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Abstract

This paper analyzes the core of cooperative games generated by asymmetric aggregative normal-form games, i.e., games where the payoff of each player depends on his strategy and the sum of the strategies of all players. We assume that each coalition calculates its worth presuming that the outside players stand alone and select individually optimal strategies (Chander & Tulkens 1997). We show that under some mild monotonicity assumptions on payoffs, the resulting cooperative game is balanced, i.e. it has a non-empty γ-core. Our paper thus offers an existence result for a core notion that is considered quite often in the theory and applications of cooperative games with externalities.

Keywords: cooperative game; aggregative game; balancedness; core

JEL Classification: C71

1 Introduction

The core is one of the most widely used solution concepts in cooperative games. This notion describes all the ways of splitting the benefits created by the entire set of players that are immune to coalitional rejections. To define the core, one needs to first define the characteristic function of a coalition, i.e., its worth. For a cooperative game with externalities, namely a game where the worth of a coalition depends on the actions of the players outside the coalition, the specification of the characteristic function requires a forecast or a belief about the behavior of the outsiders, and in particular about their coalition structure.

One of the most often encountered beliefs (in theoretical and applied works) is provided by the so called γ-beliefs. According to this approach the members of a coalition postulate

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that the outsiders play *individual* best replies to the formation of the coalition, i.e., they form singleton coalitions. These beliefs, the origins of which go back to Hart & Kurz (1983) and also to Chander & Tulkens (1997), give rise to the γ-*characteristic function* of a coalition. The γ-*core* is the set of all allocations of the worth of the grand coalition that no other coalition has incentive to block, given that its worth is defined by the γ-characteristic function.[1]

Chander & Tulkens (1997) defined the γ-core for the case of an economy where agents produce private goods. The production processes in this economy generate externalities, the total value of which affects the agents negatively. Chander & Tulkens showed that, under certain assumptions, a specific allocation of the worth of the grand coalition is in the core. Helm (2001) extended their result by showing that the cooperative game defined by this economy is balanced, which means its core is non-empty. Helm's balancedness result silently relies on the aggregative structure of the model.[2]

Since then, the γ-core formulation has been studied in various contexts. Currarini & Marini (2003) analyzed the γ-core for symmetric cooperative games. They showed that the core is non-empty under the assumption of strategic complementarity, positing also that each coalition acts as a Stackelberg leader. Lardon (2012, 2014) applied the γ-core framework to various Cournot and Bertrand oligopolistic markets and derived conditions under which the corresponding cooperative games have non-empty core. Chander & Wooders (2012) went one step further by defining and analyzing the γ-core for extensive-form cooperative games. In another interesting extension, Lardon (2017) defined the γ-core of interval oligopolistic cooperative games, i.e., games where each coalition is assigned an interval of possible worths. Finally we note that Chander (2017) justified the use of the concept by showing that each allocation in the γ-core corresponds to an equilibrium of an underlying infinitely repeated coalition formation game.

One important gap in the literature has to do with the lack of general existence results of γ-core, as most of the works focus on either applications or on symmetric general games. The goal of the current paper is to make a step towards filling this gap. In particular it examines the non-emptiness of γ-core for cooperative games generated by an important class of normal-form games that are encountered frequently in applications: the class of asymmetric aggregative games. Many economic models fit this class, such as oligopolic markets, economies with public goods, economies with environmental externalities, contest games, cost-sharing games, etc.

To derive our results we impose basically two conditions on the underlying aggregative normal-form game: (i) the payoff function of each player is monotonic in the sum of the strategies of all players; (ii) the marginal payoff of each player is decreasing in own strategy and in the sum of the strategies of all players. The first condition allows us to analyze both the decreasing and increasing monotonicity cases. A typical example in the first case is a Cournot oligopoly; and a typical example in the second case is a public goods economy. The second condition is the so called "strong concavity" assumption (Corchon 1994) and it helps us to compare the equilibrium strategies of the players in environments with and

[1] Other well-known assumptions on beliefs give core notions such as the α and β-core (Aumann 1959); the δ-core (Hart & Kurz 1983); the sequential γ-core (Currarini & Marini 2003); the recursive core (Huang & Sjostrom 2003; Koczy 2007); or the core of games with multiple externalities (Nax 2014).

