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Abstract

We introduce two-period general equilibrium models with heterogeneous producers and financial frictions. Any agent can borrow to realize their productive project but the debt repayment does not exceed a fraction (so-called *credit limit*) of the project's value. Our framework allows us to investigate the aggregate and distributional effects of credit limits and heterogeneity of agents. The connection between credit limits, welfare and efficiency is also explored.

JEL Classifications: D3, D5, E44, G10, G28.
Keywords: General equilibrium, credit limits, welfare, efficiency, wealth distribution.

1 Introduction

The great recession reminds that financial frictions play a key role in economic fluctuations. This issue has received much attention during the last three decades (see Bernanke and Gertler (1989), Kiyotaki and Moore (1997)). Among others, credit constraints are an important kind of financial frictions. Many papers (see Kiyotaki and Moore (1997), Geanakoplos and Zame (2002) for instance) investigate the roles of collateral constraints in general equilibrium contexts.[1] However, finding robust equilibrium properties or/and equilibrium computation is a challenge, especially when we work in multi-period and/or stochastic models (see Brumm, Kubler and Scheidegger (2017) for an excellent survey). There are two main difficulties: (1) agents are heterogeneous in many dimensions, and (2) credit constraints may occasionally bind (and hence, solutions may be on the boundary). Because of these difficulties, most of the existing literature has focused on balanced-growth equilibria or recursive equilibria or around steady-state analyses. In general equilibrium models, figuring out all possible

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[1]See also Gottardi and Kubler (2015), Guerrieria and Iacoviello (2017).

equilibria and their implications is far from trivial. Motivated by this point, we introduce a tractable many-agent, two-period general equilibrium model with financial frictions in order to figure out the whole set of equilibria and, thanks to this, be able to see the general picture. We investigate the aggregate and distributional effects of different parameters, including credit limits.

Our model has a finite number m of agents who differ in three dimensions: initial wealth, production function, and credit limit (see infra). They have two ways of investing: buying capital (to realize their production project) or buying a financial asset. Agents can borrow and then pay back in the next period. However, the debtor is required to put her project as collateral in order to borrow: in case that she does not repay, the creditor can seize the collateral. Due to the lack of commitment, the creditor can only obtain a fraction of the value of the project. Anticipating the possibility of default, the creditor limits the amount of credit so that the debt repayment of agent i will not exceed a fraction f_i of the debtor's project value. This fraction can be interpreted as the *credit limit* of this agent and is different across agents.[2] Our setup is in line with Kiyotaki and Moore (1997), Kiyotaki (1998).

In our economy, any agent can become a borrower (entrepreneur) or lender. Thanks to the simplicity of our model, we can categorize all possible equilibria and then explore properties of equilibrium outcomes. We identify who become entrepreneurs and lenders and provide a necessary and sufficient condition (based on fundamentals) under which there are n lenders and $m - n$ borrowers. This condition and hence the number of entrepreneurs depend not only on agents' productivities but also on the distribution of initial wealth and credit limit of agents. We also prove that the equilibrium interest rate increases when wealth or/and productivity or/and credit limit of borrowers increase.

After determining all possible equilibria, we investigate the effects of fundamentals, including credit limits.

First, our model indicates that an increase of productivity of an entrepreneur will increase the consumption of this agent and of all lenders. However, it makes decrease the consumption of other borrowers. Therefore, the inequality level (defined by the Gini coefficient) may increase or decrease when productivities increase.

Interestingly, there is a regime where an increase in an entrepreneur's productivity may have a negative impact on the aggregate output. This happens if this agent is the least productive entrepreneur. Indeed, in such a case, when this entrepreneur's productivity increases, she absorbs more capital and produces more. However, other producers (who are more productive than this agent) get less capital and they produce less. The net effect depends on the level of productivities. To sum up, increasing productivity of an agent does not necessarily have a positive effect on the economic growth.

Second, we focus on the distributional and aggregate effects of credit limits.

We point out that increasing credit limits of different borrowers (or producers) may have opposite effects on the agents' consumption. Indeed, when the credit limit of a borrower is relaxed, the interest rate increases, and hence makes the profit of other borrowers decrease but of lenders increase. However, the consumption of this borrower may be decreasing in her credit limit. This happens if her repayment increases faster

[2] This assumption is supported by the Enterprise Surveys (2018) panel datasets.

than her production value. As a result, the consumption of this borrower may display an inverted U-shape as a function of her credit limit.

One can imagine that relaxing credit limits would have positive impact on the aggregate output.However, the story is more complicated. Indeed, in our framework, we show that increasing credit limit of the most productive agent will increase the aggregate output but an increase of credit limit of an entrepreneur who is not the most productive agent, may decrease the aggregate output. The intuition is the following. When credit limit of a less productive entrepreneur increases, this agent can borrow more. However, the aggregate fund is finite. By consequence, more productive agents can get less fund. Therefore, the aggregate output may decrease. Although the aggregate output is not necessarily monotone in credit limits, it does not exceed that in the frictionless economy. It should be noticed that the non-monotone impact of financial development on economic growth is supported by Arcand et al. (2015). Our paper helps us to better understand the relationship between finance and growth because it shows a big picture by providing conditions under which the aggregate output is increasing or decreasing in credit limit which is one of the measures of the financial development of the economy.

To sum up, the effects of credit limits on individual consumptions and the aggregate output as well as are neither monotone nor linear. Our findings suggest that financial regulations (i.e., setting $(f_i)_i$ in our model) should take into account the actual structure of the economy, precisely the distribution of initial wealth and of productivities of agents.

Our last avenue of contribution concerns the equilibrium efficiency. Thanks to the tractability of the model, we can provide a necessary and sufficient condition under which an equilibrium is efficient. Given a distribution of productivity, the economy is more likely to be efficient if credit limit and/or initial wealth of the most productive agents are sufficiently high. Two points should be mentioned: (1) an equilibrium with binding borrowing constraints may be efficient or inefficient, and (2) a credit limit may lead to an efficient equilibrium but does not necessarily maximize the social welfare.

Our finding on the efficiency of equilibrium outcomes has a link with results in Section 3.1 in Gottardi and Kubler (2015) who consider stochastic exchange economies and provide a necessary and sufficient condition for the existence[3] of a Pareto-efficient equilibrium in two cases: (i) there is no aggregate uncertainty and (ii) consumers have identical CRRA utility. Although our model is deterministic, we introduce heterogeneous producers (any agent can produce by using their technology). However, in their results, the collateral requirements play no role while the credit limit in our setting role plays a crucial role on the equilibrium efficiency. It should be noticed that we can, furthermore, fully characterize all economies where efficient or inefficient equilibria arise.

Other related literature. Our paper is related to several strands of literature.

The present paper concerns the literature on the welfare effects of financial constraints. Jappelli and Pagano (1994, 1999) consider overlapping generations models with liquidity constraints and households living for three periods and argue that liquidity constraints may increase or decrease welfares. The central point in Jappelli and

[3] Gottardi and Kubler (2015) say that Pareto-efficient equilibria exist for an economy if there are initial distributions for which the competitive equilibrium is Pareto efficient.

Pagano (1994, 1999) is that liquidity constraints have two opposite effects on welfare: "they force the consumption of young below the unconstrained level but raise their permanent income by fostering capital accumulation". Obiols-Homs (2011) considers a general equilibrium with heterogeneous households (who borrowing is bounded by an exogenous limit) and a representative firms. He argues that the borrowing limit has a negative on the welfare of borrower if its *quantity effect* dominates its *price effect*. However, as in Jappelli and Pagano (1994, 1999), the mechanism of Obiols-Homs (2011) relies on the role of supply of credit to households who need to smooth their consumption while our mechanism focuses on credit to firms who need credit to finance their productive investment.

Catherine et al. (2017) build a dynamic general equilibrium model with heterogeneous firms and collateral constraints. They focus on the steady state and provide estimates suggesting that lifting financial frictions (modeled by collateral constraints) would increase aggregate welfare by 9.4% and aggregate output by 11%. Our paper differs from Catherine et al. (2017) in two aspects. First, although we also find that the aggregate output in the frictionless economy is higher than that in the economy with financial frictions, it is not a monotone function of the degree of financial friction. Second, both individual and social welfares may not monotone in the degree of financial friction. Interestingly, lifting credit constraint may decrease the welfare of some agent.

Our paper is also related to a growing literature on general equilibrium models with heterogeneous producers and financial frictions.[4] Let us mention some of them. Midrigan and Xu (2014) consider a two-sector model with a collateral constraint that requires the debt of producer does not exceed a fraction of its capital stock. They focus on balanced growth equilibrium to study the role of collateral constraint in determining TFP. Their parameterizations consistent with the data imply fairly small losses from misallocation, but potentially sizable losses from inefficiently low levels of entry and technology adoption. Khan and Thomas (2013) develop a dynamic stochastic general equilibrium with a representative household and heterogeneous firms facing a borrowing constraint (slightly different from ours) and focus on recursive equilibrium. They find that a negative shock to borrowing conditions can generate a large and persistent recession through disruptions to the distribution of capital. Buera and Shin (2013) develop a model with individual-specific technologies and collateral constraints to investigate the role of the misallocation and reallocation of resources in macroeconomic transitions. Buera and Shin (2013) find that collateral constraints have a large impact along the transition to the steady state. Moll (2014) studies the effect of collateral constraints on capital misallocation and aggregate productivity in a general equilibrium with a continuum of heterogeneous firms and financial frictions (modeled by a collateral constraint). Proposition 1 in Moll (2014) shows that the aggregate TFP is increasing in the leverage ratio which is the common across firms.[5]

[4]The reader is referred to Matsuyama (2007), Quadrini (2011), Brunnermeier, Eisenbach and Sannikov (2013) for more complete reviews on the macroeconomic effects of financial frictions and to Buera et al. (2015) for the relationship between entrepreneurship and financial frictions.

[5]In both Buera and Shin (2013), Moll (2014), the collateral constraint, which is slightly different from ours, states that the capital of a firm does not exceed a leverage ratio of its financial wealth.

Our contribution with respect to this literature is to point out that the aggregate TFP and the aggregate output in our model may not be monotone functions of the credit limits which are different across agents. One can also prove that when we set the same credit limit for every agent, the aggregate output and the aggregate TFP are increasing functions of this common credit limit.

The present paper differs from the existing literature in that the credit limit is individualized in our model while all credit variables in the above studies are common across firms. As we have just discussed, this credit heterogeneity plays an important role in the distribution of capital and of income as well as the in the aggregate output.

Our paper has a link with the literature on financial friction and economic inequality (Demirg-Kunt and Levine, 2009; Allub and Erosa, 2016; de Haan and Sturm, 2017). As mentioned in de Haan and Sturm (2017), the Gini coefficient is widely used in many empirical studies as a proxy of the inequality level. Our analyses on distributional effects of credit limits help us to identify winners and losers when credit limits are relaxed. An implication of our analyses, illustrated by numerical simulations, is that the Gini coefficient may not be a monotone function of credit limits. Whether relaxing credit limit increases or decreases the Gini coefficient depends on the economy's initial distribution of resources.

In terms of methodology, our contribution is to point out that the equilibrium outcomes of two-agent and more than two agents models are significantly different. In other words, the number of agents matters for the general equilibrium analysis. Let us mention two differences.

- First, relaxing credit limit in our two-agent model always increases the aggregate output. However, as we have mentioned, it can have a negative impact on the output if there are more than two agents.
- Second, as we have mentioned, the effects of productivities may not be monotone in the many-agent model while they are monotone in the two-agent model.

While we focus on firm heterogeneity, there is a growing literature studying the roles of household heterogeneity in macroeconomics. The reader is refereed to Kaplan and Violante (2018) for an excellent review.

The rest of this paper is organized as follows. Section 2 presents our framework. In Sections 3 and 4, we compute equilibria and then explore comparative statics. Section 5 studies the equilibrium efficiency. Section 6 compares outcomes of two models: one with credit constraints and another with exogenous borrowing limits. Section 7 concludes. Technical proofs are gathered in Appendices.

2 Framework

We consider a deterministic two-period economy with a finite number of heterogeneous agents. There is a single good (numéraire) which can be consumed or used to produce.

Each agent i has exogenous initial wealth (S_i units of good) at the initial date. To keep the model as simple as possible, we assume that agents just maximize their consumption in the second period.

Agents decide how much good for production and investment in the financial market. On the one hand, if agent i wants to realize her productive project, she buys k_i units of physical capital at the initial date to produce $F(k_i)$ units of good at the second date, where F_i is her production function.

On the other hand, she can invest in a financial asset with real return r. Denote a_i the amount that the agent i invests in the financial asset. She can also borrow and then pay back ra_i in the next period. However, the debtor is required to put her project as collateral in order to borrow: If she does not repay, the creditor can seize the collateral. Due to the lack of commitment (or just because the debtor is not willing to help the creditor take the whole value of the debtor's project), the creditor can only obtain a fraction f_i of the total value of the project. Anticipating the possibility of default, the creditor limits the amount of credit so that the debt repayment will not exceed a fraction f_i of the debtor's project value (see Kiyotaki (1998)).

To sum up, the maximization problem of agent i can be described as follows:

$$(P_i): \quad \pi_i = \max_{(k_i,a_i)} [F_i(k_i) - ra_i] \tag{1a}$$

$$\text{subject to: } 0 \leq k_i \leq S_i + a_i \tag{1b}$$

$$ra_i \leq f_i F_i(k_i). \tag{1c}$$

The better the commitment, the higher value of f_i, the larger the set of feasible allocations of the agent i. Kiyotaki (1998) interprets f_i as the collateral value of investment. In our paper, we call f_i *credit limit* and condition (1c) *credit constraint* or *borrowing constraint* of agent i.

The following table from the Enterprise Surveys panel datasets suggests that collateral constraints matter for the development of firms.[6]

Economy	Proportion of loans requiring collateral (%)	Value of collateral needed for a loan (% of the loan amount)	Percent of firms not needing a loan	Percent of firms whose recent loan application was rejected	Proportion of investments financed internally (%)
All Countries	79.1	205.8	46.4	11.0	71.0
East Asia & Pacific	82.6	238.4	50.7	6.4	77.8
Europe & Central Asia	78.7	191.9	54.3	10.9	72.4
Latin America & Caribbean	71.3	198.5	45.0	3.1	62.7
Middle East & North Africa	77.4	183.0	51.8	10.2	71.1
South Asia	81.1	236.0	44.7	14.4	73.9
Sub-Saharan Africa	85.3	214.8	37.4	15.3	73.9

According to Enterprise Surveys (2018), in the average level, 53.6% of firms need a loan and 79.1% of loans require collateral. Hence, it is natural to make the following assumption.

Assumption 1. $f_i \in [0, 1)\ \forall i$.

Notice that the value of collateral needed for a loan (% of the loan amount) corresponds to $F_i(k_i)/ra_i$ ($= 1/f_i$ if credit constraint is binding).

[6] The Enterprise Surveys (2018) panel datasets are conducted by the World Bank and its partners, and provide a database of firms in 139 countries.

Let us compare our formulation of borrowing constraint with those in the literature. Matsuyama (2007) (Section 2) considers a model with heterogeneous agents, which corresponds to our model with $k_i = 1$, $S_i = w, a_i = 1 - w$. However, different from our setup, investment projects in Matsuyama (2007) are non-divisible.

It should be noticed that constraint (1c) is different from condition (3) in Kiyotaki and Moore (1997). Indeed, Kiyotaki and Moore (1997) assume that the borrower's repayment does not exceed the market value of her land quantity while we assume that the repayment does not exceed the market value of the borrower's project.

Some authors (Buera and Shin, 2013; Moll, 2014) set $k_i \leq \theta w_i$, where $w_i \geq 0$ is the agent i's wealth and interpret that θ measures the degree of credit frictions (credit markets are perfect if $\theta = \infty$ while $\theta = 1$ corresponds to financial autarky, where all capital must be self-financed by entrepreneurs). In our framework, S_i plays a similar role of wealth w_i in Buera and Shin (2013), Moll (2014). Another way to introduce credit constraint is to set $a_i \leq \theta k_i$. This corresponds to constraint (3) in Midrigan and Xu (2014). Other authors (Kocherlakota, 1992; Obiols-Homs, 2011) consider exogenous borrowing limits by imposing a short sales constraint: $a_i \leq B$ for any i. Under these three settings, the asset holding a_i is bounded from above by an upper bound which does not depend on the interest rate r.

In Section 6, we present a model with exogenous borrowing limits

Definition 1. *Let us consider the economy $\mathcal{E}$, characterized by a list of fundamentals:*

$$\mathcal{E} \equiv (F_i, f_i, S_i)_{i=1,\dots,m}.$$

A list $(r, (a_i, k_i)_i)$ is an equilibrium if the following conditions are satisfied:

1. *Agents' optimality: for each i, given r, (a_i, k_i) is a solution of the problem (P_i).*
2. *Financial market clearing: $\sum_i a_i = 0$.*

Under standard specifications, the existence of equilibrium is guaranteed (see, for instance, Bosi, Le Van and Pham (2018)).

Remark 1 (Asset price vs interest rate)**.** *Denote $q = 1/r$, $b_i = -ra_i$. The problem (P_i) is equivalent to the following problem.*

$$(B_i) \qquad \pi_i = \max_{(k_i, b_i)} [F_i(k_i) + b_i] \tag{2a}$$

$$\textit{subject to}: \qquad 0 \leq k_i + qb_i \leq S_i \tag{2b}$$

$$b_i + f_i F_i(k_i) \geq 0. \tag{2c}$$

In this setup, at initial date agent can buy b_i units of the asset with price q, which will deliver b_i units of the consumption good at date 1. Note that the relation between the asset price and the interest rate is represented by $q = 1/r$.

In this setup, if we impose $k_i \geq \kappa b_i$, then this constraint can be interpreted as a collateral constraint in Kubler and Schmedders (2003), Gottardi and Kubler (2015): Agent i can borrow but for each unit of the asset sold short by this agent, she is required to hold κ units of capital.

3 Linear technologies

In this section, we provide equilibrium analysis under the following assumption.

Assumption 2. *Assume that* $F_i(K) = A_iK$ $\forall i$ *and* $A_1 < \ldots < A_m$.

This specification allows us to have a tractable model without losing main economic insights. The case with general concave technologies will be considered in Section 4.

3.1 Partial equilibrium

Before computing equilibrium, we study the individual problem where the interest rate is taken as given. At optimal, we have $k_i = S_i + a_i \geq 0$. So, the problem of agent i is equivalent to

$$\pi_i = \max_{(a_i)} [A_iS_i + a_i(A - r)]$$
$$\text{subject to: } a_i \geq -S_i, \quad a_i(r - f_iA_i) \leq f_iA_iS_i$$

If $r \leq f_iA_i$ (which implies that $r < A_i$ because $f_i < 1$), then there is no solution. Indeed, in this case then the constraint (1c) is always satisfied for any $a_i \geq 0$. Consequently, the agent i may choose $a_i = +\infty$ and $k_i = +\infty$ and have $\pi_i = +\infty$. Therefore, we only consider the case where $r > f_iA_i$. The solution of the agent i's problem is characterized by the following result.

