-\omega)\left(\Delta+\frac{\Delta_3}{n}\right)\left[1+\frac{\Delta_6}{n\omega}+\frac{\Delta_7}{n(1-\omega)}-\frac{2\Delta_8}{n}-\frac{\Delta_9}{n}+O\left(\frac{1}{n^2}\right)\right] \\
= & \omega(1-\omega)\Delta+\frac{\omega(1-\omega)}{n}\left[\Delta_3+\Delta\left(\frac{\Delta_6}{\omega}+\frac{\Delta_7}{1-\omega}-2\Delta_8-\Delta_9\right)\right]+O\left(\frac{1}{n^2}\right) \\
\to & \omega(1-\omega)(\omega+\tau-\delta\omega-1), \qquad \text{as } n \to \infty.
\end{aligned}
$$

To minimize $\lim_{n\to\infty} \text{VAR}(\sqrt{n}\tilde{p}_{n,\omega}) = \omega(1-\omega)\Delta$ while $\lim_{n\to\infty} \text{VAR}(\sqrt{n}\tilde{p}_{n,\omega}) \geq 0$, we have to set $\Delta = 0$. Thus, $\operatorname{argmin}_{\tau} \lim_{n\to\infty} \text{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}\right) = 1 - \omega + \delta\omega$.

However, when applying $\tau = 1 - \omega + \delta\omega$ to a labor market with finite n, we have $\Delta = 0$, $\Delta_1 = 1$ and $\Delta_3 = -\omega(1-\omega)(1-\delta)^2$. And Lemma 1 reduces to

$$
\begin{cases}
\theta = \frac{n^2\omega+n\Delta_2-\omega(1-\omega)(1-\delta)^2}{-\omega(1-\omega)(1-\delta)^2}, \\
\rho = \frac{n^2(1-\omega)+n\Delta_4-\omega(1-\omega)(1-\delta)^2}{-\omega(1-\omega)(1-\delta)^2},
\end{cases}
$$

which converge to $-\infty$ as $n \to \infty$. In theory, therefore, for a large but finite n, we need to choose τ to be $1 - \omega + \delta\omega$ plus a small positive number to ensure that $\theta > 0$ and $\rho > 0$. To estimate the small positive number, let us first try the higher-order approximation of

$$\tau = 1 - \omega + \delta\omega + \frac{c}{n},$$

for some constant $c > 0$. Then

$$
\begin{aligned}
\Delta &= \frac{c}{n}, \\
\Delta_3 &= \left[\omega(1-\delta) - \frac{c}{n}\right]\left[-(1-\omega)(1-\delta) - \frac{c}{n}\right] \qquad (15) \\
&= -\omega(1-\omega)(1-\delta)^2 + O\left(\frac{1}{n}\right).
\end{aligned}
$$

By the next to the last step in (14),

$$\text{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega} | \tau = 1 - \omega + \delta\omega + \frac{c}{n}\right) = \frac{\omega(1-\omega)c - \omega^2(1-\omega)^2(1-\delta)^2}{n} + O\left(\frac{1}{n^2}\right).$$

To minimize the above variance, we let $c = \omega(1-\omega)(1-\delta)^2$. Thus, $\tau = 1-\omega+\delta\omega+\frac{\omega(1-\omega)(1-\delta)^2}{n}$ is a higher-order approximation for $1-\omega+\delta\omega$. Further high-order approximations, if necessary, could be found similarly.

When $\tau = 1-\omega+\delta\omega+\frac{c}{n}$, let us use (15) to calculate

$$\begin{aligned} n\Delta + \Delta_3 &= c + \left[\omega(1-\delta) - \tfrac{c}{n}\right]\left[-(1-\omega)(1-\delta) - \tfrac{c}{n}\right] \\ &= \tfrac{(1-2\omega)(1-\delta)c}{n} + \tfrac{c^2}{n^2} \end{aligned} \tag{16}$$

which is negative when $\omega > .5$ and $n \geq \left|\frac{\omega(1-\omega)(1-\delta)}{1-2\omega}\right|$. As the numerators of θ and ρ in Lemma 1 are both positive for a large n, θ and ρ are negative when n is large and $\omega > .5$.