[2] We shall return to this paper later on in the introduction.

without coalition formation (this comparison plays significant role in our analysis). Under these assumptions we show that the resulting cooperative game is balanced and thus its core is non-empty.

We note that among the papers discussed earlier in the introduction, the work of Helm (2001) is the closest to the current paper, even though Helm deals with a specific application. Essentially our paper shows that the tools and the balancedness result of Helm go beyond the simple model of economies with production externalities. What allows for this generalization is the aggregative structure that exists in both papers (and the simplifications that this structure carries with it). Finally, Stamatopoulos (2015) uses also aggregative games, but it restricts attention to the symmetric case only. That paper shows that the γ-core is non-empty by showing that the equal split of the worth of the grand coalition is a core allocation.

The paper is organized as follows. Section 2 introduces the framework and Section 3 present and proves the results. Finally, the last section concludes by offering a brief discussion of the paper.

2 Framework

Consider the collection of objects $\{N, (X_i, u_i)_{i \in N}\}$, where $N = \{1, 2, \ldots, n\}$ is a set of players, X_i is the strategy set of player i and u_i is i's payoff function. We assume an aggregative payoff structure, i.e., the payoff of each player depends on his strategy and the sum of the strategies of all players. Hence i's payoff is of the form $u_i(x_i, x)$, where $x_i \in X_i$ and $x = \sum_{k \in N} x_k$.

We examine environments where players can form coalitions by signing contracts and transferring utilities among themselves. Our focus is on the formation of the grand coalition, N. This event may be blocked by the formation of smaller coalitions. Denote by S such a coalition. Should S deviate from the rest of the society, its payoff (i.e., the sum of the payoffs of its members) will depend on the partition of the outsiders. So S needs a forecast about the partition that the outsiders will form. Once such forecast was proposed by Hart & Kurz (1983) and Chander and Tulkens (1997) according to which S believes that the outsiders will all stay separate and will select individual best strategies. Let us denote the resulting normal form game (where the members of S act as a coalition and all outsiders form singleton coalitions) by Γ^S.

The optimization problems in Γ^S are described by

$$\max_{\{x_i\}_{i \in S}} \sum_{i \in S} u_i(x_i, x) \tag{1}$$

$$\max_{x_j} u_j(x_j, x), \quad j \notin S \tag{2}$$

The equilibrium strategies[3] of $i \in S$ and $j \notin S$ in Γ^S are denoted by x_i^S and x_j^S; denote $x^S = \sum_{k \in N} x_k^S$. The worth of S is then given by the γ-*characteristic function*,

[3] In the next paragraphs we discuss the issue of existence of equilibrium.

$$v_\gamma(S) = \sum_{i \in S} u_i(x_i^S, x^S) \tag{3}$$

The worth of N, $v(N)$, is defined in the standard way. All the above result into the cooperative game (N, v_γ). The core of this game is the set of all allocations of $v(N)$ that no coalition S can block given the characteristic function in (3). This core is known as the *γ-core* and it is defined by the set

$$\mathcal{W}_\gamma = \{(w_1, w_2, \ldots, w_n) \in \mathbb{R}^n\colon \sum_{i \in N} w_i = v(N) \text{ and } \sum_{i \in S} w_i \geq v_\gamma(S), \text{ all } S\}$$

To analyze the non-emptiness of $\mathcal{W}_\gamma$ we will frequently compare the equilibrium outcome of Γ^S with the equilibrium outcome of the game where no coalitions form and *all* players act alone. We denote the latter game by Γ. The equilibrium strategies of players[4] $i \in S$ and $j \notin S$ in Γ will be denoted by x_i^N and x_j^N respectively; moreover $x^N = \sum_{k \in N} x_k^N$.

Our analysis rests on certain assumptions. The first assumes that payoffs are differentiable and that equilibrium strategies lie in the interior of the strategy sets (X_i is, as usual, a convex and compact subset of $\mathbb{R}$, $i \in N$).

B0 $u_i(x_i, x)$ is continuously differentiable in each argument; further, the equilibrium strategies in Γ^S and Γ satisfy the standard first-order conditions.