Lemma 1 (Individual problem)**.** *Assume that* $F_i(K) = A_iK$. *Let* $r > 0$ *be given. The solution for agent i's maximization problem is described as follows.*

1. *If* $A_i > r > f_iA_i$, *then agent i borrows from the financial market and the borrowing constraint is binding.*

$$k_i = \frac{r}{r - f_iA_i}S_i, \quad a_i = \frac{f_iA_i}{r - f_iA_i}S_i$$
$$\pi_i = A_ik_i - ra_i = \frac{r(1 - f_i)}{r - f_iA_i}A_iS_i$$

2. *If* $A_i = r$, *then the solutions for the agent's problem include all sets* (k_i, a_i) *such that* $-S_i \leq a_i \leq \frac{f_i}{1-f_i}S_i$ *and* $k_i = a_i + S_i$.

3. *If* $A_i < r$, *then agent i does not produce goods and invest all her initial wealth in the financial market:* $k_i = 0, a_i = -S_i$.

In Lemma 1 we draw the relationship between productivity and identification of the borrower/lender: An agent borrows from the financial market if and only if her productivity is high enough, in the sense that $A_i > r$. Moreover, she borrows the maximum level imposed on her, i.e, the borrowing constraint is binding.

Notice that, in the case $r > f_iA_i$ we have

$$a_i \leq \frac{f_iA_iS_i}{r - f_iA_i}, \quad \pi_i = \frac{r(1 - f_i)}{r - f_iA_i}A_iS_i. \tag{3}$$

Corollary 1. *Assume that $A_i > r > f_iA_i$. At optimum we have*

$$\frac{\partial k_i}{\partial f_i} > 0, \quad \frac{\partial a_i}{\partial f_i} > 0, \quad \frac{\partial \pi_i}{\partial f_i} > 0$$
$$\frac{\partial^2 k_i}{\partial^2 f_i} < 0, \quad \frac{\partial^2 a_i}{\partial^2 f_i} < 0, \quad \frac{\partial^2 \pi_i}{\partial^2 f_i} < 0$$

3.2 Determining general equilibrium

To compute equilibrium, we have to find determine the interest rate (then Lemma 1 allows us to compute agents' allocations). In general equilibrium, the interest rate is endogenous and depends on fundamentals. The key point is to identify agents whose credit constraints are binding. Since $A_1 < A_2 < \cdots < A_m$, if credit constraint of agent n is binding, then so does that of agent $n+1$. Moreover, we notice that $r > \max_i(f_iA_i)$. Hence, we have that:

Lemma 2. *Assume that $A_1 < A_2 < \cdots < A_m$. If $\max_i(f_iA_i) \geq A_n$ and there exists an equilibrium, then $r > A_n$.*

Figuring out all possible equilibria requires several steps. First, we need to investigate the properties of equilibrium interest rate. To simplify the exposition, we introduce additional notations.

$$M \equiv \max_i(f_iA_i), \quad d_n \equiv \sum_{i=n}^{m} \frac{A_nS_i}{A_n - f_iA_i} \ \forall n \geq 1, \quad b_n \equiv \sum_{i=n+1}^{m} \frac{A_nS_i}{A_n - f_iA_i} \ \forall n \geq 1. \quad (4)$$

where we denote, by convention, $\sum_{i=n}^{m} x_i = 0$ if $n > m$. We observe that

$$\frac{S_m}{1 - f_m} = d_m < \cdots < d_{n+1} < b_n < d_n < b_{n-1} < \cdots < b_1 = \sum_{i=2}^{m} \frac{A_1S_i}{A_1 - f_iA_i}. \quad (5)$$

b_n is higher than the aggregate capital demand if the interest rate is higher than A_n while d_n is lower than the aggregate capital demand if the interest rate is lower than A_n. By comparing b_n, d_n with the aggregate capital supply $S = \sum_{i=1}^{m} S_i$, we obtain the following result.

Lemma 3. *Assume that $A_1 < A_2 < \cdots < A_m$. Denote $S \equiv \sum_{i=1}^{m} S_i$ the aggregate capital. Consider an equilibrium.*

1. *If $A_n > M$ and $r > A_n$, then $b_n > S$. Consequently, if $A_n > M$ and $b_n \leq S$, then $r \leq A_n$.*

2. *If $A_n > M$ and $r < A_n$, then $S > d_n$. Consequently, if $A_n > M$ and $S \leq d_n$, then $r \geq A_n$.*

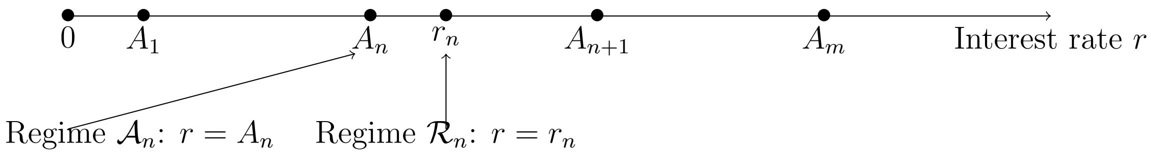
3. *If $r \in (A_n, A_{n+1})$, then $A_{n+1} > M$ and $r = r_n$ (hence $r_n \in (A_n, A_{n+1})$), where r_n is the greatest solution of the equation*

$$\underbrace{\sum_{i=n+1}^{m} \frac{f_iA_i}{r - f_iA_i} S_i}_{\textit{Asset demand}} = \underbrace{\sum_{i=1}^{n} S_i}_{\textit{Asset supply}} \quad \textit{or equivalently} \quad \underbrace{\sum_{i=n+1}^{m} \frac{rS_i}{r - f_iA_i}}_{\textit{Capital demand}} = \underbrace{S}_{\textit{Capital supply}} \quad (6)$$

Proof. See Appendix A.1. □

It should be noticed that the aggregate demand function $f(x) \equiv \sum_{i=n+1}^{m} \frac{xS_i}{x - f_i A_i}$ is not continuous at $f_i A_i$ with $i \geq n+1$. However, it is continuous and decreasing on the interval $(\max_{i \geq n+1}(f_i A_i), \infty)$. Then, the equation $f(x) = S$ has a unique solution, r_n, on such interval.

Point 3 of Lemma 3 also indicates that the equilibrium interest rate must be in the set $\{A_1, \ldots, A_m, r_1, \ldots, r_{m-1}\}$ in the interval $[A_1, A_m]$. We now identify the necessary and sufficient conditions under which $r = A_n$ or $r = r_n$.



The following result provides a necessary and sufficient condition under which the interest rate equals the TFP of agent n.

Lemma 4. *$r = A_n$ if and only if $A_n > M$ and $b_n \leq S \leq d_n$.*

Proof. See Appendix A.1. □

We need condition $A_n > \max_i(f_i A_i)$ because that $r > \max_i(f_i A_i)$. Condition $\sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - f_i A_i} \leq S$ ensures that $r \leq A_n$ while condition $S \leq \sum_{i=n}^{m} \frac{A_n S_i}{A_n - f_i A_i}$ ensures that $r \geq A_n$.

Lemma 5. *$r = r_n \neq A_n$ if and only if one of the following conditions is satisfied:*

1. *$M < A_n < r_n < A_{n+1}$, or equivalently $M < A_n$ and $d_{n+1} < S < b_n$*
2. *$A_n \leq M < r_n < A_{n+1}$, or equivalently $A_n \leq M < r_n$ and $d_{n+1} < S$.*

In any case, we have that $r_n \in [A_n, A_{n+1})$.

Proof. See Appendix A.1. □

We can now categorize all possible structures of the economy. Let us denote $\mathbf{f} \equiv (f_i)_i, \mathbf{A} \equiv (A_i)_i, \mathbf{S} \equiv (S_i)_i$. Then, an economy is characterized by a list $(\mathbf{f}, \mathbf{A}, \mathbf{S})$, and $\mathbf{E}$ denotes the set of all economies $(\mathbf{f}, \mathbf{A}, \mathbf{S})$.

Definition 2 (Categorization of economy)**.** *Consider the economy with linear technologies $\mathcal{E} \equiv (A_i, f_i, S_i)_{i=1,\ldots,m}$.*

For each $n \in \{1, \ldots, m\}$, the regime $\mathcal{A}_n$ is the set of all economies such that conditions in Lemma 4 is satisfied.

For each $n \in \{1, \ldots, m-1\}$, the regime $\mathcal{R}_n$ is the set of all economies such that one of the two conditions in Lemma 5 is satisfied.

Our main result in this section can be stated as follows.

Theorem 1 (Existence, uniqueness and computation of general equilibrium)**.** *We have that* $\{\mathcal{A}_1, \ldots, \mathcal{A}_m, \mathcal{R}_1, \ldots, \mathcal{R}_{m-1}\}$ *is a partition of* $\mathbf{E}$ *in the sense that*

$$\mathbf{E} = \cup_{i=1}^{m} \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i \tag{7a}$$

$$\mathcal{X} \cap \mathcal{Y} = \emptyset \ \forall X, Y \in \{\mathcal{A}_1, \ldots, \mathcal{A}_m, \mathcal{R}_1, \ldots, \mathcal{R}_{m-1}\} \text{ and } X \neq Y. \tag{7b}$$

Consequently, there exists a unique equilibrium. Moreover, the equilibrium interest rate is determined by the following:

$$r = \begin{cases} A_i & \text{in the regime } \mathcal{A}_i. \\ r_i & \text{in the regime } \mathcal{R}_i. \end{cases} \tag{8}$$

Proof. See Appendix A.2. □

The equilibrium existence is guarantied by condition (7a) while the uniqueness is ensured by (7a). According to Lemma 1 and Theorem 1, we can compute equilibrium outcomes:

- In the regime $\mathcal{A}_n$ where $r = A_n$, allocations of all agents are given by

$$k_i = 0, \quad a_i = S_i, \quad \pi_i = A_i k_i - r a_i = r A_i = A_n S_i \quad \forall i \leq n-1$$
$$k_n = \sum_{i=1}^{n} S_i - \sum_{i=n+1}^{m} \frac{f_i A_i S_i}{A_n - f_i A_i}, \quad \pi_n = A_n S_n$$
$$k_i = \frac{r S_i}{r - f_i A_i}, \quad a_i = \frac{f_i A_i S_i}{r - f_i A_i}, \quad \pi_i = \frac{A_n(1-f_i)}{A_n - f_i A_i} A_i S_i \quad \forall i \geq n+1$$

  while the aggregate output is

$$Y = A_n \sum_{i=1}^{n} S_i + \sum_{i=n+1}^{m} \frac{A_n(1-f_i)}{A_n - f_i A_i} A_i S_i. \tag{9}$$

- In the regime $\mathcal{R}_n$, agents $1, \ldots, n$ are lenders and agents $n+1, \ldots, m$ are borrowers. Moreover, we have $k_i = 0, a_i = -S_i, \pi_i = r S_i \ \forall i = 1, \ldots, n$ and

$$k_i = \frac{r}{r - f_i A_i} S_i, \quad a_i = \frac{f_i A_i}{r - f_i A_i} S_i, \quad \pi_i = \frac{A_i r(1-f_i)}{r - f_i A_i} S_i \quad \forall i \geq n+1 \tag{10a}$$

$$Y = \sum_{i=n+1}^{m} A_i S_i \frac{r}{r - f_i A_i}. \tag{10b}$$

Theorem 1 shows how distributions of fundamentals such as initial wealth, productivity and credit limit matter for the equilibrium outcomes. Before exploring equilibrium properties in the general case, let us provide some implications of Theorem 1. Theorem 1 allows us to identify who are borrowers and who are lenders. In model with two agents, there are one borrower and one lender (see Appendix C). In the general case, the number of borrowers (lenders) is endogenously determined. The most productive agent borrows and the least productive agent lends. However, a middle-productive agent becomes a borrower or not depends on other fundamentals, including credit limit. By the way, our findings have an implication in international macroeconomics as we show in the following example.

Example 1 (Implication: capital flows and the role of credit limits)**.** Assume that there are three agents and $A_1 < A_2 < A_3$. This economy can be viewed as a globalized economy with three countries $i = 1, 2, 3$. In this case, country 3 is borrower (capital inflows) and country 1 is lender (capital outflows). Country 2 attracts capital inflows (i.e. $a_2 \geq 0$) if and only if $A_2 \geq f_3 A_3(1 + S_3/S_1)$ (it means that country 2 is sufficiently productive and/or country 3 is not very productive and/or country 3's credit limit is not sufficiently high and/or country 1's initial wealth (S_1) is sufficiently high).

Proof. Agent 2 is borrower if and only if $r \leq A_2$ and $a_2 \geq 0$. According to our computation in Appendix C, $r \leq A_2$ if and only if $A_2 > \max_i(f_3 A_3)$ and $S_1 \geq \frac{f_3 A_3}{A_2 - f_3 A_3} - S_2$. Moreover, in the regime $\mathcal{A}_2$ where $r = A_2$, we have $a_2 = S_1 - \frac{f_3 A_3}{A_2 - f_3 A_3}$. Therefore, agent 2 borrows if and only if $A_2 > \max_i(f_3 A_3)$ and $S_1 \geq \frac{f_3 A_3}{A_2 - f_3 A_3}$, or equivalently $(A_2 - f_3 A_3)S_1 \geq f_3 A_3 S_3$ (because $A_2 > \max_i(f_3 A_3) \Leftrightarrow A_2 > f_3 A_3$). $\square$

Remark 2 (Number of borrowers)**.** *Given $A_1 < \ldots < A_m$, the number of agents whose credit constraints are binding in the economy depends on fundamentals $(S_i, f_i)_i$. The higher level of initial wealth and credit limit s of less productive agents, the higher number of these entrepreneurs in the economy.*

Remark 3 (Economy without credit constraints)**.** *Consider an economy without credit constraints (in the sense that constraint (1c) is removed). Under Assumptions 1, 2, there exists a unique equilibrium determined by*

$$k_i^* = 0, a_i^* = -S_i, \pi_i^* = A_m S_i \ \forall i < m$$
$$k_m^* = S_m, a_m^* = 0, \pi_m^* = A_m S_m.$$

The aggregate output is $Y^* = A_m \sum_i S_i$.

3.3 Equilibrium analysis

This subsection aims to explore equilibrium analysis. There are two main kinds of regime: (i) the regime $\mathcal{A}_n$ where the equilibrium outcomes depend neither on $(A_i)_{i \leq n-1}$ nor $(f_i)_{i \leq n}$ and (ii) the regime $\mathcal{R}_n$ where the equilibrium outcomes do not depend on $(A_i, f_i)_{i \leq n}$. We will present our analysis in each regime.

It is useful to understand the behavior of the interest rate. First, the interest rate is continuous in each parameter. Second, we observe that $r > \max_i(f_i A_i)$. Third, the interest rate has different properties in different regimes. In the regime $\mathcal{A}_n$, we have $r = A_n$.

The more interesting case is in the regime $\mathcal{R}_n$ where $r = r_n$–the greatest solution of equation (6). In this case, for lenders ($i \leq n$), the interest rate r decreases if S_i increases. However, for borrowers ($i \geq n+1$), the interest rate r increases if S_i increases. Indeed, agents i with $i \leq n$ are lenders, so if their initial wealths increase, they lend more and hence the equilibrium interest rate decreases. By contrast, if borrowers' initial wealths increases, their credit limit will be relaxed and they can borrow more which makes the interest rate increase. We also see that, when f_i increases, then r increases. Hence $\sum_{j \geq n+1, j \neq i} \frac{f_j A_j S_j}{r - f_j A_j}$ decreases which implies that $\frac{f_i A_i S_i}{r - f_i A_i}$ (the borrowing amount of agent i) increases, hence, $\frac{r}{f_i A_i}$ decreases. To sum up, we have the following result.

Lemma 6 (Equilibrium interest rate)**.**

1. *In the regime $\mathcal{A}_n$, we have $r = A_n$.*
2. *In the regime $\mathcal{R}_n$, we have*
   (a) *Lenders ($i \leq n$): $\frac{\partial r}{\partial S_i} < 0$.*
   (b) *Borrowers ($i > n$): $\frac{\partial r}{\partial S_i} > 0, \quad \frac{\partial r}{\partial f_i} > 0, \quad \frac{\partial r}{\partial A_i} > 0$.*

Moreover, from the equation (6) determining the interest rate, we can compute, for $j = n+1, \ldots, m$,

$$\frac{\partial r}{\partial f_j} = \frac{\frac{rA_jS_j}{(r-f_jA_j)^2}}{\sum_{i=n+1}^m \frac{f_iA_iS_i}{(r-f_iA_i)^2}} > 0, \text{ and } \sum_{j=n+1}^m \frac{\partial r}{\partial f_j}\frac{f_j}{r} = 1$$

$$\frac{\partial r}{\partial A_j} = \frac{\frac{rf_jS_j}{(r-f_jA_j)^2}}{\sum_{i=n+1}^m \frac{f_iA_iS_i}{(r-f_iA_i)^2}} > 0 \text{ and } \sum_{j=n+1}^m \frac{\partial r}{\partial A_j}\frac{A_j}{r} = 1.$$

Consequently, the elasticity of the interest rate with respect to productivity and credit limit is less than 1.

3.3.1 Effects of productivity

The following result is a direct consequence of our equilibrium computation.

Proposition 1 (Distributional and aggregate effects of productivity)**.**

1. *In the regime $\mathcal{A}_n$, the equilibrium outcomes do not depend on $(A_i)_{i \leq n-1}$ and we have that:*
   (a) *Lenders ($i < n$) and agent n: $\frac{\partial \pi_i}{\partial A_n} > 0 \ \forall i \leq n$*
   (b) *Borrowers ($i > n$): $\frac{\partial \pi_i}{\partial A_i} > 0 > \frac{\partial \pi_i}{\partial A_n}$; $\frac{\partial \pi_i}{\partial A_j} = 0 \ \forall j > n$ and $j \neq i$*
   (c) *Aggregate output : $\frac{\partial Y}{\partial A_j} > 0 \ \forall j > n$, but $\frac{\partial Y}{\partial A_n} = \sum_{i=1}^n S_i - \sum_{i=n+1}^m \frac{(1-f_i)f_iA_i^2S_i}{(A_n-f_iA_i)^2}$ may have any sign.*
2. *In the regime $\mathcal{R}_n$, the equilibrium outcomes do not depend on $(A_i)_{i \leq n}$ and we have that:*
   (a) *Lenders ($i \leq n$): $\frac{\partial \pi_i}{\partial A_j} > 0 \ \forall j > n$.*
   (b) *Borrowers ($i > n$): $\frac{\partial \pi_i}{\partial A_i} > 0$; $\frac{\partial \pi_i}{\partial A_j} < 0 \ \forall j > n$ and $j \neq i$.*
   (c) *Aggregate output : $\frac{\partial Y_n}{\partial A_j} > 0 \ \forall j > n$.*

Comments. At the individual level, the effects of an increase of the TFP A_j ($j \geq n$) on agents' income depend on the type of agents. Indeed, an increase of a TFP of any entrepreneur has a positive impact on its income and the income of all lenders. However, it will decrease the income of other borrowers (entrepreneurs) because they can get less capital.

At the aggregate level, point (2c) of Proposition 1 shows that the output is an increasing function of productivities in the regime $\mathcal{R}_n$. Indeed, if A_i (with $i > n$) increases, then r increases, so $\sum_{j\geq n+1, j\neq i} \frac{f_j A_j S_j}{r - f_j A_j}$ decreases which implies that $\frac{f_i A_i S_i}{r - f_i A_i}$ increases, hence, $\frac{r}{f_i A_i}$ decreases. Therefore, the aggregate output Y is increasing in A_i. This is in line with the existing literature on the impact of productivity on economic growth.