Let us make another try at $\tau = 1-\omega+\delta\omega+\frac{2c}{n}$. Similar to (16),

$$\begin{aligned} n\Delta + \Delta_3 &= 2c + \left[\omega(1-\delta) - \tfrac{2c}{n}\right]\left[-(1-\omega)(1-\delta) - \tfrac{2c}{n}\right] \\ &= c + \tfrac{2(1-2\omega)(1-\delta)c}{n} + \tfrac{4c^2}{n^2}. \end{aligned}$$

In this case, both θ and ρ are larger than $\frac{1}{n}$ when n is large. Note from the last step of (14) that $\text{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}\right)$ is an increasing function of τ when n is large; thus, the small positive number to be added to $1-\omega+\delta\omega$ is between $\frac{c}{n}$ and $\frac{2c}{n}$. In practice, however, as the number is too small for a large n, there is no necessity to exactly calculate it and add it to $1-\omega+\delta\omega$.

Let $\tilde{\tau}_n = \underset{\tau}{\text{argmin}}\, \text{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}|\theta,\rho \geq \frac{1}{n}\right)$. We add the restriction $\theta, \rho \geq \frac{1}{n}$ to ensure the existence of $\tilde{\tau}_n$. As $\text{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}\right)$ is an increasing function of τ when n is large, $\tilde{\tau}_n$ is unique and less than $1-\omega+\delta\omega+\frac{2c}{n}$ when n is large. By (14),

$$\text{VAR}(\sqrt{n}\tilde{p}_{n,\omega}|\tau = \tilde{\tau}_n) = \omega(1-\omega)(\omega + \tilde{\tau}_n - \delta\omega - 1) + O\left(\frac{1}{n}\right).$$

Finally, we use the relation

$$0 \leq \text{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}|\tau = \tilde{\tau}_n\right) \leq \text{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}|\tau = 1-\omega+\delta\omega+\frac{2c}{n}\right)$$

to get

$$0 \leq \omega(1-\omega)(\omega + \tilde{\tau}_n - \delta\omega - 1) + O\left(\frac{1}{n}\right) \leq O\left(\frac{1}{n}\right).$$

Letting $n \to \infty$ in the above inequality, we get $\omega(1-\omega)(\omega+\tilde{\tau}_n-\delta\omega-1) = O(\frac{1}{n})$, i.e., $\lim_{n\to\infty} \operatorname*{argmin}_{\tau} \mathrm{VAR}\left(\sqrt{n}\tilde{p}_{n,\omega}|\theta, \rho \geq \frac{1}{n}\right) = 1-\omega+\delta\omega$.

A5. Proof of Theorem 4

In this proof, we constantly apply the identities in the proof of Lemma 1. Let $\mu_n = \frac{\theta+s}{\theta+\rho+n}$ be the mean of $\tilde{p}_{n,\omega}$, and let σ_n^2 be the variance of $\tilde{p}_{n,\omega}$. The lower semivariance of $\tilde{p}_{n,\omega}$ is calculated as

$$\sigma_{n-}^2 = \int_0^{\mu_n} (x-\mu_n)^2 \frac{x^{\theta+s-1}(1-x)^{\rho+n-s-1}}{\beta(\theta+s, \rho+n-s)} \mathrm{d}x.$$

We apply Chebychev's inequality in terms of the lower semivariance (e.g., Berck and Hihn[1982]) to get

$$\mathbb{P}\left(\tilde{p}_{n,\omega} \leq \mu_n - a_n\sigma_{n-}\right) \leq \frac{1}{a_n^2}, \quad \forall\ a_n > 0.$$

By the proof of Theorem 3, $\tilde{p}_{n,\omega}$ has the variance $\sigma_n^2 = \frac{\omega(1-\omega)\Delta}{n} + O(\frac{1}{n^2})$. We let $a_n = \frac{\sqrt{\frac{\omega(1-\omega)\Delta}{n}}}{\sigma_{n-}}$, then

$$\mathbb{P}\left(\tilde{p}_{n,\omega} \leq \mu_n - \sqrt{\frac{\omega(1-\omega)\Delta}{n}}\right) \leq \frac{n\sigma_{n-}^2}{\omega(1-\omega)\Delta}. \tag{17}$$