To state the second assumption, define the function

$$a_i(x_i, x) = \frac{\partial u_i(x_i, x)}{\partial x_i} + \frac{\partial u_i(x_i, x)}{\partial x}$$

B1 $a_i(x_i, x)$ strictly decreases in each argument, for all i.

The above is the strong concavity assumption of Corchon (1994) and it will help us to perform comparisons between the equilibrium strategies in Γ^S and Γ. Notice, finally, that the above conditions jointly ensure that a unique Nash equilibrium exists in both Γ^S and Γ.

3 Analysis

Assumptions B0-B1 will hold throughout the paper. Our analysis will also use a monotonicity assumption of $u_i(x_i, x)$ over x. In particular, we will show that $\mathcal{W}_\gamma$ is non-empty if $u_i(x_i, x)$ is either decreasing or increasing in x. In accordance to this we will break the analysis into two cases. Subsection 3.1 will cover the decreasing case and subsection 3.2 will cover the case the increasing case.

[4] Although all players in Γ stand alone, we continue to categorize them in accordance to Γ^S to facilitate the comparisons that will follow.

3.1 Decreasing case

The following holds throughout the current subsection.

B2 $u_i(x_i, x)$ *is decreasing in* x*, for all* i*.*

Given the above, the first result compares the equilibrium strategies of the players in Γ^S and Γ.

Lemma 1 *The following hold:*

(i) $x_j^S \geq x_j^N$ $j \notin S$.

(ii) $\sum_{i \in S} x_i^S \leq \sum_{i \in S} x_i^N$.

Proof (i) We'll first show that $x^N \geq x^S$. Observe first that for $j \notin S$ we have the first-order conditions in Γ^S and Γ

$$\frac{\partial u_j(x_j^S, x^S)}{\partial x_j} + \frac{\partial u_j(x_j^S, x^S)}{\partial x} = 0 \tag{4}$$

$$\frac{\partial u_j(x_j^N, x^N)}{\partial x_j} + \frac{\partial u_j(x_j^N, x^N)}{\partial x} = 0 \tag{5}$$

Assume momentarily that $x^N < x^S$. Then from (4) and B1 we get

$$\frac{\partial u_j(x_j^S, x^N)}{\partial x_j} + \frac{\partial u_j(x_j^S, x^N)}{\partial x} > 0 \tag{6}$$

The above expression combined with (5) and B1 imply that $x_j^N > x_j^S$, for all $j \notin S$.

Observe next that for $i \in S$ we have

$$\frac{\partial u_i(x_i^S, x^S)}{\partial x_i} + \frac{\partial u_i(x_i^S, x^S)}{\partial x} + \sum_{\substack{k \in S \\ k \neq i}} \frac{\partial u_k(x_k^S, x^S)}{\partial x} = 0 \tag{7}$$

$$\frac{\partial u_i(x_i^N, x^N)}{\partial x_i} + \frac{\partial u_i(x_i^N, x^N)}{\partial x} = 0 \tag{8}$$

Condition B2 combined with (7) implies

$$\frac{\partial u_i(x_i^S, x^S)}{\partial x_i} + \frac{\partial u_i(x_i^S, x^S)}{\partial x} > 0 \tag{9}$$

Hence by the momentary assumption that $x^S > x^N$ and by B1 we get

$$\frac{\partial u_i(x_i^S, x^N)}{\partial x_i} + \frac{\partial u_i(x_i^S, x^N)}{\partial x} > 0 \tag{10}$$

Hence to restore (8) we need by B1 the inequality $x_i^N > x_i^S$. But then all the above imply that $\sum_{j \notin S} x_j^N + \sum_{i \in S} x_i^N \geq x^S$, which contradicts the momentary assumption $x^S > x^N$. We conclude that $x^N \geq x^S$ (this inequality will cause no further contradictions). We can then use the latter inequality and repeat the argument after expressions (4)-(5) to conclude that $x_j^S \geq x_j^N$, for all $j \notin S$.