However, an interesting point, following (1c), is that $\partial Y/\partial A_n$ may be negative in the regime $\mathcal{A}_n$ (i.e. when $A_n > \max_i(f_i A_i)$ and $\sum_{i=n+1}^{m} \frac{f_i A_i S_i}{A_n - f_i A_i} \leq \sum_{i=1}^{n} S_i \leq \frac{S_n}{1-f_n} + \sum_{i=n+1}^{m} \frac{f_i A_i S_i}{A_n - f_i A_i}$). The intuition is the following. Any agent $i < n$ does not produce. When A_n increases, agent n absorbs more capital and produces more. However, other producers (who are more productive than agent n) get less capital and they produce less (the capital amount of other producers is $\sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - f_i A_i}$ which is decreasing in A_n). The net effect depends on the level of A_n. Notice that $\partial Y/\partial A_n$ is increasing in A_n. So, when if A_n is high enough, the output is more likely to increase when A_n increase.

Example 2. To illustrate our insight, let us consider an economy having three agents with parameters $S_1 = 10, S_2 = S_3 = 5,\ f_1 = f_2 = f_3 = 0.4,\ A_1 = 1, A_3 = 2$. The economy is in the regime $\mathcal{A}_2$ if and only if $3.2/3 \leq A_2 \leq 1.4$. We can compute that

$$Y = 5(3A_2 + \frac{1.2A_2}{A_2 - 0.8}), \quad \frac{\partial Y}{\partial A_2} = 15 - \frac{4.8}{(A_2 - 0.8)^2}$$

In this regime, $\frac{\partial Y}{\partial A_2} \leq 0$ if $A_2 \in [3.2/3, 1.366]$, and $\frac{\partial Y}{\partial A_2} \geq 0$ if $A_2 \in [1.366, 1.4]$.

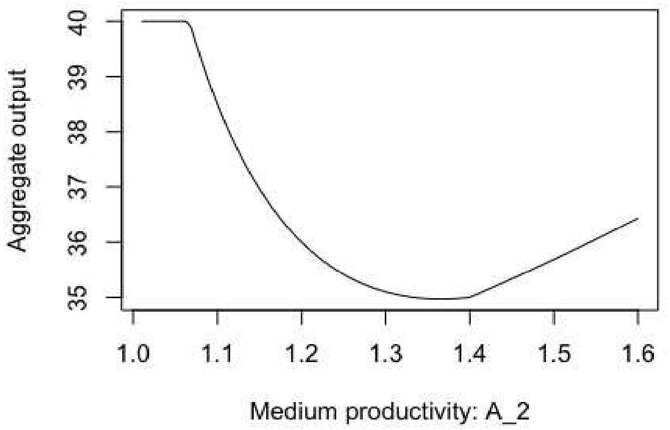


Figure 1: The effect of medium productivity A_2 on the output.

Summing up, increasing productivity of an agent does not necessarily have a positive effect on the economic growth in the short-run if the capital supply is fixed. Our finding complements the literature on the impact of productivity on the economic growth.

Entrepreneurs are important actors in modern economies, specially in developing countries (see Buera et al. (2015) for a review). In terms of implications, our result suggests that if the developing of some firms does not necessarily have a positive impact on the aggregate output. We would be careful with the development of entrepreneurship.

3.3.2 Distributional effects of credit limits

This subsection theoretically studies the distributional effects of credit limits on individual allocations. Numerical simulations will be presented in subsection 3.4.

Proposition 2 (Distributional effects of credit limits)**.**

1. *In the regime $\mathcal{A}_n$, the equilibrium outcomes do not depend on $(f_1, \dots, f_n)$ but depend on $(f_{n+1}, \dots, f_m)$. We also have*
   (a) *Lenders ($i < n$) and agent n ($i = n$): $\frac{\partial \pi_i}{\partial f_j} = 0 \ \forall j > n$.*
   (b) *Borrowers ($i > n$): $\frac{\partial \pi_i}{\partial f_i} > 0, \quad \frac{\partial \pi_i}{\partial f_j} = 0 \ \forall j \neq i$.*
2. *In the regime $\mathcal{R}_n$, the equilibrium outcomes do not depend on $(f_1, \dots, f_n)$ but depend on $(f_{n+1}, \dots, f_m)$.*
   (a) *Lenders ($i \leq n$): $\frac{\partial \pi_i}{\partial f_j} > 0 \ \forall j > n$.*
   (b) *Borrowers ($i \geq n+1$):*

$$\frac{\partial k_j}{\partial f_i} < 0, \quad \frac{\partial a_j}{\partial f_i} < 0, \quad \frac{\partial \pi_j}{\partial f_i} < 0 \ \forall j > n \text{ and } j \neq i \tag{12a}$$

$$\frac{\partial k_i}{\partial f_i} > 0, \quad \frac{\partial a_i}{\partial f_i} > 0, \text{ and} \tag{12b}$$

$$\frac{\partial \pi_i}{\partial f_i} > 0 \Longleftrightarrow A_i - r - (1 - f_i)A_i \frac{\frac{f_i A_i S_i}{(r - f_i A_i)^2}}{\sum_{i=n+1}^{m} \frac{f_i A_i S_i}{(r - f_i A_i)^2}} > 0. \tag{12c}$$

Proof. See Appendix A.3. □

Interpretations. In the regime $\mathcal{A}_n$, the interest rate $r = A_n$ does not depend on credit limits. So, an increase of credit limit of a borrower f_j ($j > n$) does not have any effect on the consumption of other agents. However, it increases the income of this borrower because she can borrow more and get more capital in order to produce more and make more profit.

In the regime $\mathcal{R}_n$, an increase of credit limit of a borrower f_j ($j > n$) have positive effects on the lenders' consumption (point 2.a) but negative effects on other borrowers' consumption (see (12a)).

The surprising point is that the consumption of an borrower does not necessarily increase when her credit limit is relaxed, i.e. $\frac{\partial \pi_i}{\partial f_i}$ may have any sign in the regime $\mathcal{R}_n$ (this is different from the regime $\mathcal{A}_n$). To see the point, by using the fact that $\frac{\partial k_i}{\partial f_i} = \frac{\partial a_i}{\partial f_i}$, we obtain the following decomposions

$$\frac{\partial \pi_i}{\partial f_i} = \underbrace{A_i \frac{\partial k_i}{\partial f_i}}_{\text{Added production}} - \underbrace{(r \frac{\partial a_i}{\partial f_i} + a_i \frac{\partial r}{\partial f_i})}_{\text{Added repayment}} \tag{13a}$$

$$= \underbrace{(A_i - r)\frac{\partial k_i}{\partial f_i}}_{\text{Expansion effect}} - \underbrace{a_i \frac{\partial r}{\partial f_i}}_{\text{Interest effect}}. \tag{13b}$$

When agent i's credit limit is relaxed, this agent can borrow more and hence get more physical capital to produce more (see (12b)). So, both production and repayment of this agent will increase. By consequence, the profit π_i of this agent will increase if her production increases faster than the repayment (this argument is formalized by (13a)).

The decomposition (13b) provides another interpretation. A change in the credit limit of agent i creates two effects. The first one, so-called *expansion effect*, represents the added profit of this entrepreneur given the interest rate (which is lower than the marginal productivity A_i) because agent i can get more capital. The second one, so-called *interest effect*, represents the added repayment created by the increasing of the interest rate. Consequently, the profit π_i increases in the credit limit f_i if the expansion effect domines the interest effect. This second interpretation is closely related to Obiols-Homs (2011) where he decomposes the effect of an exogenous borrowing limit on the welfare of consumers into *quantity effect* and *price effect* which correspond to expansion effect and interest effect respectively.

The interesting question now is to understand whether π_i is increasing in f_i. Since $\frac{\frac{f_i A_i S_i}{(r-f_i A_i)^2}}{\sum_{i=n+1}^{m} \frac{f_i A_i S_i}{(r-f_i A_i)^2}}$ is increasing in the threshold n, condition (12c) implies that $\frac{\partial \pi_i}{\partial f_i}$ is more likely to be positive when n is low (or equivalently the number of borrowers is high). Let us consider a particular case when $n+1=m$. In such a case, we see that $A_m - r - (1-f_m)A_m = f_m A_m - r < 0$. Combining with (12c), we have the following result.

Corollary 2. *In the regime $\mathcal{R}_{m-1}$ (i.e., $n+1=m$), we have $\partial\pi_m/\partial f_m < 0$.*

In the regime $\mathcal{R}_{m-1}$, it is easy to find that the interest rate $r = r_{m-1} = f_m A_m \big(1 + \frac{S_m}{\sum_{i=1}^{m-1} S_i}\big)$ and $k_m = S$. The aggregate output equals $A_m S$ and the consumption of agent m is $\pi_m = A_m k_m - r a_m = A_m(1-f_m)S$, which is decreasing in her credit limit f_m.

In the regime $\mathcal{R}_{m-1}$, the most productive agent borrows all wealths of the less productive ones, and then her production level ($A_m k_m = A_m S$) cannot increase any more. By contrast, the repayment $r a_m$ always increases in the credit limit once the credit constraint binds. Therefore, the most productive agent's consumption is decreasing in her credit limit in this regime.

3.3.3 Effects of credit limits on the aggregate output

A meaningful question is whether developing financial market has positive effects on the economic growth. In our model, relaxing credit limit (i.e., increasing f_i) can be interpreted as reduction of financial friction or improvement of the financial sector.

In our model, we manage to investigate the effects of credit limits on the aggregate production, which help us to understand better the relationship between finance and economic growth.

Proposition 3 (Effects of credit limits on the aggregate output)**.**

1. *In this regime $\mathcal{A}_n$, we have $\frac{\partial Y}{\partial f_i} = \frac{-(A_n - f_i A_i) + (1-f_i)A_i}{(A_n - f_i A_i)^2} > 0 \; \forall i \geq n+1$.*

2. *In the regime $\mathcal{R}_n$, we have*

$$\frac{\partial Y_n}{\partial f_i} = \frac{\partial r}{\partial f_i}\Big(A_i \sum_{j=n+1}^{m} \frac{f_j A_j S_j}{(r - f_j A_j)^2} - \sum_{j=n+1}^{m} \frac{f_j S_j A_j^2}{(r - f_j A_j)^2}\Big). \tag{14}$$

Proof. See Appendix A.4. □

From this and the fact that $A_{n+1} < \cdots < A_m$, we obtain the following result.

Corollary 3. *In the regime $\mathcal{R}_n$, we have $\frac{\partial Y_n}{\partial f_{n+1}} \leq 0 \leq \frac{\partial Y_n}{\partial f_m}$. Consequently, if $n+1 = m$, then $\frac{\partial Y_n}{\partial f_m} = 0$. Moreover, if $m > n+1$, then $\frac{\partial Y_n}{\partial f_{n+1}} < 0 < \frac{\partial Y_n}{\partial f_m}$.*

Our above results lead to several implications which we will discuss as follows.

Non-monotone effects of credit limits

One can imagine that relaxing credit limits would have positive impact on the aggregate output as shown in Khan and Thomas (2013) (section VI. C), Midrigan and Xu (2014) (section II.B), Moll (2014) (Proposition 1), and Catherine et al. (2017). However, the story is more complicated.

1. In the regime $\mathcal{A}_n$, relaxing credit limit of any borrower has a positive impact on the aggregate output.

2. The more interesting case is found in the regime $\mathcal{R}_n$. Indeed, the property $\partial Y_n/\partial f_{n+1} < 0$ if $m > n+1$ in Corollary 3 means that an increase in credit limit of a borrower may have a negative impact on the aggregate output. The intuition is the following. Agent $n+1$ is the least productive borrower. When credit limit of this agent increases, she will borrow more and hence more productive agents can get less funds (see (12a) in Proposition 2). By consequence, the aggregate output is decreasing in f_{n+1}.

3. By contrast the aggregate output increases in the most productive agent's credit limit ($\frac{\partial Y_n}{\partial f_m} \geq 0$). It should be recalled, however, that if f_m increases, consumptions of any other borrowers will decrease (see (12a) in Proposition 2).

Our article is related to the literature on the relationship between entrepreneurship and financial frictions (see Buera et al. (2015) for a review). The entrepreneurs play a crucial role on economic growth. However, many of them face difficulty when finding credit. Indeed, according to Enterprise Surveys (2018), in the average level, 53.6% of firms need a loan and 79.1% of loans require collateral. Improving access to credit of entrepreneurs is expected to boost economic growth. However, our result above suggests that relaxing credit limits of less productive entrepreneurs may have a negative impact on the economic growth.

We can also compare the aggregate outputs in economies with and without credit constraints.

Proposition 4 (With versus without financial frictions)**.** *Consider the economy $\mathcal{E}$ with credit constraints. We have $Y \leq Y^* \equiv A_m \sum_i S_i$. Moreover, $Y = Y^*$ if and only if $f_m A_m \geq A_{m-1}(1 - S_m/S)$.*

Proof. See Appendix A.5 □

Propositions 4 indicates that the aggregate output in the economy with frictions does not exceed than that in the economy without frictions (this is consistent with Midrigan and Xu (2014), Catherine et al. (2017)). However, as we have discussed, the effects of financial frictions are neither monotone nor linear. They depend not only on the distribution of resources in the economy but also on the credit limits of all agents. Notice that the non-monotonicity effect of financial depth on economic growth is supported by Arcand et al. (2015) where they empirically show that financial depth starts having a negative effect on output growth when credit to the private sector reaches 100% of GDP. Our contribution is to provide a general picture to understand whether relaxing credit limit has a positive/negative effect on the aggregate output.

Remark 4 (The aggregate TFP)**.** *Since the aggregate capital is S and the technology of agents is linear, we can define the aggregate TFP, denoted by A, by the following equation $Y = AS$. In the frictionless economy, the aggregate TFP equals A_m the highest TFP of agents. We see that $A \leq A_m$. This is consistent with the result in Catherine et al. (2017).*

Since the aggregate output Y is endogenous, the aggregate TFP is endogenously determined and depends on the distribution of resources as well as the credit limits of agents. Proposition 1 in Moll (2014) shows that, for given wealth shares, the aggregate TFP is always increasing in the leverage ratio of capital to wealth (an index of the quality of credit markets). We complement his results by providing conditions under which the TFP is non monotone in the maximum ratio of debt over the value of the collateral (another index of the quality of credit markets)

Remark 5 (Heterogeneous vs homogeneous credit limits)**.** *We now assume that $f_i = f$ $\forall i$ and let f vary. Intuitively, producers can get more capital.*[7] *Hence, the aggregate output is an increasing function of the homogeneous credit limit f, and so does the aggregate TFP. This result still holds in the general technology case (see Remark 11). This finding is consistent with those in Buera and Shin (2013), Khan and Thomas (2013), Midrigan and Xu (2014), Moll (2014), Catherine et al. (2017).*

This observation indicates that the heterogeneity of credit limit matters for the equilibrium outcome.

Remark 6 (**Number of agents matters**)**.** *When there are two agents, we have $m = n + 1$. Consequently, $\partial Y_1/\partial f_2 = 0$. By combining with point 1 of Proposition 3, we see that $\partial Y/\partial f_2 \geq 0$, i.e., the output is an increasing function of credit limits of both agents.*

[7] Indeed, one can prove that $\partial Y/\partial f > 0$ in the regime $\mathcal{A}_n$. The intuition is that borrowers can borrow more from the least productive entrepreneur (agent n) who is indifferent between two choices: producing or investing (because the interest rate equals her marginal productivity. We can also prove that $\partial Y_n/\partial f = 0$ in the in the regime $\mathcal{R}_n$. The intuition is that when f increases, every borrower can borrow more given the interest rate but the interest rate will also increase. The net effect is zero because the elasticity of the interest rate with respect to the credit limit is one.

When $m > n+1$, there are at least two borrowers. In this case, Corollary 3 shows that $\frac{\partial Y_n}{\partial f_{n+1}} < 0 < \frac{\partial Y_n}{\partial f_m}$. This means that increasing credit limit of different borrowers may have opposite effects on the aggregate output.

Our results suggests that the number of agents matters when we want to observe the impacts of credit limits $(f_i)_i$ in general equilibrium models.

3.4 Numerical simulations

In this subsection, we illustrate our analytical analyses by providing numerical simulations. To get explicit solutions, let us consider an economy with three agents and we focus on the case $A_1 \geq \max_i(f_i A_i)$ so that the interest rate may have any value in the interval $[A_1, A_3]$. Detailed computations are presented in Appendix C.

Effects of credit limit f_3 and TFP A_3

Figure 2 shows the effects of the most productive agent's credit limit f_3 and productivity A_3. Several points related to the effects of f_3 deserve mentions:

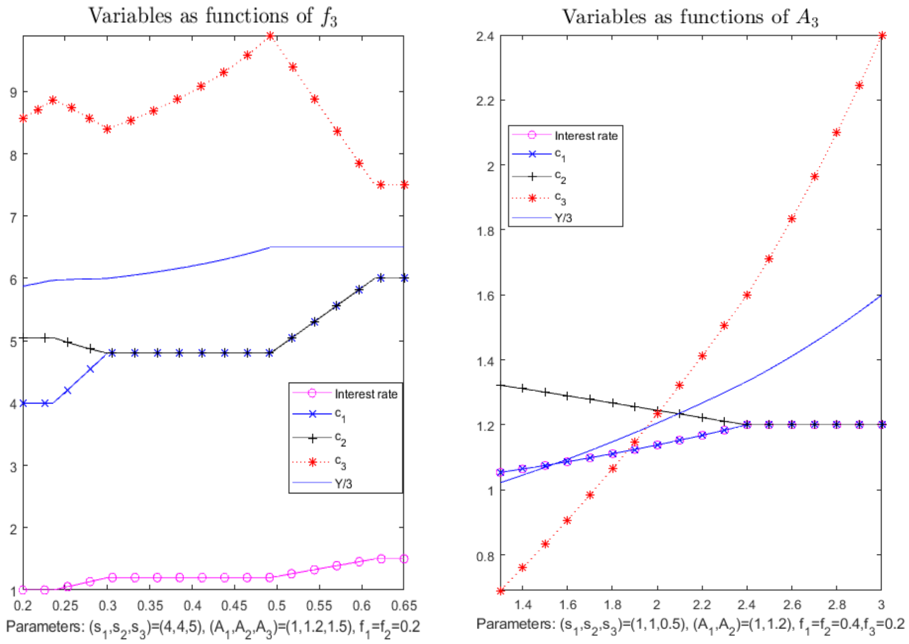


Figure 2: Effects of credit limit f_3 and productivity A_3.

- An increase in f_3 may make the consumption of agent 1 increase but that of agent 2 decrease and then increase. When $r \geq A_2$ (this happens if and only if $S_1 \leq \frac{f_2 S_2}{1-f_2} + \frac{f_3 A_3 S_3}{A_2 - f_3 A_3}$), they will have the same level of consumption $rS_1 = rS_2$. The consumption of agent 3 firstly increases, secondly decreases, thirdly increases, fourthly decreases and lastly is unchanged in f_3.

- An increase in f_3 in our **competitive market structure** decreases income inequality between agents 1 and 2. Moreover, it can increase the sum of consumptions of these two agents as well as the aggregate output (this illustrates our theoretical results in Proposition 3 and Corollary 3).

We now look at the effects of the most productive agent's productivity A_3. While agents 1 and 3's consumption is increasing in A_3, agent 2's consumption is not. Indeed, as shown in Figure 2, the consumption of agent 2 decreases when the most productive agent's productivity A_3 increases so that the interest rate r passes from $A_1 = 1$ to $A_2 = 1.2$.