Let $\kappa_n = \frac{\theta+s-1}{\theta+\rho+n-2}$ and ε_n be the mode and median of $\tilde{p}_{n,\omega}$, respectively. The mode κ_n maximizes the density function $\frac{x^{\theta+s-1}(1-x)^{\rho+n-s-1}}{\beta(\theta+s,\rho+n-s)}, 0 < x < 1$. As the median lies between the mean and mode, we have

$$\begin{aligned}
|\mu_n - \varepsilon_n| &\leq |\mu_n - \kappa_n| = \left|\tfrac{\theta+s}{\theta+\rho+n} - \tfrac{\theta+s-1}{\theta+\rho+n-2}\right| \\
&= \left|\tfrac{n+\rho-\theta-2s}{(\theta+\rho+n)(\theta+\rho+n-2)}\right| \\
&\leq \left|\tfrac{\theta+s}{(\theta+\rho+n)(\theta+\rho+n-2)}\right| + \left|\tfrac{n+\rho-s}{(\theta+\rho+n)(\theta+\rho+n-2)}\right| \\
&= O(\tfrac{1}{n}).
\end{aligned}$$

In the following lower-bound estimation of (17), we use Gamma function, denoted by $\Gamma(\cdot)$, and its Stirling's approximation $\Gamma(z+1) \approx \sqrt{2\pi z}\left(\frac{z}{e}\right)^z$.

$$
\begin{aligned}
& \mathbb{P}\left(\tilde{p}_{n,\omega} \leq \mu_n - \sqrt{\tfrac{\omega(1-\omega)\Delta}{n}}\right) \\
= \;& \mathbb{P}\left(\tilde{p}_{n,\omega} \leq \varepsilon_n\right) - \int\limits_{\mu_n - \sqrt{\frac{\omega(1-\omega)\Delta}{n}}}^{\varepsilon_n} \tfrac{x^{\theta+s-1}(1-x)^{\rho+n-s-1}}{\beta(\theta+s,\rho+n-s)} \mathrm{d}x \\
\geq \;& \tfrac{1}{2} - \left|\varepsilon_n - \left(\mu_n - \sqrt{\tfrac{\omega(1-\omega)\Delta}{n}}\right)\right| \tfrac{\kappa_n{}^{\theta+s-1}(1-\kappa_n)^{\rho+n-s-1}}{\beta(\theta+s,\rho+n-s)} \\
= \;& \tfrac{1}{2} - \left|\sqrt{\tfrac{\omega(1-\omega)\Delta}{n}} + O(\tfrac{1}{n})\right| \tfrac{\left(\frac{\theta+s-1}{\theta+\rho+n-2}\right)^{\theta+s-1}\left(1-\frac{\theta+s-1}{\theta+\rho+n-2}\right)^{\rho+n-s-1}}{\frac{\Gamma(\theta+s)\Gamma(\rho+n-s)}{\Gamma(\theta+\rho+n)}} \\
= \;& \tfrac{1}{2} - \left|\sqrt{\tfrac{\omega(1-\omega)\Delta}{n}} + O(\tfrac{1}{n})\right| \tfrac{\frac{(\theta+s-1)^{\theta+s-1}}{\Gamma(\theta+s)} \frac{(\rho+n-s-1)^{\rho+n-s-1}}{\Gamma(\rho+n-s)}}{\frac{(\theta+\rho+n-2)^{\theta+\rho+n-2}}{(\theta+\rho+n-1)\Gamma(\theta+\rho+n-1)}} \\
\approx \;& \tfrac{1}{2} - \sqrt{\tfrac{\omega(1-\omega)\Delta}{n}} \tfrac{\frac{e^{\theta+s-1}}{\sqrt{2\pi(\theta+s-1)}} \frac{e^{\rho+n-s-1}}{\sqrt{2\pi(\rho+n-s-1)}}}{\frac{e^{\theta+\rho+n-2}}{(\theta+\rho+n-1)\sqrt{2\pi(\theta+\rho+n-2)}}} \\
= \;& \tfrac{1}{2} - \sqrt{\tfrac{\omega(1-\omega)\Delta}{n}} \tfrac{(\theta+\rho+n-1)\sqrt{\theta+\rho+n-2}}{\sqrt{2\pi(\theta+s-1)(\rho+n-s-1)}} \\
\approx \;& \tfrac{1}{2} - \sqrt{\tfrac{\omega(1-\omega)\Delta}{n}} \tfrac{\frac{n}{\Delta}\sqrt{\frac{n}{\Delta}}}{\sqrt{2\pi \frac{n\omega}{\Delta} \frac{n(1-\omega)}{\Delta}}} \\
= \;& \tfrac{1}{2} - \tfrac{1}{\sqrt{2\pi}}.
\end{aligned}
$$