(ii) Since $x^N \geq x^S$ and $\sum_{j \notin S} x_j^S \geq \sum_{j \notin S} x_j^N$ it must be that $\sum_{i \in S} x_i^N \geq \sum_{i \in S} x_i^S$. ■

To show the non-emptiness of the core we will utilize the Shapley-Bondareva balancedness theorem. Let $\mathcal{C}$ be the set of all coalitions that can be formed by the players in N; and let $\mathcal{C}_i$ be the set of all coalitions that include i. The vector $(\delta_S)_{S \in \mathcal{C}}$, where $\delta_S \in [0,1]$ for all S, is a balanced collection of weights if for all $i \in N$, $\sum_{S \in \mathcal{C}_i} \delta_S = 1$.

Proposition *(Bondareva 1963, Shapley 1967) Let (N, v) be a cooperative game, where v denotes its characteristic function. The core of (N, v) is non-empty iff for every balanced collection of weights, $\sum_{S \in \mathcal{C}} \delta_S v(S) \leq v(N)$.*

In order to use this result we need an intermediate step. Take a balanced collection of weights $(\delta_S)_{S \in \mathcal{C}}$. Given this collection, we will show that the sum of the "average" equilibrium strategies of all players, where the average is computed with respect to all coalitions that each players belongs to, i.e., $\sum_{j \in N} \sum_{S \in \mathcal{C}_j} \delta_S x_j^S$, is not higher than the sum of the "average" equilibrium strategies with respect to all coalitions that a certain player, say i, belongs to, i.e., $\sum_{j \in N} \sum_{S \in \mathcal{C}_i} \delta_S x_j^S$.

Lemma 2 $\sum_{j \in N} \sum_{S \in \mathcal{C}_j} \delta_S x_j^S \leq \sum_{j \in N} \sum_{S \in \mathcal{C}_i} \delta_S x_j^S$, *for any $i \neq j$ and any balanced collection of weights $(\delta_S)_{S \in \mathcal{C}}$.*

Proof Let N' be the set of all players but i. Then to prove the stated inequality we just need to sum over all $j \in N'$.

Notice that

$$\sum_{j \in N'} \sum_{S \in \mathcal{C}_j} \delta_S x_j^S = \sum_{j \in N'} \Big[\sum_{S \in \mathcal{C}_j \cap \mathcal{C}_i} \delta_S x_j^S + \sum_{S \in \mathcal{C}_j \setminus \mathcal{C}_j \cap \mathcal{C}_i} \delta_S x_j^S \Big] \tag{11}$$

and

$$\sum_{j \in N'} \sum_{S \in \mathcal{C}_i} \delta_S x_j^S = \sum_{j \in N'} \Big[\sum_{S \in \mathcal{C}_i \cap \mathcal{C}_j} \delta_S x_j^S + \sum_{S \in \mathcal{C}_i \setminus \mathcal{C}_i \cap \mathcal{C}_j} \delta_S x_j^S \Big] \tag{12}$$

Expressions (11) and (12) imply that

$$\sum_{j\in N'}\sum_{S\in\mathcal{C}_j}\delta_S x_j^S \leq \sum_{j\in N'}\sum_{S\in\mathcal{C}_i}\delta_S x_j^S \Leftrightarrow \sum_{j\in N'}\sum_{S\in\mathcal{C}_j\setminus\mathcal{C}_j\cap\mathcal{C}_i}\delta_S x_j^S \leq \sum_{j\in N'}\sum_{S\in\mathcal{C}_i\setminus\mathcal{C}_i\cap\mathcal{C}_j}\delta_S x_j^S \tag{13}$$

Observe next that

$$\sum_{j\in N'}\sum_{S\in\mathcal{C}_j\setminus\mathcal{C}_j\cap\mathcal{C}_i}\delta_S x_j^S = \sum_{S\in\mathcal{C}\setminus\mathcal{C}_i}\delta_S\sum_{j\in S}x_j^S \leq \sum_{S\in\mathcal{C}\setminus\mathcal{C}_i}\delta_S\sum_{j\in S}x_j^N \tag{14}$$