Effects of the middle-productive agent's credit limit f_2

Figure 3 shows the effects of the agent 2's credit limit f_2 on the equilibrium interest rate, on her consumption, and on the aggregate output. In this example, we set $S_1 = 4$, $S_2 = 4$, $S_3 = 3$, $f_1 = 0.2$, $f_3 = 0.2$, $A_1 = 1$, $A_2 = 1.2$, $A_3 = 1.5$ and we let f_2 vary.

When f_2 varies from 0.1 to 0.6, the interest rate varies from $A_1 = 1$ to $A_2 = 1.2$. The agent 2's consumption and the aggregate output are not monotone functions of f_2. Precisely, they are are increasing in f_2 in the regime $\mathcal{A}_1$ (i.e., $r = A_1$) but decreasing in f_2 in the regime $\mathcal{R}_1$ (i.e., $r = r_1$). These simulations complement our findings in Proposition 3 and Corollary 3.

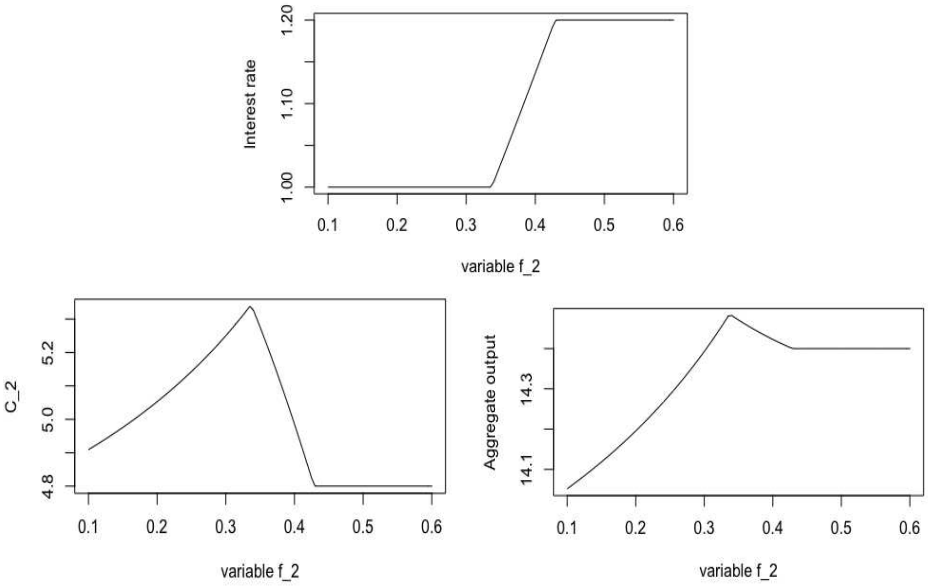


Figure 3: Effects of credit limit f_2.

Gini coefficient

We complement our distributional analyses in subsection 3.3.2 by showing some properties of the Gini coefficient. To see the main point, let us consider a particular case where all agents has the same initial wealth $S_1 = S_2 = S_3 = 4$ but different productivity $A_1 = 1, A_2 = 1.2, A_3 = 1.4$. Assume that $f_1 = f_2 = 0.2$. Figure 4 shows that when f_3 varies from 0.2 to 0.7, the Gini coefficient is not necessarily a monotone function of credit limit f_3.

Let us look at the curve of the Gini coefficient. The first part corresponds to the regime $\mathcal{A}_1$ (i.e., $r = A_1$); the Gini coefficient increases in f_3. The second part corresponds to the regime $\mathcal{R}_1$ (i.e., $r = r_1$); the Gini coefficient decreases. The third part corresponds to the regime $\mathcal{A}_2$ (i.e., $r = A_2$); the Gini coefficient increases. The fourth part corresponds to the regime $\mathcal{R}_2$ (i.e., $r = r_2$); the Gini coefficient decreases. The last part corresponds to the regime $\mathcal{A}_3$ (i.e., $r = A_3$); the Gini coefficient is constant.

Let us provide the intuition. In the regime $\mathcal{A}_n (n = 1, 2)$, the interest rate equals A_n. By consequence, when the credit limit f_3 of agent 3 increases, only consumption of this agent increases while that of other agents rests unchanged. So, the Gini coefficient increases in these regimes. However, in the regime $\mathcal{A}_3$, the equilibrium outcomes do not depend on f_3, so the Gini coefficient is constant. In the regime $\mathcal{R}_n (n = 1, 2)$, the interest rate equals r_n which is increasing in f_3. Since the interest rate increases in f_3, an increase of f_3 benefit lenders (point 2.a of Proposition 2) who are the poorest people. Hence, the Gini coefficient decreases in these regimes.

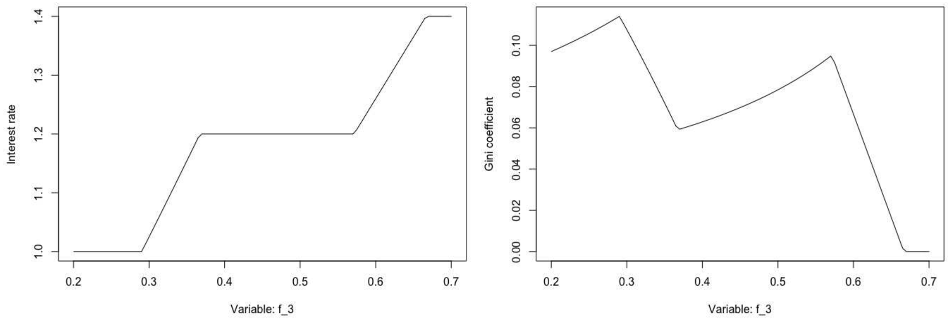


Figure 4: Gini coefficient (model with the same initial wealth).

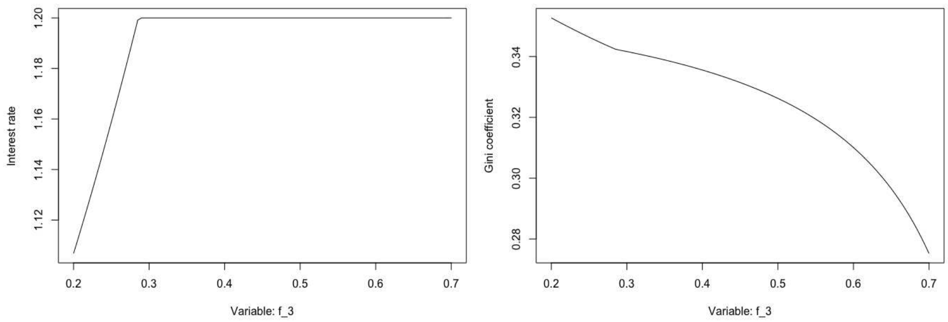


Figure 5: Gini coefficient (when the most productive agent is ex ante the poorest).

When the most productive agent has a very low initial wealth, an increase in her credit limit f_3 would decrease the Gini coefficient. Indeed, consider parameters as in the above example except that $S_1 = 2, S_2 = 6, S_3 = 1$. It means that the middle-productive agent is ex ante the richest and the most productive agent is the poorest at the beginning. In this case, relaxing the credit limit of agent 3 (the most productive entrepreneur) reduces the Gini coefficient. This is illustrated in Figure 5.

4 General concave technologies

This part focuses on the general case where technologies are concave. Given interest rate r, the problem of agent i is the following.

$$\begin{aligned} (P_i) \qquad & \pi = \max_{k_i, a_i} [F_i(k_i) - r a_i] && (15a) \\ \text{subject to :} \qquad & 0 \leq k_i \leq S_i + a_i && (15b) \\ & r a_i \leq f_i F_i(k_i) && (15c) \end{aligned}$$

We make standard conditions on F_i. Different from the linear technology case, Inada condition $F_i'(0) = \infty$ is required.

Assumption 3. *We assume that F_i is in C^1, concave and strictly increasing, $F_i(0) = 0$, $F_i(\infty) = \infty$, $F_i'(0) = \infty$, $F_i'(\infty) = 0$.*

4.1 Partial equilibrium

Given the interest rate r, this subsection studies the optimal allocations of agents.

Denote $x(F_i, r_i)$ the solution of $F_i'(x) = r$. To simplify, when there is no confusion, we write $x(r_i)$ instead of $x(F_i, r_i)$. By definition of $x(r_i)$, we have $F_i''(x)x'(r_i) = 1$.

Denote $G_i(k) \equiv \frac{k}{F_i(k)} - \frac{S_i}{F_i(k)}$. It is easy to prove that $G_i(k)$ is strictly increasing in k. Since $F_i(0) = 0$, we have $G_i(0) = -\infty$. Given r, f_i, F_i, S_i, denote $k(F_i, S_i, f_i, r)$ the unique solution of the equation $rG_i(k) = f_i$.

We summarize properties of $x(F_i, r_i)$ and $k(F_i, S_i, f_i, r)$ which will be useful for our analyses.

Lemma 7. *We have the following properties.*

1. *$x(F_i, r)$ is decreasing in r.*
2. *$k(F_i, S_i, f_i, r)$ is increasing in f_i, S_i and decreasing in r.*
3. *For $A > 0$, we have that $x(AF_i, r) = x(F_i, \frac{r}{A})$ and $k(AF_i, S_i, f_i, r) = k(F_i, S_i, f_i, \frac{r}{A})$ are increasing in A.*

The following result provides a characterization of the solution of problem (P_i).

Lemma 8. *Let Assumption 3 be satisfied and consider the problem (P_i).*

1. *If $rG_i(x(F_i, r)) \geq f_i$, then credit constraint is binding and $k = k(F_i, S_i, f_i, r)$. Moreover, $k = k(F_i, S_i, f_i, r) \leq x(F_i, r)$.*
2. *If $rG_i(x(F_i, r)) < f_i$, then credit constraint is not binding and $k = x(F_i, r)$. In this case, we have $x(F_i, r) = k < k(F_i, S_i, f_i, r)$.*

Agent i borrows if and only if $x(F_i, r) > S_i$ or equivalently $F_i'(S_i) > r$. This happens when her initial wealth is low and/or interest rate is low and/or her productivity is high.

Proof. See Appendix B.1. □

Remark 7. 1. *When $F_i(k) = Ak^\alpha$, then $H_i(r) \equiv rG_i(x(F_i, r)) = \alpha\Big(1 - \big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\big)^{\frac{1}{1-\alpha}}\Big)$ is decreasing in r and $H_i(0) = \alpha$ in this case.*

2. *If $H_i(r) \equiv rG_i(x(F_i, r))$ is decreasing in r and $H_i(0) < f_i$, then borrowing constraint is not binding. So, under Cobb-Douglas production function $F_i(k) = A_i k^{\alpha_i}$, if $\alpha_i < f_i$, then borrowing constraint is not binding.*

In the following, we work under the following assumptions.

Assumption 4. *For any i, the function $rG_i(x(F_i, r))$ is decreasing in r.*

Under these assumptions, $rG_i(x(F_i, r)) \geq f_i$ is equivalent to $r \leq R(F_i, f_i, S_i)$ where $R(F_i, f_i, S_i)$ is a positive function. It means that credit constraint is binding if and only if $r \leq R(F_i, f_i, S_i)$. The level $R(F_i, f_i, S_i)$ is the subjective interest rate of agent below which agent borrows so that borrowing constraint is binding. It is natural to assume that $R(F_i, f_i, S_i)$ is decreasing in f_i and in S_i, and increasing in productivity (in the sense that $R(AF_i, f_i, S_i) < R(BF_i, f_i, S_i)$ for any $A > B$). To sum up, the credit constraint of agent i is more likely to bind if the interest rate, her initial wealth and credit limit are low, and/or her productivity is high.

Remark 8. 1. *With Cobb-Douglas technology $F_i(k_i) = A_i k_i^\alpha$, with $\alpha > f_i$, we have*

$$R(F_i, f_i, S_i) = \alpha A_i S_i^{\alpha-1}\big(1 - \frac{f_i}{\alpha}\big)^{1-\alpha}.$$

2. *With AK technology $F_i(k_i) = A_i k_i$, we have $R(F_i, f_i, S_i) = A_i$.*

4.2 Determining general equilibrium

In this subsection, we will determine equilibrium. The key point is to identify all agents whose credit constraints are binding because agents whose credit constraints are not binding have similar behavior. Recall that borrowing constraint of agent i is binding if and only if $r \leq R(F_i, f_i, S_i)$. Hence, we have to rank $R(F_i, f_i, S_i)$.

Assumption 5. *Assume that $R_1 < R_2 < \cdots < R_m$ where*

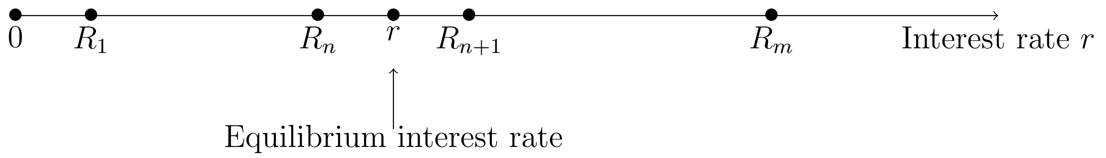
$$R_i \equiv R(F_i, S_i, f_i). \tag{16}$$

Remark 9. *By definition, we have $r_i G_i(x(F_i, R_i)) = f_i$ and $x(F_i, R_i) = k(F_i, S_i, f_i, R_i)$.*

With this ranking, we have an important property: if agent i's credit constraint is binding, then so does that of agent $i+1$. One can interpret that agent $i+1$ needs credit more than agent i.

In a particular case where $f_i = f$, $S_i = S$, $F_i = A_i F$ $\forall i$ with $A_1 < A_2 \ldots < A_m$, agent $i+1$ needs credit mores than agent i because she is more productive than agent i. However, in general case, it should be noticed that $R_i > R_j$ does not necessarily mean that agent i is more productive than agent j.

Since there is at least one agent whose credit constraint is not binding, we have $r > R_1$. So the equilibrium interest rate must be either in the interval $(R_n, R_{n+1}]$ for some index $n < m$ or higher than r_m. We will find conditions under which such property is satisfied.



To simplify our exposition, we introduce additional notations:

$$B_n(d) \equiv \begin{cases} \sum_{i=1}^{n} x(F_i, d) + \sum_{i=n+1}^{m} k(F_i, S_i, f_i, d) & \text{if } n \leq m-1 \\ \sum_{i=1}^{m} x(F_i, d) & \text{if } n = m. \end{cases}$$

Remark 10. $B_n(R_n) > B_{n+1}(R_{n+1}) = B_n(R_{n+1})$.[8]

It should be noticed that $B_n(R_n)$ is the aggregate capital demand when the interest rate equals R_n. We next introduce some notations of the economy without credit constraints.

Definition 3 (Economy without credit constraints)**.** *The unique equilibrium interest rate, denoted by r^*, of the economy without credit constraints is determined by*

$$\sum_i x(F_i, r^*) = S \equiv \sum_i S_i. \tag{17}$$

In such case, the capital of agent i is $k_i^* = x(F_i, r^*)$ and asset holding $a_i^* = x(F_i, r^*) - S_i$. The aggregate output is $Y^* = \sum_i F_i(k_i^*)$.

By the construction of (R_i) and function $B_n(\cdot)$, we obtain a result similar to Lemma 3, which is useful to characterize all possible equilibria.

Lemma 9. *Let Assumptions 3, 4, 5 be satisfied. Consider an equilibrium $((k_i, a_i)_i, r)$ and an index $n \in \{1, \ldots, m-1\}$.*

[8]Indeed, since $R_n < R_{n+1}$, we notice that

$$\begin{aligned} B_n(R_n) &\equiv \sum_{i=1}^{n} x(F_i, R_n) + \sum_{i=n+1}^{m} k(F_i, S_i, f_i, R_n) \\ &> B_n(R_{n+1}) = \sum_{i=1}^{n} x(F_i, R_{n+1}) + \sum_{i=n+1}^{m} k(F_i, S_i, f_i, R_{n+1}) = \sum_{i=1}^{n+1} x(F_i, R_{n+1}) + \sum_{i=n+2}^{m} k(F_i, S_i, f_i, R_{n+1}) \end{aligned}$$

where the last equality follows $k(F_{n+1}, S_{n+1}, f_{n+1}, R_{n+1}) = x(F_{n+1}, R_{n+1})$. Therefore, $B_n(R_n) > B_{n+1}(R_{n+1}) = B_n(R_{n+1})$ $\forall n$.

1. *If $r > R_m$, Lemma 8 implies that credit constraint of any agent is not binding. So, the equilibrium coincides to that of the economy without credit constraints. Therefore, we have $r = r^* > R_m$.*

2. *If $r > R_n$, then credit constraint of any agent $i \leq n$ is not binding. Hence $k_i = x(F_i, r) < x(F_i, R_n)\ \forall i \leq n$. Condition $r > R_n$ also implies that $k(F_i, S_i, f_i, r) < k(F_i, S_i, f_i, R_n)$. Therefore, we have*
$$\sum_i S_i < B_n(R_n).$$

3. *If $r \leq R_{n+1}$, then credit constraint of any agent $i \geq n+1$ is binding, and hence $k_i = k(F_i, S_i, f_i, r) \geq k(F_i, S_i, f_i, R_{n+1})\ \forall i \geq n+1$. Moreover, we have $k_i \geq x(F_i, r) \geq x(F_i, R_{n+1})$ Therefore, we have*
$$\sum_i S_i \geq B_n(R_{n+1}).$$

To avoid a taxonomical exposition, let us focus on Lemma 9's point 2. It indicates that if the interest rate r is higher than the threshold R_n, then the aggregate capital demand associated to the equilibrium interest rate R_n is higher than the aggregate capital supply.

We are now ready to figure out all possible structures of the economy, which will be essential to identify the whole set of equilibria.

Definition 4 (Categorization of economy)**.** *Consider the economy with linear technologies. The regime $\mathcal{R}_m$ is the set of all economies such that $B_m(R_m) > S$ ($\Longleftrightarrow \sum_{i=1}^m x(F_i, R_m) > S \Longleftrightarrow r^* > R_m$).*

For each $n \in \{1, \dots, m-1\}$, the regime $\mathcal{R}_n$ is the set of all economies such that
$$B_n(R_n) > S \geq B_{n+1}(R_{n+1}).$$

Our main result is stated as follows.

Theorem 2 (Existence, uniqueness, and computation of general equilibrium)**.** *Let Assumptions 3, 4, 5 be satisfied. $\{\mathcal{R}_1, \dots, \mathcal{R}_m\}$ is a partition of $\mathbf{E}$ in the sense that*
$$\mathbf{E} = \cup_{i=1}^m \mathcal{R}_i \text{ and } \mathcal{X} \cap \mathcal{Y} = \emptyset\ \forall X, Y \in \{\mathcal{R}_1, \dots, \mathcal{R}_m\} \text{ with } X \neq Y.$$
All possible cases are described as follows.

1. *In the regime $\mathcal{R}_m$, credit constraint of any agent is not binding. In this case, the equilibrium coincides to that of the economy without credit constraints. Agent i borrows ($k_i \geq S_i$) if and only if $F_i'(S_i) \leq r^*$.*

2. *In the regime $\mathcal{R}_n$ (with $1 \leq n \leq m-1$), the equilibrium interest rate, denoted by r_n, and agents' capital are determined by*
$$\sum_{i=1}^n x(F_i, r_n) + \sum_{i=n+1}^m k(F_i, S_i, f_i, r_n) = S \equiv \sum_i S_i \tag{18a}$$
$$k_i = \begin{cases} x(F_i, r_n) & \text{if } i \leq n \\ k(F_i, S_i, f_i, r_n) & \text{if } i \geq n+1. \end{cases} \tag{18b}$$

Notice that $R_n < r_n \leq R_{n+1}$ in this case.