Finally, we re-write (17) as

$$
\left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\right)\omega(1-\omega)\Delta + O\left(\frac{1}{\sqrt{n}}\right) \leq n\sigma_{n-}^2 \leq n\sigma_n^2 = \omega(1-\omega)\Delta + O\left(\frac{1}{n}\right).
$$

Letting $n \to \infty$, we get

$$
\left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\right)\omega(1-\omega)\Delta \leq \liminf_{n\to\infty} n\sigma_{n-}^2 \leq \limsup_{n\to\infty} n\sigma_{n-}^2 \leq \omega(1-\omega)\Delta.
$$

Therefore, $\Delta = 0$ minimizes the limit of lower semivariance of $\sqrt{n}\tilde{p}_{n,\omega}$.

We can apply similar arguments to the upper semivariance of $\sqrt{n}\tilde{p}_{n,\omega}$.

A6. Proof of Theorem 5

Note that $\Delta_6 - \omega\Delta_8 = (1-\tau)(2\omega-1) + \omega^2(\delta-1)$. By the proof of Lemma 1,

$$
\begin{aligned}
\mathbb{E}[\tilde{p}_{n,\omega}] &= \tfrac{\theta+s}{\theta+\rho+n} = \tfrac{n^2\omega+n\Delta_6+\Delta_3}{n^2+n\Delta_8+2\Delta_3} \\
&= \omega + \tfrac{n(\Delta_6-\omega\Delta_8)+(1-2\omega)\Delta_3}{n^2+n\Delta_8+2\Delta_3} \\
&= \omega + \tfrac{(1-\tau)(2\omega-1)+\omega^2(\delta-1)}{n} + O(\tfrac{1}{n^2}).
\end{aligned}
$$

The median of $\tilde{p}_{n,\omega}$ can be approximated by (e.g., Kerman [2011])

$$\begin{aligned}
\frac{\theta+s-\frac{1}{3}}{\theta+\rho+n-\frac{2}{3}} &= \frac{\frac{n^2\omega+n\Delta_6+\Delta_3}{n\Delta+\Delta_3}-\frac{1}{3}\frac{n\Delta+\Delta_3}{n\Delta+\Delta_3}}{\frac{n^2+n\Delta_8+2\Delta_3}{n\Delta+\Delta_3}-\frac{2}{3}\frac{n\Delta+\Delta_3}{n\Delta+\Delta_3}} = \frac{3(n^2\omega+n\Delta_6+\Delta_3)-(n\Delta+\Delta_3)}{3(n^2+n\Delta_8+2\Delta_3)-2(n\Delta+\Delta_3)} \\
&= \frac{3n^2\omega+n(3\Delta_6-\Delta)+2\Delta_3}{3n^2+n(3\Delta_8-2\Delta)+4\Delta_3} = \omega + \frac{n[3\Delta_6-3\omega\Delta_8+(2\omega-1)\Delta]}{3n^2+n(3\Delta_8-2\Delta)+4\Delta_3} + O\left(\frac{1}{n^2}\right) \\
&= \omega + \frac{n\Big[(2\omega-1)[3(1-\tau)+\Delta]+3\omega^2(\delta-1)\Big]}{3n^2+n(3\Delta_8-2\Delta)+4\Delta_3} + O\left(\frac{1}{n^2}\right) \\
&= \omega + \frac{n[2(1-\tau)(2\omega-1)-(\omega+1)\omega(1-\delta)]}{3n^2+n(3\Delta_8-2\Delta)+4\Delta_3} + O\left(\frac{1}{n^2}\right) \\
&= \omega + \frac{2(1-\tau)(2\omega-1)-(\omega+1)\omega(1-\delta)}{3n} + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