where the inequality is due to Lemma 1(ii). Moreover, Lemma 1(i) implies

$$\sum_{j\in N'}\sum_{S\in\mathcal{C}_i\setminus\mathcal{C}_i\cap\mathcal{C}_j}\delta_S x_j^S \geq \sum_{j\in N'}\sum_{S\in\mathcal{C}_i\setminus\mathcal{C}_i\cap\mathcal{C}_j}\delta_S x_j^N, \tag{15}$$

since, for each j, we sum over coalitions in which j is an outsider. Before continuing let us write

$$\sum_{S\in\mathcal{C}\setminus\mathcal{C}_i}\delta_S\sum_{j\in S}x_j^N = \sum_{j\in N'}\sum_{S\in\mathcal{C}_j\setminus\mathcal{C}_j\cap\mathcal{C}_i}\delta_S x_j^N \tag{16}$$

Since we work with a balanced collection of weights we have

$$\sum_{S\in\mathcal{C}_j\setminus\mathcal{C}_j\cap\mathcal{C}_i}\delta_S = 1 - \sum_{S\in\mathcal{C}_j\cap\mathcal{C}_i}\delta_S, \quad \sum_{S\in\mathcal{C}_i\setminus\mathcal{C}_i\cap\mathcal{C}_j}\delta_S = 1 - \sum_{S\in\mathcal{C}_i\cap\mathcal{C}_j}\delta_S$$

Hence

$$\sum_{S\in\mathcal{C}_j\setminus\mathcal{C}_j\cap\mathcal{C}_i}\delta_S = \sum_{S\in\mathcal{C}_i\setminus\mathcal{C}_i\cap\mathcal{C}_j}\delta_S \tag{17}$$

Expression (17) implies

$$\sum_{j\in N'}\sum_{S\in\mathcal{C}_j\setminus\mathcal{C}_j\cap\mathcal{C}_i}\delta_S x_j^N = \sum_{j\in N'}\sum_{S\in\mathcal{C}_i\setminus\mathcal{C}_i\cap\mathcal{C}_j}\delta_S x_j^N \tag{18}$$

since x_j^N is independent of the weights. The combination of (13)-(18) proves the result. ■

We can now state and prove the following.

Proposition 1 (N, v_γ) *is balanced and thus* $\mathcal{W}_\gamma$ *is non-empty.*

Proof Take a balanced collection of weights $(\delta_S)_{S\in\mathcal{C}}$. For $i \in N$ define $x_i^{\mathcal{C}} = \sum\limits_{S\in\mathcal{C}_i} \delta_S x_i^S$. Then:

$$\begin{aligned}
v(N) &\geq \sum_{i\in N} u_i(x_i^{\mathcal{C}}, \sum_{k\in N} x_k^{\mathcal{C}}) = \sum_{i\in N} u_i(x_i^{\mathcal{C}}, \sum_{k\in N}\sum_{S\in\mathcal{C}_k} \delta_S x_k^S) \\
&\geq \sum_{i\in N} u_i(x_i^{\mathcal{C}}, \sum_{k\in N}\sum_{S\in\mathcal{C}_i} \delta_S x_k^S) \text{ [by Lemma 2 and B2]} \\
&= \sum_{i\in N} u_i(\sum_{S\in\mathcal{C}_i} \delta_S x_i^S, \sum_{k\in N}\sum_{S\in\mathcal{C}_i} \delta_S x_k^S) \\
&\geq \sum_{i\in N}\sum_{S\in\mathcal{C}_i} \delta_S u_i(x_i^S, \sum_{k\in N} x_k^S) \text{ [by concavity (see B1)]} \\
&= \sum_{S\in\mathcal{C}} \delta_S \sum_{i\in S} u_i(x_i^S, \sum_{k\in N} x_k^S) = \sum_{S\in\mathcal{C}} \delta_S v_\gamma(S)
\end{aligned}$$

■

We note here that the concavity assumption is not uncommon when dealing with the non-emptiness of the core of normal-form games. For example, it is used by Zhao (1999) and Uyanik (2015) to show the non-emptiness of the α-core of cooperative games.

3.2 Increasing case

Let's now examine the case where the payoff of each player increases in the sum of the strategies of all players. So we assume the following throughout the current subsection.

B2′ $u_i(x_i, x)$ is increasing in x, for all i.