Any agent $i (i \geq n+1)$ borrows and her credit constraint is binding. The credit constraint ofany agent $i \leq n$ is not binding. Moreover, agent $i (i \leq n)$ borrows if and only if $F_i'(S_i) \leq r_n$.

Proof. See Appendix B.2. □

Since $\{\mathcal{R}_1, \dots, \mathcal{R}_m\}$ is a partition of $\mathbf{E}$, the existence and the uniqueness of equilibrium are ensured. This is consistent with Theorem 1 for the linear technology case

The equilibrium interest rate is determined by the following:

$$r = \begin{cases} r^* \text{ in the regime } \mathcal{R}_m \\ r_i \text{ in the regime } \mathcal{R}_i \quad \forall i = 1, \dots, m-1. \end{cases} \tag{19}$$

Notice that the equilibrium interest rate is lower that in the frictionless economy, i.e., $r_n \leq r^* \; \forall n$.

4.3 Equilibrium analysis

In this subsection, we investigate the effects of credit limits (f_i). We focus on the regime $\mathcal{R}_n$ with $n \leq m-1$. Notice that in this regime, the equilibrium outcomes do not depend on $(f_i)_{i=1}^n$.

Conventional notation: The equilibrium in this regime is denoted $(r_n, (k_i^n, a_i^n)_i)$. Recall that $k_i^n = k(F_i, S_i, f_i, r_n) \; \forall i \geq n$ and $k_i^n = x(F_i, r_n) \; \forall i \geq n$. When there is no confusion, we write $x_i(\cdot)$ and $k_i(\cdot)$ instead of $x(F_i, \cdot)$ and $k(F_i, S_i, \cdot, \cdot)$ respectively.

We present useful properties of the equilibrium interest rate r. It is easy to obtain the first-order conditions.

$$F_i'(k_i^n) = r_n \forall i \leq n \tag{20a}$$
$$\lambda_i^n = (1 + \mu_i^n f_i) F_i'(k_i^n) \tag{20b}$$
$$\lambda_i^n = (1 + \mu_i^n) r_n \tag{20c}$$
$$\mu_i^n \geq 0, \text{ and } f_i F_i(k_i^n) - r_n a_i^n = 0. \tag{20d}$$

where λ_i^n and μ_i^n are respectively the multiplier associated to constraints (15b) and (15c). Since $f_i < 1$, we obtain that $F_i'(k_i^n) \geq r_n > f_i F_i'(k_i^n)$.

According to equation (18a), we obtain the following result which is similar to Lemma 6.

Lemma 10. *Consider the regime $\mathcal{R}_n$ with $n \leq m-1$. Then we have that r_n is increasing in $f_j \; \forall j > n$ and*

$$0 = \sum_{i=1}^{n} x_i'(r_n) \frac{\partial r_n}{\partial f_j} + \sum_{i=n+1}^{m} \frac{\partial k_i}{\partial r_n} \frac{\partial r_n}{\partial f_j} + \frac{\partial k_j}{\partial f_j}, \; i.e., \; \frac{\partial r_n}{\partial f_j} = \frac{\frac{\partial k_j}{\partial f_j}}{-\sum\limits_{i=1}^{n} x_i'(r_n) - \sum\limits_{i=n+1}^{m} \frac{\partial k_i}{\partial r_n}} \tag{21}$$

where $\partial k_j / \partial f_j > 0$, $x_i'(r_n) < 0$ and $\partial k_i / \partial r_n < 0$ (following Lemma 7).

4.3.1 Individual allocations

The following result, which is corresponding to Proposition 2 in Section 3, shows the effects of credit limits on agents' consumption.

Proposition 5 (Effects of credit limits on individual allocations)**.** *Assume that we are in the regime $\mathcal{R}_n$. In this case, the equilibrium outcomes do not depend on $(f_1, \dots, f_n)$ but depend on $(f_{n+1}, \dots, f_m)$. Let us consider f_i with $i \geq n+1$.*

1. *For agents ($j \leq n$) whose credit constraints are not binding, we have that:*

$$f_i \uparrow \implies k_j^n \equiv x(F_j, r_n) \downarrow, \quad a_j^n \downarrow \tag{22a}$$

$$\frac{\partial \pi_j^n}{\partial f_i} = -a_j^n \tag{22b}$$

   Consequently, $\partial \pi_j^n / \partial f_i \geq 0$ if and only if agent j is lender ($a_j^n \leq 0$).

2. *For agents ($j > n$) whose credit constraints are binding, we have that:*

$$\forall j > n,\ j \neq i\text{: } f_i \uparrow \implies k_j^n \equiv k(F_j, S_j, f_j, r_n) \downarrow, \quad a_j^n \downarrow, \text{ and } \quad \pi_j^n \downarrow \tag{23a}$$

$$\text{Agent } i\text{: } \begin{cases} \frac{\partial k_i^n}{\partial f_i} > 0, \quad \frac{\partial a_i^n}{\partial f_i} > 0 \\ \frac{\partial \pi_i}{\partial f_i} \text{ may have any sign.} \end{cases} \tag{23b}$$

Proof. See Appendix B.3. □

Interpretations. When f_i increases, both k_i^n and a_i^n increases. So, agent i's consumption $\pi_i^n = F_i(k_i^n) - r_n a_i^n$ may increase or decrease in f_i. When the production $F_i(k_i^n)$ increases faster than the repayment $r_n a_i^n$, agent i's consumption is increasing in f_i. Our argument is formalized by the following decompositions which follow from definition of π_i and noticing that $\frac{\partial k_i^n}{\partial f_i} = \frac{\partial a_i^n}{\partial f_i}$

$$\frac{\partial \pi_i^n}{\partial f_i} = \underbrace{F_i'\big(k_i(F_i, S_i, f_i, r_n)\frac{\partial k_i^n}{\partial f_i}}_{\text{Added production}} - \underbrace{\Big(r_n \frac{\partial a_i^n}{\partial f_i} + a_i^n \frac{\partial r_n}{\partial f_i}\Big)}_{\text{Added repayment}} \tag{24a}$$

$$= \underbrace{\Big(F_i'\big(k_i(F_i, S_i, f_i, r_n) - r_n\Big)\frac{\partial k_i^n}{\partial f_i}}_{\text{Expansion effect}} - \underbrace{a_i^n \frac{\partial r_n}{\partial f_i}}_{\text{Interest effect}} \tag{24b}$$

When $F_i'\big(k_i(F_i, S_i, f_i, r_n) = r_n$, the expansion effect is vanished. So, the net effect is negative, i.e., $\partial \pi_i^n / \partial f_i \leq 0$

4.3.2 Aggregate output

The aggregate output equals

$$Y_n = \sum_{i=1}^{m} F_i(k_i) = \sum_{i=1}^{n} F_i(x(F_i, r_n)) + \sum_{i=n+1}^{m} F_i(k(F_i, S_i, f_i, r_n)).$$

We firstly look at **the effects of credit limits** $(f_i)_i$. In the regime $\mathcal{R}_n$, the equilibrium outcomes do not depend on $(f_i)_{i\leq n}$. For $j > n$, we have

$$\frac{\partial Y_n}{\partial f_j} = \underbrace{F_j'(k(F_j, S_j, f_j, r_n))\Big(\frac{\partial k_j}{\partial r_n}\frac{\partial r_n}{\partial f_j} + \frac{\partial k_j}{\partial f_j}\Big)}_{\text{Added production of agent } j}$$
$$+ \underbrace{\sum_{i=1}^{n} F_i'(x_i(r_n))\underbrace{x_i'(r_n)}_{<0}\frac{\partial r_n}{\partial f_j} + \sum_{i\geq n+1, i\neq j}^{m} F_i'(k(F_i, S_i, f_i, r_n))\underbrace{\frac{\partial k_i}{\partial r_n}}_{<0}\frac{\partial r_n}{\partial f_j}}_{\text{Production losses of other agents}} \tag{25}$$

When f_j increases, the interest rate increases. By consequence, $k(F_j, S_j, f_j, r_n)$ will increase while $k(F_i, S_i, f_i, r_n)$ with $i \neq j$ and $x(F_i, r_n)$ with $i \leq n$ will decrease.

The first term represents the marginal added production of agent j when her credit limit f_j is relaxed while the two last terms represent the marginal production loss of other agents. The aggregate output increases in f_j if the marginal added production exceeds the marginal production loss. This happens under the conditions in the following result which is consistent with Corollary 3.

Corollary 4. *Consider the regime $\mathcal{R}_n$. Denote $\mathcal{I}_n = \arg\max_{i>n}\{F_j'(k_j)\}$. The, we have $\partial Y_n/\partial f_j \geq 0$ $\forall j \in \mathcal{I}_n$. In particular, if there are two agents, then the aggregate output is increasing in f_2.*

The first statement can be easily proved by using Lemma 10 and noticing that $F_i'(x_i(r_n))) = r_n \leq F_j(k_j)$ $\forall j > n$. Our result indicates that relaxing credit limit of agents having the highest marginal productivity, will have a positive effect on the aggregate output.

Remark 11 (Homogeneous credit limit)**.** *As in Remark 5, if we set $f_i = f$ $\forall i$ and let f vary, we can prove that the aggregate output is an increasing function of f. The intuition is very simple: all credit-constrained firms, who have higher marginal productivity, can borrow more from other agents who have lower marginal productivity. A formal proof is presented in Appendix B.4.*

The effects of productivities

Let $F_j(\cdot) = A_j Z_j(\cdot)$, where $Z_j(\cdot)$ is a strictly increasing, concave production function, we investigate the effects of A_j. To simplify the notations and without loss of generality, we can assume that $Z_j = F$ $\forall j$.

According to Lemma 7, we have $x(A_jF, r) = x(F, \frac{r}{A_j})$ and $k(A_jF, S, f, r) = k(F, S, f, \frac{r}{A_j})$. So, we write $x_j(\frac{r}{A_j}), k_j(\frac{r}{A_j})$ instead of $x(F, \frac{r}{A_j}), k(F, S_j, f_j, \frac{r}{A_j})$. With these notations, we can compute, for each $j \leq n$,

$$\frac{\partial Y_n}{\partial A_j} = F\Big(x\big(\frac{r_n}{A_j}\big)\Big) + A_j F'\Big(x_j\big(\frac{r_n}{A_j}\big)\Big) \underbrace{x_j'\big(\frac{r_n}{A_j}\big)}_{<0} \frac{\frac{\partial r_n}{\partial A_j} A_j - r_n}{A_j^2} \tag{26}$$
$$+ \sum_{i=1 \leq n, i \neq j} F_i'(x_i(r_n)) \underbrace{x_i'(r_n)}_{<0} \frac{\partial r_n}{\partial A_j} + \sum_{i \geq n+1}^{m} F_i'(k(F_i, S_i, f_i, r_n)) \underbrace{\frac{\partial k_i}{\partial r_n}}_{<0} \frac{\partial r_n}{\partial A_j}.$$

It is easy to prove that $\frac{\partial r_n}{\partial A_j} > 0$. So, the above decomposition suggests that $\frac{\partial Y_n}{\partial A_j}$ may have any sign.

With versus without financial frictions

With general technologies, we can also compare the equilibrium outcomes of our model with that of the frictionless framework.

Proposition 6 (With versus without financial frictions)**.** *Consider the economy $\mathcal{E}$ with credit constraints. We have $Y \leq Y^*$. Moreover, $Y = Y^*$ if and only if $r^* \geq R_m$.*

Proof. See Appendix B.5. □

5 Efficiency and welfare analysis

This section aims to explore the efficiency of equilibrium by providing a necessary and sufficient condition under which the equilibrium is efficient. First, following Malinvaud (1953), Alvarez and Jermann (2000), Becker, Dubey and Mitra (2014) we introduce some notions of efficiency.

Definition 5. *Consider an economy characterized by production functions and wealths $(F_i, S_i)_{i=1,\dots,m}$.*

1. *(Efficient production plan) A plan $(k_i)_i$ is said to be efficient if (1) it is feasible in the sense that $\sum_i k_i \leq \sum_i S_i$ and (2) there does not exist another feasible production plan $(k_i')_i$ such that $\sum_i F_i(k_i') > \sum_i F_i(k_i)$.*

2. *(Efficient allocation). An allocation $(c_i)_i$ is said to be efficient if (1) it is feasible in the sense that $\sum_i c_i \leq \sum_i F_i(k_i)$ with some feasible plan (k_i) and (2) there does not exist another feasible allocation $(c_i')_i$ which dominates $(c_i)_i$ in the sense of Pareto.*

3. *(Constrained efficient allocation). An allocation $(c_i)_i$ is said to be constrained efficient if (1) it is efficient and (2) $\pi_i \geq A_i S_i$ $\forall i = 1, 2$.*

Let us consider equilibrium of our economy $\mathcal{E} = (F_i, f_i, S_i)$ with credit constraints. One can prove that $(k_i)_i$ is an efficient production plan if and only if it is a solution of the following problem

$$(PP): \quad F(S) \equiv \max_{(k_i)\geq 0} \sum_i F_i(k_i) \tag{27a}$$

$$\text{subject to}: \quad \sum_i k_i \leq S \equiv \sum_i S_i. \tag{27b}$$

The consumption allocation $(\pi_i)_i$ is efficient if and only if $\sum_i \pi_i = F(S)$. It is constrained efficient if and only if $\sum_i \pi_i = F(S)$ and $\pi_i \geq A_i S_i \; \forall i$.

The simplicity of our framework allows us to easily characterize the efficient production plans and Pareto efficient allocations of equilibrium.

Proposition 7 (Efficiency)**.** *Consider the economy with credit constraints and linear technology (resp., concave technology). Let Assumptions 1, 2 (resp., Assumptions 1, 3, 4, 5) be satisfied. The following statements are equivalent:*

1. *The production plan of equilibrium is efficient.*
2. *The consumption allocation of equilibrium is efficient*
3. *The consumption allocation of equilibrium is constrained efficient.*
4. $Y = A_m S$*, or equivalently* $f_m A_m \geq A_{m-1}(1 - S_m/S)$
   (resp., $r^* \geq \max_i r_i = r_m$*, i.e.,* $Y = Y^*$*).*

Proof. This is a consequence of Proposition 4 (resp., Proposition 6). □

Corollary 5. *Assume that there are two agents having Cobb-Douglas technology:* $F_i(k) = A_i k_i^\alpha$ *with* $\alpha > f_i \; \forall i$*. Then, we have*

$$r^* = \bar{r} \equiv \frac{\alpha\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1 + S_2)^{1-\alpha}}, \quad r_i \equiv \alpha A_i S_i^{\alpha-1}\big(1 - \frac{f_i}{\alpha}\big)^{1-\alpha}.$$

By consequence, the equilibrium is efficient if and only if $r^* \geq \max_i r_i$*, i.e.,*

$$\frac{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}{A_i^{\frac{1}{1-\alpha}}} \frac{S_i}{S_1 + S_2} + \frac{f_i}{\alpha} \geq 1 \; \forall i. \tag{28}$$

Intuitions. In the linear technology case, the main insight is clear: the equilibrium is efficient if and only if the TFP and/or credit limit and/or initial wealth of the most productive agent are sufficiently high. It should be noticed that even $f_m = 1$ (the highest level), the equilibrium efficiency still requires conditions on productivity and initial wealth $A_m \geq A_{m-1}(1 - S_m/S)$.

In the general concave technology case, let us provide the intuition of the condition $r^* \geq \max_i r_i = r_m$. Since $r_m = \max_i r_i$, agent m needs credit more than any other agents: her credit constraint is binding for any interest rate lower than r_m. Condition $r^* \geq r_m$ ensures that agent m's credit constraint is not binding, and so do credit constraints of any other agents. By consequence, the equilibrium is efficient. The inverse is also true: when credit constraint of some agent is binding, the efficiency fails.

Since r^* does not depend on credit limits $(f_i)_i$, and r_m is decreasing in f_m, the higher the credit limit f_m, the bigger room for the equilibrium efficiency. This is illustrated by condition (28) in the two-agent economy.

An interesting feature of our model is that when the equilibrium is inefficient (i.e., when $r^* < r_m$), the level of the efficiency, defined by the difference between the output of this equilibrium and that of frictionless economy $Y - Y^*$, is not monotone in credit limits.

Our result is related to Gottardi and Kubler (2015) who consider an exchange economy with complete markets and collateral constraints. Section 3.1 in Gottardi and Kubler (2015) provides a necessary and sufficient condition for *the existence of a Pareto-efficient equilibrium* in two cases: (i) there is no aggregate uncertainty and (ii) consumers have identical CRRA utility. Their conditions are based on agents' endowments and Gottardi and Kubler (2015) require the Lucas tree's dividend in every state to be sufficiently large so that collateral constraints never bind.

Several differences between Gottardi and Kubler (2015) and the present paper should be mentioned.

1. In terms of setting, our model is deterministic and has exogenous wealths while Gottardi and Kubler (2015)'s model has uncertainty and endogenous wealths. However, Gottardi and Kubler (2015) considers an exchange economy while in our paper we focus on heterogeneous producers (any agent can produce by using their own technology).

2. Second, our necessary and sufficient condition is based on all parameters including credit limits and productivities while in results in Section 3.1 of Gottardi and Kubler (2015) credit limits play no role.

3. Third, Theorem 2 in Gottardi and Kubler (2015) only considers the existence of a Pareto-efficient equilibrium while our Proposition 7 characterizes all economies where efficient and/or inefficient equilibria arise.

6 Models with exogenous borrowing limits

This section compares models with credit constraints and with exogenous borrowing limits. We now consider borrowing constraint $a_i \leq \bar{a}_i$ instead of constraint (1c). The problem of agent i now becomes

$$(Q_i) \qquad \pi_i = \max_{(k_i,a_i)} [F_i(k_i) - ra_i] \tag{29a}$$

$$\text{subject to :} \qquad 0 \leq k_i \leq S_i + a_i \tag{29b}$$

$$a_i \leq \bar{a}_i. \tag{29c}$$

Remark 12. *If we replace $a_i \leq \bar{a}_i$ by condition $a_i \leq \theta_i k_i$ which corresponds to constraint (3) in Midrigan and Xu (2014). Then, at optimal, we must have $S_i + a_i = k_i \geq$*

$\frac{a_i}{\theta_i}$, so $a_i \leq \bar{b}_i \equiv \xi_i S_i$ where $\xi_i \equiv \frac{1}{\frac{1}{\theta_i}-1}$, the problem (Q_i) becomes

$$(Q_{i1}) \qquad \pi_i = \max_{(k_i,a_i)} [F_i(k_i) - ra_i] \tag{30a}$$
$$\textit{subject to}: \qquad 0 \leq k_i \leq S_i + a_i \tag{30b}$$
$$a_i \leq \bar{b}_i. \tag{30c}$$

If we replace $a_i \leq \bar{a}_i$ by condition $a_i \leq \theta_i S_i$. Then, the problem (Q_i) becomes

$$(Q_{i2}) \qquad \pi_i = \max_{(k_i,a_i)} [F_i(k_i) - ra_i] \tag{31a}$$
$$\textit{subject to}: \qquad 0 \leq k_i \leq S_i + a_i \tag{31b}$$
$$a_i \leq \bar{d}_i \equiv \theta_i S_i. \tag{31c}$$

We observe that with both setting $a_i \leq \theta_i S_i$ and $a_i \leq \theta_i k_i$, agent i has exogenous borrowing limit. So, we can apply our results in this section.