In the above two approximations, both the mean and the median react negatively with an increasing τ, when n is finitely large and $\omega > .5$. To maximize the mean and median, we thus minimize $\tau \in (1-\omega+\delta\omega)$ such that $\theta > 0$ and $\rho > 0$. Particularly, using the proof of Theorem 3, we can make it smaller than $1-\omega+\delta\omega+\frac{2c}{n}$ when n is large enough, i.e.

$$\begin{cases}
1-\omega+\delta\omega < \underset{\tau}{\operatorname{argmax}} \operatorname{MEAN}\left(\tilde{p}_{n,\omega}|\theta,\rho \geq \frac{1}{n}\right) \leq 1-\omega+\delta\omega+\frac{2c}{n}, \\
1-\omega+\delta\omega < \underset{\tau}{\operatorname{argmax}} \operatorname{MEDIAN}\left(\tilde{p}_{n,\omega}|\theta,\rho \geq \frac{1}{n}\right) \leq 1-\omega+\delta\omega+\frac{2c}{n}.
\end{cases}$$

Finally, let $n \to \infty$ in the above inequalities to get

$$\begin{cases}
\lim_{n\to\infty} \underset{\tau}{\operatorname{argmax}} \operatorname{MEAN}\left(\tilde{p}_{n,\omega}|\theta,\rho \geq \frac{1}{n}\right) = 1-\omega+\delta\omega, \\
\lim_{n\to\infty} \underset{\tau}{\operatorname{argmax}} \operatorname{MEDIAN}\left(\tilde{p}_{n,\omega}|\theta,\rho \geq \frac{1}{n}\right) = 1-\omega+\delta\omega,
\end{cases}$$

for any $\omega \in (.5, 1)$.

A7. Proof of Theorem 6

When n is large, $\theta, \rho \geq \frac{1}{n}$ implies $\tau(\omega) \in (1-\omega+\delta\omega, 1)$. By the proof of Lemma 1, both $\theta+s \to \infty$ and $\rho+n-s \to \infty$ as $n \to \infty$. Applying Stirling's formula, Johnson et al. [1995, page 219] derive the following approximation for the ratio of the variance and the squared MAD around the mean:

$$\lim_{\theta+s\to\infty,\rho+n-s\to\infty} \frac{\left(\mathbb{E}\left[|\tilde{p}_{n,\omega}-\mathbb{E}(\tilde{p}_{n,\omega})|\right]\right)^2}{\operatorname{VAR}(\tilde{p}_{n,\omega})} = \frac{2}{\pi}.$$

Thus, minimizing the MAD around the mean is equivalent to minimizing the variance of $\tilde{p}_{n,\omega}$ when n is large. By Theorem 3, we have proved Theorem 6.

A8. Proof of Theorem 7

When n is large, $\theta, \rho \geq \frac{1}{n}$ implies $\tau(\omega) \in (1-\omega+\delta\omega, 1)$. Thus, $\theta + s \to \infty$, $\rho + n - s \to \infty$ and $\theta+\rho+n \to \infty$ as $n \to \infty$. In this proof, we use $\frac{\theta+s-\frac{1}{3}}{\theta+\rho+n-\frac{2}{3}}$ to approximate the median, say m, of $\tilde{p}_{n,\omega}$ (e.g., Kerman [2011]). By (10), (11), and the proof of Lemma 1,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n\mathbb{E}[|\tilde{p}_{n,\omega}-m|]}{\mathbb{E}[|\tilde{p}_{n,\omega}-\mathbb{E}(\tilde{p}_{n,\omega})|]} &= \lim_{n\to\infty} \frac{\frac{2n\left(\frac{\theta+s-\frac{1}{3}}{\theta+\rho+n-\frac{2}{3}}\right)^{\theta+s}\left(1-\frac{\theta+s-\frac{1}{3}}{\theta+\rho+n-\frac{2}{3}}\right)^{\rho+n-s}}{(\theta+\rho+n)\beta(\theta+s,\rho+n-s)}}{\frac{2(\theta+s)^{\theta+s}(\rho+n-s)^{\rho+n-s}}{\beta(\theta+s,\rho+n-s)(\theta+\rho+n)^{\theta+\rho+n}}} \\
&= \lim_{n\to\infty} \frac{n\left(\frac{\theta+s-\frac{1}{3}}{\theta+\rho+n-\frac{2}{3}}\right)^{\theta+s}\left(\frac{\rho+n-s-\frac{1}{3}}{\theta+\rho+n-\frac{2}{3}}\right)^{\rho+n-s}(\theta+\rho+n)^{\theta+\rho+n}}{(\theta+\rho+n)(\theta+s)^{\theta+s}(\rho+n-s)^{\rho+n-s}} \\
&= \lim_{n\to\infty} \frac{n\left(\frac{\theta+s-\frac{1}{3}}{\theta+s}\right)^{\theta+s}\left(\frac{\rho+n-s-\frac{1}{3}}{\rho+n-s}\right)^{\rho+n-s}}{(\theta+\rho+n)\left(\frac{\theta+\rho+n-\frac{2}{3}}{\theta+\rho+n}\right)^{\theta+\rho+n}} \\
&= \frac{\lim\limits_{\theta+s\to\infty}\left(1-\frac{1}{3(\theta+s)}\right)^{\theta+s}\lim\limits_{\rho+n-s\to\infty}\left(1-\frac{1}{3(\rho+n-s)}\right)^{\rho+n-s}}{\lim\limits_{n\to\infty}\frac{\theta+\rho+n}{n}\lim\limits_{\theta+\rho+n\to\infty}\left(1-\frac{2}{3(\theta+\rho+n)}\right)^{\theta+\rho+n}} \\
&= \frac{\Delta e^{-\frac{1}{3}}e^{-\frac{1}{3}}}{e^{-\frac{2}{3}}} \\
&= \Delta.
\end{aligned}
$$