We first note that under the latter assumption, Lemma 1 will be modified as follows.

Lemma 3 *The following hold:*

(i) $x_j^S \leq x_j^N$ $j \notin S$.

(ii) $\sum\limits_{i\in S} x_i^S \geq \sum\limits_{i\in S} x_i^N$.

Proof (i) The proof is similar to the proof of Lemma 1(i), but with reverse inequalities. This allows us to present the proof skipping some steps. We first show that $x^S \geq x^N$. To show this, we momentarily assume that the opposite holds, i.e., $x^S < x^N$. Then from (4) and B1 we get

$$\frac{\partial u_j(x_j^S, x^N)}{\partial x_j} + \frac{\partial u_j(x_j^S, x^N)}{\partial x} < 0 \tag{19}$$

Hence from (5) and B1 we need $x_j^N < x_j^S$, for all $j \notin S$.

Condition B2′ and (7) imply

$$\frac{\partial u_i(x_i^S, x^S)}{\partial x_i} + \frac{\partial u_i(x_i^S, x^S)}{\partial x} < 0 \tag{20}$$

Hence by the momentary assumption that $x^N > x^S$ and by B1 we get

$$\frac{\partial u_i(x_i^S, x^N)}{\partial x_i} + \frac{\partial u_i(x_i^S, x^N)}{\partial x} < 0 \tag{21}$$

Hence to restore (8) we need by B1 the inequality $x_i^S > x_i^N$, for all $i \in S$. But then $\sum\limits_{j \notin S} x_j^S + \sum\limits_{i \in S} x_i^S \geq x^N$, which contradicts the momentary assumption $x^S < x^N$. We conclude that $x^S \geq x^N$. Then, with an argument like the one in the last part of the proof of Lemma 1(i) we have that $x_j^N \geq x_j^S$, for all $j \notin S$.

(ii) As in the proof of Lemma 1(ii) after reversing the inequalities. ■

Not surprisingly, Lemma 3 implies the following (which is the analogue of Lemma 2).

Lemma 4 $\sum\limits_{j \in N} \sum\limits_{S \in \mathcal{C}_j} \delta_S x_j^S \geq \sum\limits_{j \in N} \sum\limits_{S \in \mathcal{C}_i} \delta_S x_j^S$, *for any* $i \neq j$ *and any balanced collection of weights* $(\delta_S)_{S \in \mathcal{C}}$.

Proof Similarly to the proof of Lemma 2 after reversing the inequalities. ■

Finally we have the following result.

Proposition 2 (N, v_γ) *is balanced and thus* $\mathcal{W}_\gamma$ *is non-empty.*

Proof Using the notation of Proposition 1 we have:

$$v(N) \geq \sum_{i \in N} u_i(x_i^{\mathcal{C}}, \sum_{k \in N} x_k^{\mathcal{C}}) = \sum_{i \in N} u_i(x_i^{\mathcal{C}}, \sum_{k \in N} \sum_{S \in \mathcal{C}_k} \delta_S x_k^S) \geq \sum_{i \in N} u_i(x_i^{\mathcal{C}}, \sum_{k \in N} \sum_{S \in \mathcal{C}_i} \delta_S x_k^S)$$

where the inequality follows from Lemma 4 and B2′. Then the rest of the proof is identical to the corresponding part of the proof of Proposition 1. ■

4 Discussion

This paper analyzed the γ-core of asymmetric aggregative normal-form games. It presented a set of conditions under which such games are balanced and have non-empty core. The analysis and results have taken, we believe, the literature on γ-core one step ahead: the framework presented in the paper is general enough to encompass many frequently encountered economic applications.

One can think of two further extensions that could take things even further. The one is to drop the monotonicity of the payoffs over the sum of the strategies of all players (although this monotonicity is natural in many economic applications). The other, and most interesting, extension would be to drop altogether the aggregative structure of the normal-form game. Then to compare the two resulting games Γ^S and Γ, and to reproduce results like those of Propositions 1 and 2, we would need to compare the equilibrium strategies of *each* member of S across the two games, and not just the sums of the equilibrium strategies of all members of S, as we did in the current paper. Both tasks are left for future research.

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