Proposition 8 (General equilibrium: Finite agent case)**.** *Assume that there are m agents with production function $F_i(k) = A_i k_i$ and $A_1 < A_2 < \cdots < A_m$.*

1. *If $S_1 - \sum_{i=2}^{m} \bar{a}_i > 0$, then $r = A_1$.*
2. *$\bar{a}_m > \sum_{i=1}^{m-1} S_i$, then $r = A_m$.*
3. *For $2 \leq n \leq m-1$:*
   - *(a) If $\sum_{i \geq n} \bar{a}_i > \sum_{i \leq n-1} S_i$ and $\sum_{i \leq n} S_i > \sum_{i \geq n+1} \bar{a}_i$, then $r = A_n$.*
   - *(b) If $\sum_{i \leq n-1} S_i = \sum_{i \geq n} \bar{a}_i$, then any $r \in [A_{n-1}, A_n]$ is an equilibrium interest rate.*

Proof. See Appendix D. □

Proposition 8 computes the equilibrium interest rate in all economies. From this, we can find the whole set of equilibria (see Appendix D for more details). Some points deserve to mention:

1. Different from the model with credit constraint, our model with exogenous borrowing limit may have multiple equilibria (point 3.b). In this case, the aggregate output does not depend on the interest rate r but the distribution of consumptions does. The consumption of lenders (resp., borrowers) is increasing (resp., decreasing) in the interest rate r. Since there are multiple equilibrium interest rate, there are multiple equilibrium consumption distributions, including the Gini coefficient.

2. In any case, the aggregate output the aggregate output is increasing in borrowing limits and productivities. This is different from our model with credit constraint. The reason is that the asset holding a_i in the problem (Q_i) is bounded from above by an exogenous limit while the upper bound of a_i in the model with credit limit is endogenous and depends on the interest rate r.

3. Any economy such that $\bar{a}_m \geq \sum_{i=1}^{m-1} S_i$ is efficient (in the sense in Definition 5). Notice that when $\bar{a}_m = \sum_{i=1}^{m-1} S_i$, the economy is efficient but multiple equilibria arise (point 3.b).

7 Concluding remarks

In our model, we have fully characterized the whole set of equilibria and provided comparative statics. The distributional and aggregate effects of both productivity and credit limit are neither monotone nor linear.

Our paper suggests that the number of agents and the credit heterogeneity in general equilibrium models matter for the equilibrium implications, especially when analyzing the effects of productivities and credit limits on the production and the income distribution.

It would be interesting to extend our analysis in a dynamic framework and investigate the effects of credit limits and the distribution of productivity on the evolution of the distributions of important variables (income, wealth, ...).

Appendices

A Proofs: linear technology case

A.1 Proof of Lemmas 3, 4, 5

Proof of Lemma 3. We present proofs of points 1 and 2. Point 3 is a direct consequence of points 1 and 2, and the fact that $r > \max_i(f_iA_i)$.

1. Since $r > A_i$ for any $i = 1, \dots, n$, Lemma 1 implies that $k_i = 0, a_i = -S_i \; \forall i = 1, \dots, n$. Hence, we have, by using market clearing condition,

$$\sum_{i=1}^{n} S_i = -\sum_{i=1}^{n} a_i = \sum_{i=n+1}^{m} a_i \leq \sum_{i=n+1}^{m} \frac{f_iA_i}{r - f_iA_i} S_i < \sum_{i=n+1}^{m} \frac{f_iA_i}{A_n - f_iA_i} S_i \tag{A.1}$$

   where the first inequality follows (3) while the last inequality follows $r > A_n > \max_i(f_iA_i)$ and the fact that the function $H(r) \equiv \sum_{i=n+1}^{m} \frac{f_iA_i}{r - f_iA_i} S_i$ is decreasing in $(\max_i(f_iA_i), +\infty)$. Notice that this function is not decreasing in the interval $(0, \infty)$.

2. Since $r < A_n$, again Lemma 1 implies that

$$k_i = \frac{r}{r - f_iA_i} S_i, \quad a_i = \frac{f_iA_i}{r - f_iA_i} S_i \quad \forall i \geq n. \tag{A.2}$$

   We have

$$\sum_{i=n}^{m} \frac{A_nS_i}{A_n - f_iA_i} S < \sum_{i=n}^{m} \frac{rS_i}{r - f_iA_i} = \sum_{i=n}^{m} k_i \leq \sum_{i=1}^{m} S_i = S \tag{A.3}$$

   where the first inequality follows $A_n > r > \max_i(f_iA_i)$.

□

Proof of Lemma 4. If $r = A_n$, we have

$$k_i = 0 \; \forall i \leq n - 1 \tag{A.4}$$

$$k_i = \frac{rS_i}{r - f_iA_i} \; \forall i \geq n + 1. \tag{A.5}$$

This implies that $A_n = r > \max_i(f_i A_i)$. Since $0 \leq k_n \leq \frac{rS_i}{r - f_n A_n}$, we have

$$\sum_{i=n+1}^{m} \frac{rS_i}{r - f_i A_i} \leq \sum_i k_i = \sum_{i=n}^{m} k_i \leq \sum_{i=n}^{m} \frac{rS_i}{r - f_i A_i} = \sum_{i=n} \frac{A_n S_i}{A_n - f_i A_i} \tag{A.6}$$

By converse, suppose that $A_n > \max_i(f_i A_i)$ and $\sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - f_i A_i} \leq S \leq \sum_{i=n}^{m} \frac{A_n S_i}{A_n - f_i A_i}$. Applying points 1 and 2, we have $r \geq A_n$ and $r \leq A_n$. Hence $r = A_n$. □

Proof of Lemma 5. Part 1. Assume that $r = r_n \neq A_n$. By definition of r and r_n, we have $\sum_{i=n+1}^{m} \frac{rS_i}{r - f_i A_i} = S$, and $r_n > \max_i(f_i A_i)$.

We will prove that $r = r_n \in (A_n, A_{n+1})$.

If $r \leq A_n$, then $r < A_{n+1}$, and hence $k_i = \frac{rS_i}{r - f_i A_i}$ $\forall i \geq n+1$. Since $\sum_{i=n+1}^{m} \frac{rS_i}{r - f_i A_i} = S = \sum_i k_i$. We have $k_i = 0$ $\forall i \leq n$, and hence $k_n = 0$. This implies that $r \geq A_n$. Therefore, we have $r = A_n$, a contradiction. Thus, we have $r > A_n$.

If $r \geq A_{n+1}$, we have $k_i = 0$ $\forall i \leq n$. Hence $S = \sum_i k_i \leq \sum_{i=n+1}^{m} \frac{rS_i}{r - f_i A_i}$. Since $\sum_{i=n+1}^{m} \frac{rS_i}{r - f_i A_i} = S$, we have $k_i = \frac{rS_i}{r - f_i A_i}$ $\forall i \geq n+1$. Hence $A_{n+1} \geq r$. So, $r = A_{n+1}$. We have just proved that $r \leq A_{n+1}$. By definition of r, we get that $A_{n+1} > \max_i(f_i A_i)$. If $r_n = A_{n+1}$, then applying Lemma 4, we have $\sum_{i=n+2}^{m} \frac{A_{n+1} S_i}{A_{n+1} - f_i A_i} = b_{n+1} \leq S$. However, by definition of r_n, we have $\sum_{i=n+1}^{m} \frac{A_{n+1} S_i}{A_{n+1} - f_i A_i} = S$, contradiction. Therefore, we obtain $r_n < A_{n+1}$.

We have just proved that $r_n \in (A_n, A_{n+1})$. Applying point 2 of Lemma 3, we have $S > d_{n+1}$. There are two cases:

1. $\max_i(f_i A_i) \geq A_n$. In this case, we have $A_n \leq \max_i(f_i A_i) < r_n < A_{n+1}$.
2. $\max_i(f_i A_i) < A_n$. We get $\max_i(f_i A_i) < A_n < r_n < A_{n+1}$. Notice that, in this case, $r_n \in (A_n, A_{n+1})$ is equivalent to $d_{n+1} < S < b_n$.

Part 2. Conversely, assume that (i) $A_n \leq \max_i(f_i A_i) < r_n < A_{n+1}$ or (ii) $\max_i(f_i A_i) < A_n < r_n < A_{n+1}$.

1. If $A_n \leq \max_i(f_i A_i) < r_n < A_{n+1}$. Condition $A_n \leq \max_i(f_i A_i)$ implies that $r > A_n$. Then $k_i = 0$ $\forall i \leq n$, and hence $S = \sum_{i=n+1}^{m} k_i \leq \sum_{i=n+1}^{m} \frac{rS_i}{r - f_i A_i}$

   By definition r_n, we have $S = \sum_{i=n+1}^{m} \frac{r_n S_i}{r_n - f_i A_i}$

   Since the function $f(X) \equiv \sum_{i=n+1}^{m} \frac{XS_i}{X - f_i A_i}$ is decreasing on the interval $(\max_{i \geq n+1}(f_i A_i), \infty)$ and $r, r_n > \max_i(f_i A_i)$, we have $r \leq r_n$. This implies that $r \in (A_n, A_{n+1})$. Therefore, point 3 of Lemma 3 implies that $r = r_n$.

2. If $\max_i(f_i A_i) < A_n$ and $d_{n+1} < S < b_n$. We have $S < d_n$ because $d_n > b_n$. According to point 2 of Lemma 3, we have $r \geq A_n$.

   Condition $S > d_{n+1}$ implies that $S > b_{n+1}$ because $d_{n+1} > b_{n+1}$. According to point 1 of Lemma 3, we have $r \leq A_{n+1}$.

   If $r = A_{n+1}$, then Lemma 4 implies that $S \leq d_{n+1}$. This is a contradiction because $S > d_{d+1}$.

   If $r = A_n$, Lemma 4 implies that $S \in [b_n, d_n]$. However, $S \leq b_n$. Thus, we have $S = b_n = \sum_{i=n+1}^{m} \frac{A_n S_i}{A_n - f_i A_i}$. Since $A_n > \max_i(f_i A_i)$, then $A_n = r_n$, a contradiction.

   Summing up, we have $r \in (A_n, A_{n+1})$. By applying point 3 of Lemma 3, we have $r = r_n$.

□

A.2 Proof of Theorem 1

Equation (8) is a consequence of Lemma 4 and Lemma 5. Let us prove (7a) and (7b).

Denote $M \equiv \max_i(f_i A_i)$. According to Definition 2, we see that:

1. The economy $\mathcal{E} \equiv (F_i, f_i, S_i)_{i=1,\dots,m} \in \mathcal{A}_1$ if and only if $A_1 > M$ and $S > b_1$.
2. $\mathcal{E} \in \mathcal{A}_m$ if and only if $S \leq d_m$.
3. $\mathcal{E} \in \mathcal{A}_n$ with $n \in \{2, \dots, m-1\}$ if and only if $A_n > M$ and $b_n \leq S \leq d_n$.
4. $\mathcal{E} \in \mathcal{R}_n \equiv \mathcal{R}_{n,1} \cup \mathcal{R}_{n,2}$ with $n \in \{1, \dots, m-1\}$ where
   (a) $\mathcal{R}_{n,1}$ is the set of economies such that $A_n > M$ and $d_{n+1} < S < b_n$.
   (b) $\mathcal{R}_{n,2}$ is the set of economies such that $A_{n+1} > M \geq A_n$ and $d_{n+1} < S$.

Equilibrium Existence–Proof of (7a). In order to prove (7a), we verify that $\mathbf{E} \subset \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i$. Let us consider an economy $\mathcal{E}$. There are only two cases.

1. $M < A_1$. In this case, we have $M < A_n \ \forall n$. Therefore, it is easy to see that $\mathbf{E} \in \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_{i,1} \subset \cup_{i=1}^m \mathcal{A}_i \cup \cup_{i=1}^{m-1} \mathcal{R}_i$.
2. There exists $n \in \{1, \dots, m-1\}$ such that $A_{n+1} > M \geq A_n$. There are two sub-cases.
   (a) $S > d_{n+1}$. In this case, $\mathcal{E} \in \mathcal{R}_{n+1,2}$.
   (b) $S \leq d_{n+1}$. Recall that $M < A_{n+1}$. In this case, we will prove that $\mathcal{E} \in \cup_{i=n+1}^m \mathcal{A}_i \cup \cup_{i=n+1}^{m-1} \mathcal{R}_i$. Indeed, since $S \leq d_{n+1}$, there are $2(m-n)-1$ cases.
      i. If there exists $i \in \{n+1, m-1\}$ such that $b_i \leq S \leq d_i$. Then $\mathcal{E} \in \mathcal{A}_i$ (because $A_i > M$).
      ii. If there exists $i \in \{n+1, m-1\}$ such that $d_{i+1} \leq S \leq b_i$. Then $\mathcal{E} \in \mathcal{R}_{i,1}$ (because $A_i > M$).
      iii. Last, if $S \leq d_m$, then $\mathcal{E} \in \mathcal{R}_m$.

Equilibrium Uniqueness–Proof of (7b). There exists a unique equilibrium interest rate if and only if (7b) holds. We have to prove that:

$$\mathcal{A}_n \cap \mathcal{A}_h = \emptyset \ \forall n \neq h \tag{A.7a}$$
$$\mathcal{A}_n \cap \mathcal{R}_{h,1} = \emptyset \ \forall n, h \tag{A.7b}$$
$$\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h \tag{A.7c}$$
$$\mathcal{R}_n \cap \mathcal{R}_h = \emptyset \ \forall n \neq h. \tag{A.7d}$$

Following (5), it is easy to see that the two firsts equalities hold.

We now prove that $\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h$. Suppose that there exists $\mathcal{E} \in \mathcal{A}_n \cap \mathcal{R}_{h,2}$. It means that (1) $A_n > M$ and $b_n \leq S \leq d_n$, and (ii) $A_{h+1} > M \geq A_h$ and $d_{h+1} < S$. From these conditions we get $A_n > A_h$, and hence $n \geq h+1$. Thus, we obtain $S > d_{h+1} \geq d_n \geq S$, a contradiction. Therefore, we have $\mathcal{A}_n \cap \mathcal{R}_{h,2} = \emptyset \ \forall n, h$.

Last, we prove $\mathcal{R}_n \cap \mathcal{R}_h = \emptyset$, or equivalently $\mathcal{R}_{n,i} \cap \mathcal{R}_{h,j} = \emptyset \ \forall i, j \in \{1, 2\}$, $\forall n \neq h$. Without loss of generality, we can assume that $n < h$. It is easy to see that $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,1} = \emptyset$ and $\mathcal{R}_{n,2} \cap \mathcal{R}_{h,2} = \emptyset$. We now prove that $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,2} = \emptyset$ and $\mathcal{R}_{n,2} \cap \mathcal{R}_{h,1} = \emptyset$.

1. Suppose that there exists $\mathcal{E} \in \mathcal{R}_{n,1} \cap \mathcal{R}_{h,2}$. It means that $A_n > M$; $d_{n+1} < S < b_n$; $A_{h+1} > M \geq A_h$; $d_{h+1} < S$. Since $h > n$, then $A_h > A_n > M$. This is a contradiction because $M \geq A_h$. So, we have $\mathcal{R}_{n,1} \cap \mathcal{R}_{h,2} = \emptyset$.

2. Suppose that there exists $\mathcal{E} \in \mathcal{R}_{n,2} \cap \mathcal{R}_{h,1}$. It means that $A_{n+1} > M \geq A_n$; $d_{n+1} < S$;$A_h > M$; $d_{h+1} < S < b_h$.

   Since $h \geq n+1$, we have $b_h \leq b_{n+1} < d_{n+1} < S < b_h$, a contradiction.

Another presentation of equilibrium computation. According to Theorem 1, we obtain the following results.

Lemma 11. *Assume that* $\max(f_iA_i) < A_1 < A_2 < \cdots < A_m$. *Then* $r \in [A_1, A_m]$ *and there are* $(2m-1)$ *different cases determined by the following.*

1. *If* $\frac{f_mS_m}{1-f_m} \geq \sum_{i=1}^{m-1} S_i$ *or equivalently* $\frac{S_m}{1-f_m} \geq S$, *then* $r = A_m$.
2. *For all* $1 < n < m$: *If* $\sum_{i=n+1}^{m} \frac{f_iA_iS_i}{A_n - f_iA_i} \leq \sum_{i=1}^{n} S_i \leq \frac{S_n}{1-f_n} + \sum_{i=n+1}^{m} \frac{f_iA_iS_i}{A_n-f_iA_i}$, *or equivalently* $\sum_{i=n+1}^{m} \frac{A_nS_i}{A_n-f_iA_i} \leq S \leq \sum_{i=n}^{m} \frac{A_nS_i}{A_n-f_iA_i}$, *then* $r = A_n$.
3. *For all* $1 < n < m$: *If* $\sum_{i=n+1}^{m} \frac{f_iA_iS_i}{A_{n+1}-f_iA_i} < \sum_{i=1}^{n} S_i < \sum_{i=n+1}^{m} \frac{f_iA_iS_i}{A_n-f_iA_i}$, *or equivalently* $\sum_{i=n+1}^{m} \frac{A_{n+1}S_i}{A_{n+1}-f_iA_i} < S < \sum_{i=n+1}^{m} \frac{A_nS_i}{A_n-f_iA_i}$ *then* $r = r_n$, *and* $r \in (A_n, A_{n+1})$.
4. *If* $\sum_{i=2}^{m} \frac{f_iA_iS_i}{A_1-f_iA_i} \leq S_1$, *or equivalently* $\sum_{i=2}^{m} \frac{A_iS_i}{A_1-f_iA_i} \leq S$, *then* $r = A_1$.

Lemma 12. *Assume that there exists* $n_0 \in \{1, \ldots, m-1\}$ *such that* $\max_i(f_iA_i) \in [A_{n_0}, A_{n_0+1})$. *Then* $r \in (A_{n_0}, A_m]$ *and there are* $2(m-n_0)$ *possible cases determined by the following.*

1. *If* $\frac{f_mS_m}{1-f_m} \geq \sum_{i=1}^{m-1} S_i$ *or equivalently* $\frac{S_m}{1-f_m} \geq S$, *then* $r = A_m$.
2. *For all* $n_0 + 1 \leq n < m$: *If* $\sum_{i=n+1}^{m} \frac{f_iA_iS_i}{A_n-f_iA_i} \leq \sum_{i=1}^{n} S_i \leq \frac{S_n}{1-f_n} + \sum_{i=n+1}^{m} \frac{f_iA_iS_i}{A_n-f_iA_i}$ *or equivalently* $\sum_{i=n+1}^{m} \frac{A_nS_i}{A_n-f_iA_i} \leq \sum_{i=1}^{m} S_i \leq \sum_{i=n}^{m} \frac{A_nS_i}{A_n-f_iA_i}$, *then* $r = A_n$.
3. *For all* $n_0 + 1 \leq n < m$: *If* $\sum_{i=n+1}^{m} \frac{f_iA_iS_i}{A_{n+1}-f_iA_i} < \sum_{i=1}^{n} S_i < \sum_{i=n+1}^{m} \frac{f_iA_iS_i}{A_n-f_iA_i}$, *or equivalently* $\sum_{i=n+1}^{m} \frac{A_{n+1}S_i}{A_{n+1}-f_iA_i} < S = \sum_{i=1}^{m} S_i < \sum_{i=n+1}^{m} \frac{A_nS_i}{A_n-f_iA_i}$, *then* $r = r_n$, *and* $r \in (A_n, A_{n+1})$.
4. *If* $\sum_{i=n_0+1}^{m} \frac{f_iA_iS_i}{A_{n_0+1}-f_iA_i} < \sum_{i=1}^{n_0} S_i$, *or equivalently* $\sum_{i=n_0+1}^{m} \frac{A_{n_0+1}S_i}{A_{n_0+1}-f_iA_i} < S$, *then* $r < A_{n_0+1}$, *and hence* $A_{n_0} \leq \max_i(f_iA_i) < r < A_{n_0+1}$ *and* $r = r_{n_0}$.