By the proof of Theorem 6,

$$
\lim_{n\to\infty} \frac{\left(\mathbb{E}\left[n^{\frac{3}{2}}|\tilde{p}_{n,\omega}-m|\right]\right)^2}{\mathrm{VAR}(\sqrt{n}\tilde{p}_{n,\omega})} = \lim_{n\to\infty} \frac{(n\mathbb{E}[|\tilde{p}_{n,\omega}-m|])^2}{(\mathbb{E}[|\tilde{p}_{n,\omega}-\mathbb{E}(\tilde{p}_{n,\omega})|])^2}\frac{(\mathbb{E}[|\tilde{p}_{n,\omega}-\mathbb{E}(\tilde{p}_{n,\omega})|])^2}{\mathrm{VAR}(\tilde{p}_{n,\omega})} = \frac{2\Delta^2}{\pi}.
$$

Therefore, by the proof of Theorem 3,

$$
\lim_{n\to\infty}\left(\mathbb{E}\left[n^{\frac{3}{2}}|\tilde{p}_{n,\omega}-m|\right]\right)^2 = \frac{2\Delta^2}{\pi}\lim_{n\to\infty}\mathrm{VAR}(\sqrt{n}\tilde{p}_{n,\omega}) = \frac{2\omega(1-\omega)\Delta^3}{\pi}.
$$

The above limit is non-negative and has a minimum value at $\Delta = 0$.

A9. Proof of Theorem 8

For any $i, j \in N$ and $i \neq j$, if i uniformly outperforms j and they have equal employment opportunity, then

$$\begin{aligned}
\gamma_j[v] &= \sum_{T \subseteq N: j \in T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{j})] \\
&= \sum_{T \subseteq N: j \in T, i \in T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{j})] \\
&\quad + \sum_{T \subseteq N: j \in T, i \notin T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{j})] \\
&\leq \sum_{T \subseteq N: j \in T, i \in T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{i})] \\
&\quad + \sum_{T \subseteq N: j \in T, i \notin T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{j})] \\
&\overset{Z = T \setminus \bar{j}}{=} \sum_{T \subseteq N: j \in T, i \in T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{i})] \\
&\quad + \sum_{Z \subseteq N: j \notin Z, i \notin Z} \mathbb{P}(\mathbf{S} = Z \cup \bar{j})[v(Z \cup \bar{j}) - v(Z)] \\
&\leq \sum_{T \subseteq N: j \in T, i \in T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{i})] \\
&\quad + \sum_{Z \subseteq N: j \notin Z, i \notin Z} \mathbb{P}(\mathbf{S} = Z \cup \bar{i})[v(Z \cup \bar{i}) - v(Z)] \\
&\overset{T = Z \cup \bar{i}}{=} \sum_{T \subseteq N: j \in T, i \in T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{i})] \\
&\quad + \sum_{T \subseteq N: j \notin T, i \in T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{i})] \\
&= \sum_{T \subseteq N: i \in T} \mathbb{P}(\mathbf{S} = T)[v(T) - v(T \setminus \bar{i})] \\
&= \gamma_i[v].
\end{aligned}$$