A.3 Proof of Proposition 2

Let $i \geq n+1, j \geq n+1$, and $j \neq i$, we can compute and see that (by noticing that r is increasing in f_i),

$$\frac{\partial k_i}{\partial f_j} = S_i \frac{\partial r}{\partial f_j} \frac{\partial\Big(\dfrac{r}{r - f_iA_i}\Big)}{\partial r} < 0$$

$$\frac{\partial a_i}{\partial f_j} = -\frac{A_if_iS_i}{(r-f_iA_i)^2} \frac{\partial r}{\partial f_j} < 0$$

$$\frac{\partial \pi_i}{\partial f_j} = A_i(1-f_i)S_i \frac{\partial\Big(\dfrac{r}{r - f_iA_i}\Big)}{\partial f_j} < 0.$$

Since $k_i - a_i = S_i$, we can compute that

$$\frac{\partial a_i}{\partial f_i} = \frac{\partial k_i}{\partial f_i} = \frac{S_i}{(r - f_iA_i)^2}\Big(\frac{\partial r}{\partial f_i}(r - f_iA_i) - r(\frac{\partial r}{\partial f_i} - A_j)\Big)$$
$$= \frac{S_i}{(r - f_iA_i)^2}A_i\Big(r - f_i\frac{\partial r}{\partial f_i}\Big) > 0$$
$$\text{because } r - f_i\frac{\partial r}{\partial f_i} = r\Big(1 - \frac{\frac{f_iA_iS_i}{(r-f_iA_i)^2}}{\sum_{i=n+1}^m \frac{f_iA_iS_i}{(r-f_iA_i)^2}}\Big) > 0.$$

We now look at $\frac{\partial \pi_i}{\partial f_i}$ with $i \geq n+1$.

$$\frac{1}{A_iS_i}\frac{\partial \pi_i}{\partial f_i} = \frac{\partial\Big(\dfrac{r(1-f_i)}{r - f_iA_i}\Big)}{\partial f_i} = \frac{\big((1-f_i)r'(f_i) - r\big)(r - f_iA_i) - r(1-f_i)\big(r'(f_i) - A_i\big)}{(r - f_iA_i)^2}. \tag{A.8}$$

The numerator is

$$\begin{aligned}
&\big((1-f_i)r'(f_i) - r\big)(r - f_iA_i) - r(1-f_i)\big(r'(f_i) - A_i\big)\\
=&r'(f_i)\Big((1-f_i)(r - f_iA_i) - r(1-f_i)\Big) - r(r - f_iA_i) + rA_i(1-f_i)\\
=&-r'(f_i)(1-f_i)f_iA_i + r(A_i - r).
\end{aligned}$$

We now consider

$$\frac{r(A_i - r) - r'(f_i)(1-f_i)f_iA_i}{r} = A_i - r - (1-f_i)A_i\frac{\frac{f_iA_iS_i}{(r-f_iA_i)^2}}{\sum_{i=n+1}^m \frac{f_iA_iS_i}{(r-f_iA_i)^2}}. \tag{A.9}$$

A.4 Proof of Proposition 3

It is easy to see that $\frac{\partial Y_n}{\partial f_i} = 0\ \forall i \leq n$. For $i \geq n+1$, we have

$$\begin{aligned}
\frac{\partial Y_n}{\partial f_i} &= \sum_{j=n+1}^m A_jS_j\frac{\partial\Big(\dfrac{r}{r - f_jA_j}\Big)}{\partial f_i}\\
&= \sum_{i\neq j=n+1}^m A_jS_j\frac{\partial\Big(\dfrac{r}{r - f_jA_j}\Big)}{\partial r}\frac{\partial r}{\partial f_i} + A_iS_i\frac{r'(f_i)(r - f_iA_i) - r(r'(f_i) - A_i)}{(r - f_iA_i)^2}\\
&= \sum_{i\neq j=n+1}^m A_jS_j\frac{-f_jA_j}{(r - f_jA_j)^2}\frac{\partial r}{\partial f_i} + A_iS_i\frac{-f_iA_ir'(f_i) + rA_i}{(r - f_iA_i)^2}\\
&= \sum_{j=n+1}^m A_jS_j\frac{-f_jA_j}{(r - f_jA_j)^2}\frac{\partial r}{\partial f_i} + \frac{rS_iA_i^2}{(r - f_iA_i)^2}\\
&= \sum_{j=n+1}^m A_jS_j\frac{-f_jA_j}{(r - f_jA_j)^2}\frac{\frac{rA_iS_i}{(r-f_iA_i)^2}}{\Big(\sum_{j=n+1}^m \frac{f_jA_jS_j}{(r-f_jA_j)^2}\Big)} + \frac{rS_iA_i^2}{(r - f_iA_i)^2}\\
&= \frac{\partial r}{\partial f_i}\Big(A_i\sum_{j=n+1}^m \frac{f_jA_jS_j}{(r - f_jA_j)^2} - \sum_{j=n+1}^m \frac{f_jS_jA_j^2}{(r - f_jA_j)^2}\Big).
\end{aligned}$$

A.5 Proof of Proposition 4

We firstly consider the regime $\mathcal{R}_n$ with $n \leq m-1$. In this regime, we have

$$Y = Y_n = \sum_{i=n+1}^{m} \frac{rA_iS_i}{r - f_iA_i} \leq \sum_{i=n+1}^{m} A_m \frac{rS_i}{r - f_iA_i} = A_mS. \tag{A.10}$$

Notice that $Y = A_mS$ if and only if $n+1 = m$.

We now consider the regime $\mathcal{A}_n$ with $n \leq m$. In this regime, we have

$$\begin{aligned} Y &= A_n \sum_{i=1}^{n} S_i + \sum_{i=n+1}^{m} \frac{A_n(1-f_i)A_iS_i}{A_n - f_iA_i} = A_n \sum_{i=1}^{m} S_i + A_n \sum_{i=n+1}^{m} \frac{A_n(A_i - A_n)}{A_n - f_iA_i} \\ &\leq A_nS + (A_m - A_n) \sum_{i=n+1}^{m} \frac{A_n}{A_n - f_iA_i} \leq A_nS + (A_m - A_n)S = A_mS. \end{aligned}$$

where the last inequality is from the condition $\sum_{i=n+1}^{m} \frac{A_nS_i}{A_n - f_iA_i}$ in the regime $\mathcal{A}_n$. It is easy to see that $Y = A_mS$ if and only if either $n+1 > m$ or $n+1 = m$ and $\frac{A_{m-1}}{A_{m-1} - f_mA_m} S_m = S$.

Combining these two cases, we obtain our result.

B Proofs: concave technology case

B.1 Proof of Lemma 8

Assume that $F'(0) = 0$, then at optimum, $k > 0$. The Lagrange function is

$$L = F(k) - ra + \lambda(S + a - k) + \mu(fAk^{\alpha} - ra)$$

It is easy to see that (k, a) is a solution if and only if there exists (λ, μ) such that

$$\begin{aligned} &[k] : (1 + \mu f)F'(k) = \lambda \\ &[a] : (1 + \mu)r = \lambda \\ &\mu_i \geq 0, \text{ and } \mu_i(fF(k) - a) = 0. \end{aligned}$$

These equations imply that:

$$F'(k) = r\frac{1 + \mu}{1 + f\mu} \geq r. \tag{B.1}$$

Denote $x(F, r)$ the solution of $F'(x) = r$. We see that $x(F, r)$ is decreasing in r.

Since F' is decreasing, we have $k \leq x(F, r)$.

Case 1: The credit constraint is binding: $fF(k) = ra$. In this case, (k, a) is the solutions of the following equations:

$$a = k - S \tag{B.2}$$

$$fF(k) = r(k - S), \quad i.e., \quad \frac{f}{r} = \frac{k}{F(k)} - \frac{S}{F(k)}. \tag{B.3}$$

Consider $k/F(k)$. Its derivative is $\frac{F(k) - kF'(k)}{(F(k))^2} \geq 0$ because F is concave. So, $G(k) \equiv \frac{k}{F(k)} - \frac{S}{F(k)}$ is strictly increasing in k. Moreover, $G(0) < f/r$ and $G(\infty) > f/r$ (because $F'(\infty) < 1$). Therefore, there exists a unique solution k of equation (B.3), and this is positive.

We have to now verify that such solution satisfies $k \leq x(F, r)$. Since $G(k) = r$, this condition is equivalent to $G(x(F, r)) \geq f/r$.

Case 2: $fF(k) > ra$. In this case, we have $\mu = 0$, and hence $F'(k) = r$, i.e, $k = x(F, r)$. It remains to check that this value of k_i satisfies the condition: $fF(k) > ra = r(S - k)$, i.e., $f/r > G(x(F, r))$.

In this case, agent borrows (i.e., $a > 0$) if and only if $k > S$ or equivalently $x(F, r) > S$. This means that her wealth is low and/or interest rate is low and/or her productivity is high.

Notice that if $rG(x(F, r)) < (\geq)f$, then $G(x(F, r)) < (\geq)f/r = G(k(F, S, f, r))$, which implies that $x(F, r) < (\geq)k(F, S, f, r)$.

B.2 Proof of Theorem 2

Let us consider an equilibrium. Since there is at least one agent whose credit constraint is not binding, we have $r > R_1$.

- Suppose that $r \in (R_n, R_{n+1}]$. So, credit constraint of any agent $i \geq n + 1$ is binding and that of any agent $i \leq n$ is not binding. Hence, the capital demand is determined by

$$\sum_i k_i = \sum_{i=1}^{n} x(F_i, r) + \sum_{i=n+1}^{m} k(F_i, S_i, f_i, r). \tag{B.4}$$

  In this case, the equilibrium interest rate is determined by

$$\sum_{i=1}^{n} x(F_i, r) + \sum_{i=n+1}^{m} k(F_i, S_i, f_i, r) = S \equiv \sum_i S_i. \tag{B.5}$$

  According to Lemma 7, the left-hand side is decreasing in r, and hence this equation has a unique solution.

  Since $r \in (R_n, R_{n+1}]$, Lemma 7 implies that

$$\sum_{i=1}^{n} x(F_i, R_n) + \sum_{i=n+1}^{m} k(F_i, S_i, f_i, R_n) > \sum_i S_i \geq \sum_{i=1}^{n} x(F_i, R_{n+1}) + \sum_{i=n+1}^{m} k(F_i, S_i, f_i, R_{n+1}).$$

  Conversely, if this condition holds, using Lemma 7 we can easily prove that $r \in (R_n, R_{n+1}]$. Indeed, if $r > R_{n+1}$, then point 2 of Lemma 9 implies that $S < B_{n+1}(R_{n+1})$. This contradicts to $S \geq B_{n+1}(R_{n+1})$. If $r \leq R_n$, then point 3 of Lemma 9 implies that $S \geq B_{n-1}(R_n) = B_n(R_n)$. This contradicts to $S < B_n(R_n)$. Therefore, we obtain $r \in (R_n, R_{n+1}]$.

- We now suppose that $r^* > R_m$. We will prove that credit constraint of any agent is not binding. Suppose that $\mathcal{B} = \{i \in \{1, \ldots, m\} : (1c) \text{ is binding}\} \neq \emptyset$. Let $n : 1 \leq n \leq m-1$ be the highest element in $\mathcal{B}$, i.e., credit constraint of any agent $i \geq n+1$ is binding while that of any agent $i \leq n$ is not. We have $r \in (R_n, R_{n+1}]$. So, $k(F_i, S_i, f_i, r) \geq k(F_i, S_i, f_i, R_n + 1) > k(F_i, S_i, f_i, R_m)$ and $x(F_i, r) \geq x(F_i, R_{n+1}) \geq x(F_i, R_m)$. Hence, we get that

$$\sum_i S_i = \sum_{i=1}^{n} x(F_i, r) + \sum_{i=n+1}^{m} k(F_i, S_i, f_i, r) \geq \sum_{i=1}^{m} x(F_i, R_m). \tag{B.6}$$

However, by definition of r^*, we have

$$\sum_i S_i = \sum_{i=1}^m x(F_i, r^*) < \sum_{i=1}^m x(F_i, R_m). \tag{B.7}$$

This is a contradiction.

We now prove that $r_n \leq r^*$ $\forall n \leq m-1$. Indeed, in the regime $\mathcal{R}_n$, for any $i \geq n+1$, agent i's credit constraint is binding. Hence, Lemma 8 follows that $k(F_i, S_i, f_i, r_n) \leq x(F_i, r_n)$ $\forall i \geq n+1$. Consequently, we get that

$$\sum_{i=1}^m x(F_i, r^*) = S = \sum_{i=1}^n x(F_i, r_n) + \sum_{i=n+1}^m k(F_i, S_i, f_i, r_n) \leq \sum_{i=1}^m x(F_i, r_n)$$

which implies that $r^* \geq r_n$.

B.3 Proof of Proposition 5

For $i \leq n$, the agent i's consumption is equal to $F_i(x(F_i, r_n))$ while for $i > n$ the agent i's consumption is $\pi_i F_i(k(F_i, S_i, f_i, r_n)) - r_n a_i = (1-f_i)F_i(k_i(F_i, S_i, f_i, r_n))$.

1. For $j \leq n$, when f_i increases, r_n increases. According to Lemma 10, $k_j = x(F_j, r_n)$ and $a_j = k_j - S_j$ will decrease. It is easy to compute that

$$\begin{aligned}\frac{\partial \pi_j}{\partial f_i} &= \frac{\partial\Big((F_j'(x_j(r_n)) - r_n\big(x_j(r_n) - S_j\big)\Big)}{\partial f_i} \\ &= \big(F_j'(x_j(r_n)) - r_n\big)x_j'(r_n) - (x_j(r_n - S_j)) = -a_j.\end{aligned}$$

2. We now take $j > n$ and $i \neq j$. When f_i increases, r_n increases. According to Lemma 10, $k_j = x(F_j, r_n)$ and $a_j = k_j - S_j$ will decrease. However, π_j will decrease because

$$\frac{\partial \pi_j}{\partial f_i} = (1-f_j)F_j'\big(k_j(F_j, S_j, f_j, r_n)\big)\frac{\partial k_j}{\partial r_n}\frac{\partial r_n}{\partial f_i} < 0.$$

We now look at the allocation of agent i. Following (21), we have

$$\frac{\partial k_i}{\partial r_n}\frac{\partial r_n}{\partial f_i} + \frac{\partial k_i}{\partial f_i} = -\Big(\sum_{i=1}^n x_i'(r_n) + \sum_{i \geq n+1, i \neq j} \frac{\partial k_i}{\partial r_n}\Big)\frac{\partial r_n}{\partial f_j} > 0.$$

It means that $k_i = k(F_i, S_i, f_i, r_n)$, and hence $a_i = k_i - S_i$ are increasing in f_i. We can also compute that

$$\begin{aligned}\frac{\partial \pi_i}{\partial f_i} &= (1-f_i)F_i'\big(k_i(F_i, S_i, f_i, r_n)\big)\Big(\frac{\partial k_i}{\partial r_n}\frac{\partial r_n}{\partial f_i} + \frac{\partial k_i}{\partial f_i}\Big) - F_i\big(k_i(F_i, S_i, f_i, r_n)\big) \\ \frac{\partial \pi_i}{\partial f_i} &= F_i'\big(k_i(F_i, S_i, f_i, r_n)\frac{\partial k_i}{\partial f_i} - r\frac{\partial a_i}{\partial f_i} - a_i\frac{\partial r_n}{\partial f_i} \\ &= \Big(F_i'\big(k_i(F_i, S_i, f_i, r_n) - r\Big)\frac{\partial k_i}{\partial f_i} - a_i\frac{\partial r_n}{\partial f_i}.\end{aligned}$$

B.4 Proof of Remark 11 (homogeneous credit limit)

Notice that $\frac{\partial k_i^n}{\partial f} = \frac{\partial k_i}{\partial r_n}\frac{\partial r_n}{\partial f} + \frac{\partial k_i}{\partial f}$. First, from equation (18a), we have $\sum_{i=1}^n x_i'(r_n)\frac{\partial r_n}{\partial f} + \sum_{i=n+1}^m \frac{\partial k_i^n}{\partial f} = 0$. Since $x_i(\cdot)$ is decreasing for any $i \leq n$ and r_n is increasing in f, we have $\sum_{i=n+1}^m \frac{\partial k_i^n}{\partial f} > 0$.

We now claim that $\frac{\partial k_i^n}{\partial f} = \left(\frac{\partial k_i}{\partial r_n}\frac{\partial r_n}{\partial f} + \frac{\partial k_i}{\partial f}\right) > 0 \; \forall i \geq n+1$, i.e., any credit-constrained agent gets more capital. Taking the derivative with respect to f of both sides of the equation $fF_i(k_i^n) - r_n(k_i^n - S_i)$, we have

$$F_i(k_i^n) + fF_i'(k_i^n)\frac{\partial k_i^n}{\partial f} = \frac{\partial r_n}{\partial f}(k_i^n - S_i) + r_n\frac{\partial k_i^n}{\partial f} \tag{B.8}$$

$$\text{i.e.,} \quad \frac{\partial k_i^n}{\partial f} = \Big(\frac{\partial r_n}{\partial f}\frac{f}{r_n} - 1\Big)\frac{F_i(k_i^n)}{r_n - fF_i'(k_i^n)}. \tag{B.9}$$

By summing with respect to i from $n+1$ to m and noticing that $\sum_{i=n+1}^m \frac{\partial k_i^n}{\partial f} > 0$ and $r_n - fF_i'(k_i^n) > 0 \; \forall i \geq n+1$, we get that $\frac{\partial r_n}{\partial f}\frac{f}{r_n} - 1 \geq 0$. From this and (B.9), we obtain $\frac{\partial k_i^n}{\partial f} > 0 \; \forall i \geq n+1$.

We now observe that

$$\frac{\partial Y_n}{\partial f} = \sum_{j=n+1}^m F_j'(k_i^n)\frac{\partial k_i^n}{\partial f} + \sum_{i=1}^n F_i'(x_i(r_n))x_i'(r_n)\frac{\partial r_n}{\partial f} \tag{B.10}$$

$$\geq r_n\Big(\sum_{j=n+1}^m \frac{\partial k_i^n}{\partial f} + \sum_{i=1}^n x_i'(r_n)\frac{\partial r_n}{\partial f}\Big) = 0. \tag{B.11}$$

B.5 Proof of Proposition 6

The output in the frictionless economy is $Y^* = \sum_i F_i(k_i^*) = \sum_i F_i(x(F_i, r^*))$. Since $F_i'(k_i^*) = r^* \; \forall i$ and $\sum_{i=1}^m k_i^* = S$, we have

$$Y^* \equiv \sum_i F_i(k_i^*) = \max_{(k_i)\geq 0; \sum_{i=1}^m k_i \leq S} \sum_{i=1}^m F_i(k_i). \tag{B.12}$$

By consequence, $Y \leq Y^*$ in equilibrium. Since $(k_i^*)_i$ is the unique solution of the above maximization problem, it is easy to see that $Y = Y^*$ if and only if $r = r^*$.

We now prove that $Y = Y^*$ if and only if $r^* \geq R_m$.

Suppose that $r^* \geq R_m$, we will prove that $r = r^*$ which implies that $Y = Y^*$.

Indeed, if $r^* > R_m$, then the economy is in the regime $\mathcal{R}_m$. Then $r = r^*$, and hence $Y = Y^*$.

If $r^* = R_m$, then $S = \sum_{i=1}^m x(F_i, R_m)$. The economy is in the regime $\mathcal{R}_{m-1}$ and we have

$$S = \sum_{i=1}^{m-1} x(F_i, r) + k(F_m, S_m, f_m, r).$$

If $S = \sum_{i=1}^{m-1} x(F_i, R_m) + k(F_m, S_m, f_m, R_m)$, then we have By definition of R_m, we have $k(F_m, S_m, f_m, R_m) = x(F_m, R_m)$ which in turns implies that

$$\sum_{i=1}^{m-1} x(F_i, r) + k(F_m, S_m, f_m, r) = S = \sum_{i=1}^{m-1} x(F_i, R_m) + k(F_m, S_m, f_m, R_m).$$

Since both functions $x(F_i, \cdot)$ with $i \leq m-1$, and $k(F_m, S_m, f_m, \cdot)$ are decreasing, we get that $r = R_m = r^*$.

C Two- and three-agent economies

Two-agent economy: equilibrium computation

In the case of two agents, the linear form helps us to find closed-form solutions (see Pham and Pham (2018) for more analyses in this case). Let Assumption 1, 2 be satisfied. Assume that $A_2 > A_1$. There are three cases, each having a unique equilibrium and with the productive agent being the borrower in any cases.

1. If $f_2 \leq \frac{A_1}{A_2}\frac{S_1}{S_1+S_2}$, then the credit constraint of agent 2 is binding and there exists a unique equilibrium characterized by:

$$\begin{aligned} \text{Interest rate: } & r = A_1 \\ \text{Physical capital: } & k_2 = \frac{A_1}{A_1 - f_2A_2}S_2, \quad k_1 = -\frac{f_2A_2}{A_1 - f_2A_2}S_2 + S_1 \\ \text{Financial asset: } & a_2 = \frac{f_2A_2}{A_1 - f_2A_2}S_2, \quad a_1 = -\frac{f_2A_2}{A_1 - f_2A_2}S_2; \end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_1S_1 + A_2S_2\frac{A_1(1-f_2)}{A_1 - f_2A_2}, \quad c_2 = A_2S_2\frac{A_1(1-f_2)}{A_1 - f_2A_2}, \quad c_1 = A_1S_1.$$

2. If $\frac{A_1}{A_2}\frac{S_1}{(S_1+S_2)} < f_2 < \frac{S_1}{S_1+S_2}$, then the credit constraint of agent 2 is binding, and there exists a unique equilibrium characterized by:

$$\begin{aligned} \text{Interest rate: } & r = f_2A_2\Big(1 + \frac{S_2}{S_1}\Big), \text{ or equivalently } \frac{f_2A_2}{r - f_2A_2}S_2 = S_1 \\ \text{Physical capital: } & k_2 = S_1 + S_2, \quad k_1 = 0 \\ \text{Financial asset: } & a_2 = S_1, \quad a_1 = -S_1. \end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_2(S_1 + S_2), \quad c_2 = A_2(1 - f_2)(S_1 + S_2), \quad c_1 = f_2A_2(S_1 + S_2).$$

3. If $f_2 \geq \frac{S_1}{S_1+S_2}$, then the credit constraint is not binding, and there exists a unique equilibrium characterized by:

$$\begin{aligned} \text{Interest rate: } & r = A_2 \\ \text{Physical capital: } & k_2 = S_2 + S_1, \quad k_1 = 0 \\ \text{Financial asset: } & a_2 = S_1; \quad a_1 = -S_1 \end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_2(S_1 + S_2), \quad c_2 = A_2S_2, \quad c_1 = A_2S_1.$$

Three-agent economy: equilibrium computation

General results are presented in Theorem 1, Lemma 11 and Lemma 12. In this appendix, we focus on the case $max(f_2A_2, f_3A_3) < A_1$ because in this case the interest rate r may take any value in $[A_1, A_m]$. There are 5 different cases. In any case, we can explicitly compute equilibrium outcomes.

1. If $S_1 \geq \frac{f_3 A_3}{A_1 - f_3 A_3} S_3 + \frac{f_2 A_2}{A_1 - f_2 A_2} S_2$, then there exists a unique equilibrium where agent 1 lends, agents 2 and 3 borrow in the financial market, with:

$$\text{Interest rate:} \quad r = A_1$$
$$\text{Financial assets:} \quad a_1 = -\frac{f_2 A_2}{A_1 - f_2 A_2} S_2 - \frac{f_3 A_3}{A_1 - f_3 A_3} S_3, \quad a_i = \frac{f_i A_i}{A_1 - f_i A_i} S_i, \quad \forall i \in \{2,3\}$$
$$\text{Capital allocation:} \quad k_1 = S_1 - \frac{f_2 A_2}{A_1 - f_2 A_2} S_2 - \frac{f_3 A_3}{A_1 - f_3 A_3} S_3,$$
$$k_i = \frac{A_1}{A_1 - f_i A_i} S_i, \quad \forall i \in \{2,3\}$$

The aggregate output and consumption of each agent:

$$Y = A_1 S_1 + \frac{1 - f_2}{A_1 - f_2 A_2} A_1 A_2 S_2 + \frac{1 - f_3}{A_3 - f_3 A_3} A_1 A_3 S_3$$
$$c_1 = A_1 S_1, \quad \pi_i = \frac{(1 - f_i)}{A_1 - f_i A_i} A_1 A_i S_i \quad \forall i \in \{2,3\}$$

2. If $\frac{f_3 A_3}{A_2 - f_3 A_3} S_3 + \frac{f_2}{1 - f_2} S_2 < S_1 < \frac{f_3 A_3}{A_1 - f_3 A_3} S_3 + \frac{f_2 A_2}{A_1 - f_2 A_2} S_2$, then there exists a unique equilibrium under which agent 1 lends, agents 2 and 3 borrow from the financial market, with:

$$\text{Interest rate:} \quad r \in (A_1, A_2); \quad \frac{f_2 A_2}{r - f_2 A_2} S_2 + \frac{f_3 A_3}{r - f_3 A_3} S_3 = S_1 \tag{C.1a}$$
$$\text{Capital allocation:} \quad k_1 = 0, \quad k_2 = \frac{r}{r - f_2 A_2} S_2, \quad k_3 = \frac{r}{r - f_3 A_3} S_3 \tag{C.1b}$$
$$\text{Financial assets:} \quad a_1 = -S_1, \quad a_2 = \frac{f_2 A_2}{r - f_2 A_2} S_2, \quad a_3 = \frac{f_3 A_3}{r - f_3 A_3} S_3 \tag{C.1c}$$

The aggregate output and consumption of each agent:

$$Y = \frac{r}{r - f_2 A_2} A_2 S_2 + \frac{r}{r - f_3 A_3} A_3 S_3$$
$$c_1 = r S_1, \quad c_2 = \frac{r(1 - f_2)}{r - f_2 A_2} A_2 S_2, \quad c_3 = \frac{r(1 - f_3)}{r - f_3 A_3} A_3 S_3$$

By using (C.1a), we can compute the equilibrium interest rate. Indeed, (C.1a) implies that $r(S_2(r - f_3 A_3) + S_3(r - f_2 A_2)) = S(r - f_2 A_2)(r - f_3 A_3)$, or equivalently

$$S_1 r^2 - r\Big((S_1 + S_2) f_2 A_2 + (S_1 + S_3) f_3 A_3\Big) + S f_2 A_2 f_3 A_3 = 0 \tag{C.2a}$$
$$\text{so, } r = \frac{(S_1 + S_2) f_2 A_2 + (S_1 + S_3) f_3 A_3 + \sqrt{\Delta}}{2 S_1} \tag{C.2b}$$
$$\text{where } \Delta \equiv \big((S_1 + S_2) f_2 A_2 + (S_1 + S_3) f_3 A_3\big)^2 - 4 S_1 S f_2 A_2 f_3 A_3 \tag{C.2c}$$

Moreover, we have $\frac{\partial Y}{\partial f_3} > 0 > \frac{\partial Y}{\partial f_2}$.

3. If $\frac{f_3 A_3}{A_2 - f_3 A_3} S_3 - S_2 \leq S_1 \leq \frac{f_3 A_3}{A_2 - f_3 A_3} S_3 + \frac{f_2}{1 - f_2} S_2$, then there exists a unique equilibrium, under which agent 1 lends, agent 2 borrows in the financial market.

$$\text{Interest rate:} \quad r = A_2$$
$$\text{Capital allocation:} \quad k_1 = 0, \quad k_3 = \frac{A_2}{A_2 - f_3 A_3} S_3, \quad k_2 = S_2 + S_1 - \frac{f_3 A_3}{A_2 - f_3 A_3} S_3$$
$$\text{Financial assets:} \quad a_1 = -S_1, \quad a_3 = \frac{f_3 A_3}{A_2 - f_3 A_3} S_3, \quad a_2 = S_1 - \frac{f_3 A_3}{A_2 - f_3 A_3} S_3$$

The aggregate output and consumption of each agent:

$$Y = \frac{1-f_3}{A_2 - f_3 A_3} A_2 A_3 S_3 + A_2(S_2 + S_1)$$
$$c_1 = A_2 S_1, \quad c_2 = A_2 S_2, \quad c_3 = \frac{(1-f_3)A_2}{A_2 - f_3 A_3} A_3 S_3$$

4. If $\frac{f_3}{1-f_3} S_3 - S_2 < S_1 < \frac{f_3 A_3}{A_2 - f_3 A_3} - S_2$, then there exists a unique equilibrium under which agents 1 and 2 lend, and agent 3 borrows in the financial market, with:

$$\text{Interest rate:} \quad r \in (A_2, A_3) \quad \textit{and} \quad r = f_3 A_3 \Big(1 + \frac{S_3}{S_1 + S_2}\Big)$$
$$\text{Capital allocation:} \quad k_1 = 0, \quad k_2 = 0, \quad k_3 = \frac{r}{r - f_3 A_3} S_3 = \sum_i S_i$$
$$\text{Financial assets:} \quad a_1 = -S_1, \quad a_2 = -S_2, \quad a_3 = \frac{f_3 A_3}{r - f_3 A_3} S_3$$

The aggregate output and consumption of each agent:

$$Y = \frac{r}{r - f_3 A_3} A_3 S_3 = A_3 S$$
$$\pi_i = r S_i \quad \forall i \in \{1, 2\}, \qquad c_3 = \frac{r(1-f_3)}{r - f_3 A_3} A_3 S_3 = (1 - f_3) A_3 S.$$

5. If $S_1 \leq \frac{f_3}{1-f_3} S_3 - S_2$, there exists a unique equilibrium under which agents 1 and 2 lend, and agent 3 borrows from the financial market, with:

$$\text{Interest rate:} \quad r = A_3$$
$$\text{Capital allocation:} \quad k_1 = 0, \quad k_2 = 0, \quad k_3 = S_1 + S_2 + S_3$$
$$\text{Financial asset:} \quad a_1 = -S_1, \quad a_2 = -S_2, \quad a_3 = S_1 + S_2$$

The aggregate output and consumption of each agent are:

$$Y = A_3(S_1 + S_2 + S_3)$$
$$\pi_i = A_3 S_i \quad \forall i \in \{1, 2, 3\}$$

D Proofs: Models with exogenous borrowing limits

To prove Proposition 8, we need several intermediate steps. First, we solve th problem (Q_i). Notice that $a_i \geq -S_i \; \forall i$. At optimal, we must have $k_i = S_i + a_i$. Then, $\pi_i = A_i S_i + (A_i - r) a_i$.

Lemma 13 (Individual problem)**.** *The solution of the problem (Q_i) is given by*

1. *If $A_i < r$, then agent i does not produce goods and invest all her wealth in the financial market: $k_i = 0, a_i = -S_i$.*
2. *If $A_i > r$, then agent i borrows from the financial market and the borrowing constraint is binding: $a_i = \bar{a}_i, k_i = S_i + \bar{a}_i$.*
3. *If $A_i = r$, then the solutions for the agent's problem include all sets (k_i, a_i) such that $-S_i \leq a_i \leq \bar{a}_i$ and $k_i = a_i + S_i$.*

We now investigate the properties of the equilibrium interest rate which is useful to figure out all possible equilibria.

Lemma 14. *1. If $r < A_n$, then $\sum_{i\geq n} \bar{a}_i \leq \sum_{i\leq n-1} S_i$.*

2. If $r > A_n$, then $\sum_{i\leq n} S_i \leq \sum_{i\geq n+1} \bar{a}_i$.

3. If $r \in (A_n, A_{n+1})$, then $\sum_{i\leq n} S_i = \sum_{i\geq n+1} \bar{a}_i$.

4. If $r = A_n$, then $S_n + \sum_{i\geq n} \bar{a}_i \geq \sum_{i\leq n} S_i \geq \sum_{i\geq n+1} \bar{a}_i$.

Proof. Point 1. If $r < A_n$, then, according to Lemma 13, $a_i = \bar{a}_i$, $k_i = S_i + \bar{a}_i$ $\forall i \geq n$. The financial market clearing condition implies that

$$\sum_{i\geq n} \bar{a}_i = \sum_{i\geq n} a_i = -\sum_{i\leq n-1} a_i \leq \sum_{i\leq n-1} S_i$$

Point 2. If $r > A_n$, then, according to Lemma 13, $a_i = -S_i$ $\forall i \leq n$. The financial market clearing condition implies that

$$\sum_{i\leq n} S_i = -\sum_{i\leq n} a_i = \sum_{i\geq n+1} a_i \leq \sum_{i\geq n+1} \bar{a}_i$$

where the last inequality is from the fact that $a_i + S_i \geq 0$ $\forall i$.

Point 3. If $r \in (A_n, A_{n+1})$, then $a_i = \bar{a}_i$, $k_i = S_i + \bar{a}_i$ $\forall i \geq n+1$ and $a_i = -S_i, k_i = 0$ $\forall i \leq n$. By using market clearing condition, we have $\sum_{i\leq n} S_i = \sum_{i\geq n+1} \bar{a}_i$.

Point 4: If $r = A_n$, we have

$$a_i = -S_i, \qquad k_i = 0 \quad \forall i = 1, \dots, n-1 \tag{D.1}$$

$$a_i = \bar{a}_i, \quad k_i = \bar{a}_i + S_i \quad \forall i = n+1, \dots, m \tag{D.2}$$

$$a_n = \sum_{i=1}^{n-1} S_i - \sum_{i=n+1}^{m} \bar{a}_i, \quad k_n = \sum_{i=1}^{n} S_i - \sum_{i=n+1}^{m} \bar{a}_i. \tag{D.3}$$

Since $a_n \leq \bar{a}_n$, we have $\sum_{i\geq n} \bar{a}_i \geq \sum_{i\leq n-1} S_i$. Condition $k_n \geq 0$ implies that $\sum_{i\leq n} S_i \geq \sum_{i\geq n+1} \bar{a}_i$. □

Proof of Proposition 8. Since $\sum_i a_i = 0$, we have $r \in [A_1, A_m]$.

Point 1. If $r = A_1$, then $r < A_i$ $\forall i \geq 2$ which implies that $a_i = \bar{a}_i$ and $k_i = S_i + \bar{a}_i$. By using market clearing condition, we have $a_1 = -\sum_{i=2}^m \bar{a}_i$, and hence $k_1 = S_1 - \sum_{i=2}^m \bar{a}_i$. Therefore, we need condition $S_1 - \sum_{i=2}^m \bar{a}_i > 0$. Conversely, if $S_1 - \sum_{i=2}^m \bar{a}_i > 0$, point 2 of Lemma 14 implies that $r = A_1$.

Point 2. If $\bar{a}_m > \sum_{i=1}^{m-1} S_i$, point 1 of Lemma 14 implies that $r = A_m$.

Point 3 is consequence of Lemma 14. □

By combining Proposition 8 and Lemma 14, we can compute equilibrium allocations and the aggregate output.

1. If $S_1 - \sum_{i=2}^m \bar{a}_i > 0$, then $r = A_1$. In this case, the equilibrium allocations and aggregate output are

$$a_i = \bar{a}_i, \; k_i = \bar{a}_i + S_i \; \forall i = 2, \dots, m, \quad a_1 = -\sum_{i=2}^m \bar{a}_i, \; k_1 = S_1 - \sum_{i=2}^m \bar{a}_i \tag{D.4a}$$

$$Y = \sum_i A_i k_i = A_1(S_1 - \sum_{i=2}^m \bar{a}_i) + \sum_{i=2}^m A_i(\bar{a}_i + S_i). \tag{D.4b}$$

2. $\bar{a}_m > \sum_{i=1}^{m-1} S_i$, then $r = A_m$. In this case, the equilibrium allocations and aggregate output are

$$a_i = -S_i,\ k_i = 0\ \forall i = 1, \ldots, m-1, \quad a_m = \sum_{i=1}^{m-1} S_i,\ k_m = \sum_{i=1}^{m} S_i \tag{D.5a}$$

$$Y = \sum_i A_i k_i = A_m \sum_{i=1}^{m} S_i. \tag{D.5b}$$

3. For $2 \leq n \leq m-1$:

   (a) If $\sum_{i \geq n} \bar{a}_i > \sum_{i \leq n-1} S_i$ and $\sum_{i \leq n} S_i > \sum_{i \geq n+1} \bar{a}_i$, then $r = A_n$. In this case, the equilibrium allocations are

$$a_i = -S_i,\ k_i = 0\ \forall i < n, \quad a_i = \bar{a}_i,\ k_i = \bar{a}_i + S_i\ \forall i > n \tag{D.6a}$$

$$a_n = \sum_{i=1}^{n-1} S_i - \sum_{i=n+1}^{m} \bar{a}_i, \quad k_n = \sum_{i=1}^{n} S_i - \sum_{i=n+1}^{m} \bar{a}_i, \tag{D.6b}$$

$$Y = \sum_i A_i k_i = A_n(\sum_{i=1}^{n} S_i - \sum_{i=n+1}^{m} \bar{a}_i) + \sum_{i=n+1}^{m} A_i(\bar{a}_i + S_i) \tag{D.6c}$$

   Condition $\sum_{i \geq n} \bar{a}_i > \sum_{i \leq n-1} S_i$ ensures that $a_n < \bar{a}_n$ while condition $\sum_{i \leq n} S_i > \sum_{i \geq n+1} \bar{a}_i$ ensures that $k_n > 0$.

   (b) If $\sum_{i \leq n-1} S_i = \sum_{i \geq n} \bar{a}_i$, then any $r \in [A_{n-1}, A_n]$ is an equilibrium interest rate. If $r \in (A_{n-1}, A_n)$, the equilibrium allocations and aggregate output are

$$a_i = -S_i, \qquad k_i = 0, \quad \pi_i = rS_i \quad \forall i = 1, \ldots, n-1 \tag{D.7a}$$

$$a_i = \bar{a}_i, \quad k_i = \bar{a}_i + S_i, \quad \pi_i = A_i(\bar{a}_i + S_i) - r\bar{a}_i \quad \forall i = n, \ldots, m \tag{D.7b}$$

$$Y = \sum_i A_i k_i = \sum_{i=n}^{m} A_i(\bar{a}_i + S_i) \tag{D.7c}$$

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