The Beta-Binomial distribution and (3) are not required in the proof. Also, the equality of employment opportunity is not required for other players in N, except that $\mathbb{P}(\mathbf{S} = Z \cup \bar{i}) = \mathbb{P}(\mathbf{S} = Z \cup \bar{j})$ for any $Z \subseteq N \setminus \bar{i} \setminus \bar{j}$. This identity means i and j have equal chance to be hired by Z, when both are unemployed.

Similar arguments can be used to prove $\lambda_j[v] \leq \lambda_i[v]$. In this case, we use the identity $\mathbb{P}(\mathbf{S} = Z \setminus \bar{i}) = \mathbb{P}(\mathbf{S} = Z \setminus \bar{j})$ for any $Z \subseteq N$ such that $i, j \in Z$. This identity implies both i and j have equal opportunity to be laid off from Z, when both are employed in Z.

A10. Proof of Theorem 9

When the employed individuals are symmetric in v, Theorem 8 claims that each employed person receives $\frac{(1-\tau)v(\mathbf{S})}{|\mathbf{S}|}$ as his employment benefits. Similarly, any unemployed person receives $\frac{(\tau-\delta)v(\mathbf{S})}{n-|\mathbf{S}|}$ as his unemployment welfare.

Finally,

$$\frac{(1-\tau)v(\mathbf{S})}{|\mathbf{S}|} = \frac{(\tau-\delta)v(\mathbf{S})}{n-|\mathbf{S}|}$$

is equivalent to

$$\frac{(1-\tau)}{\omega} = \frac{(\tau-\delta)}{1-\omega},$$

which itself is equivalent to

$$\tau = 1-\omega+\delta\omega.$$

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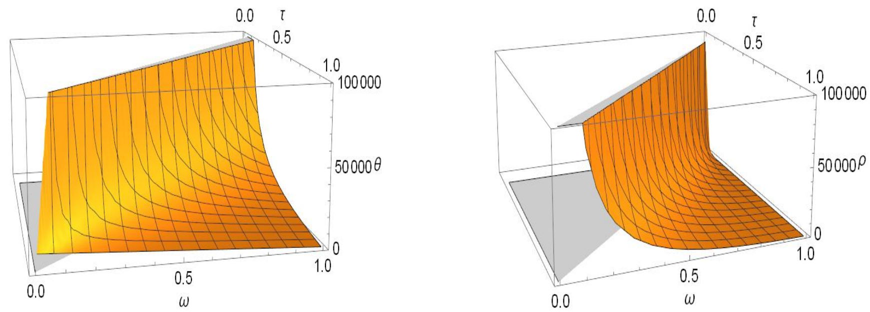
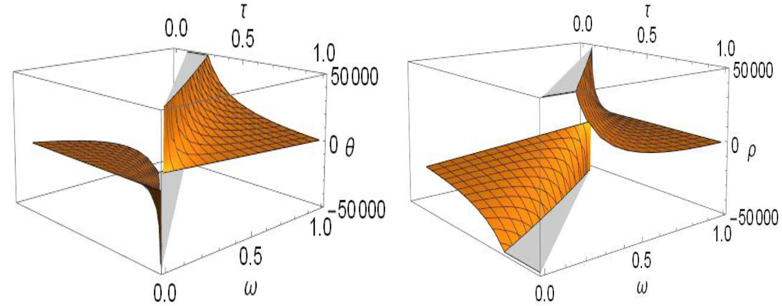


(a) (ω, τ) for $0 < \theta \leq 100000$ (b) (ω, τ) for $0 < \rho \leq 100000$



(c) (ω, τ) for $-50000 < \theta \leq 50000$ (d) (ω, τ) for $-50000 < \rho \leq 50000$

Figure I

The Feasible Solution Sets $\Omega_{10000,\omega,.1}$ for $\omega \in (0, 1)$.